This assignment is optional. Each question is worth a certain number of points. If you solve the problem correctly, you will get these points added to your assignments average. Partial solution of a problem may lead to zero points for this problem (in particular, True/False guesses with no explanation).

**Problem 1.** (2 points)

Let $C$ be the Hamming code defined by the matrix by the check matrix
\[
\begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1
\end{pmatrix}.
\]

(a) Is 010101 a word of $C$?

(b) Is 101010 a word of $C$?

Explain your answers.

**Problem 2.** (2 points)

Let $C_1, C_2 \subseteq V^n$ be two linear codes. Let $C_3$ be the set of sequences in $V^n$ that are words both in $C_1$ and in $C_2$. Let $C_4$ be the set of sequences in $V^n$ that are words in $C_1$ or in $C_2$ (or in both). True or false?

(a) $C_3$ must be a linear code.

(b) $C_4$ must be a linear code.

If a claim is false, provide a counterexample. If a claim is true, explain why.

**Problem 3.** (3 points)

In the following, $a + b$ is the usual bitwise addition mod 2. The operation $a * b$ places $b$ at the end of $a$. For example, $11 * 01 = 1101$ and $00 * 11 = 0011$.

Let $C_1 \subseteq V^n$ be a linear code with check matrix $H_1$, dimension $k_1$, and distance $d$. Let $C_2 \subseteq V^n$ be a linear code with check matrix $H_2$, dimension $k_2$, and distance $d$. (Both codes have the same distance.)

(a) We define the new code $C \subseteq V^{2n}$ as

$C = \{ c_1 * c_2 : c_1 \in C_1 \text{ and } c_2 \in C_2 \}$.

In other words, for every $c_1$ of $C_1$ and $c_2$ of $C_2$, the sequence $c_1 * c_2$ is a word of $C$. Show that $C$ is a linear code, find its dimension and distance, and describe its check matrix.

(b) We define the new code $C^* \subseteq V^{2n}$ as

$C^* = \{ c_1 * (c_1 + c_2) : c_1 \in C_1 \text{ and } c_2 \in C_2 \}$.

Show that $C^*$ is a linear code.
**Problem 4.** (3 points)

True or false?

a) Let \( C \subset V^n \) be a linear code. Let \( C_2 \subset C \) be the set of words of \( C \) that have an even weight. Then \( C_2 \) must be a linear code.

b) Let \( C_3 \subset V^n \) be the set of all sequences in \( V^n \) that have an even weight. Then \( C_3 \) must have a check matrix with a single row.

c) There does not exist a code \( C_4 \subset V^n \) with the following properties. \( C_4 \) consists of \( 0_n \) and of sequences of odd weight. (There is at least one sequence of odd weight and no nonzero sequences of even weight.) \( C_4 \) has a check matrix with at most three rows.

If a claim is false, provide a counterexample. If a claim is true, explain why.

**Problem 5.** (3 points) Very difficult flow problem. Only for students who want a challenge.

Adam decides to quit math and move to Hollywood to make a movie. He of course receives offers from \( m \) different producers, where the \( i \)'th producer wants to give a funding of \( X_i \) dollars. A producer is willing to invest this money only if all her favorite actors are in the movie. Every producer has a list of favorite actors, and some actors may be on several lists. In total, the lists contain the names of \( n \) distinct actors, and the \( i \)'th actor requires a fee of \( Y_i \) dollars for participating. Since Adam is only doing this for the money, he would like to maximize the funding gained minus the salaries paid. Describe an efficient algorithm for maximizing this quantity.

Hint: build a flow network out of a bipartite graph. Study the cuts in this network and ignore the flows. What is the meaning of the size of a cut?