NORMAL MEASURES ON A TALL CARDINAL

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ABSTRACT. We study the number of normal measures on a tall cardinal. Our main results are that:

- The least tall cardinal may coincide with the least measurable cardinal and carry as many normal measures as desired.
- The least measurable limit of tall cardinals may carry as many normal measures as desired.

1. INTRODUCTION

A classical question in set theory concerns the size and structure of the set of normal measures on a measurable cardinal κ. Some of the most notable results include:

- (Solovay [16]) If κ is 2^κ-supercompact then κ carries 2^{2^κ} normal measures.
- (Kunen [12]) If κ is measurable then there exists an inner model in which κ is the unique measurable cardinal and carries exactly one normal measure.
- (Friedman and Magidor [5]) By forcing over Kunen's model we may obtain generic extensions where κ carries any reasonable prescribed number of normal measures.

Another classical question concerns the relationship between the large cardinal properties of *strong compactness* and *supercompactness*. Strong compactness has many of the same combinatorial consequences as supercompactness, but the two properties can be very different:

- (Solovay [16]) If κ is supercompact then κ is strongly compact and there are κ measurable cardinals less than κ.
- (Magidor [13]) If κ is strongly compact then there is a generic extension in which κ remains strongly compact and is the least measurable cardinal.
- (Magidor [13]) If κ is supercompact then there is a generic extension in which κ remains supercompact and is the least strongly compact cardinal.

Hamkins [10] introduced the concept of a *tall cardinal*, which is a natural weakening of the notion of a *strong cardinal*. Part of the intuition is that "tall is to strong as strongly compact is to supercompact".

- κ is λ-*strong* if there is $j : V \longrightarrow M$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $V_\lambda \subseteq M$, and is *strong* if it is λ-strong for every λ.
- κ is λ-*tall* if there is $j : V \longrightarrow M$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and ${}^\kappa M \subseteq M$, and is *tall* if it is λ-tall for every λ.

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In the definition of "tall cardinal" it is important that we make the demand "${}^\kappa M \subseteq M$"; if κ is measurable then by iterating ultrapowers we may obtain embeddings with $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \lambda$ for arbitrary λ, however the target model will not in general be closed even under ω-sequences. To see that strong cardinals are tall we note that there are unboundedly many strong limit cardinals λ such that $\mathrm{cf}(\lambda) > \kappa$, and that if we take an embedding witnessing that κ is λ-strong for such λ and form the associated (κ, λ)-extender E, then $V_\lambda \subseteq \mathrm{Ult}(V, E)$ and ${}^\kappa\, \mathrm{Ult}(V, E) \subseteq \mathrm{Ult}(V, E)$.

Hamkins [10] made a detailed study of tall cardinals, and proved among other things that a measurable limit of tall cardinals is tall and that starting with a strong cardinal we may produce a model where the least measurable cardinal is tall. We note that these results are in line with the intuition that the properties of tall cardinals parallel those of strongly compact cardinals. Apter and Gitik [2] continued the study of tall cardinals and answered some questions left open by Hamkins' work.

There is one important respect in which tall cardinals are more tractable than strongly compact cardinals, and this is that the methods of inner model theory extend to the level of tall cardinals. It is an easy application of the core model for a strong cardinal to show that tall cardinals are equiconsistent with strong cardinals. Schindler [14] has proved that in canonical inner models for large cardinals, every tall cardinal is either a strong cardinal or a measurable limit of strong cardinals. It is worth noting that if a class of cardinals has a measurable accumulation point then by an easy reflection argument the least such point κ is not even $(\kappa + 2)$-strong, so that measurable limits of strong cardinals can give an easy example of a non-strong tall cardinal.

Solovay's proof that a 2^κ-supercompact cardinal κ carries the maximal number 2^{2^κ} of normal measures can easily be adapted (replacing supercompactness measures by extenders) to show that the same holds true for a $(\kappa + 2)$-strong cardinal κ. In this paper we will show that we can control the number of normal measures on a non-strong tall cardinal; the parallel questions for non-supercompact strongly compact cardinals remain open.

Our main results (roughly speaking) state that:

- The least tall cardinal may coincide with the least measurable cardinal and carry as many normal measures as desired.
- The least measurable limit of tall cardinals may carry as many normal measures as desired.

Our arguments use and extend ideas from prior work of Magidor (unpublished, but see an account in [3]), Hamkins [10], and Friedman and Magidor [5].

- Magidor observed that we can produce embeddings witnessing a high degree of strong compactness but not 2^κ-supercompactness for κ by forming a composition $j^M_U \circ j$, where $j : V \longrightarrow M$ witnesses a high degree of supercompactness for κ and $j^M_U : M \longrightarrow Ult(M, U)$ is the ultrapower of M by a normal measure U on κ with $U \in M$ having Mitchell order zero. Hamkins made the parallel observation that if j witnesses a high degree of strongness for κ then $j^M_U \circ j$ witnesses tallness but not $(\kappa + 2)$-strongness.
- To produce a model where the least measurable cardinal is tall, Hamkins started with a tall cardinal κ and iterated with Easton support to add non-reflecting stationary subsets in each measurable cardinal less than κ. He

then lifted embeddings of the form $j^M_U \circ j$ where j is a witness to some degree of strongness to show that κ is still tall in the extension.

- Friedman and Magidor produced models with fine control over the number of normal measures on a measurable cardinal κ. Among the key features are the use of iterations with non-stationary supports, the use of Sacks forcing together with a version of Jensen coding, and the use of canonical inner models for cardinals up to the level $o(\kappa) = \kappa^{++} + 1$ as ground models for the various forcing constructions. All of this is important to ensure that a measure U in V has only a small number of extensions $\bar{U}$ in $V[G]$; the heart of the argument is that (by inner model theory and properties of the forcing) $j^{V[G]}_{\bar{U}}$ must be a lift of j^V_U and that (by non-stationary support and coding) there are only so many possible liftings.

In our setting we generally kill unwanted instances of measurability by non-stationary support iteration of forcing to add a non-reflecting stationary set; this requires some care since our iterands have slightly less closure than was the case for Friedman and Magidor. Since our starting hypotheses are at or above the level of a strong cardinal we need to use comparatively large canonical inner models as the ground models for our forcing constructions, which complicates the analysis of the measures in the generic extension. To produce embeddings to witness tallness we must lift embeddings of the form $j^M_U \circ j$ to generic extensions obtained by non-stationary support iteration, which is substantially harder than lifting j^V_U for some normal measure U in V.

The paper is organised as follows:

- Section 2 contains some technical background needed for the main results.
- Section 3 is concerned with the situation in which the least measurable cardinal is tall. We show that in this situation the least measurable cardinal can carry any specified number of normal measures.
- Section 4 is concerned with the least measurable limit of tall cardinals. In parallel with the results of Section 3, we show that the least measurable limit of tall cardinals can carry any specified number of normal measures.

1.1. **Notation and conventions.** When $\mathbb{P}$ is a forcing poset and $p, q \in \mathbb{P}$, we write "$p \leq q$" for "p is stronger than q". When α is an ordinal of uncountable cofinality and $X \subseteq \alpha$, we will say that some statement $P(\beta)$ holds for *almost all β in X* if and only if there is a club set $D \subseteq \alpha$ such that $P(\beta)$ holds for all $\beta \in D \cap X$. Since the notion "contains a club set" is not absolute between models of set theory, we will sometimes write phrases like *for V-almost all β in X* to indicate that the witnessing club set may be chosen in V. As usual, ON is the class of ordinals and REG is the class of regular cardinals.

For κ regular and λ a cardinal, we write "$\mathrm{Add}(\kappa, \lambda)$" for the standard poset to add λ Cohen subsets of κ. The conditions are partial functions from $\kappa \times \lambda$ to 2 of cardinality less than κ, and the ordering is extension.

When $\mathbb{P}_\beta$ is a forcing iteration of length β, we write "G_β" for a typical $\mathbb{P}_\beta$-generic object, "$\mathbb{Q}_\alpha$" for the iterand at stage α in $\mathbb{P}_\beta$ and "g_α" for the $\mathbb{Q}_\alpha$-generic object added by G_β. If $p \in \mathbb{P}_\beta$ then the *support* $\mathrm{supp}(p)$ of the condition p is the set of α such that $p(\alpha) \neq \dot{1}_\alpha$, where $\dot{1}_\alpha$ is a fixed $\mathbb{P}_\alpha$-name for the trivial condition in $\mathbb{Q}_\alpha$. The *support* of $\mathbb{P}_\beta$ is the set of α such that $\dot{\mathbb{Q}}_\alpha$ is not the canonical $\mathbb{P}_\alpha$-name for the trivial forcing.

When β is an ordinal we write "β^+" for the least cardinal greater than β. That is $\beta^+ = |\beta|^+$.

If M is an inner model of set theory and $E \in M$ with $M \models$ "E is an extender", we write $j^M_E : M \longrightarrow \mathrm{Ult}(M, E)$ for the ultrapower of M by E. As a special case we write $j^M_U : M \longrightarrow \mathrm{Ult}(M, U)$ when $U \in M$ with $M \models$ "U is a normal measure".

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2. Some technical results

2.1. **Adding non-reflecting stationary sets.** For a Mahlo cardinal α, we let $\mathrm{NR}(\alpha)$ be the standard poset for adding a non-reflecting stationary subset of α consisting of regular cardinals. Formally speaking the conditions in $\mathrm{NR}(\alpha)$ are functions r from some ordinal less than α to 2, such that $\{\eta : r(\eta) = 1\} \subseteq REG$ and r is identically zero on a club subset of β for every $\beta \leq \mathrm{dom}(r)$ of uncountable cofinality. The ordering on $\mathrm{NR}(\alpha)$ is end-extension.

The following facts are standard:

Fact 2.1. *$\mathrm{NR}(\alpha)$ is α-strategically closed in the sense that player II wins the game of length α, where players I and II collaborate to build a decreasing sequence of conditions and II plays at limit stages.*

Fact 2.2. *For every $\beta < \alpha$, the set of conditions $r \in \mathrm{NR}(\alpha)$ such that $\mathrm{dom}(r) > \beta$ is dense and β-closed.*

Fact 2.3. *If G is $\mathrm{NR}(\alpha)$-generic, then in $V[G]$ the function $\bigcup G$ is the characteristic function of a non-reflecting stationary set of regular cardinals.*

2.2. **Reducing dense sets.** Let $\mathbb{P}_\beta$ be an iterated forcing poset of length β, D a dense set in $\mathbb{P}_\beta$ and $\alpha < \beta$. Then a condition p in $\mathbb{P}_\beta$ *reduces D to α* if and only if for every $q \leq p$ with $q \in D$, $q \restriction \alpha ^\frown p \restriction [\alpha, \beta) \in D$.

The following facts are straightforward:

Fact 2.4. *If p reduces D to α then $\{r \in \mathbb{P}_\alpha : r ^\frown p \restriction [\alpha, \beta) \in D\}$ is dense below $p \restriction \alpha$. Hence if $p \restriction \alpha \in G_\alpha$ there is $r \in G_\alpha$ such that $r \leq p \restriction \alpha$ and $r ^\frown p \restriction [\alpha, \beta) \in D$.*

Fact 2.5. *If the "tail forcing" $\mathbb{P}/G_\alpha$ is forced to have a $|\mathbb{P}_\alpha|^+$-closed dense subset, then for every dense set D and $p \in \mathbb{P}$ there is $q \leq p$ such that $q \restriction \alpha = p \restriction \alpha$ and q reduces D to α. More generally if $\mathcal{D}$ is a family of dense sets and $\mathbb{P}/G_\alpha$ is forced to have a $\max(|\mathcal{D}|^+, |\mathbb{P}_\alpha|^+)$-closed dense subset, then for every $p \in \mathbb{P}$ there is $q \leq p$ such that $q \restriction \alpha = p \restriction \alpha$ and q reduces every $D \in \mathcal{D}$ to α.*

Fact 2.6. *If $\dot\tau$ is a $\mathbb{P}$-name for an object in V, D is the dense set of conditions which decide the value of $\dot\tau$ and p reduces D to α, then $p \Vdash \dot\tau = \dot\sigma$ for some $\mathbb{P}_\alpha$-name $\dot\sigma$.*

2.3. **NS support iterations.** We will use the technology of iterated forcing with non-stationary (NS) support, which was introduced by Jensen [4] in his work on the celebrated Coding Theorem. Friedman and Magidor [5] used NS support iterations in their work on controlling the number of measures on a measurable cardinal, and we will use many of the same ideas. We are iterating forcing posets with rather less closure than was available to Friedman and Magidor, so we give the proofs in some detail.

Definition 2.7. *An iterated forcing poset $\mathbb{P}_\eta$ of length η is an* iteration with non-stationary supports (NS iteration) *if for every $\gamma \leq \eta$:*

(1) *If γ is not inaccessible, then $\mathbb{P}_\gamma$ is the inverse limit of $(\mathbb{P}_\alpha)_{\alpha<\gamma}$.*
(2) *If γ is inaccessible, then $\mathbb{P}_\gamma$ is the set of conditions in the inverse limit of $(\mathbb{P}_\alpha)_{\alpha<\gamma}$ whose support is a non-stationary subset of γ.*

We note that supports are larger in an NS iteration than in an Easton iteration, so that in general NS iterations have better closure and worse chain condition than Easton iterations.

We will need some technical lemmas about NS iterations.

Lemma 2.8. *Let κ be inaccessible and let $\mathbb{P}_\kappa$ be an NS iteration of length κ such that:*

- *The support of $\mathbb{P}_\kappa$ is an unbounded set $I \subseteq \kappa$ consisting of inaccessible cardinals.*
- *For each $\alpha \in I$ it is forced by $\mathbb{P}_\alpha$ that:*
  - $|\mathbb{Q}_\alpha| < \min(I \setminus (\alpha+1))$.
  - *$\mathbb{Q}_\alpha$ has a decreasing sequence of dense sets $(\mathbb{Q}_{\alpha,\beta})_{\beta<\alpha}$, where $\mathbb{Q}_{\alpha,\beta}$ is $|\beta|$-closed and $\mathbb{Q}_{\alpha,0} = \mathbb{Q}_\alpha$.*

Then for every $\gamma < \kappa$:

- $|\mathbb{P}_{\gamma+1}| < \min(I \setminus (\gamma+1))$.
- *For every $\beta < \min(I \setminus (\gamma+1))$, it is forced that $\mathbb{P}_\kappa/G_{\gamma+1}$ has a $|\beta|$-closed dense subset.*

Proof. We prove by induction on $\gamma < \kappa$ that $|\mathbb{P}_\gamma| < \min(I \setminus (\gamma+1))$. Since it is forced by $\mathbb{P}_\gamma$ that $\mathbb{Q}_\gamma$ is either trivial or of size less than $\min(I \setminus (\gamma+1))$, and $\min(I \setminus (\gamma+1))$ is inaccessible, it will follow immediately that $|\mathbb{P}_{\gamma+1}| < \min(I \setminus (\gamma+1))$.

If γ is a successor ordinal then $\gamma \notin I$, $\mathbb{P}_{\gamma+1} \simeq \mathbb{P}_\gamma$ and we are done by induction. If γ is a limit ordinal then $\mathbb{P}_\gamma$ is contained in the inverse limit of $\langle \mathbb{P}_\beta : \beta < \gamma \rangle$, we have $|\mathbb{P}_\beta| < \min(I \setminus (\beta+1)) \leq \min(I \setminus (\gamma+1))$ by induction, and so $|\mathbb{P}_\gamma| \leq \prod_{\beta<\gamma} |\mathbb{P}_\beta| < \min(I \setminus (\gamma+1))$ because $\min(I \setminus (\gamma+1))$ is inaccessible.

For the second claim, we work in $V[G_{\gamma+1}]$ and define D to be the set of conditions $q \in \mathbb{P}_\kappa/G_{\gamma+1}$ such that $\Vdash_{\mathbb{P}_\eta/G_{\gamma+1}} q(\eta) \in \mathbb{Q}_{\eta,\beta}$ for all $\eta \in \operatorname{supp}(q)$. Since $\mathbb{Q}_{\eta,\beta}$ is $|\beta|$-closed, the main point is to check that if $\nu < |\beta|$ and $(q_i)_{i<\nu} \in V[G_{\gamma+1}]$ is a decreasing sequence from D then $\bigcup_{i<\nu} \operatorname{supp}(q_i)$ can be covered by a set $Y \in V$ such that $Y \cap \delta$ is non-stationary for all inaccessible $\delta \in (\gamma, \kappa)$.

Let $\lambda = \min(I \setminus (\gamma+1))$. Since $|\mathbb{P}_{\gamma+1}| < \lambda$, there is a set $\mathcal{Y} \in V$ such that:

(1) $|\mathcal{Y}| < \lambda$.
(2) $\{\operatorname{supp}(q_i) : i < \nu\} \subseteq \mathcal{Y}$.
(3) For every $Z \in \mathcal{Y}$, $Z \subseteq I$ and $Z \cap \delta$ is non-stationary for all inaccessible $\delta \in (\gamma, \kappa)$.

We set $Y = \bigcup \mathcal{Y}$ and claim that Y is as required. To see this suppose that $\gamma < \delta < \kappa$ and δ is inaccessible. If I is bounded in δ then Y is bounded in δ because $Y \subseteq I$. On the other hand, if I is unbounded in δ then $\delta > \lambda$ because $\delta > \gamma$ and λ is the least point of I above γ, and so Y is non-stationary in δ because $Y \cap \delta$ is the union of fewer than δ non-stationary subsets of δ. This concludes the proof of Lemma 2.8. □

In our applications $\mathbb{Q}_\alpha$ will either be α-closed (in which case we may set $\mathbb{Q}_{\alpha,\beta} = \mathbb{Q}_\alpha$) or will be of the form $\mathrm{NR}(\alpha)$ (in which case we may set $\mathbb{Q}_{\alpha,\beta} = \{r : \beta < \mathrm{dom}(r)\}$). When $p \in \mathbb{P}_\kappa$ and C is a club subset of κ, we say that the pair (p, C) is "well-groomed" if:

- The club set C is disjoint from the support of p.
- For all $\alpha \in \mathrm{supp}(p)$ such that $\alpha > \min(C)$, $\Vdash_\alpha p(\alpha) \in \mathbb{Q}_{\alpha,\max(\alpha\cap C)^+}$.

Lemma 2.9. *Let $\mathbb{P}_\kappa$ be an NS iteration as in the hypotheses of Lemma 2.8. For every $p \in \mathbb{P}_\kappa$ and every club set $C \subseteq \kappa$ which is disjoint from* $\mathrm{supp}(p)$*, there is $q \leq p$ such that:*

- $\mathrm{supp}(p) = \mathrm{supp}(q)$.
- $q \restriction \min(C) = p \restriction \min(C)$.
- *(q, C) is well-groomed.*

Proof. For $\alpha \in \mathrm{supp}(p)$ with $\alpha < \min(C)$ we set $q(\alpha) = p(\alpha)$. For each $\beta \in C$, let β^* be the immediate successor of β in C. For $\alpha \in \mathrm{supp}(p) \cap (\beta, \beta^*)$ we note that α is inaccessible, so that $\beta^+ < \alpha$, and choose $q(\alpha)$ so that $q(\alpha)$ is forced to be an extension of $p(\alpha)$ lying in the dense set $\mathbb{Q}_{\alpha,\beta^+}$. This concludes the proof of Lemma 2.9. □

The following "fusion lemma" is modeled on [5, Lemma 4].

Lemma 2.10. *Let $\mathbb{P}_\kappa$ be an NS iteration as in the hypotheses of Lemma 2.8, and let $p \in \mathbb{P}_\kappa$. For each $\alpha < \kappa$ let $\mathcal{D}_\alpha$ be a family of dense sets in $\mathbb{P}_\kappa$ such that $|\mathcal{D}_\alpha| < \min(I \setminus (\alpha+1))$. Then there is $q \leq p$ such that for almost every α, q reduces every $D \in \mathcal{D}_\alpha$ to $\alpha + 1$.*

Proof. We build sequences $(p_i)_{i<\kappa}$, $(\alpha_i)_{i<\kappa}$, and $(C_i)_{i<\kappa}$ such that:

- $p_0 \leq p$.
- $(p_i)_{i<\kappa}$ is a decreasing sequence of conditions, and $(\mathrm{supp}(p_i))_{i<\kappa}$ is a continuous increasing sequence of sets.
- $(\alpha_i)_{i<\kappa}$ is an increasing and continuous sequence of ordinals less than κ.
- For $i < j < \kappa$, $p_i \restriction \alpha_i + 1 = p_j \restriction \alpha_i + 1$.
- For $i < \kappa$, p_{i+1} reduces every $D \in \mathcal{D}_\alpha$ to $\alpha_i + 1$.
- For all $i < \kappa$, the pair (p_i, C_i) is well-groomed.
- $(C_i)_{i<\kappa}$ is decreasing and continuous.
- For all $j < \kappa$, $\alpha_j \in C_j$.

At successor stages we first choose $p'_{i+1} \leq p_i$ such that $p'_{i+1} \restriction \alpha_i + 1 = p_i \restriction \alpha_i + 1$ and p'_{i+1} reduces every $D \in \mathcal{D}_\alpha$ to $\alpha_i + 1$, which is possible by Lemma 2.8 and Fact 2.5. We then choose a club set $C_{i+1} \subseteq C_i$ which is disjoint from $\mathrm{supp}(p'_{i+1})$ with $\min(C_{i+1}) > \alpha_i + 1$, use Lemma 2.9 to extend p'_{i+1} to p_{i+1} such that $p_{i+1} \restriction \alpha_i + 1 = p'_{i+1} \restriction \alpha_i + 1$ and (p_{i+1}, C_{i+1}) is well-groomed, and finally choose $\alpha_{i+1} \in C_{i+1}$.

For limit j we define $\alpha_j = \sup_{i<j} \alpha_i$ and $C_j = \bigcap_{i<j} C_i$. We will choose p_j in such a way that $\mathrm{supp}(p_j) = \bigcup_{i<j} \mathrm{supp}(p_i)$. Two key points are that:

- By construction, the club set C_j is disjoint from $\mathrm{supp}(p_j)$.
- Since $(C_i)_{i<j}$ is decreasing and $\alpha_k \in C_k$ for all $k < j$, we have that $\alpha_k \in C_i$ whenever $i \leq k < j$. Since C_i is club, it follows that $\alpha_j \in C_i$ for all $i < j$, so that $\alpha_j \in C_j$. In particular $\alpha_j \notin \mathrm{supp}(p_j)$.

For $\beta \in \mathrm{supp}(j) \cap \alpha_j$ it is clear that $(p_i(\beta))_{i<j}$ is constant for large i, so we set $p_j(\beta)$ equal to the eventual value of that sequence. We now fix $\beta \in \mathrm{supp}(p_j)$

with $\beta > \alpha_j$. We have $\beta \in \mathrm{supp}(p_i)$ for all large $i < j$, so we fix $i < j$ to be least such that $\beta \in \mathrm{supp}(p_i)$. By construction the pair (p_k, C_k) is well-groomed and $\beta \in \mathrm{supp}(p_k)$ for all k with $i \le k < j$ so that $\Vdash_\beta p_k(\beta) \in \mathbb{Q}_{\beta,\max(C_k\cap\beta)^+}$. Now $C_j \subseteq C_k$ and hence $\max(C_j \cap \beta) \le \max(C_k \cap \beta)$ for $i \le k < j$, and since $\mathbb{Q}_{\beta,\eta}$ is forced to decrease as η increases we have $\Vdash_\beta p_k(\beta) \in \mathbb{Q}_{\beta,\max(C_j\cap\beta)^+}$ for $i \le k < j$.

Since $\alpha_j \in C_j$, $\alpha_j < \beta$ and $(\alpha_k)_{k<j}$ is strictly increasing, we see that $j \le \alpha_j \le \max(C_j \cap \beta)$. We may now use the closure of $\mathbb{Q}_{\beta,\max(C_j\cap\beta)^+}$ to choose $p_j(\beta)$ so that $p_j \restriction \beta$ forces it to be a lower bound for $(p_k(\beta))_{k<j}$ lying in $\mathbb{Q}_{\beta,\max(C_j\cap\beta)^+}$.

To ensure that p_j is a legitimate condition we should verify that $\mathrm{supp}(p_j \restriction \gamma)$ is non-stationary in γ for every $\gamma \le \kappa$. For $\gamma < \alpha_j$ we just choose $i < j$ so large that $p_j \restriction \gamma = p_i \restriction \gamma$, and use the fact that p_i is a condition. For $\gamma = \alpha_j$ we note that $\alpha_i \notin \mathrm{supp}(p_i)$ implies by agreement that $\alpha_i \notin \mathrm{supp}(p_j)$ for all $i < j$, so that the sequence $(\alpha_i)_{i<j}$ enumerates a club set disjoint from $\mathrm{supp}(p_j \restriction \alpha_j)$. For $\gamma > \alpha_j$ the club filter on γ is j^+-complete, and we are done since $\mathrm{supp}(p_j) = \bigcup_{i<j} \mathrm{supp}(p_i)$.

It is routine to check that (p_j, C_j) is well-groomed, and so we have completed the inductive construction. We let q be the unique condition such that $q \restriction \alpha_i + 1 = p_i \restriction \alpha_i + 1$ for every $i < \kappa$. We note that $\mathrm{supp}(q)$ is non-stationary in every inaccessible $\gamma < \kappa$ because $q \restriction \gamma = p_i \restriction \gamma$ for all large $i < \gamma$, and $\mathrm{supp}(q)$ is non-stationary in κ because $(\alpha_i)_{i<\kappa}$ is strictly increasing and continuous with $\alpha_i \notin \mathrm{supp}(q)$. This concludes the proof of Lemma 2.10. □

To illustrate the use of the fusion lemma we prove some properties of NS iterations.

Lemma 2.11. *Let $\mathbb{P}_\kappa$ be an NS iteration as in the hypotheses of Lemma 2.8. Let $f : \kappa \to ON$ be a function in $V[G_\kappa]$. Then there is a function $F \in V$ such that $\mathrm{dom}(F) = \kappa$, $F(\gamma)$ is a set of ordinals with $|F(\gamma)| < \min(I\setminus(\gamma+1))$ and $f(\gamma) \in F(\gamma)$ for V-almost all $\gamma < \kappa$.*

Proof. Let $\dot f$ name f. By Lemma 2.10 and Fact 2.6, there is a dense set of conditions q such that q reduces $\dot f(\gamma)$ to a $\mathbb{P}_{\gamma+1}$-name $\dot\sigma_\gamma$ for almost all γ. Now let $F(\gamma) = \{\alpha : \exists r \le q \restriction \gamma \; r \Vdash \dot\sigma_\gamma = \alpha\}$. By Lemma 2.8 we have $|F(\gamma)| < \min(I \setminus (\gamma+1))$, and for each γ such that $q \Vdash \dot f(\gamma) = \dot\sigma_\gamma$ we see that $q \Vdash \dot f(\gamma) \in F(\gamma)$. This concludes the proof of Lemma 2.11. □

Applying Lemma 2.11 to the increasing enumeration of a club set, we obtain a useful corollary.

Corollary 2.12. *Let $\mathbb{P}_\kappa$ be an NS iteration as in the hypotheses of Lemma 2.8, and let $C \in V[G_\kappa]$ be a club subset of κ. Then there is $D \in V$ such that D is club in κ and $D \subseteq C$.*

Lemma 2.13. *Let $\mathbb{P}_\kappa$ be an NS iteration as in the hypotheses of Lemma 2.8. Let N be an inner model of V such that ${}^\kappa N \subseteq N$ (so that $\mathbb{P}_\kappa \in N$). Then $V[G_\kappa] \models {}^\kappa N[G_\kappa] \subseteq N[G_\kappa]$.*

Proof. Since $N[G_\kappa]$ is a model of ZFC it suffices to show that $V[G_\kappa] \models {}^\kappa ON \subseteq N[G_\kappa]$. Let $\dot f$ name a function from κ to ON. By Lemma 2.10 and Fact 2.6, there is a dense set of conditions q such that q reduces $\dot f \restriction \gamma$ to a $\mathbb{P}_{\gamma+1}$-name $\dot\sigma_\gamma$ for almost all γ. Since ${}^\kappa N \subseteq N$ we see that $(\dot\sigma_\gamma)_{\gamma<\kappa} \in N$. It follows easily that $q \Vdash \dot f = \dot g$, where $\dot g$ denotes the union of the realisations of the names $\dot\sigma_\gamma$ and $\dot g \in N$. This concludes the proof of Lemma 2.13. □

2.4. **Sacks forcing and coding forcing.** Let α be inaccessible. We will use a version of Sacks forcing at α which was introduced by Friedman and Thompson [6]. Conditions in $\mathrm{Sacks}(\alpha)$ are perfect $< \alpha$-closed trees $T \subseteq {}^{<\alpha}2$ which "split on a club" in the following sense: there is a club set of levels such that all nodes on levels in this set have two immediate successors, while nodes on levels outside this set have only one immediate successor.

$\mathrm{Sacks}(\alpha)$ is α-closed, satisfies a version of the fusion lemma and preserves α^+. It also preserves stationary subsets of $\alpha^+ \cap \mathrm{cof}(\alpha)$. Following Friedman and Magidor we will use Sacks forcing at α in conjunction with a version of Jensen coding at α^+. We refer the reader to the discussion in [5], particularly Lemma 8 of that paper. Given a partition $(T_i)_{i<\alpha^+}$ of $\alpha^+ \cap \mathrm{cof}(\alpha)$ into pairwise disjoint stationary sets, we will consider forcing with $\mathrm{Sacks}(\alpha) * \mathrm{Code}(\alpha)$, where $\mathrm{Code}(\alpha)$ is a certain forcing defined using $(T_i)_{i<\alpha^+}$ which codes the Sacks generic object and its own generic object in a robust way. $\mathrm{Code}(\alpha)$ is α-closed and adds no α-sequences of ordinals. The key point is that in the generic extension by $\mathrm{Sacks}(\alpha) * \mathrm{Code}(\alpha)$ the generic object for this forcing poset is unique in a very strong sense:

Fact 2.14. *Let H be $\mathrm{Sacks}(\alpha) * \mathrm{Code}(\alpha)$-generic over V and let W be an outer model of $V[H]$ in which stationary subsets of α^+ from $V[H]$ remain stationary. Then H is the unique element of W which is $\mathrm{Sacks}(\alpha) * \mathrm{Code}(\alpha)$-generic over V.*

The following fact, which is implicit in the work of Friedman and Magidor, is easily proved by a fusion argument along the same lines as Lemmas 2.11 and 2.13.

Fact 2.15. *Let $\mathbb{P}_\kappa$ be an NS iteration satisfying the hypotheses of Lemma 2.8, let G_κ be $\mathbb{P}_\kappa$-generic over V and let g be $\mathrm{Sacks}(\kappa)^{V[G_\kappa]}$-generic over $V[G_\kappa]$. Then:*

- *If $f : \kappa \to ON$ is a function in $V[G_\kappa * g]$, there is a function $F \in V$ such that $\mathrm{dom}(F) = \kappa$, $F(\gamma)$ is a set of ordinals with $|F(\gamma)| < \min(I \setminus (\gamma + 1))$ and $f(\gamma) \in F(\gamma)$ for V-almost all $\gamma < \kappa$.*
- *If N is an inner model of V such that ${}^\kappa N \subseteq N$, then $V[G_\kappa * g] \models {}^\kappa N[G_\kappa * g] \subseteq N[G_\kappa * g]$.*

2.5. **Strongly unfoldable cardinals.** We will use the large cardinal concept of *strong unfoldability*, which was introduced by Villaveces [18]. We recall that if κ is inaccessible then a *κ-model* is a transitive model M of ZFC minus Powerset such that $\kappa = |M| \in M$ and ${}^{<\kappa}M \subseteq M$.

Definition 2.16. *A cardinal κ is* strongly unfoldable *(resp.* strongly unfoldable up to μ*) if and only if κ is inaccessible and for every κ-model M and every λ (resp. every $\lambda < \mu$) there is $\pi : M \longrightarrow N$ an elementary embedding into a transitive set N such that $\mathrm{crit}(\pi) = \kappa$, $\pi(\kappa) > \lambda$ and $V_\lambda \subseteq N$.*

Roughly speaking strongly unfoldable cardinals bear the same relation to strong cardinals that weakly compact cardinals bear to measurable cardinals.

Fact 2.17. *If κ is strong then κ is strongly unfoldable.*

Proof. Let M be an arbitrary κ-model, let j witness that κ is λ-strong, and then set $N = j(M)$ and $\pi = j \restriction M$. □

Fact 2.18. *If λ is strongly unfoldable, $\kappa < \lambda$ and κ is strong (resp. strongly unfoldable) up to λ, then κ is strong (resp. strongly unfoldable).*

Proof. Let κ be strong up to λ where λ is strongly unfoldable, and let $\mu > \lambda$ be arbitrary. Let M be some λ-model, and note that $V_\lambda \subseteq M$ and $M \models$ "κ is strong up to λ". Now let $\pi : M \longrightarrow N$ be such that $\mathrm{crit}(\pi) = \lambda$, $\pi(\lambda) > \mu$ and $V_\mu \subseteq N$. Then $N \models$ "κ is strong up to $\pi(\lambda)$", and so easily κ is μ-strong. It follows that κ is strong. The argument in case κ is strongly unfoldable up to λ is very similar. □

Fact 2.19. *If κ is measurable and strongly unfoldable, then any normal measure on κ concentrates on strongly unfoldable cardinals. In particular:*

- *This is true for κ strong.*
- *For any embedding $i : V \longrightarrow N$ with $\mathrm{crit}(i) = \kappa$, in N the cardinal κ is an inaccessible limit of strongly unfoldable cardinals.*
- *Any normal measure on κ concentrates on inaccessible limits of strongly unfoldable cardinals.*

Proof. Let U be some normal measure on the measurable and strongly unfoldable cardinal κ. We claim that κ is strongly unfoldable in $\mathrm{Ult}(V, U)$, from which it follows that U concentrates on strongly unfoldable cardinals. Let $M \in \mathrm{Ult}(V, U)$ be a κ-model and let $\lambda > \kappa$. Let $\pi : M \longrightarrow N$ be such that $\mathrm{crit}(\pi) = \kappa$, $\pi(\kappa) > \mu$ and $V_\mu \subseteq N$. Now $j_U \restriction M \in \mathrm{Ult}(V, U)$ by κ-closure, and $j_U \restriction M$ is an elementary embedding from M to $j_U(M)$, so that $j_U(\pi) \circ (j_U \restriction M)$ is an elementary embedding from M to $\pi_U(N)$ lying in $\mathrm{Ult}(V, U)$. Since $j_U(\pi)(j_U(\kappa)) = j_U(\pi(\kappa)) > j_U(\mu)$ and $j_U(V_\mu) = V^{\mathrm{Ult}(V,U)}_{j_U(\mu)} \subseteq j_U(N)$, we see readily that κ is strongly unfoldable in $\mathrm{Ult}(V, U)$. This concludes the proof of Fact 2.19. □

Fact 2.20. *If a class of cardinals has a measurable accumulation point, then the least such point is not strongly unfoldable.*

Proof. If κ is strongly unfoldable and κ is a measurable accumulation point of the class X, let M be a κ-model with $X \cap \kappa \in M$. Let $\pi : M \longrightarrow N$ with $\pi(\kappa) > \kappa$ and $V_{\kappa+2} \subseteq N$. Then in N we have that κ is a measurable accumulation point of $\pi(X \cap \kappa)$, so that in M there is measurable $\alpha < \kappa$ such that α is a limit of X. But $V_\kappa \subseteq M$, so that α truly is a measurable accumulation point of X. This concludes the proof of Fact 2.20. □

2.6. **Some inner model theory.** We will need some ideas from inner model theory. Since some of our results are concerned with measurable limits of strong cardinals, we will need to use fairly large inner models. We refer the reader to Steel's survey [17] and the paper by Jensen and Steel on the core model for one Woodin cardinal [11]. Henceforth we write K for the core model for one Woodin cardinal. We need to analyse normal measures in generic extensions of K.

Lemma 2.21. *Suppose that K exists and U is a normal measure on κ in some set generic extension $V[G]$. Then $j_U^{V[G]} \restriction K$ is an iteration of K. Furthermore if U concentrates on K-non-measurable cardinals, and there is a unique total extender E on K's extender sequence such that $\mathrm{crit}(E) = \kappa$ and κ is not measurable in $\mathrm{Ult}(K, E)$, then E is the first extender used in the iteration of K induced by $j_U^{V[G]}$.*

Proof. Let $N = \mathrm{Ult}(V[G], U)$. By the definability of K [11, Theorem 1.1], $j_U^{V[G]} \restriction K^{V[G]}$ is an elementary embedding from $K^{V[G]}$ to K^N. Since N is an inner model of $V[G]$ which is closed under ω-sequences, it follows from results of Schindler [15] that K^N is an iterate of $K^{V[G]}$ and that $j_U^{V[G]} \restriction K^{V[G]}$ is the iteration map. By the generic absoluteness of K [11, Theorem 1.1] we have that $K^{V[G]} = K$. In summary $j_U^{V[G]} \restriction K$ is an iteration of K, where we note that in general this iteration may only exist in $V[G]$.

If U concentrates on K-non-measurables then κ is not measurable in K^N, because the core model K is uniformly definable. By the agreement among models in an iteration, it follows that κ is not measurable after one step of the iteration of K induced by $j_U^{V[G]}$, so that the first extender which is used must be E. This concludes the proof of Lemma 2.21. □

Remark. If E is the unique total extender on K's sequence such that κ is not measurable in $\mathrm{Ult}(K, E)$, then there is a unique measure U of order zero on κ in K and U is equivalent to E. To see this let W be any measure of order zero, and note that j_W^K is an iteration of K whose first step must be an application of E. If $i : \mathrm{Ult}(K, E) \longrightarrow \mathrm{Ult}(K, W)$ is the rest of the iteration map then by normality $i : j_E^K(f)(\kappa) \mapsto j_W^K(f)(\kappa)$, so that $\mathrm{rge}(i) = \mathrm{Ult}(K, W)$; it follows that i is the identity and $j_E^K = j_W^K$.

Assuming that $V = K$ and that $V[G]$ is a sufficiently mild extension of V we can get finer information about normal measures in $V[G]$. The following lemma is a more general version of a result by Friedman and Magidor [5, Lemma 18].

Lemma 2.22. *Let $V = K$, let κ be the largest measurable cardinal, and let $V[G]$ be a generic extension of V by some poset $\mathbb{P}$ such that for every $f : \kappa \to \kappa$ with $f \in V[G]$ there is $g \in V$ such that:*

- *For all $\alpha < \kappa$, $g(\alpha)$ is a subset of κ and $|g(\alpha)|$ is less than the least measurable cardinal greater than α.*
- *For almost all $\alpha < \kappa$, $f(\alpha) \in g(\alpha)$.*

Let $W \in V[G]$ be a normal measure on κ. Then:

- *The iteration of V induced by $j_W^{V[G]}$ has exactly one step.*
- *If $i : V \longrightarrow N$ is the one-step iteration of V induced by $j_W^{V[G]}$, then there exists a unique filter H such that:*
  - *H is $i(\mathbb{P})$-generic over N with $i``G \subseteq H$.*
  - $\mathrm{Ult}(V[G], W) = N[H]$.
  - *$j_W^{V[G]}$ is the standard lifting of i using H, that is $j_W^{V[G]} : i_G(\dot\tau) \mapsto i_H(i(\dot\tau))$ for all $\mathbb{P}$-terms $\dot\tau$.*

Proof. Let i be the first step of the iteration and let k be the rest of it. Suppose for contradiction that k is not the identity, and let $\mathrm{crit}(k) = \mu$. Then $\kappa < \mu \leq i(\kappa)$ because the iteration is normal (in the sense that the critical points are increasing) and $i(\kappa)$ is the largest measurable cardinal in $\mathrm{dom}(k)$. Therefore $\mu < k(\mu) \leq j_W^{V[G]}(\kappa)$, so that $\mu = [f]_W$ for some function $f : \kappa \to \kappa$ with $f \in V[G]$. Find $g \in V$ such that $|g(\alpha)|$ is less than the next measurable cardinal greater than α, and $f(\alpha) \in g(\alpha)$ for V-almost all α. Then

$$\mu = j_W^{V[G]}(f)(\kappa) \in j_W^G(g)(\kappa) = k(i(g)(\kappa)).$$

By the choice of g we have that $|i(g)(\kappa)| < \mu$, so $\mu \in k``i(g)(\kappa)$, which is an immediate contradiction since $\mu = \mathrm{crit}(k)$.

For the second part we set $H = j_W^{V[G]}(G)$. It is routine to verify that H has all the properties listed. Moreover if $j_U^{V[G]}$ is the standard lift of i via some object H' then by definition $j_U^{V[G]}(G) = H'$, so that $H' = H$. This concludes the proof of Lemma 2.22. □

2.7. **Strong cardinals, Laver functions and extenders.** We will use a fact proved by Gitik and Shelah [7] in their work on indestructibility for strong cardinals.

Fact 2.23. *Let κ be strong. Then there is a* Laver function for κ*: that is a function $L : \kappa \to V_\kappa$ such that for every $x \in V$ and every μ there is $j : V \longrightarrow M$ such that j witnesses "κ is μ-strong" and $j(L)(\kappa) = x$.*

In our applications we only need to anticipate ordinals, so we will use the term *ordinal Laver function* for a function $l : \kappa \to \kappa$ such that for every $\eta \in ON$ and every μ there is $j : V \longrightarrow M$ such that j witnesses "κ is μ-strong" and $j(l)(\kappa) = \eta$. Clearly the existence of Laver functions implies the existence of ordinal Laver functions.

For technical reasons, it will be convenient to have the strongness of a strong cardinal κ witnessed by extenders of a special type.

Lemma 2.24. *Let κ be a strong cardinal and assume that there are no measurable cardinals greater than κ. Let $\lambda > \kappa$ be a strong limit cardinal with $\kappa < \mathrm{cf}(\lambda) < \lambda$. Then there exist a (κ, λ)-extender E and a function $h : \kappa \to \kappa$ such that:*

- $V_\lambda \subseteq \mathrm{Ult}(V, E)$ *and* ${}^\kappa \mathrm{Ult}(V, E) \subseteq \mathrm{Ult}(V, E)$.
- $j_E(\kappa) > \lambda = j_E(h)(\kappa)$.
- λ *is singular in* $\mathrm{Ult}(V, E)$.
- *In* $\mathrm{Ult}(V, E)$ *there are no measurable cardinals in the half-open interval* $(\kappa, \lambda]$.
- *For every $\eta < \kappa$, there are no measurable cardinals in the half-open interval* $(\eta, h(\eta)]$.

Proof. Let $l : \kappa \to \kappa$ be an ordinal Laver function. Let $\mu = \lambda + 2$ and let $j' : V \longrightarrow M'$ be an embedding such that j' witnesses "κ is μ-strong" and $j'(l)(\kappa) = \lambda$. By the agreement between V and M', in M' the cardinal λ is singular and there are no measurable cardinals in the interval $(\kappa, \lambda]$.

Now let E be the (κ, λ)-extender approximating j'. By the choice of λ we have that $V_\lambda \subseteq \mathrm{Ult}(V, E)$ and ${}^\kappa \mathrm{Ult}(V, E) \subseteq \mathrm{Ult}(V, E)$. As usual there is an elementary embedding $k' : \mathrm{Ult}(M, E) \longrightarrow M'$ given by $k' : j_E(f)(a) \mapsto j'(f)(a)$, and $j' = k' \circ j_E$. Since $j'(l)(\kappa) = \lambda$ we see that $\lambda + 1 \subseteq \mathrm{rge}(k')$ and hence $\mathrm{crit}(k') > \lambda$.

Now $k'(j_E(l)(\kappa)) = j'(l)(\kappa) = \lambda$, so that easily $j_E(l)(\kappa) = \lambda$. By the elementarity of k' and the facts that $\mathrm{crit}(k') > \lambda$ and in M' there are no measurable cardinals in the interval $(\kappa, \lambda]$, we see that in $\mathrm{Ult}(V, E)$ the cardinal λ is singular and there are no measurable cardinals in the half-open interval $(\kappa, \lambda]$.

Define h by setting $h(\eta) = l(\eta)$ for η such that there are no measurable cardinals in $(\eta, l(\eta)]$, and $h(\eta) = \eta$ for other values of η. Clearly $j_E(h)(\kappa) = j_E(l)(\kappa) = \lambda$, so that the extender E and the function h have all the properties required. This concludes the proof of Lemma 2.24. □

As we mentioned in the introduction, we will sometimes consider elementary embeddings of the form $j_U^{\mathrm{Ult}(V,E)} \circ j_E^V$ where E is an extender witnessing that κ is at least $(\kappa + 2)$-strong and U is a measure on κ of Mitchell order zero (so that κ is not measurable in $\mathrm{Ult}(\mathrm{Ult}(V, E), U)$). The idea is that embeddings of this type can witness any prescribed degree of tallness for κ, and can sometimes be lifted onto generic extensions in which all V-measurable cardinals below κ have been rendered non-measurable.

We record some useful information about embeddings of this general form.

Lemma 2.25. *Let $\lambda > \kappa + 1$ with $\mathrm{cf}(\lambda) > \kappa$, and let E be a (κ, λ)-extender witnessing that κ is λ-strong. Let $U \in \mathrm{Ult}(V, E)$ be a normal measure on κ. Then:*

- $j^V_U \restriction \mathrm{Ult}(V, E) = j^{\mathrm{Ult}(V,E)}_U$.
- $j^{\mathrm{Ult}(V,E)}_U \circ j^V_E = j^{\mathrm{Ult}(V,U)}_{j^V_U(E)} \circ j^V_U$.

Proof. Since $\mathrm{cf}(\lambda) > \kappa$ we have ${}^\kappa \mathrm{Ult}(V, E) \subseteq \mathrm{Ult}(V, E)$, and it follows that $j^V_U \restriction \mathrm{Ult}(V, E) = j^{\mathrm{Ult}(V,E)}_U$. By this fact and the elementarity of j^V_U,

$$j^{\mathrm{Ult}(V,E)}_U (j^V_E(x)) = j^V_U (j^V_E(x)) = j^{\mathrm{Ult}(V,U)}_{j_U(E)} (j^V_U(x))$$

for all x, so that $j^{\mathrm{Ult}(V,E)}_U \circ j^V_E = j^{\mathrm{Ult}(V,U)}_{j^V_U(E)} \circ j^V_U$ as claimed. This concludes the proof of Lemma 2.25.

□

2.8. **Generic transfer.** One of the basic techniques in the area of forcing and large cardinals is the transfer of generic objects for sufficiently distributive forcing posets. For example the following easy fact is very often useful:

Fact 2.26. *Let $i : M \longrightarrow N$ be an elementary embedding between transitive models of* ZFC, *and suppose $N = \{i(f)(a) : \mathrm{dom}(f) \in [\mu]^{<\omega}, a \in \mathrm{dom}(i(f))\}$ for some M-cardinal μ. Let $\mathbb{P} \in M$ be a poset such that forcing with $\mathbb{P}$ over M adds no μ-sequence of ordinals, and let G be $\mathbb{P}$-generic over M. Then i"G generates a filter which is $i(\mathbb{P})$-generic over N.*

We note that the $i(\mathbb{P})$-generic object in the conclusion is the only generic filter H which is compatible with G and i in the sense that i"$G \subseteq H$. We will need some results with a similar flavour, which generalise results by Friedman and Magidor [5] and Friedman and Thompson [6].

Lemma 2.27. *Let $j : V \longrightarrow M$ be an elementary embedding with critical point κ, and let $\mathbb{P}_\kappa$ be an NS iteration of length κ satisfying the hypotheses of Lemma 2.8. Suppose that for for every dense subset $D \subseteq j(\mathbb{P}_\kappa)$ in M there is a sequence $\vec{\mathcal{D}} = (\mathcal{D}_\alpha)_{\alpha<\kappa}$ such that:*

- *For each α, $\mathcal{D}_\alpha$ is a family of dense subsets of $\mathbb{P}_\kappa$ with $|\mathcal{D}_\alpha| < \min(I \setminus (\alpha + 1))$.*
- *$D \in j(\vec{\mathcal{D}})_\kappa$.*

Let G_κ be $\mathbb{P}_\kappa$-generic over V. Then:

1. *If κ is not in the support of $j(\mathbb{P}_\kappa)$, then there is a unique filter H such that H is $j(\mathbb{P}_\kappa)$-generic over M and j"$G_\kappa \subseteq H$.*
2. *If κ is in the support of $j(\mathbb{P}_\kappa)$, $j(\mathbb{P}_\kappa) \restriction \kappa + 1 = \mathbb{P}_\kappa * \mathbb{Q}$ and g is $\mathbb{Q}$-generic over $M[G_\kappa]$ then there is a unique filter H such that H is $j(\mathbb{P}_\kappa)$-generic over M, j"$G_\kappa \subseteq H$ and $G_\kappa * g = H \restriction \kappa + 1$.*

Proof. We start with some analysis which is common to the proofs of both claims. Let $D \subseteq j(\mathbb{P}_\kappa)$ be dense with $D \in M$, find a sequence $\vec{\mathcal{D}}$ as in the hypotheses, and then use Lemma 2.10 to find a dense set of conditions $q \in \mathbb{P}_\kappa$ such that for almost all α, q reduces all dense sets $E \in \mathcal{D}_\alpha$ to $\alpha + 1$. The key point is that for any such q, $j(q)$ reduces D to $\kappa + 1$.

- If κ is not in the support of $j(\mathbb{P}_\kappa)$, then we claim that $j``G_\kappa$ generates a filter which is $j(\mathbb{P}_\kappa)$-generic over M. Towards this end, let $D \in M$ be a dense open subset of $j(\mathbb{P}_\kappa)$. Arguing as above we may find $q \in G_\kappa$ such that $j(q)$ reduces D to $\kappa + 1$, and since κ is not in the support of $j(\mathbb{P}_\kappa)$ in fact $j(q)$ reduces D to κ.

  By Fact 2.4 and the fact that $q = j(q) \restriction \kappa \in G_\kappa$, we may find $r \leq q$ such that $r \in G_\kappa$ and $r^\frown j(q) \restriction (\kappa, j(\kappa)) \in D$. Since D is open and clearly $j(r) \leq r^\frown j(q) \restriction (\kappa, j(\kappa))$, we have that $j(r) \in D$ and hence $j``G_\kappa \cap D \neq \emptyset$.
- If κ is in the support of $j(\mathbb{P}_\kappa)$, $j(\mathbb{P})_{\kappa+1} \restriction \kappa + 1 = \mathbb{P}_\kappa * \mathbb{Q}$ and g is $\mathbb{Q}$-generic over $M[G_\kappa]$ then define a filter H as follows. For $r \in j(\mathbb{P}_\kappa)$, $r \in H$ if and only if:
  - $r \restriction \kappa + 1 \in G_\kappa * g$.
  - There is $p \in G_\kappa$ such that $r \restriction (\kappa, j(\kappa)) = j(p) \restriction (\kappa, j(\kappa))$.

  We need to show that H is generic over M, so let $D \in M$ be dense open. Arguing essentially as before we find $q \in G_\kappa$ such that $j(q)$ reduces D to $\kappa + 1$. The key new point is that since q has non-stationary support, κ is not in $\mathrm{supp}(j(q))$. So $j(q) \restriction \kappa + 1 \in G_\kappa * g$ (with a trivial entry at κ) and we may find $q' \in G_\kappa * g$ such that $q' \restriction \kappa \leq q$ and $q'^\frown j(q) \restriction (\kappa, j(\kappa)) \in D$. Clearly $q'^\frown j(q) \restriction (\kappa, j(\kappa)) \in H$, and we are done.

This concludes the proof of Lemma 2.27. □

In applications of Lemma 2.27, j will typically be j_E for some extender E and we will use information about E to make a suitable choice of $\vec{\mathcal{D}}$ for each relevant D.

The other transfer fact which we will need is a version of the "tuning fork" argument of Friedman and Thompson [6] for the forcing Sacks(κ) discussed in Section 2.4.

Lemma 2.28. *Let $j : V \longrightarrow M \subseteq V[G]$ be a generic elementary embedding, where G is $\mathbb{P}$-generic over V for some poset $\mathbb{P}$. Let g be* Sacks(κ)*-generic over V, and assume that:*

1. *$g \in M$.*
2. *For every ordinal η in the interval $(\kappa, j(\kappa))$ there is in V a club set $E \subseteq \kappa$ such that $j(E) \cap (\kappa, \eta) = \emptyset$.*
3. *For every dense set $D \subseteq j(\mathrm{Sacks}(\kappa))$ with $D \in M$, there is a sequence $\vec{\mathcal{D}} = (\mathcal{D}_\alpha)_{\alpha<\kappa}$ such that:*
   - *$\mathcal{D}_\alpha$ is a family of dense subsets of* Sacks(κ) *with $|\mathcal{D}_\alpha| < \kappa$.*
   - *$D \in j(\vec{\mathcal{D}})_\kappa$.*

Then there are exactly two filters h such that h is $j(\mathrm{Sacks}(\kappa))$-generic over M and $j``g \subseteq h$.

Proof. Let $b : \kappa \to 2$ be the generic function added by g, that is to say b is the unique function such that $b \restriction \zeta \in T$ for all $T \in g$. By a routine density argument and our hypothesis, for each $\eta \in (\kappa, j(\kappa))$ there is some condition $T \in g$ such that $j(T)$ has no splitting levels between κ and η. Since $g \in M$, and conditions are closed trees, we see that $b \in \mathrm{Lev}_\kappa(j(T))$. Since T has a club set of splitting levels, κ is a splitting level of $j(T)$. We may define functions $c_\eta^{\mathrm{left}} : \eta \to 2$ (resp. $c_\eta^{\mathrm{right}} : \eta \to 2$)

by setting $c_\eta^{\rm left}$ to be the unique element $t \in \mathrm{Lev}_\eta(j(T))$ such that $t \upharpoonright \kappa+1 = b^\frown 0$ (resp. $t \upharpoonright \kappa+1 = b^\frown 1$). Clearly the functions $c_\eta^{\rm left}$ do not depend on the choice of T and cohere with each other.

Now we define $h^{\rm left}$ to be the filter $\{T \in j(\mathrm{Sacks}(\kappa)) : \forall \eta\ c_\eta^{\rm left} \in T\}$, with a similar definition for $h^{\rm right}$. We claim that $h^{\rm left}$ and $h^{\rm right}$ are M-generic. To this end, let $D \in M$ be a dense open set and let $\vec{\mathcal{D}}$ be as in the hypotheses. By a standard fusion argument there is a dense set of Sacks conditions q such that for every $\alpha < \kappa$, every $E \in \mathcal{D}_\alpha$ and every $t \in \mathrm{Lev}_{\alpha+1}(q)$, $q_t \in E$. So there is a condition $q \in g$ such that $j(q)_{b^\frown 0} \in D$, and it follows that $h^{\rm left} \cap D \neq \emptyset$. By the same argument $h^{\rm right}$ is generic, and it is clear that these are the only possible generic filters containing j“g. This concludes the proof of Lemma 2.28. □

3. The least measurable cardinal

3.1. **The least measurable cardinal is tall with a unique normal measure.**

Theorem 1. *It is consistent (modulo the consistency of a strong cardinal) that the least measurable cardinal is tall and carries a unique normal measure.*

Proof. Before giving the details, we outline the main idea. As discussed in the introduction we will destroy measurable cardinals below a strong cardinal κ by an NS support iteration $\mathbb{P}_\kappa$, where we force with $\mathrm{NR}(\alpha)$ for measurable $\alpha < \kappa$. The argument that κ carries a unique normal measure in the extension is a fairly straightforward application of Lemma 2.22. To show that κ is still tall we lift embeddings of the form $j_U^{Ult(V,E)} \circ j_E^V$ where E is a carefully chosen extender witnessing some degree of strongness for κ and $U \in Ult(V,E)$ with U a measure of order zero. The careful choice of E gives information about the supports of the iterations obtained by applying the embeddings j_U^V, j_E^V and $j_U^{Ult(V,E)} \circ j_E^V$ to the iteration $\mathbb{P}_\kappa$; this information will be used at various points to ensure that we are in the scope of Lemma 2.27.

By standard arguments in inner model theory we may assume that:

- $V = K$ (so that in particular GCH holds).
- There is a unique total extender E_0 on the sequence for K such that κ is not measurable in $\mathrm{Ult}(K, E_0)$.
- The extender E_0 is equivalent to some normal measure U on κ, which is the unique such measure of order zero.
- κ is the unique strong cardinal and the largest measurable cardinal.

Let $\mathbb{P}_\kappa$ be the iteration with NS support where we force with $\mathrm{NR}(\alpha)$ for each V-measurable $\alpha < \kappa$. We note that by the analysis of Lemma 2.21 (or the theory of gap forcing [9]) no new measurable cardinals can appear in the course of the iteration, so that after forcing with $\mathbb{P}_\kappa$ there are no measurable cardinals below κ. Let G_κ be $\mathbb{P}_\kappa$-generic over V. We note that if W is a normal measure on κ in $V[G_\kappa]$ then by the usual reflection arguments κ must concentrate on cardinals which reflect stationary sets, and hence W concentrates on cardinals which are non-measurable in V.

Claim 1.1. *The cardinal κ is tall in $V[G_\kappa]$.*

Proof. Let λ be a strong limit cardinal with $\kappa < \mathrm{cf}(\lambda) < \lambda$. Appealing to Lemma 2.24 we fix a (κ, λ)-extender E and a function $h : \kappa \to \kappa$ such that:

- $V_\lambda \subseteq \mathrm{Ult}(V, E)$ and ${}^\kappa \mathrm{Ult}(V, E) \subseteq \mathrm{Ult}(V, E)$.
- $j_E(\kappa) > \lambda = j_E(h)(\kappa)$.
- In $\mathrm{Ult}(V, E)$ there are no measurable cardinals in the half-open interval $(\kappa, \lambda]$.
- For every $\eta < \kappa$, there are no measurable cardinals in the half-open interval $(\eta, h(\eta)]$.

Let $M_1 = \mathrm{Ult}(V, E)$, and note that $U \in M_1$ where U is the unique measure of order zero on κ. Let $M_2 = \mathrm{Ult}(M_1, E)$, $i = j_U^{M_1}$, and $j = i \circ j_E^V$. In order to show that κ is tall in $V[G_\kappa]$, we will construct in $V[G_\kappa]$ a lifting of the embedding j onto $V[G_\kappa]$. For this it will suffice to find $G_{j(\kappa)}$ which is $j(\mathbb{P}_\kappa)$-generic over M_2 and is such that $j``G_\kappa \subseteq G_{j(\kappa)}$.

Let $i^* = j_U^V$ and $N = \mathrm{Ult}(V, U)$. Recall from Lemma 2.25 that $i^* \restriction M_1 = i$, and that $j = j^N_{i^*(E)} \circ i^*$. We will find this information useful in lifting j onto $V[G_\kappa]$.

Subclaim 1.1.1. *The filter $H \in V[G_\kappa]$ generated by $i^*``G_\kappa$ is $i^*(\mathbb{P}_\kappa)$-generic over N.*

Proof. We will appeal to Lemma 2.27 to transfer G_κ along the ultrapower map $i^* : V \longrightarrow N$. Note that κ is not measurable in N, so it is not in the support of $i^*(\mathbb{P})_\kappa$ and we are in the situation of part 1 of the conclusion of the lemma.

To check that the hypotheses of Lemma 2.27 are satisfied, let $D \in N$ be a dense subset of $i^*(\mathbb{P}_\kappa)$ and write $D = i^*(d)(\kappa)$, where we may assume that $d(\alpha)$ is a dense subset of $\mathbb{P}_\kappa$ for all α. Then set $\mathcal{D}_\alpha = \{d(\alpha)\}$. This concludes the proof of Subclaim 1.1.1. □

By Lemma 2.13 $V[G_\kappa] \models {}^\kappa N[G_\kappa] \subseteq N[G_\kappa]$, and so easily $V[G_\kappa] \models {}^\kappa N[H] \subseteq N[H]$.

Subclaim 1.1.2. *There is a filter $g \in V[G_\kappa]$ which is* $\mathrm{NR}(i^*(\kappa))$ *generic over $N[H]$.*

Proof. By GCH we see that $|i^*(\kappa^+)| = \kappa^+$, so that working in $V[G_\kappa]$ we may enumerate the dense subsets of $\mathrm{NR}(i(\kappa))$ which lie in $N[H]$ in order type κ^+. Since $V[G_\kappa] \models {}^\kappa N[H] \subseteq N[H]$, and $\mathrm{NR}(i^*(\kappa))$ is κ^+-strategically closed in $N[H]$ by Fact 2.1, we may then build a suitable generic object g in the standard way. This concludes the proof of Subclaim 1.1.2. □

Subclaim 1.1.3. *There is a filter $G_{j(\kappa)} \in V[G_\kappa]$ which is $j(\mathbb{P}_\kappa)$-generic over M_2 and is such that $j``G_\kappa \subseteq G_{j(\kappa)}$.*

Proof. We start by showing that for every dense subset D of $j_E^V(\mathbb{P}_\kappa)$ lying in M_1, there exists in V a sequence $\vec{\mathcal{D}} = (\mathcal{D}_\alpha)_{\alpha<\kappa}$ such that $|\mathcal{D}_\alpha|$ is less than the least measurable cardinal greater than α, and $D \in j_E^V(\vec{\mathcal{D}})_\kappa$.

To see this we recall that $j_E(h)(\kappa) = \lambda$ and that $h(\alpha)$ is less that the least measurable cardinal greater than α. We fix $a \in [\lambda]^{<\omega}$ and d such that $D = i(d)(a)$, where d is a function with domain $[\kappa]^{|a|}$ such that $d(x)$ is a dense subset of $\mathbb{P}_\kappa$ for all x, and then set

$$\mathcal{D}_\alpha = \{d(x) : x \in [h(\alpha)]^{|a|}\}.$$

We now apply the elementary embedding i^* to see that the hypotheses of Lemma 2.27 are satisfied in N by the iteration $i^*(\mathbb{P}_\kappa)$ and the embedding $j^N_{i^*(E)}$. Since κ is measurable in M_1, $i^*(\kappa)$ is measurable in M_2, and we are in the situation of part 2 of the conclusion of the lemma. We use $H * g$ to build $G_{j(\kappa)}$ such that $j^N_{i^*(E)}$"$H \subseteq G_{j(\kappa)}$. Since i^*"$G_\kappa \subseteq H$ we see that j"$G_\kappa \subseteq G_{j(\kappa)}$ as required. This concludes the proof of Subclaim 1.1.3. $\square$

Using the preceding results we may obtain an embedding $j : V[G_\kappa] \longrightarrow M[G_{j(\kappa)}]$, where $j(\kappa) > \lambda$ and $V[G_\kappa] \models {}^\kappa M[G_{j(\kappa)}] \subseteq M[G_{j(\kappa)}]$. This concludes the proof of Claim 1.1. $\square$

Claim 1.2. *In $V[G_\kappa]$ the cardinal κ carries a unique normal measure.*

Proof. By standard arguments, the lifted map $i^* : V[G_\kappa] \longrightarrow N[H]$ has the form $j^{V[G]}_{\bar{U}}$, where $\bar{U}$ is the induced normal measure on κ. We will show that this is the only normal measure on κ.

Let $W \in V[G_\kappa]$ be an arbitrary normal measure on κ. Since W concentrates on cardinals which reflect stationary sets, W must concentrate on cardinals which are not measurable in V. By the analysis in Lemma 2.22, there exists $H' \in V[G_\kappa]$ which is generic over N for $i^*(\mathbb{P}_\kappa)$, and is such that i^*"$G_\kappa \subseteq H'$ and $j^{V[G_\kappa]}_W$ is the result of lifting i^* to an embedding from $V[G_\kappa]$ to $N[H']$. Since i^*"G_κ generates H we see that $H = H'$, hence $j^{V[G_\kappa]}_W = j^{V[G_\kappa]}_{\bar{U}}$ and $W = \bar{U}$. This concludes the proof of Claim 1.2 $\square$

This concludes the proof of Theorem 1. $\square$

3.2. **The least measurable cardinal is tall with several normal measures.**

Theorem 2. *It is consistent (modulo the consistency of a strong cardinal) that the least measurable cardinal is tall and carries exactly two normal measures.*

Proof. The proof can be viewed as a "pushout" of Theorem 1 from this paper and the main theorem of [5]. Following Friedman and Magidor we use Sacks forcing and Jensen coding to increase the number of measures on κ in a controlled way, while simultaneously killing measurable cardinals below κ in the same way as in Theorem 1. To show that κ is still tall we do a lifting argument of the same general kind as in the proof of Theorem 1, but new technical issues arise because of the presence of the Sacks and coding forcings in the iteration. The analysis of the measures on κ in the final model is parallel to that in Theorem 1, here the presence of the $\mathrm{NR}(\alpha)$ forcing in the iteration does not cause major new difficulties.

Using inner model theory, we make the same assumptions about V as we did in the proof of Theorem 1, but we add one extra assumption:

- There is a sequence $(S^\alpha)_{\alpha<\kappa}$ such that:
  - For each α, $S^\alpha = (S^\alpha_i)_{i<\alpha^+}$ is a partition of $\alpha^+ \cap \mathrm{cof}(\alpha)$ into disjoint stationary sets.
  - $[\alpha \mapsto S^\alpha]_U = S^\kappa = (S_i)_{i<\kappa^+}$ where the sets S_i form a partition of $\kappa^+ \cap \mathrm{cof}(\kappa)$ into sets which are stationary *in* V (not just in $\mathrm{Ult}(V, U)$ as guaranteed by Los' theorem).

Now we define an NS iteration $\mathbb{P}_{\kappa+1}$ of length $\kappa+1$ where for inaccessible $\alpha \le \kappa$:

- If α is measurable we force with $\mathrm{Sacks}(\alpha) * \mathrm{Code}(\alpha) * \mathrm{NR}(\alpha)$.
- If α is non-measurable or $\alpha = \kappa$, and h"$\alpha \subseteq \alpha$, then we force with $\mathrm{Sacks}(\alpha) * \mathrm{Code}(\alpha)$.

Here the coding forcing Code(α) is defined using S^α. We note that forcing with the α-closed forcing poset Sacks(α) $*$ Code(α) does not change the definition of NR(α), so in the first case we may view the iterand at α as the product (Sacks(α) $*$ Code(α)) $\times$ NR(α).

Note that for every inaccessible α, it is forced that $\mathbb{P}/G_{\alpha+1}$ has a dense $h(\alpha)^+$-closed subset: this is true by the properties of h (which handle the measurable cardinals in the support) plus the fact that non-measurable cardinals in the support must be closed under h.

Following our usual convention, we will denote the generic object at stage α by "g_α". We write "g_α^{Sacks}" for the Sacks(α)-generic component and "g_α^{Code}" for the Code(α)-generic component.

Claim 2.1. *The cardinal κ is tall in $V[G_{\kappa+1}]$.*

Proof. As in the proof of Claim 1.1, we fix λ strong limit with $\kappa < \text{cf}(\lambda) < \lambda$. We choose E, h, $M_1 = \text{Ult}(V, E)$, $M_2 = \text{Ult}(M_1, U)$, $N = \text{Ult}(V, U)$, $i = j_U^{M_1}$, $i^* = j_U^V$, and $j = i \circ j_E^V = j_{i^*(E)}^N \circ i^*$ exactly as before.

We collect some information for use in the various transfer arguments:

- The iteration $i^*(\mathbb{P}_\kappa)$ has $\mathbb{P}_{\kappa+1}$ as an initial segment, because κ is not measurable in N.
- The iteration $j(\mathbb{P}_\kappa)$ has $i^*(\mathbb{P}_\kappa)$ as an initial segment, and has $i^*(\text{Sacks}(\kappa) * \text{Code}(\kappa) \times \text{NR}(\kappa))$ as the iterand at coordinate $i^*(\kappa)$.

Subclaim 2.1.1. *There is a filter $H \in V[G_{\kappa+1}]$ which is generic over N for the poset $i^*(\mathbb{P}_\kappa)$ and is such that i^*"$G_\kappa \subseteq H$.*

Proof. As in the proof of Subclaim 1.1.1, we appeal to Lemma 2.27. The hypotheses are satisfied exactly as before. We are now in the situation of part 2 of the conclusion, and we will build the generic object H using $G_{\kappa+1}$. □

As usual, we may lift $i^* : V \longrightarrow N$ to an embedding $i^* : V[G_\kappa] \longrightarrow N[H]$. Note that since we built H using G_κ, we have $g_\kappa \in N[H]$.

The following result is easy, but we include a proof for the sake of completeness.

Subclaim 2.1.2. *For every η with $\kappa < \eta < i^*(\kappa)$ (resp. $\kappa < \eta < j_E^V(\kappa)$) there is $C \in V$ a club subset of κ such that $i^*(C)$ (resp. $j_E^V(C)$) is disjoint from (κ, η).*

Proof. In case $\kappa < \eta < i^*(\kappa)$, let $\eta = i(f)(\kappa)$ for some $f : \kappa \to \kappa$ and let $C = \{\alpha : f``\alpha \subseteq \alpha\}$. In case $\kappa < \eta < j_E^V(\kappa)$, let $\eta = j_E^V(f)(a)$ for $a \in [\lambda]^{<\omega}$ and $f : [\kappa]^{|a|} \to \kappa$, and let $C = \{\alpha : f``[h(\alpha)]^{|a|} \subseteq \alpha\}$. □

Subclaim 2.1.3. *There is a filter $h_0^{\text{Sacks}} \in V[G_{\kappa+1}]$ which is $i^*(\text{Sacks}(\kappa))$-generic over $N[H]$, and is such that i^*"$g_\kappa^{\text{Sacks}} \subseteq h_0^{\text{Sacks}}$.*

Proof. This follows from Lemma 2.28 applied to the embedding $i^* : V[G_\kappa] \longrightarrow N[H]$, and the generic object g_κ^{Sacks}. Hypothesis 1 holds by the construction of H, hypothesis 2 holds by Subclaim 2.1.2, and hypothesis 3 holds by the same analysis as we used in Subclaim 1.1.1. □

Remark. We actually have two options for choosing h_0^{Sacks}, since κ is not measurable in N, but this is irrelevant for the current claim. In the proof of Claim 2.2 below this point becomes crucial.

We may now lift the embedding $i^* : V[G_\kappa] \longrightarrow N[H]$ to obtain an embedding $i^* : V[G_\kappa * g_\kappa^{\rm Sacks}] \longrightarrow N[H * h_0^{\rm Sacks}]$.

Subclaim 2.1.4. *There is a filter $h_0^{\rm Code} \in V[G_{\kappa+1}]$ which is $i^*({\rm Code}(\kappa))$-generic over $N[H * h_0^{\rm Sacks}]$, and is such that i^*"$g_\kappa^{\rm Code} \subseteq h_0^{\rm Code}$.*

Proof. Since Code(κ) adds no κ-sequences of ordinals, this is immediate from Fact 2.26. □

Let $h_0 = h_0^{\rm Sacks} * h_0^{\rm Code}$, and lift i^* once again to obtain $i^* : V[G_{\kappa+1}] \longrightarrow N[H * h_0]$.

Subclaim 2.1.5. *There is a filter $g \in V[G_{\kappa+1}]$ which is ${\rm NR}(i^*(\kappa))$-generic over $N[H * h_0]$.*

Proof. The argument is similar to that for Subclaim 1.1.2, only this time we work in $V[G_{\kappa+1}]$. By Fact 2.15 , $V[G_{\kappa+1}] \models {}^\kappa N[G_{\kappa+1}] \subseteq N[G_{\kappa+1}]$, from which it follows that $V[G_{\kappa+1}] \models {}^\kappa N[H * h_0] \subseteq N[H * h_0]$. Now we build g in the same way as before. □

Subclaim 2.1.6. *There is a filter $G_{j(\kappa)} \in V[G_{\kappa+1}]$ which is $j(\mathbb{P}_\kappa)$-generic over M_2 and is such that j"$G_\kappa \subseteq G_{j(\kappa)}$.*

Proof. The argument is exactly parallel to that for Subclaim 1.1.3. We will build $G_{j(\kappa)}$ using $H * (h_0 * g)$. □

As usual we may lift $j : V \longrightarrow M_2$ to obtain $j : V[G_\kappa] \longrightarrow M_2[G_{j(\kappa)}]$. We note that by construction $j^N_{i^*(E)}$"$H \subseteq G_{j(\kappa)}$ so that we also have a lifted map $j^N_{i^*(E)} : N[H] \longrightarrow M_2[G_{j(\kappa)}]$.

To finish the argument we need to transfer g_κ. Since we already did the work of transferring g_κ along i^* to obtain the generic object h_0, our remaining task is to transfer h_0 along $j^N_{i^*(E)} : N[H] \longrightarrow M_2[G_{j(\kappa)}]$.

Subclaim 2.1.7. *There is a filter $h^{\rm Sacks} \in V[G_{\kappa+1}]$ which is $j({\rm Sacks}(\kappa))$-generic over $M_2[G_{j(\kappa)}]$, and is such that j"$g_\kappa^{\rm Sacks} \subseteq h^{\rm Sacks}$.*

Proof. This follows from Lemma 2.28 applied to the embedding $j^N_{i^*(E)} : N[H] \longrightarrow M_2[G_{j(\kappa)}]$ and the generic object $h_0^{\rm Sacks}$. Hypothesis 1 holds because we made sure to include $h_0^{\rm Sacks}$ in $G_{j(\kappa)}$, hypothesis 2 holds by Subclaim 2.1.2, and hypothesis 3 holds by the same analysis as was used in Subclaim 1.1.3. □

We may therefore lift j to obtain $j : V[G_\kappa * g_\kappa^{\rm Sacks}] \longrightarrow M_2[G_{j(\kappa)} * h^{\rm Sacks}]$, and we may also obtain a lifted embedding $j^N_{i^*(E)} : N[H * h_0^{\rm Sacks}] \longrightarrow M_2[G_{j(\kappa)} * h^{\rm Sacks}]$.

Subclaim 2.1.8. *There is a filter $h^{\rm Code} \in V[G_{\kappa+1}]$ which is $j({\rm Code}(\kappa))$-generic over $M_2[G_{j(\kappa)} * h^{\rm Sacks}]$, and is such that j"$g_\kappa^{\rm Code} \subseteq h^{\rm Code}$.*

Proof. To see this we apply Fact 2.26 to the embedding $j^N_{i^*(E)} : N[H * h_0^{\rm Sacks}] \longrightarrow M_2[G_{j(\kappa)} * h^{\rm Sacks}]$ and the generic object $h_0^{\rm Code}$, using the fact that this is generic for a poset which adds no $i^*(\kappa)$-sequences. □

With these results in hand we may now work in $V[G_{\kappa+1}]$, and lift $j : V \longrightarrow M_2$ to obtain $j : V[G_{\kappa+1}] \longrightarrow M_2[G_{j(\kappa)} * g_{j(\kappa)}]$, where $g_{j(\kappa)} = h^{\rm Sacks} * h^{\rm Code}$. To finish the verification that κ is tall in $V[G_{\kappa+1}]$, we should check that the target model is closed under κ-sequences. This is immediate because $V[G_{\kappa+1}] \models {}^{\kappa}M_2[G_{\kappa+1}] \subseteq M_2[G_{\kappa+1}]$, and the part of $G_{j(\kappa)} * g_{j(\kappa)}$ above κ adds no κ-sequences of ordinals. This concludes the proof of Claim 2.1. $\square$

Claim 2.2. *The cardinal κ carries exactly two normal measures in $V[G_{\kappa+1}]$.*

Proof. Our argument is very similar to that of [6, Lemmas 9 and 10]. We revisit the proof of Subclaims 2.1.3 and 2.1.4, and use the splitting at level κ in the Sacks part to produce h_0^{left} and h_0^{right} which are both generic for $i^*(\text{Sacks}(\kappa) * \text{Code}(\kappa))$ over $N[H]$, and which both contain $i^{*}``g_\kappa$ as a subset. Now we may lift i^* to get embeddings from $V[G_{\kappa+1}]$ to each of $N[H * h_0^{\text{left}}]$ and $N[H * h_0^{\text{right}}]$, and derive normal measures U^{left} and U^{right} on κ in $V[G_{\kappa+1}]$.

We now claim that U^{left} and U^{right} are the only two normal measures on κ in $V[G_{\kappa+1}]$. We assume that W is such a normal measure, and consider the ultrapower map $j_W^{V[G_{\kappa+1}]}$. Using Fact 2.15 to get the necessary bounding property, it follows from Lemma 2.22 that $j_W^{V[G_{\kappa+1}]} \restriction V = i^*$, and that $j_W^{V[G_{\kappa+1}]}$ is a lift of i^* which is completely determined by $j_W^{V[G_{\kappa+1}]}(G_{\kappa+1})$.

It remains to analyse the possibilities for $j_W^{V[G_{\kappa+1}]}(G_{\kappa+1})$. Exactly as in [5, Lemma 9]:

- Since $\text{crit}(j_W^{V[G_{\kappa+1}]}) = \kappa$, the restriction of $j_W^{V[G_{\kappa+1}]}(G_\kappa)$ to κ is G_κ.
- By Fact 2.14, the only element of $V[G_{\kappa+1}]$ which is $\text{Sacks}(\kappa) * \text{Code}(\kappa)$-generic over $N[G_\kappa]$ is g_κ, hence the restriction of $j_W^{V[G_{\kappa+1}]}(G_\kappa)$ to $\kappa+1$ is $G_{\kappa+1}$.
- By the definition of H, and using the fact that $i^{*}``G_\kappa \subseteq j_W^{V[G_{\kappa+1}]}(G_\kappa)$, we see that H must agree with $j_W^{V[G_{\kappa+1}]}(G_{\kappa+1})$ in the interval $(\kappa, i^*(\kappa))$, so that $j_W^{V[G_{\kappa+1}]}(G_\kappa) = H$.
- By Lemma 2.28 and Fact 2.26, the only two possibilities for $j_W^{V[G_{\kappa+1}]}(g_\kappa)$ are h^{left} and h^{right}.

This analysis shows that there are only two possibilities for $j_W^{V[G_{\kappa+1}]}(G_{\kappa+1})$, namely $H * h^{\text{left}}$ and $H * h^{\text{right}}$. This implies that W is either U^{left} or U^{right}. This concludes the proof of Claim 2.2. $\square$

This concludes the proof of Theorem 2. $\square$

Suppose now we wish to have the least measurable cardinal κ be tall and carry μ normal measures for $2 < \mu \leq \kappa^+$. Following Friedman and Magidor we will simply replace $\text{Sacks}(\alpha)$ by a variation in which, at each splitting level β, a node on level β has $h_\mu(\beta)$ successors where h_μ is the μ^{th} canonical function from κ to κ. The argument is exactly as for Theorem 2, the key point is that now $j(h_\mu)(\kappa) = \mu$ so that the "tuning fork" argument as in Subclaim 2.1.3, using a suitably modified version of Lemma 2.28, provides us with exactly μ distinct compatible generic objects for Sacks forcing at $j(\kappa)$.

3.3. The least measurable cardinal is tall with many normal measures.

Theorem 3. *Let κ be strong with no measurable cardinals above κ and $2^\kappa = \kappa^+$. Let μ be a cardinal with $\text{cf}(\mu) > \kappa^+$. Then there is a generic extension in which $2^\kappa = \kappa^+$, $2^{\kappa^+} = \mu$, κ is the least measurable cardinal, κ is tall and κ carries the maximal number μ of normal measures.*

Proof. Compared with Theorems 1 and 2, the proof is rather straightforward. The reason is that here we *want* an explosion of normal measures in the generic extension, so we do not need any elaborate machinery to control them. We use Hamkins' proof that the least measurable cardinal can be tall [10, Theorem 4.1], the only difference being that we work over a ground model where κ is strong and 2^{κ^+} is large.

We start by forcing to make the strongness of κ indestructible under κ^+-closed forcing, using for example the indestructibility iteration from [7]. Then we force with $\mathrm{Add}(\kappa^+, \mu)$ to produce an extension V' where κ is strong, $2^\kappa = \kappa^+$ and $2^{\kappa^+} = \mu$. Working in V' we fix a measure U of order zero on κ, and let $\mathbb{P}$ be the iteration of $\mathrm{NR}(\alpha)$ with Easton support for each measurable $\alpha < \kappa$. Let G be $\mathbb{P}$-generic over V', then by Hamkins' arguments κ is tall and the least measurable cardinal in $V'[G]$.

Let $i : V' \longrightarrow M' = \mathrm{Ult}(V', U)$ be the ultrapower map. Since $2^\kappa = \kappa^+$, κ is not measurable in M' and $V'[G] \models {}^\kappa M'[G] \subseteq M'[G]$, we see that:

- The forcing poset $i(\mathbb{P})/G$ is κ^+-closed in $V'[G]$.
- The set of antichains of $i(\mathbb{P})/G$ which lie in $M'[G]$ has size κ^+ in $V'[G]$.

Since every condition has two incompatible extensions, we may build a complete binary tree of height κ^+ where each branch generates a distinct generic object for $i(\mathbb{P})/G$ over $M'[G]$. This implies that there are μ distinct generic objects.

For each $H \in V'[G]$ which is $i(\mathbb{P})/G$-generic over $M'[G]$, we may lift i to obtain a map $i_H : V'[G] \longrightarrow M'[G][H]$. By the usual arguments i_H is the ultrapower map obtained from a measure U_H, and we can recover H from U_H because $G * H = j_{U_H}^{V'[G]}(G)$, so that κ carries μ distinct normal measures in $V'[G]$. This concludes the proof of Theorem 3. □

4. The least measurable limit of tall cardinals

As we mentioned in the introduction, measurable limits of tall cardinals are tall and the least measurable limit κ of tall cardinals is not even $(\kappa + 2)$-strong, providing a simple example of a non-strong tall cardinal. As we also mentioned, Schindler has shown that in canonical inner models for large cardinals every tall cardinal is either strong or a measurable limit of strong cardinals. In this section we investigate the number of measures on tall cardinals of this type.

Arguments from inner model theory show that if it is consistent that there is a measurable limit of strong cardinals, then it is consistent that the least measurable limit of tall cardinals carries a unique normal measure. For example, this will be the case in the ground model for the forcing construction that proves Theorem 4 below.

4.1. **The least measurable limit of tall cardinals with several normal measures.**

Theorem 4. *It is consistent (modulo the consistency of a measurable limit of strong cardinals) that the least measurable limit of tall cardinals carries exactly two normal measures.*

Proof. The construction is rather similar to that for Theorem 2, but is simpler in that we do not need to kill measurable cardinals by adding non-reflecting stationary sets. The main novel point is that we need to choose the support of the iteration rather carefully, so that we can lift enough embeddings to argue that the strong cardinals below κ remain tall in the final model.

Using inner model theory, we may assume that V satisfies the following list of properties:

- $V = K$.
- There is a cardinal κ which is the least measurable limit of strong cardinals and the largest measurable cardinal.
- The cardinal κ has a unique normal measure U of order zero.
- The measure U is equivalent to some total extender with critical point κ on K's extender sequence.
- There is a sequence $(A_\alpha)_{\alpha<\kappa}$ such that A_α is a partition of $\alpha^+ \cap \mathrm{cof}(\alpha)$ into α^+ disjoint stationary pieces, and if we let $A_\kappa = [\alpha \mapsto A_\alpha]_U$ then A_κ is a partition of $\kappa^+ \cap \mathrm{cof}(\kappa)$ into pieces that are stationary in V.

By the result of Schindler [14] mentioned above, the only tall cardinals in V are κ and the strong cardinals below κ. In particular κ is the least measurable limit of tall cardinals. Our assumption that κ is the least measurable limit of strong cardinals implies that κ carries no measures of order greater than zero, so that in fact U is the unique normal measure on κ.

Let B be the set of $\alpha < \kappa$ such that α is an inaccessible limit of strongly unfoldable cardinals. By Fact 2.19 U concentrates on B. Let $\mathbb{P}_{\kappa+1}$ be an NS iteration where we first add a Cohen subset of ω, and then force with $\mathrm{Sacks}(\alpha) * \mathrm{Code}(\alpha)$ for all $\alpha \in B \cup \{\kappa\}$, using the partition A_α as the parameter in the definition of $\mathrm{Code}(\alpha)$. The Cohen set at the start of the iteration will make the iteration into a "forcing with a very low gap" in the sense of Hamkins [9], which will be useful in Claim 4.3 below. We note that if U is the unique normal measure on κ then κ is an inaccessible limit of strongly unfoldable cardinals in $\mathrm{Ult}(V, U)$, so that κ is in the support of the iteration $j_U(\mathbb{P}_\kappa)$.

Claim 4.1. *For every strong $\lambda < \kappa$, λ is strong in $V[G_{\lambda+1}]$, and is tall in $V[G_{\kappa+1}]$.*

Proof. Once we have established that λ is strong in $V[G_{\lambda+1}]$ it will follow readily that λ is tall in $V[G_{\kappa+1}]$. To see this let F be any (λ, μ) extender in $V[G_{\lambda+1}]$ for some $\mu > \lambda$. Since the tail forcing $\mathbb{P}_{\kappa+1}/G_{\lambda+1}$ adds no λ-sequences of ordinals, we may lift the ultrapower map $j_F^{V[G_{\lambda+1}]}$ onto $V[G_{\kappa+1}]$ by transferring the tail-generic. In particular we may lift extenders witnessing that λ is strong in $V[G_{\lambda+1}]$ and obtain extenders witnessing that λ is tall in $V[G_{\kappa+1}]$.

To show that λ is strong in $V[G_{\lambda+1}]$, let μ be strong limit such that $\mu > \mathrm{cf}(\mu) > \lambda$. By Lemma 2.24 we may fix a (λ, μ)-extender E witnessing that λ is μ-strong with μ singular in $\mathrm{Ult}(V, E)$.

The key point is that the support of $j_E(\mathbb{P}_{\lambda+1})$ is empty in the interval $(\lambda, \mu]$. To see this suppose for contradiction that in $\mathrm{Ult}(V, E)$ this interval contains a point α which is an inaccessible limit of strongly unfoldable cardinals, and let δ be such that $\lambda < \delta < \alpha$ and δ is strongly unfoldable in $\mathrm{Ult}(V, E)$. Since $V_\mu \subseteq \mathrm{Ult}(V, E)$, λ is strong up to δ in $\mathrm{Ult}(V, E)$, so that by Fact 2.18 λ is strong in $\mathrm{Ult}(V, E)$. By the usual reflection arguments it follows that in V the cardinal λ is a strong limit of strong cardinals, contradicting our assumption that κ is the least measurable limit of strong cardinals.

To lift j_E onto $V[G_{\lambda+1}]$ we will use the ideas from the proof of Theorems 1 and 2. The situation here is simpler because we are only lifting the extender ultrapower map j_E, rather than the composition of j_E and a subsequent measure ultrapower.

Subclaim 4.1.1. *There is $H \in V[G_{\lambda+1}]$ such that H is $j_E(\mathbb{P}_\lambda)$-generic over* $\mathrm{Ult}(V, E)$ *and j_E“$G_\lambda \subseteq H$.*

Proof. This follows from Lemma 2.27, by an argument very similar to that for Subclaim 1.1.3. □

We may now lift j_E to obtain an embedding from $V[G_\lambda]$ to $\mathrm{Ult}(V, E)[H]$. In a mild abuse of notation we also denote the lifted embedding by “j_E”.

Subclaim 4.1.2. *For every η with $\lambda < \eta < j_E(\lambda)$, there is a club set $D \subseteq \lambda$ such that $j_E(D)$ is disjoint from the interval (λ, η).*

Proof. The proof is similar to that for Subclaim 2.1.2. □

Subclaim 4.1.3. *There is $h \in V[G_{\lambda+1}]$ such that h is $j_E(\mathrm{Sacks}(\lambda) * \mathrm{Code}(\lambda))$-generic over* $\mathrm{Ult}(V, E)[H]$ *and j_E“$g_\lambda \subseteq h$.*

Proof. The proof is similar to the proofs of Subclaims 2.1.7 and 2.1.8. □

We may now lift again to obtain an embedding $j_E : V[G_\lambda] \longrightarrow \mathrm{Ult}(V, E)[H * h]$. Since the iteration $j_E(\mathbb{P}_{\lambda+1})$ has empty support in $(\lambda, \mu]$, and $G_{\lambda+1}$ is an initial segment of H, it is easy to see that $V_\mu^{V[G_{\lambda+1}]} \subseteq \mathrm{Ult}(V, E)[H * h]$. It follows that the lifted map $j_E : V[G_\lambda] \longrightarrow \mathrm{Ult}(V, E)[H * h]$ witnesses that λ is μ-strong in $V[G_{\lambda+1}]$.

This concludes the proof of Claim 4.1. □

Claim 4.2. *The cardinal κ is measurable and carries exactly two normal measures in $V[G_{\kappa+1}]$.*

Proof. The argument that the unique measure on κ in V extends in exactly two ways is essentially that of Friedman and Magidor [5, Lemmas 9 and 10], see also the proof of Claim 2.2. □

Claim 4.3. *κ is the least measurable limit of tall cardinals in $V[G_{\kappa+1}]$.*

Proof. Since the iteration is a forcing with a very low gap, by results of Hamkins [9] it does not create any new measurable, tall or strong cardinals. □

This concludes the proof of Theorem 4. □

As in the remarks following Theorem 2, we may vary the form of Sacks forcing used and prove versions of Theorem 4 where the cardinal κ carries exactly μ normal measures for $1 < \mu \leq \kappa^+$.

4.2. **The least measurable limit of tall cardinals with many normal measures.**

Theorem 5. *Let κ be the least measurable limit of strong cardinals and the largest inaccessible cardinal. Let $V = K$. Let μ be a cardinal with $\mathrm{cf}(\mu) > \kappa^+$. Then there is a generic extension in which $2^\kappa = \kappa^+$, $2^{\kappa^+} = \mu$, κ is the least measurable limit of tall cardinals and κ carries the maximal number μ of normal measures.*

Proof. As with Theorem 3, the construction here is simple because we are not at pains to bound the number of normal measures. One subtle point is now that we need many strong cardinals to be somewhat indestructible, but fortunately we can appeal to results of the first author [1] to arrange this.

By the result of Schindler [14] mentioned earlier, the only tall cardinals in V are κ and the strong cardinals below κ, in particular κ is the least measurable limit of tall cardinals. The first step is to force with an Easton iteration of length κ with support contained in the set of measurable cardinals less than κ, which satisfies the conditions of Hamkins' gap forcing theorem [9] and makes each V-strong cardinal $\delta < \kappa$ indestructible under δ^+-closed forcing. See [1] for the details of the construction.

Let V_1 be the resulting model. If we choose U to be a measure of order zero on κ in V then the image of the indestructibility iteration under j_U does not have κ in its support, so that by standard arguments (using the fact that $2^\kappa = \kappa^+$) the measurability of κ is preserved in V_1. By the gap forcing theorem we have not created any new measurable, tall or strong cardinals. Hence κ is still the least measurable limit of tall cardinals, and the strong and tall cardinals still coincide below κ.

The next step is to force with $\mathrm{Add}(\kappa^+, \mu)$ over V_1, obtaining a model V_2 where $2^\kappa = \kappa^+$ and $2^{\kappa^+} = \mu$. By the indestructibility we arranged in step one, the strong cardinals below κ are still strong, and clearly κ is still measurable. We claim that no new strong (resp. tall) cardinals have been created: if $\delta < \kappa$ and δ is strong (resp. tall) in V_2 then by closure δ is strong (resp. tall) up to the next V_1-strong cardinal in V_1, so that δ is strong (resp. tall) in V_1. So it remains true in V_2 that κ is the least measurable limit of tall cardinals, and the strong and tall cardinals coincide below κ.

The final step is to force with an Easton iteration of length κ, where we force with $\mathrm{Add}(\alpha, 1)$ for each inaccessible cardinal $\alpha < \kappa$ which is not a limit of inaccessible cardinals. We obtain a model V_3. By arguments of Hamkins [8, Theorem 3.6] the strong cardinals below κ are preserved, and by the gap forcing theorem we have again not created any new measurable, tall or strong cardinals. Exactly as in the proof of Theorem 3, κ is still measurable and carries μ normal measures in V_3. This concludes the proof of Theorem 5. □

Some open questions

As we mentioned in the introduction, there is a strong analogy between the concepts of tallness and strong compactness. Many of the results we have proved in this paper for tallness are analogous to open questions about strong compactness:

- In a situation where the least measurable cardinal is strongly compact, what can we say about the number of normal measures on this cardinal?
- What can we say about the number of normal measures on the least measurable limit of strongly compact cardinals?

One major reason that tall cardinals are more tractable than strongly compact cardinals is that tall cardinals are within the scope of core model theory. The main property of the core model which we have used in this paper is that if $V = K$, $V[G]$ is a set-generic extension of V and $U \in V[G]$ with U a normal measure on κ, then $j_U^{V[G]} \restriction V$ is an iteration of V. In the absence of a core model theory for strongly compact cardinals, we may optimistically hope to produce models with this property by set forcing, and this leads to our final question:

- Is it possible to produce a set-generic extension V^* of V such that every ultrapower map coming from a normal measure in a generic extension of V^* induces an iteration of V^*?

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