THE FIRST MEASURABLE CARDINAL CAN BE THE FIRST UNCOUNTABLE REGULAR CARDINAL AT ANY SUCCESSOR HEIGHT

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Abstract. We use techniques due to Moti Gitik to construct models in which for an arbitrary ordinal ρ, $\aleph_{\rho+1}$ is both the least measurable and least regular uncountable cardinal.

1. Introduction

In this paper, we study cardinal patterns where the least uncountable regular cardinal is the least measurable cardinal. Jech [Jec68] and Takeuti [Tak70] independently showed that if we assume the consistency of ZFC+ "There is a measurable cardinal", then the theory ZF+DC+"ω_1 is the least measurable cardinal" is consistent, and this is an equiconsistency. We can also ask whether it is consistent for the least uncountable regular cardinal κ to be the least measurable cardinal and also be such that $\kappa > \omega_1$. The first author has proved that this is indeed the case. In particular, it follows from the work of [Apt96] that relative to the consistency of AD, it is consistent for $\aleph_2$ to be both the least measurable and least regular uncountable cardinal.

The methods used in the proof of this result may be extended to show that relative to the consistency of AD, it is possible to obtain models in which, e.g., $\aleph_{\omega+1}$ is both the least regular uncountable and least measurable cardinal, or $\aleph_{\omega+2}$ is both the least regular uncountable and least measurable cardinal, or there is an uncountable limit cardinal which is both the least regular and least measurable cardinal. However, these methods do not easily extend to cardinals such as $\aleph_3$ or $\aleph_{\omega+3}$.

In this paper we will prove the following theorem, which allows us to handle cardinals different from those provided by AD.

Theorem 1.1. *Let $V \models$ ZFC + "$\rho > 0$ is an ordinal" + "There is a sequence $\langle \kappa_\xi \; ; \; \xi < \rho \rangle$ of strongly compact cardinals such that each limit point of the sequence $\langle \kappa_\xi \; ; \; \xi < \rho \rangle$ is singular, and with a measurable cardinal κ_ρ above the supremum of the sequence". There is then a partial ordering in the ground model V and a symmetric model $V(G)$ of the theory ZF + "For each $1 \leq \beta \leq \rho$, $\aleph_\beta$ is singular" + "$\aleph_{\rho+1}$ is a measurable cardinal carrying a normal measure".*

Theorem 1.1 generalises earlier work of the first author (different from that found in [Apt96]). Specifically, the first author, in [Apt85] and with Henle in [AH91] and

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Date: May 23, 2014.

The authors were partially supported by the DFG-NWO cooperation grant KO 1353/5-1 "Infinitary combinatorics without the Axiom of Choice", and the Hausdorff Center for Mathematics of the University of Bonn. The first author was also partially supported by various PSC - CUNY grants. The first author wishes to thank the members of the Bonn Logic Group, and especially his coauthors, for all of the hospitality shown to him during his visits to Bonn. The contents of this paper also appeared as part of the second author's PhD thesis (see [Dim11]). We thank the referee for the conscientious and constructive reading and very helpful comments of several versions of this paper.

Magidor in [AM95], showed how to symmetrically collapse a measurable cardinal κ to be the successor of a singular cardinal of cofinality ω while preserving the measurability of κ. The large cardinal hypotheses used (instances of supercompactness in [Apt85] and [AM95] and instances of strong compactness in [AH91]) are considerably stronger than the existence of one measurable cardinal.

Note that the standard proof for the existence of a normal measure for a measurable cardinal requires the use of DC. Since $\aleph_1$ has cofinality ω in $V(G)$, both DC and AC_ω are false in this model. It is therefore especially relevant that $\aleph_{\rho+1}$ carries a normal measure in $V(G)$.

The forcing mentioned in the theorem above is a modification of Gitik's forcing in [Git80]. In addition to what Gitik proved about this forcing, we prove that in our version, none of the strongly compact cardinals collapses in the symmetric model $V(G)$, but all other regular uncountable cardinals below κ_ρ do. We modify Gitik's construction in several ways. This is because Gitik's (class sized) forcing does not focus on exactly which cardinals are preserved in his symmetric model N_G, but rather on preserving the power set axiom. In particular, the remarks found in [Git80, page 62, paragraph immediately following Theorem II] mention that the (well-ordered) cardinals of N_G are ω, the ground model strongly compact cardinals, and the (singular) limits of the ground model strongly compact cardinals. However, other than [Git80, Lemma 3.4], which indicates that every limit ordinal has cofinality ω in N_G, there is neither an explicit proof nor hint at any point in [Git80] as to how to determine the cardinal structure of N_G. By carefully reworking the definition of Gitik's forcing conditions in the context of set forcing, and by providing a detailed analysis of the nature of our partial ordering, we are able to determine precisely (see Corollary 2.9) the relevant cardinal structure of our symmetric inner model $V(G)$.

To collapse the intervals between the strongly compacts, we will choose fine ultrafilters over the $\mathcal{P}_{\kappa_\xi}(\alpha)$ for each $\alpha \in (\kappa_\xi, \kappa_{\xi+1})$ in order to make parts of the forcing be isomorphic with strongly compact Prikry forcings. The isomorphisms will ensure the relevant collapses. To ensure that an $\alpha \in (\omega, \kappa_0)$ has collapsed we will use a fine ultrafilter over $\mathcal{P}_\omega(\alpha)$ and the same proof. If there are any limit ordinals $\lambda < \rho$ (e.g., if $\rho = \omega + 1$) then similar arguments using fine ultrafilters will collapse the cardinals in the open interval $(\bigcup_{\xi<\lambda} \kappa_\xi, \kappa_\lambda)$. Forcing at each κ_ξ will be done with a κ_ξ-complete measure over κ_ξ. Finally, a small forcing argument guarantees that κ_ρ remains measurable in the symmetric forcing extension and carries a normal measure.

2. The Gitik construction

For our construction, we assume knowledge of forcing as presented in [Kun80, Ch. VII], [Jec03, Ch. 14], and of symmetric submodels as presented in [Jec03, Ch.15].

Let $\rho \in \mathsf{Ord}$. We start with an increasing sequence of cardinals,

$$\langle \kappa_\xi \ ; \ \xi < \rho \rangle,$$

such that for every $\xi < \rho$, κ_ξ is strongly compact, and such that the sequence has no regular limits. Let $\kappa_\rho > \bigcup_{\xi<\rho} \kappa_\xi$ be a measurable cardinal.

We will construct a model with a sequence of $\rho + 1$-many successive singular cardinals in which κ_ρ is the first regular uncountable cardinal, and it is still measurable. We will do this by modifying and adapting Gitik's construction of [Git80], whose notation and terminology we freely use.

Our construction is a finite support product of Prikry-like forcings which are interweaved in order to prove a Prikry-like lemma for that part of the forcing.

Call $\mathsf{Reg}^{\kappa_\rho}$ the set of regular cardinals $\alpha \in [\omega, \kappa_\rho)$ in V. For convenience call $\omega =: \kappa_{-1}$. For an $\alpha \in \mathsf{Reg}^{\kappa_\rho}$ we define a $\mathsf{cf}'\alpha$ to distinguish between the following categories.

(type 1) This occurs if there is a largest $\kappa_\xi \leq \alpha$ (i.e., $\alpha \in [\kappa_\xi, \kappa_{\xi+1})$). We then define $\mathsf{cf}'\alpha := \alpha$.

If $\alpha = \kappa_\xi$ and $\xi \neq -1$, then let Φ_{κ_ξ} be a measure for κ_ξ. If $\alpha = \omega$ then let Φ_ω be any uniform ultrafilter on ω.

If $\alpha > \kappa_\xi$ is inaccessible, then let H_α be a κ_ξ-complete fine ultrafilter over $\mathcal{P}_{\kappa_\xi}(\alpha)$ and let $h_\alpha : \mathcal{P}_{\kappa_\xi}(\alpha) \to \alpha$ be a bijection. Define

$$\Phi_\alpha := \{X \subseteq \alpha \ ; \ h_\alpha^{-1}``X \in H_\alpha\}.$$

This is a uniform κ_ξ-complete ultrafilter over α.

If $\alpha > \kappa_\xi$ is not inaccessible, then let Φ_α be any κ_ξ-complete uniform ultrafilter over α.

(type 2) This occurs if there is no largest strongly compact $\leq \alpha$. We then let β be the largest (singular) limit of strongly compacts $\leq \alpha$. Define $\mathsf{cf}'\alpha := \mathsf{cf}\beta$. Let

$$\langle \kappa_\nu^\alpha \ ; \ \nu < \mathsf{cf}'\alpha \rangle$$

be a fixed ascending sequence of strongly compacts $\geq \mathsf{cf}'\alpha$ such that $\beta = \bigcup\{\kappa_\nu^\alpha \ ; \ \nu < \mathsf{cf}'\alpha\}$.

If α is inaccessible, then for each $\nu < \mathsf{cf}'\alpha$, let $H_{\alpha,\nu}$ be a fine ultrafilter over $\mathcal{P}_{\kappa_\nu^\alpha}(\alpha)$ and $h_{\alpha,\nu} : \mathcal{P}_{\kappa_\nu}(\alpha) \to \alpha$ a bijection. Define

$$\Phi_{\alpha,\nu} := \{X \subseteq \alpha \ ; \ h_\alpha^{-1}``X \in H_{\alpha,\nu}\}.$$

Again, $\Phi_{\alpha,\nu}$ is a κ_ν^α-complete uniform ultrafilter over α.

If α is not inaccessible, then for each $\nu < \mathsf{cf}'\alpha$, let $\Phi_{\alpha,\nu}$ be any κ_ν^α-complete uniform ultrafilter over α.

This $\mathsf{cf}'\alpha$ will be used when we want to organise the choice of ultrafilters for the type 2 cardinals.

We use the fine ultrafilters H_α and $H_{\alpha,\nu}$ to make sure that in the end only the strongly compacts and their singular limits remain cardinals below κ_ρ. For type 1 ordinals we will do some tree-Prikry like forcings to singularise in cofinality ω. Type 1 cardinals in the open intervals $(\kappa_\xi, \kappa_{\xi+1})$ will be collapsed to κ_ξ because enough of these forcings will be isomorphic to strongly compact Prikry forcings (or "fake" strongly compact Prikry forcings in the case of $\xi = -1$). This is why we use fine ultrafilters for these cardinals.

To singularise type 2 ordinals Gitik used a technique he credits in [Git80] to Magidor, a Prikry-type forcing that relies on the countable cofinal sequence $\vec{c}$ that we build for $\mathsf{cf}'\alpha$ to pick a countable sequence of ultrafilters $\langle \Phi_{\vec{c}(n)} \ ; \ n \in \omega \rangle$. To show that the type 2 cardinals are collapsed, we use again the fine ultrafilters.

As usual with Prikry-type forcings, one has to prove a Prikry-like lemma (see [Git80, Lemma 5.1]). For the arguments one requires the forcing conditions to grow nicely.

These conditions can be viewed as trees. These trees will grow from "left to right" in order to ensure that a type 2 cardinal α will have the necessary information from the Prikry sequence[1] at stage $\mathsf{cf}'\alpha$. Let us take a look at the definition of the stems of the Prikry sequences to be added.

Definition 2.1. For $t \subseteq \mathsf{Reg}^{\kappa_\rho} \times \omega \times \kappa_\rho$ we define the sets

$$\mathsf{dom}(t) := \{\alpha \in \mathsf{Reg}^{\kappa_\rho} \ ; \ \exists m \in \omega \exists \gamma \in \mathsf{Ord}((\alpha, m, \gamma) \in t)\}, \text{ and}$$
$$\mathsf{dom}^2(t) := \{(\alpha, m) \in \mathsf{Reg}^{\kappa_\rho} \times \omega \ ; \ \exists \gamma \in \mathsf{Ord}((\alpha, m, \gamma) \in t)\}.$$

Let P_1 be the set of all finite subsets t of $\mathsf{Reg}^{\kappa_\rho} \times \omega \times \kappa_\rho$, such that for every $\alpha \in \mathsf{dom}(t)$, $t(\alpha) := \{(m, \gamma) \ ; \ (\alpha, m, \gamma) \in t\}$ is an injective function from some finite subset of ω into α.

To add a Prikry sequence to a type 2 cardinal α, we want to have some information on the Prikry sequence of the cardinal $\mathsf{cf}'\alpha$. We also want to make sure that these stems are appropriately ordered for the induction in the proof of the aforementioned Prikry-like lemma. So we define the following.

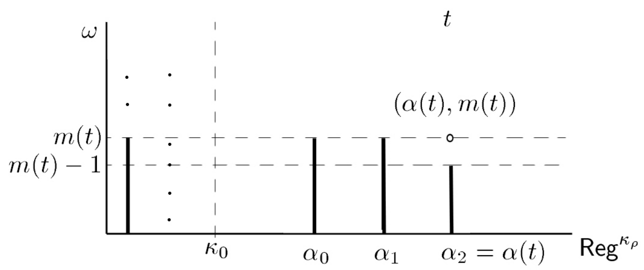
Definition 2.2. Let P_2 be the set of all $t \in P_1$ such that the following hold.

(1) For every $\alpha \in \mathsf{dom}(t)$, $\mathsf{cf}'\alpha \in \mathsf{dom}(t)$ and $\mathsf{dom}(t(\mathsf{cf}'\alpha)) \supseteq \mathsf{dom}(t(\alpha))$.
(2) If $\{\alpha_0, \ldots, \alpha_{n-1}\}$ is an increasing enumeration of $\mathsf{dom}(t) \setminus \kappa_0$, then there are $m, j \in \omega$, such that $m \geq 1$, $j \leq n-1$ with the properties that
   - for every $k < j$ we have that $\mathsf{dom}(t(\alpha_k)) = m + 1$ and
   - for every $k \in \{j, \ldots, n-1\}$ we have that $\mathsf{dom}(t(\alpha_k)) = m$.

   These m and α_j are unique for t and are denoted by $m(t) := m$ and $\alpha(t) := \alpha_j$.
(3) If $\mathsf{cf}'\alpha(t) < \alpha(t)$ then $m(t) \in \mathsf{dom}(t(\mathsf{cf}'\alpha(t)))$.

We may think of the point $(\alpha(t), m(t))$ as the point we have to fill in next, in order to extend t. As we will see in the next definition, the value of $t(\mathsf{cf}'\alpha(t))(m(t))$ will decide how the condition t will grow at the point $(\alpha(t), m(t))$.

Let us call elements of P_2 stems. In the following image we can see roughly what a stem t with a domain $\{\alpha_0, \alpha_1, \alpha_2\}$ above κ_0 looks like.



In order to add Prikry sequences, we will use the ultrafilters and define the partial ordering with which we will force.

Definition 2.3. Let P_3 be the set of pairs (t, T) such that

(1) $t \in P_2$,
(2) $T \subseteq P_2$,
(3) $t \in T$,
(4) for every $t' \in T$ we have $t' \supseteq t$ or $t' \subseteq t$, and $\mathsf{dom}(t') = \mathsf{dom}(t)$,
(5) for every $t' \in T$, if $t' = r \cup \{(\alpha(r), m(r), \beta)\}$ then $t'^{-} := r \in T$, i.e., T is tree-like,

[1]For the rest of this paper, we will abuse terminology by using phrases like "Prikry sequence" and "Prikry forcing" when referring to our Prikry-like forcing notions.

(6) for every $t' \in T$ with $t' \supseteq t$, if $\alpha(t')$ is of type 1 (i.e., $\mathsf{cf}'(\alpha(t')) = \alpha(t')$) then

$$\mathsf{Suc}_T(t') := \{\beta \ ; \ t' \cup \{(\alpha(t'), m(t'), \beta)\} \in T\} \in \Phi_{\alpha(t')}, \text{ and}$$

(7) for every $t' \in T$ with $t' \supseteq t$, if $\alpha(t')$ is of type 2 (i.e., $\mathsf{cf}'\alpha(t') < \alpha(t')$) and $m(t') \in \mathsf{dom}(t'(\mathsf{cf}'(\alpha(t'))))$ then

$$\mathsf{Suc}_T(t') := \{\beta \ ; \ t' \cup \{(\alpha(t'), m(t'), \beta)\} \in T\} \in \Phi_{\alpha(t'), t'(\mathsf{cf}'\alpha(t'))(m(t'))}.$$

For a (t, T) in P_3 and a subset $x \subseteq \mathsf{Reg}^{\kappa_\rho}$ we write $T \restriction x$ for $\{t' \restriction x \ ; \ t' \in T\}$.

We call t the trunk of (t, T).

This P_3 is the forcing we are going to use. It is partially ordered by

$$(t, T) \leq (s, S) :\iff \mathsf{dom}(t) \supseteq \mathsf{dom}(s) \text{ and } T \restriction \mathsf{dom}(s) \subseteq S.$$

A full generic extension via this P_3 adds too many subsets of ordinals and makes every ordinal in the interval (ω, κ_ρ) countable. By restricting to sets of ordinals which can be approximated with finite domains we ensure that the former strongly compacts are still cardinals in the symmetric model to be constructed, and the power sets stay small.

To build a symmetric model we need an automorphism group of the complete Boolean algebra $B = B(P_3)$ that is induced by P_3, as in [Jec03, Corollary 14.12]. We start by considering $\mathcal{G}$ to be the group of permutations of $\mathsf{Reg}^{\kappa_\rho} \times \omega \times \kappa_\rho$ whose elements a satisfy the following properties.

- For every $\alpha \in \mathsf{Reg}^{\kappa_\rho}$ there is a permutation a_α of α that moves only finitely many elements of α, and is such that for each $n \in \omega$ and each $\beta \in \alpha$,

$$a(\alpha, n, \beta) = (\alpha, n, a_\alpha(\beta)).$$

  The finite subset of α that a_α moves, we denote by $\mathsf{supp}(a_\alpha)$, which stands for "support of a_α".
- For only finitely many $\alpha \in \mathsf{Reg}^{\kappa_\rho}$ is a_α not the identity. This finite subset of $\mathsf{Reg}^{\kappa_\rho}$ we denote by $\mathsf{dom}(a)$.

We extend $\mathcal{G}$ to P_3 as follows. For $a \in \mathcal{G}$ and $(t, T) \in P_3$, define

$$a(t, T) := (a``t, \{a``t' \ ; \ t' \in T\}),$$

where $a``t := \{(\alpha, n, a_\alpha(\beta)) \ ; \ (\alpha, n, \beta) \in t\}$.

Unfortunately, in general $a(t, T)$ is not a member of P_3 because of the branching condition at type 2 cardinals. In particular, it is possible that for some $\alpha \in \mathsf{dom}(t)$ of type 2, and some $t' \in T$ with $\alpha = \alpha(t')$, we have that $a_{\mathsf{cf}'\alpha}(t'(\mathsf{cf}'\alpha)(m(t'))) = \gamma \neq t'(\mathsf{cf}'\alpha)(m(t'))$, and even though we had before $\mathsf{Suc}_T(t') \in \Phi_{\alpha, t'(\mathsf{cf}'\alpha)(m(t'))}$, it is not true that $\mathsf{Suc}_T(t') \in \Phi_{\alpha,\gamma}$.

To overcome this problem, for an $a \in \mathcal{G}$, define $P^a \subseteq P_3$ as follows.

(t, T) is in P^a iff the following hold:

(1) $\mathsf{dom}(t) \supseteq \mathsf{dom}(a)$,
(2) for every $\alpha \in \mathsf{dom}(t)$ we have that $\mathsf{dom}(t(\alpha)) = \mathsf{dom}(t(\mathsf{cf}'\alpha))$, and
(3) for every $\alpha \in \mathsf{dom}(t)$, we have that

$$\mathsf{rng}(t(\alpha)) \supseteq \{\beta \in \mathsf{supp}(a_\alpha) \ ; \ \exists q \in T(\beta \in \mathsf{rng}(q(\alpha)))\}.$$

The equality in (2) ensures that there will be no severe mixup in the requirements for membership in ultrafilters of the form $\Phi_{\alpha,\gamma}$. In (3) we require that the stem of each condition already contains all the ordinals that the a_α could move. This will prevent any trouble with membership in the ultrafilters. One may think that this requirement should be $\mathsf{supp}(a_\alpha) \subseteq \mathsf{rng}(t(\alpha))$ but this is not the case; note that there might be some γ in a_α which doesn't appear in the range of any $q \in T$.

Now, we have that $a : P^a \to P^a$ is an automorphism. Also, as mentioned in [Git80, page 68], for every $a \in \mathcal{G}$ the set P^a is a dense subset of P_3. Therefore, a can be extended to a unique automorphism of the complete Boolean algebra B. We denote the automorphism of B with the same letter, and also by $\mathcal{G}$ the automorphism group of B that consists of these extended automorphisms. By [Jec03, (14.36)], every automorphism a of B induces an automorphism of the Boolean valued model V^B.

Proceeding to the definition of our symmetric model, for every $e \subseteq \mathsf{Reg}^{\kappa_\rho}$ define

$$E_e := \{(t, T) \in P_3 \ ; \ \mathsf{dom}(t) \subseteq e\},$$
$$I := \{E_e \ ; \ e \subset \mathsf{Reg}^{\kappa_\rho} \text{ is finite and closed under } \mathsf{cf}'\},$$
$$\mathsf{fix} E_e := \{a \in \mathcal{G} \ ; \ \forall \alpha \in e(a_\alpha \text{ is the identity on } \alpha)\},$$

and let $\mathcal{F}$ be the normal filter (see [Jec03, (15.34)]) over $\mathcal{G}$ that is generated by

$$\{\mathsf{fix} E_e \ ; \ E_e \in I\}.$$

For each $\dot{x}$ in the Boolean valued model V^B, define its symmetry group

$$\mathsf{sym}(\dot{x}) := \{a \in \mathcal{G} \ ; \ a(\dot{x}) = \dot{x}\}.$$

A name $\dot{x}$ is called symmetric iff its symmetry group is in the filter $\mathcal{F}$. The class of hereditarily symmetric names HS is defined by recursion on the rank of the name, i.e.,

$$\mathsf{HS} := \{\dot{x} \in V^G \ ; \ \forall \dot{y} \in \mathsf{tc}_{\mathsf{dom}}(\dot{x})(\mathsf{sym}(\dot{y}) \in \mathcal{F})\},$$

where $\mathsf{tc}_{\mathsf{dom}}(\dot{x})$ is defined as the union of all x_n, which are defined recursively by $x_0 := \{\dot{x}\}$ and $x_{n+1} := \bigcup\{\mathsf{dom}(\dot{y}) \ ; \ \dot{y} \in x_n\}$.

We will say that an $E_e \in I$ supports a name $\dot{x} \in \mathsf{HS}$ if $\mathsf{fix} E_e \subseteq \mathsf{sym}(\dot{x})$.

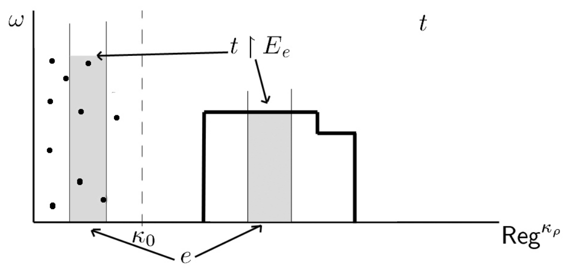
For some V-generic ultrafilter G on B we define the symmetric model

$$V(G) := \{\dot{x}^G \ ; \dot{x} \in \mathsf{HS}\}.$$

By [Jec03, Lemma 15.51], this is a model of ZF, and $V \subseteq V(G) \subseteq V[G]$.

For each $(t, T) \in P_3$ and each $E_e \in I$, define

$$(t, T) \upharpoonright^* E_e = (t \upharpoonright e, \{t' \upharpoonright e \ ; \ t' \in T\}).$$



According to [Git80, Lemma 3.3.], if ϕ is a formula with n free variables, $\dot{x}_1, \ldots, \dot{x}_n \in \mathsf{HS}$, and $E_e \in I$ is such that $\mathsf{sym}(\dot{x}_1), \ldots, \mathsf{sym}(\dot{x}_n) \supseteq \mathsf{fix} E_e$ then we have that for every $(t, T) \in P_3$

$$(t, T) \Vdash \phi(\dot{x}_1, \ldots, \dot{x}_n) \iff (t, T) \upharpoonright^* E_e \Vdash \phi(\dot{x}_1, \ldots, \dot{x}_n).$$

This implies the following lemma, which we will refer to as *the approximation lemma.*

Lemma 2.4. *If $X \in V(G)$ is a set of ordinals, then there is an $E_e \in I$ such that $X \in V[G \upharpoonright^* E_e]$, where $G \upharpoonright^* E_e := \{(t, T) \upharpoonright^* E_e \ ; \ (t, T) \in G\}$.*

Proof. Because of [Git80, Lemma 3.3.] mentioned above, because of the symmetry lemma [Jec03, (15.41)], and because canonical names $\check{\alpha}$ are not moved by automorphisms of B, we have that if $\dot{X} \in \mathsf{HS}$ is a P_3-name for X and $E_e \in I$ supports $\dot{X}$ then the set

$$\ddot{X} := \{(\check{\alpha}, (t,T) \restriction^* E_e) \ ; \ (t,T) \Vdash \check{\alpha} \in \dot{X}\}$$

is an E_e-name for X. qed

We will use the approximation lemma in all our subsequent proofs.

Theorem 2.5. *For every $0 \leq \xi \leq \rho$, κ_ξ is a cardinal in $V(G)$. Consequently, their (singular) limits are also preserved.*

The proof proceeds by construing a finite support part of P_3 as a two-step iterated forcing $\mathbb{E} * \dot{\mathbb{Q}}$, where the first component is small forcing relative to κ_ξ, and the second component satisfies a Prikry lemma and does not add bounded subsets to κ_ξ. This corresponds to the intuition behind the definition of P_3. The details of this proof, however, are largely technical, and in order to highlight the structure of this proof, we relegate them to the appendix.

Proof. Assume towards a contradiction that there is some $\delta < \kappa_\xi$ and a bijection $f : \delta \to \kappa_\xi$ in $V(G)$. Let $\dot{f}$ be a name for f with support $E_e \in I$. Note that e is a finite subset of $\mathsf{Reg}^{\kappa_\rho}$ that is closed under cf'. By the approximation lemma (Lemma 2.4), since f may be coded by a set of ordinals, there is an E_e-name for f, i.e., for this e,

$$f \in V[G \restriction^* E_e].$$

We will show that this is impossible, by taking a dense subset of E_e and showing that it is forcing equivalent to an iterated forcing construction, the first part of which has cardinality less than κ_ξ, and the second part of which does not collapse κ_ξ (by not adding bounded subsets to κ_ξ, similarly to Prikry forcing).

It's not hard to check that

$$J := \{(t,T) \in E_e \ ; \forall q \in T \forall \alpha \geq \kappa_\xi \forall n < \omega (\text{ if } (\alpha, n) \in \mathsf{dom}^2(q) \setminus \mathsf{dom}^2(t) \text{ and } \mathsf{cf}'\alpha < \alpha \text{ then the ultrafilter } \Phi_{\alpha, q(\mathsf{cf}'\alpha)(n)} \text{ is } \kappa_\xi\text{-complete})\}$$

is dense in E_e (see also Lemma A.1 in the appendix and the beginning of the proof of [Git80, Theorem 5.4]). Without loss of generality assume that $e \cap \kappa_\xi \neq \emptyset$. Define the sets

$$\mathbb{E} := \{(t,T) \restriction^* E_{e \cap \kappa_\xi} \ ; \ (t,T) \in J\}, \text{ and}$$
$$P_2^* := \{t \restriction (e \setminus \kappa_\xi) \ ; \ t \in P_2\}.$$

For $s \in P_2^*$ we can define $\alpha(s)$ and $m(s)$ as we did for the $s \in P_2$, in Definition 2.2(2).

Let G^* be an arbitrary $\mathbb{E}$-generic filter and note that for every $\alpha \in e \setminus \kappa_\xi$ such that $\mathsf{cf}'\alpha < \kappa_\xi$, the set $\langle t(\mathsf{cf}'\alpha)(m) \ ; \ \exists T((t,T) \in G^* \wedge m \in \mathsf{dom}(t(\mathsf{cf}'\alpha)))\rangle$ is the Prikry sequence that is added to $\mathsf{cf}'\alpha$ by $\mathbb{E}$.

In $V[G^*]$ we define a partial ordering $\mathbb{Q}$ by $(s, S) \in \mathbb{Q} :\Longleftrightarrow$

(1) $s \in P_2^*$,
(2) $S \subseteq P_2^*$,
(3) $s \in S$,
(4) for all $s' \in S$, $\mathsf{dom}(s') = \mathsf{dom}(s) = e \setminus \kappa_\xi$, and either $s' \supseteq s$ or $s' \subseteq s$,
(5) for every $s' \in S$ and every $s'' \in P_2^*$, if $s'' \subseteq s'$ then $s'' \in S$, i.e., S is tree-like,

(6) for every $s' \in S$ with $s' \supseteq s$, if $\alpha(s')$ is of type 1 then
$$\{\beta\ ;\ s' \cup \{(\alpha(s'), m(s'), \beta)\} \in S\} \in \Phi_{\alpha(s')},$$
(7) for every $s' \in S$ with $s' \supseteq s$, if $\alpha(s')$ is of type 2 and $\mathsf{cf}'\alpha(s') \geq \kappa_\xi$ then
$$\{\beta\ ;\ s' \cup \{(\alpha(s'), m(s'), \beta)\} \in S\} \in \Phi_{\alpha(s'), s'(\mathsf{cf}'\alpha(s'))(m(s'))},\text{ and}$$
(8) for every $s' \in S$ with $s' \supseteq s$, if $\alpha(s')$ is of type 2 and $\mathsf{cf}'\alpha(s') < \kappa_\xi$ then
$$\{\beta\ ;\ s' \cup \{(\alpha(s'), m(s'), \beta)\} \in S\} \in \Phi_{\alpha(s'), \bigcup G^*(\mathsf{cf}'\alpha(s'))(m(s'))}.$$

$\mathbb{Q}$ is partially ordered by $(s, S) \leq_{\mathbb{Q}} (r, R)$ iff $S \subseteq R$.

This definition means that $\mathbb{Q}$ is like P_3 but restricted to ordinals and ultrafilters at and above κ_ξ. Because of this we can get a canonical $\mathbb{E}$-name $\dot{\mathbb{Q}}$ for $\mathbb{Q}$ and that J densely embeds into $\mathbb{E} * \dot{\mathbb{Q}}$ (Lemma A.2 in the appendix). Therefore, $\mathbb{Q}$ can be seen as the top part of the forcing E_e, cut at κ_ξ. For the rest of the proof we will work with $\mathbb{Q}$ inside $V[G^*]$.

At this point we have that the presumed collapsing function $f : \delta \to \kappa_\xi$ is in some generic extension $V[G \restriction^* E_e]$, that E_e is forcing equivalent to J, which in turn is forcing equivalent to $\mathbb{E} * \dot{\mathbb{Q}}$. We know that $\mathbb{E}$ is too small to add a function like f, so f must be added by $\mathbb{Q}$. To derive the desired contradiction, it remains to show that $\mathbb{Q}$ cannot add a function like f. To show this, as is standard with Prikry-like forcing notions, we have, proved as Lemma A.3 in the appendix:

The Prikry lemma for $\mathbb{Q}$. In $V[G^*]$, let $\tau_1, \ldots, \tau_k$ be $\mathbb{Q}$-names, and ϕ be a formula with k free variables. Then for every forcing condition $(s, S) \in \mathbb{Q}$ there is a stronger condition $(s, W) \in \mathbb{Q}$ with the same trunk which decides $\phi(\tau_1, \ldots, \tau_k)$.

The proof of this lemma resembles the proof of Gitik's Prikry-like lemma in [Git80, Lemma 5.1], but with an application of the Lévy-Solovay theorem [LS67] in the usual way. That is, using the fact that $\mathbb{E}$ is small forcing with respect to κ_ξ, we get that all ultrafilters involved in the definition of $\mathbb{Q}$ can be extended to κ_ξ-complete ultrafilters in $V[G^*]$. Then every time we have to intersect conditions in $\mathbb{Q}$, we get a pseudo-condition with splitting sets in the extended ultrafilters. But by the definition of these extended ultrafilters we can always find subsets in the original ultrafilters of the ground model, and thus get a stronger condition that is in $\mathbb{Q}$.

Using the Prikry lemma for $\mathbb{Q}$, we get, using the standard Prikry-style arguments, the κ_ξ-completeness of the ultrafilters in the definition of $\mathbb{Q}$, and the usual application of the Lévy-Solovay theorem, that $\mathbb{Q}$ does not add bounded subsets to κ_ξ (Lemma A.4 in the appendix). Therefore E_e cannot collapse κ_ξ, and so the collapsing function $f : \delta \to \kappa_\xi$ cannot exist in $V[G \restriction^* E_e]$. This completes the proof of Theorem 2.5. qed

Next we will see that we singularised the targeted ordinals. This is similar to [Git80, Lemma 3.4].

Lemma 2.6. *Every cardinal in* $(\mathsf{Reg}^{\kappa_\rho})^V$ *has cofinality* ω *in* $V(G)$*. Thus every cardinal in the interval* (ω, κ_ρ) *is singular.*

Proof. Let $\alpha \in \mathsf{Reg}^{\kappa_\rho}$. For every $\beta < \alpha$, the set
$$D_\beta := \{(t, T) \in P_3\ ;\ \exists n < \omega(t(\alpha)(n) \geq \beta)\}$$
is dense in $(P_3, \leq)$. Hence $f_\alpha := \bigcup\{t(\alpha)\ ;\ (t, T) \in G\}$ is a function from ω onto an unbounded subset of α. This function has a symmetric name, which is supported

by E_a, where a is the smallest subset of $\mathsf{Reg}^{\kappa_\rho}$ that contains α and is closed under cf'. Therefore $f_\alpha \in V(G)$. qed

Usually in symmetric models built from ZFC-models with large cardinals, there is some combinatorial residue from the large cardinal. Such is the case also here. First we will show that in the interval (ω, κ_ρ), *only* the former strongly compact cardinals and their (singular) limits remain cardinals, i.e., that all cardinals of V that are between the κ_ξ and their (singular) limits have collapsed.

Lemma 2.7. *For every ordinal $\xi \in [-1, \rho)$ and every $\alpha \in (\kappa_\xi, \kappa_{\xi+1})$, $(|\alpha| = \kappa_\xi)^{V(G)}$.*

Proof. Fix an ordinal $\xi \in [-1, \rho)$. Since strongly compact cardinals are limits of strongly inaccessible cardinals, it suffices to show that for every strongly inaccessible $\alpha \in (\kappa_\xi, \kappa_{\xi+1})$, we have that $(|\alpha| = \kappa_\xi)^{V(G)}$.

Fix $\alpha \in (\kappa_\xi, \kappa_{\xi+1})$ strongly inaccessible. We have a bijection $h_\alpha : \mathcal{P}_{\kappa_\xi}(\alpha) \to \alpha$ (see the definition of type 1 ordinals). We will use this bijection to show that $E_{\{\alpha\}}$ is isomorphic to the following partial ordering.

Let P^s_α be the forcing that consists of all injective H_α-trees, i.e., of sets T such that

- T consists of finite injective sequences of elements of $\mathcal{P}_{\kappa_\xi}(\alpha)$,
- $(T, \trianglelefteq)$ is a tree, where $\trianglelefteq$ denotes end extension,
- T has a trunk tr_T, i.e., an element such that for every $t \in T$, either $t \trianglelefteq \mathsf{tr}_T$ or $\mathsf{tr}_T \trianglelefteq t$, and
- for every $t \in T$ such that $t \trianglerighteq \mathsf{tr}_T$, the set $\{x \in \mathcal{P}_{\kappa_\xi}(\alpha) \ ; \ t^\frown\langle x\rangle \in T\}$ of the $\trianglelefteq$-successors of t in T, is in the ultrafilter H_α.

The forcing P^s_α is partially ordered by
$$S \leq T \iff S \subseteq T.$$
Towards the isomorphism, define a function f from the injective finite sequences of elements of $\mathcal{P}_{\kappa_\xi}(\alpha)$ to P_2 by
$$f(t) := \{(\alpha, m, \beta) \ ; \ m \in \mathsf{dom}(t) \land \beta = h_\alpha(t(m))\}.$$
Define another function $i : P^s_\alpha \to E_{\{\alpha\}}$ by
$$i(T) := (f(\mathsf{tr}_T), \{f(t) \ ; \ t \in T \land t \trianglerighteq \mathsf{tr}_T\}).$$
This i is indeed a function from T to $E_{\{\alpha\}}$ because h_α is a bijection. In fact, this i is a bijection itself. It is easy to see that it also preserves the $\leq$ relation of the forcings, so P^s_α and $E_{\{\alpha\}}$ are isomorphic.

Now let $\hat{G}$ be P^s_α-generic. Because H_α is fine we have that for every $\beta < \alpha$ the set
$$D_\beta := \{T \in P^s_\alpha \ ; \ \exists m \in \omega(\beta \in \mathsf{tr}_T(m))\}$$
is dense in P_3. So
$$\alpha = \bigcup_{n<\omega} (\bigcup \hat{G})(n).$$
But each $(\bigcup \hat{G})(n)$ is in $\mathcal{P}_{\kappa_\xi}(\alpha)$, hence it has cardinality less than $\kappa_\xi < \alpha$. Therefore in any forcing extension of V via P^s_α, α has become a countable union of sets of cardinality less than κ_ξ and therefore is collapsed to κ_ξ. So there is an $E_{\{\alpha\}}$-name for a collapsing function from κ_ξ to α, which can be seen as a P_3-name in HS for such a function, supported by $E_{\{\alpha\}}$. qed

Next we show that the regular cardinals of type 2 have collapsed to the singular limit of strongly compacts below them.

Lemma 2.8. *For every α of type 2, if β is the largest limit of strongly compacts below α, then* $(|\alpha| = \beta)^{V(G)}$.

Proof. Similarly to the proof of the previous lemma, we assume that α is inaccessible and we look at each of the bijections $h_{\alpha,\nu} : \mathcal{P}_{\kappa_\nu^\alpha}(\alpha) \to \alpha$. Let e be the smallest finite subset of $\mathsf{Reg}^{\kappa_\rho}$ that contains α and is closed under cf'. Look at $V[G \restriction^* E_e]$. Let $\langle \gamma_i \ ; \ i \in \omega \rangle$ be the Prikry sequence added to $\mathsf{cf}'\alpha$, and let $\langle \alpha_i \ ; \ i \in \omega \rangle$ be the Prikry sequence added to α. For each $i \in \omega$, let

$$A_i := h^{-1}_{\alpha,\gamma_i}(\alpha_i).$$

We want to show that for each $\delta \in \alpha$, there is some $i \in \omega$ such that $\delta \in A_i$. Fix $\delta \in \alpha$. For all $i \in \omega$, the V-ultrafilter H_{α,γ_i} is fine, so

$$\{A \in \mathcal{P}_{\kappa^\alpha_{\gamma_i}}(\alpha) \ ; \ \delta \in A\} \in H_{\alpha,\gamma_i}.$$

So for every $i \in \omega$, the set

$$Z_i := \{\zeta \in \alpha \ ; \ \delta \in h^{-1}_{\alpha,\gamma_i}(\zeta)\} \in \Phi_{\alpha,\gamma_i}.$$

Define the set

$$D_\delta := \{(t,T) \in E_e \ ; \ \exists i \in \mathsf{dom}(t)(\delta \in h^{-1}_{\alpha,\gamma_i}(t(\alpha)(i)))\}.$$

This is dense in E_e and δ was arbitrary. Therefore in $V[G \restriction^* E_e]$, we have that $\alpha = \bigcup_{i\in\omega} A_i$ is a countable union of $\leq \beta$-sized sets, and thus there is a symmetric name for a collapse of α to β, supported by E_e. qed

We summarise our results on the cardinal structure of the interval (ω, κ_ρ).

Corollary 2.9. *An uncountable cardinal of $V(G)$ that is less than or equal to κ_ρ is a successor cardinal in $V(G)$ iff it is in $\{\kappa_\xi \ ; \ \xi \leq \rho\}$. Thus in $V(G)$, for every $\xi \leq \rho$ we have that $\kappa_\xi = \aleph_{\xi+1}$.*

Also, an uncountable cardinal of $V(G)$ that is less than or equal to κ_ρ is a limit cardinal in $V(G)$ iff it is a limit in the sequence $\langle \kappa_\alpha \ ; \ \alpha < \rho\rangle$ in V.

Proof. This follows inductively, using Theorem 2.5, Lemma 2.7, and Lemma 2.8. qed

Before we go into the combinatorial properties in $V(G)$, let us mention that the Axiom of Choice fails really badly in this model. The following is [Git80, Theorem 6.3].

Lemma 2.10. *In $V(G)$, countable unions of countable sets are not necessarily countable. In particular, every set in H_{κ_ρ} is a countable union of sets of smaller cardinality. Here "x has a smaller cardinality than y" means that x is a subset of y and there is no bijection between them.*

3. Results

We will now prove our main result, Theorem 1.1. We will use the approximation lemma for $V(G)$ and the Lévy-Solovay theorem [LS67], which says that measurability is preserved under small forcing. In particular, it says that if U is a normal measure over κ_ρ, then the set generated from U by taking supersets is still a normal measure after small forcing.

Proof. By Corollary 2.9, we only need to show that the measurability of κ_ρ and the existence of a normal measure over κ_ρ is preserved to $V(G)$. We will prove that if U is a normal measure for κ_ρ in the ground model, then the following set defined in $V(G)$,

$$U' := \{Y \subseteq \kappa_\rho \ ; \ \exists X \in U(X \subseteq Y)\},$$

is a normal measure over κ_ρ in $V(G)$. This U' is clearly a filter in $V(G)$, so it remains to show that it is also a κ_ρ-complete normal ultrafilter. For this, we need to use the approximation lemma for $V(G)$.

To show that U' is an ultrafilter, let $X \subseteq \kappa_\rho$, $X \in V(G)$, and let $\dot{X} \in \mathsf{HS}$ be a name for X, supported by $E_e \in I$. By the approximation lemma, we have that $X \in V[G \restriction^* E_e]$, so we can use the Lévy-Solovay theorem [LS67] to see that either $X \in U'$ or $\kappa_\rho \setminus X \in U'$.

To show that U' is κ-complete, let $\gamma < \kappa_\rho$ and $\langle X_\delta \ ; \ \delta < \gamma \rangle$ be a sequence of sets in U'. Let $\sigma \in \mathsf{HS}$ be a name for this sequence and let $E_{e'} \in I$ be a support for this sequence. Since a sequence of sets of ordinals can be coded into a set of ordinals, we can use the approximation lemma to get that the sequence is in $V[G \restriction^* E_{e'}]$. Again by the Lévy-Solovay theorem [LS67] we get that its intersection is in U'. Therefore U' is a measure for κ_ρ in $V(G)$.

To show that U' is normal, let $f : \kappa_\rho \to \kappa_\rho$, $f \in V(G)$ be regressive. Since f can be coded by a set of ordinals, by the approximation lemma, $f \in V[G \restriction^* E_{e''}]$ for some $E_{e''} \in I$. Again, by the Lévy-Solovay theorem [LS67], we will get a set in U' on which f is constant. qed

The construction in this paper is a generalised construction. For particular results, e.g., $\aleph_{\omega+3}$ becoming both the first uncountable regular cardinal and a measurable cardinal, we just put $\rho = \omega + 2$. Thus we can immediately get a theorem such as the following.

Theorem 3.1. *If V is a model of "There is an $\omega+2$-sequence of strongly compact cardinals with a measurable cardinal above this sequence", then there is a symmetric model in which $\aleph_{\omega+3}$ is both a measurable cardinal and the first regular cardinal.*

We can replace "measurable" by some large cardinal properties that are preserved under small forcing, and which are of the form "For every set of ordinals X, there is a set Y such that $\phi(X, Y)$ holds" for certain formulas ϕ with two free variables. This is because for such properties we can capture the arbitrary set of ordinals in an intermediate ZFC model that is included in the symmetric model and use small forcing arguments to prove that such a large cardinal property is preserved. This allows us to construct models in which the first ρ uncountable cardinals are singular and $\aleph_{\rho+1}$ is, e.g., weakly compact, Erdős, Ramsey, etc.

Appendix A. Details for the proof of Theorem 2.5

This appendix contains the details on the proof of Theorem 2.5. Therefore all subsequent notation is the notation in the proof of that theorem. The arguments in the proof of the next lemma are part of the beginning of the proof of [Git80, Theorem 5.4] where it is shown that the powerset axiom holds in the class version of this model.

Lemma A.1. *The following set is dense in E_e.*

$$J := \{(t,T) \in E_e \ ; \ \forall q \in T \forall \alpha \geq \kappa_\xi \forall n < \omega(\text{ if } (\alpha, n) \in \mathsf{dom}^2(q) \setminus \mathsf{dom}^2(t) \text{ and } \mathsf{cf}'\alpha < \alpha \text{ then the ultrafilter } \Phi_{\alpha, q(\mathsf{cf}'\alpha)(n)} \text{ is } \kappa_\xi\text{-complete})\}$$

Proof. First notice that the set

$$D := \{(t,T) \in E_e \ ; \ \forall \alpha \in \mathsf{dom}(t)(\mathsf{dom}(t(\alpha)) = \mathsf{dom}(t(\mathsf{cf}'\alpha)))\}$$

is dense in E_e. This proof is similar to the proof that for $a \in \mathcal{G}$, P^a is dense in P_3. Now we will prove that for every $(t,T) \in D$ there is a $T' \subseteq T$ such that $(t, T') \in J$. For every $\alpha \in \mathsf{dom}(t) \setminus \kappa_\xi$ such that $\mathsf{cf}'\alpha < \alpha$, let λ_α be the least ordinal $\nu < \mathsf{cf}'\alpha$ such that $\kappa_\nu^\alpha \geq \kappa_\xi$.

We have that $(t,T') \in J$ iff for all $q \in T'$, if $\alpha(q) \geq \kappa_\xi$ and $\mathsf{cf}'\alpha(q) < \alpha(q)$ then $q(\mathsf{cf}'\alpha(q))(m(q)) \geq \lambda_{\alpha(q)}$. This equivalence is true because the right hand side of the implication above ensures that the ultrafilter $\Phi_{\alpha,q(\mathsf{cf}'\alpha),m(q)}$ is κ_ξ-complete. Define

$$b := \{\mathsf{cf}'\alpha \ ; \ \alpha \in \mathsf{dom}(t) \setminus \kappa_\xi \wedge \mathsf{cf}'\alpha < \alpha\},$$

and for $\beta \in b$ define

$$c_\beta := \max\{\lambda_\alpha \ ; \ \alpha \in \mathsf{dom}(t) \setminus \kappa_\xi \text{ and } \mathsf{cf}'\alpha = \beta < \alpha\}.$$

Then $(t,T') \in J$ if for all $q \in T'$ and $(\alpha,m) \in \mathsf{dom}^2(q) \setminus \mathsf{dom}^2(t)$ such that $\mathsf{cf}'\alpha \in b$ we have that $q(\mathsf{cf}'\alpha)(m) \geq c_{\mathsf{cf}'\alpha}$. So let

$$\begin{aligned} T' :=\{q \in T \ ; \ &\forall \alpha' \in b \forall m < \omega (\text{if } (\alpha',m) \in \mathsf{dom}^2(q) \setminus \mathsf{dom}^2(t) \\ &\text{then } q(\alpha')(m) \geq c_{\alpha'})\}. \end{aligned}$$

Clearly, this $(t,T') \in J$. qed

Lemma A.2. *There is an $\mathbb{E}$-name $\dot{\mathbb{Q}}$ for $\mathbb{Q}$ and J densely embeds into $\mathbb{E} * \dot{\mathbb{Q}}$.*

Proof. An obvious name $\dot{\mathbb{Q}}$ for $\mathbb{Q}$ is the following. For $(t,T) \in \mathbb{E}$, $(\sigma,(t,T)) \in \dot{\mathbb{Q}}$ iff

(a) there is an $s \in P_2^*$ and an $\mathbb{E}$-name $\bar{\sigma}$ such that $s \cup t \in P_2$, $\sigma = (\check{s},\bar{\sigma})\check{}$, and for all $\pi \in \mathsf{dom}(\bar{\sigma})$ there is a $s' \in P_2^*$ such that $\check{s}' = \pi$.
(b) $(t,T) \Vdash \check{s} \in \bar{\sigma}$,
(c) $(t,T) \Vdash \forall \pi (\pi \in \bar{\sigma} \to \mathsf{dom}(\pi) = \mathsf{dom}(\check{s}) \wedge (\pi \subseteq \check{s} \vee \pi \supseteq \check{s}))$,
(d) $(t,T) \Vdash \forall \pi (\pi \in \bar{\sigma} \wedge \alpha(\pi) = \mathsf{cf}'\alpha(\pi) \to \exists \check{X}(\check{X} \in \check{\Phi}_{\alpha(\pi)} \wedge \forall \check{\beta}(\check{\beta} \in \check{X} \leftrightarrow \pi \cup \{(\check{\alpha}(\pi),\check{m}(\pi),\check{\beta})\} \in \bar{\sigma})))$,
(e) $(t,T) \Vdash \forall \pi (\pi \in \bar{\sigma} \wedge \alpha(\pi) > \mathsf{cf}'\alpha(\pi) \wedge \mathsf{cf}'\alpha(\pi) \geq \kappa_\xi \to \exists \check{X}(\check{X} \in \check{\Phi}_{\alpha(\pi),\pi(\mathsf{cf}'\alpha(\pi))(m(\pi))} \wedge \forall \check{\beta}(\check{\beta} \in \check{X} \leftrightarrow \pi \cup \{(\check{\alpha}(\pi),\check{m}(\pi),\check{\beta})\} \in \bar{\sigma})))$,
(f) $(t,T) \Vdash \forall \pi (\pi \in \bar{\sigma} \wedge \alpha(\pi) > \mathsf{cf}'\alpha(\pi) \wedge \mathsf{cf}'\alpha(\pi) < \kappa_\xi \to \exists \check{X}(\check{X} \in \check{\Phi}_{\alpha(\pi)\Gamma(\mathsf{cf}'\alpha(\pi))(m(\pi))} \wedge \forall \check{\beta}(\check{\beta} \in \check{X} \leftrightarrow \pi \cup \{(\check{\alpha}(\pi),\check{m}(\pi),\check{\beta})\} \in \bar{\sigma})))$, where Γ is the standard $\mathbb{E}$-name for $\bigcup\{t \ ; \ \exists T((t,T) \in G^*)\}$,
(g) $(t,T) \Vdash \forall \pi \forall \pi' (\pi \in \bar{\sigma} \wedge \pi' \in \check{P}_2^* \wedge \pi' \subseteq \pi \to \pi' \in \bar{\sigma})$.

The name for the ordering on $\dot{\mathbb{Q}}$ is defined as

$$((\sigma,\tau)\check{},(t,T)) \in \leq_{\dot{\mathbb{Q}}} :\iff (t,T) \Vdash \sigma \subseteq \tau.$$

From the forcing theorem we have that $\dot{\mathbb{Q}}^{G^*} = \mathbb{Q}$. For every $(t,T) \in P_3$ and $t' \in T$ such that $t' \supseteq t$ define

$$(t,T) \uparrow (t') := (t', \{t'' \in T \ ; \ t'' \subseteq t' \text{ or } t \subseteq t''\}),$$

the extension of (t,T) with trunk t'. If $t' \subseteq t$ then we conventionally take $(t,T) \uparrow (t') := (t,T)$.

Define a map $i : J \to \mathbb{E} * \dot{\mathbb{Q}}$. For $(r,R) \in J$, we take $i((r,R)) = ((r_1,R_1),\rho) :\iff$

(i) $(r_1,R_1) := (r,R) \restriction^* E_{e\cap\kappa_\xi}$,
(ii) $\rho = (\check{r}_2,\bar{\rho})\check{}$, where $r_2 := r \restriction (e \setminus \kappa_\xi)$ and for all $\pi \in \mathsf{dom}(\bar{\rho})$, there is an $r' \in R$ such that $(\pi = (r' \restriction (e \setminus \kappa_\xi))^\vee$,
(iii) For all $r' \in R$ with $r' \subsetneq r$ we have that $((r' \restriction (e \setminus \kappa_\xi))^\vee,(r_1,R_1)) \in \bar{\rho}$,
(iv) For all $r' \in R$ with $r' \supseteq r$ we have that

$$((r' \restriction (e \setminus \kappa_\xi))^\vee,(r_1,R_1) \uparrow (r' \restriction (e \cap \kappa_\xi))) \in \bar{\rho}.$$

(v) No other elements are in $\bar{\rho}$.

Claim 1. For all $(r,R)\in J$, $i((r,R))=((r_1,R_1),\rho)\in\mathbb{E}*\dot{\mathbb{Q}}$.

Proof of claim. That $(r_1,R_1)\in\mathbb{E}$ is immediate. So we must show that $(r_1,R_1)\Vdash\rho\in\dot{\mathbb{Q}}$. Requirement (a) clearly holds with $r_2:=r\restriction(e\setminus\kappa_\xi)$.

For (b) we want that $(r_1,R_1)\Vdash\check{r}_2\in\bar{\rho}$ which holds because $(\check{r}_2,(r_1,R_1))\in\bar{\rho}$.

For (c) we want that

$$(r_1,R_1)\Vdash\forall\pi(\pi\in\rho\to\mathsf{dom}(\pi)=\mathsf{dom}(\check{r}_2)\wedge(\pi\subseteq\check{r}_2\vee\pi\supseteq\check{r}_2))$$

or equivalently that

$$\forall\pi\in V^{\mathbb{E}}\forall(b,B)\leq(r_1,R_1)\exists(b',B')\leq(b,B)$$
$$((b',B')\Vdash\neg\pi\in\bar{\rho}\text{ or }(b',B')\Vdash(\mathsf{dom}(\pi)=\mathsf{dom}(\check{r}_2)\wedge(\pi\subseteq\check{r}_2\vee\pi\supseteq\check{r}_2))).$$

Let $\pi\in V^{\mathbb{E}}$ and $(b,B)\leq(r_1,R_1)$ be arbitrary and let $(b',B')\leq(b,B)$ decide the formula $\pi\in\bar{\rho}$. Assume that $(b',B')\nVdash\neg\pi\in\bar{\rho}$. Then we have that $(b',B')\Vdash\pi\in\bar{\rho}$. By the definition of ρ there is some $r'\in R$ such that

$$(b',B')\Vdash\pi=(r'\restriction(e\setminus\kappa_\xi))^\vee.$$

Since (r,R) is a condition in P_3 we get that

$$(b',B')\Vdash\mathsf{dom}(\pi)=\mathsf{dom}(\check{r}_2)\wedge(\pi\subseteq\check{r}_2\vee\pi\supseteq\check{r}_2).$$

For (d) we want to show that for every $\pi\in V^{\mathbb{E}}$,

$$(r_1,R_1)\Vdash(\pi\in\bar{\rho}\wedge\alpha(\pi)=\mathsf{cf}'\alpha(\pi)\to$$
$$\exists\check{X}(\check{X}\in\check{\Phi}_{\alpha(\pi)}\wedge\forall\check{\beta}(\check{\beta}\in\check{X}\leftrightarrow\pi\cup\{(\check{\alpha}(\pi),\check{m}(\pi),\check{\beta})\}\in\bar{\rho}))).$$

As before, let $\pi\in V^{\mathbb{E}}$ and $(b,B)\leq(r_1,R_1)$ be arbitrary and let $(b',B')\leq(b,B)$ decide the formula $\pi\in\bar{\rho}\wedge\alpha(\pi)=\mathsf{cf}'\alpha(\pi)$. Assume that

$$(b',B')\nVdash\neg(\pi\in\bar{\rho}\wedge\alpha(\pi)=\mathsf{cf}'\alpha(\pi)).$$

Then $(b',B')\Vdash(\pi\in\bar{\rho}\wedge\alpha(\pi)=\mathsf{cf}'\alpha(\pi))$.

Let $r'\in R$ be such that $(b',B')\leq(r_1,R_1)\uparrow(r'\restriction(e\cap\kappa_\xi))$ and

$$(b',B')\Vdash\pi=(r'\restriction(e\setminus\kappa_\xi))^\vee.$$

Call $r'_1:=r'\restriction(e\cap\kappa_\xi)$ and $r'_2:=r'\restriction(e\setminus\kappa_\xi)$.

We want to show that

$$(b',B')\Vdash\exists\check{X}(\check{X}\in\check{\Phi}_{\alpha(\pi)}\wedge\forall\check{\beta}(\check{\beta}\in\check{X}\leftrightarrow\pi\cup\{(\check{\alpha}(\pi),\check{m}(\pi),\check{\beta})\}\in\bar{\sigma})).$$

Case 1, if $\alpha(r')=\alpha(r'_2)\geq\kappa_\xi$. Then let

$$X:=\mathsf{Suc}_R(r')=\{\beta\ ;\ r'\cup\{(\alpha(r'),m(r'),\beta)\}\in R\},$$

and note that $X\in\Phi_{\alpha(r'_2)}$ and

$$\begin{aligned}r'\cup\{(\alpha(r'),m(r'),\beta)\}\in R&\iff((r'_2\cup\{(\alpha(r'_2),m(r'_2),\beta)\})^\vee,(r_1,R_1)\uparrow(r'_1))\in\bar{\rho}\\&\iff\beta\in X.\end{aligned}$$

Let $\check{\beta}\in V^{\mathbb{E}}$ be arbitrary. We want that

$$(b',B')\Vdash(\check{\beta}\in\check{X}\wedge\pi\cup\{(\check{\alpha}(\pi),\check{m}(\pi),\check{\beta})\}\in\bar{\rho})\vee(\check{\beta}\notin\check{X}\wedge\pi\cup\{(\check{\alpha}(\pi),\check{m}(\pi),\check{\beta})\}\notin\bar{\rho}),$$

or equivalently that

$$\forall(c,C)\le(b',B')\exists(c',C')\le(c,C)$$
$$((c',C')\Vdash(\check\beta\in\check X\wedge\pi\cup\{(\check\alpha(\pi),\check m(\pi),\check\beta)\}\in\bar\rho)\text{ or }$$
$$(c',C')\Vdash(\check\beta\notin\check X\wedge\pi\cup\{(\check\alpha(\pi),\check m(\pi),\check\beta)\}\notin\bar\rho)).$$

So let $(c,C)\le(b',B')$ be arbitrary and let $(c',C')\le(c,C)$ be stronger than $(r_1,R_1)\uparrow(r_1')$ and decide $\pi\cup\{(\check\alpha(\pi),\check m(\pi),\check\beta)\}\in\bar\rho$.

Clearly this (c',C') satisfies

$$(c',C')\Vdash(\check\beta\in\check X\wedge\pi\cup\{(\check\alpha(\pi),\check m(\pi),\check\beta)\}\in\bar\rho)\text{ or}$$
$$(c',C')\Vdash(\check\beta\notin\check X\wedge\pi\cup\{(\check\alpha(\pi),\check m(\pi),\check\beta)\}\notin\bar\rho)$$

and we're done with this case.

Case 2, if $\alpha(r')<\kappa_\xi$, then let $r''\supseteq r'$ be such that $r''\restriction(e\setminus\kappa_\xi)=r_2'$ and $\alpha(r'')=\alpha(r_2')\ge\kappa_\xi$. The rest follows as in Case 1.

For (e), (f), and (g) we proceed similarly. qed Claim 1

Claim 2. The map i is a dense embedding.

Proof of claim. Let $((t,T),\sigma)\in\mathbb{E}*\dot{\mathbb{Q}}$ be arbitrary. We want to define an $(r,R)\in J$ such that $i((r,R))\le((t,T),\sigma)$. Define $r:=t\cup s$ and let $\sigma=(\check s,\bar\sigma)$. By (a) of the definition of $\dot{\mathbb{Q}}$, $r\in P_2$.

If $r'\in P_2$ is such that $r'\subseteq r$ then let $r'\in R$. For $r'\supseteq r$ we define R recursively as follows. Let $r'\in P_2$ be such that $r'\supseteq r$ and $r'\in R$.

- If $\alpha(r')<\kappa_\xi$ then $r'\cup\{(\alpha(r'),m(r'),\beta)\}\in R:\Longleftrightarrow$
$$(r'\cup\{(\alpha(r'),m(r'),\beta)\})\restriction(e\cap\kappa_\xi)\in T.$$
- If $\alpha(r')\ge\kappa_\xi$ then $r'\cup\{(\alpha(r'),m(r'),\beta)\}\in R:\Longleftrightarrow$
$$(t,T)\uparrow(r'\restriction(e\cap\kappa_\xi))\Vdash((r'\restriction(e\setminus\kappa_\xi))\cup\{(\alpha(r'),m(r'),\beta)\})^\vee\in\bar\sigma.$$

Subclaim 1. For every $r'\in R$, call $r_1':=r'\restriction(e\cap\kappa_\xi)$ and $r_2':=r'\restriction(e\setminus\kappa_\xi)$. Then $(t,T)\uparrow(r_1')\Vdash\check r_2'\in\bar\sigma$.

Proof of subclaim. Since by the definition of R and $\dot{\mathbb{Q}}$ this holds for all $r'\subseteq r$, we'll use induction with base case $r'=r$. For $r'=r$ it holds with $(t',T')=(t,T)$ due to (b) of the definition of $\dot{\mathbb{Q}}$. So assume it holds for r' and let $r'\cup\{(\alpha(r'),m(r'),\beta)\}\in R$ be arbitrary.

If $\alpha(r')<\kappa_\xi$ then $\alpha(r')=\alpha(r_1')$ and by the definition of R we have that $(r_1'\cup\{(\alpha(r'),m(r'),\beta)\})\in T$. By the induction hypothesis we get that

$$(t,T)\uparrow(r_1'\cup\{(\alpha(r'),m(r'),\beta)\})\le(t,T)\uparrow(r_1')\Vdash\check r_2'\in\bar\sigma.$$

If $\alpha(r')\ge\kappa_\xi$ then $\alpha(r')=\alpha(r_2')$ and by the definition of R we have that $(t,T)\uparrow(r_1')\Vdash(r_2'\cup\{(\alpha(r_2'),m(r_2'),\beta)\})^\vee\in\bar\sigma$. qed Subclaim 1

Subclaim 2. $(r,R)\in J$

Proof of subclaim. To show that $(r,R)\in P_3$ we only need to verify (6) and (7) of the definition of P_3.

For (6), let $r' \in R$ with $r' \supseteq r$ and $\alpha(r')$ of type 1. Call $r'_1 := r' \restriction (e \cap \kappa_\xi)$ and $r'_2 := r' \restriction (e \setminus \kappa_\xi)$.

If $\alpha(r') < \kappa_\xi$ then $r' \cup \{(\alpha(r'), m(r'), \beta)\} \in R$ iff $r'_1 \cup \{(\alpha(r'_1), m(r'_1), \beta)\} \in T$. Since $(t, T) \in \mathbb{E}$ we get that

$$\mathsf{Suc}_R(r') = \{\beta\ ;\ r'_1 \cup \{(\alpha(r'_1), m(r'_1), \beta)\} \in T\} \in \Phi_{\alpha(r')}.$$

So assume that $\alpha(r') \geq \kappa_\xi$. We have that

$$(t, T) \Vdash \forall \pi (\pi \in \bar{\sigma} \wedge \pi \supseteq \check{s} \wedge \alpha(\pi) = \mathsf{cf}'\alpha(\pi) \rightarrow$$
$$\exists \check{X} (\check{X} \in \check{\Phi}_{\alpha(\pi)} \wedge \forall \check{\beta} (\check{\beta} \in \check{X} \leftrightarrow \pi \cup \{(\alpha(\pi), m(\pi), \beta)\} \in \bar{\sigma}))).$$

By Subclaim 1 we have that

$$(t, T) \uparrow (r'_1) \Vdash \check{r}'_2 \in \bar{\sigma} \wedge \check{r}'_2 \supseteq \check{s} \wedge \alpha(\check{r}'_2) = \mathsf{cf}'\alpha(\check{r}'_2),$$

thus $(t, T) \uparrow (r'_1) \Vdash \exists \check{X} (\check{X} \in \check{\Phi}_{\alpha(\check{r}'_2)} \wedge \forall \check{\beta} (\check{\beta} \in \check{X} \leftrightarrow \check{r}'_2 \cup \{(\alpha(\check{r}'_2), m(\check{r}'_2), \beta)\} \in \bar{\sigma}))$. So for some $(t', T') \leq (t, T) \uparrow (r'_1)$ there is some $X_{(t', T')} \in \Phi_{\alpha(r'_2)}$ such that for every $\check{\beta} \in V^{\mathbb{E}}$ we have that

$$(1) \qquad (t', T') \Vdash \check{\beta} \in \check{X}_{(t', T')} \leftrightarrow \check{r}'_2 \cup \{(\alpha(\check{r}'_2), m(\check{r}'_2), \beta)\} \in \bar{\sigma}.$$

Define the set $X := \bigcap \{X_{(t', T')}\ ;\ (t', T') \leq (t, T) \uparrow (r'_1)$ and (1) holds$\}$. Since κ_ξ is inaccessible, $|\mathbb{E}| < \kappa_\xi$, and $\Phi_{\alpha(r'_2)}$ is κ_ξ-complete, we have that $X \in \Phi_{\alpha(r'_2)}$ and (1) holds for $(t, T) \uparrow (r'_1)$ and X. Let $\beta \in X$. Then $(t, T) \uparrow (r'_1) \Vdash \check{r}'_2 \cup \{(\alpha(\check{r}'_2), m(\check{r}'_2), \check{\beta})\} \in \bar{\sigma}$ which by the definition of R means that $r' \cup \{(\alpha(r'), m(r'), \beta)\} \in R$. So $X \subseteq \mathsf{Suc}_R(r') \in \Phi_{\alpha(r')}$.

For (7), let $r' \in R$ with $r' \supseteq r$ and $\alpha(r')$ of type 2. Again, call $r'_1 := r' \restriction (e \cap \kappa_\xi)$ and $r'_2 := r' \restriction (e \setminus \kappa_\xi)$.

If $\alpha(r') \geq \kappa_\xi$ and $\mathsf{cf}'\alpha(r') < \kappa_\xi$ then we have that for every $\pi \in V^{\mathbb{E}}$,

$$(t, T) \Vdash (\pi \in \bar{\sigma} \wedge \pi \supseteq \check{s} \wedge \alpha(\pi) > \mathsf{cf}'\alpha(\pi) < \kappa_\xi \rightarrow$$
$$\exists \check{X} (\check{X} \in \check{\Phi}_{\alpha(\pi), \Gamma(\mathsf{cf}'\alpha(\pi))(m(\pi))} \wedge \forall \check{\beta} (\check{\beta} \in \check{X} \leftrightarrow \pi \cup \{(\alpha(\pi), m(\pi), \beta)\} \in \bar{\sigma}))).$$

By Subclaim 1 we have that

$$(t, T) \uparrow (r'_1) \Vdash \check{r}'_2 \in \bar{\sigma} \wedge \check{r}'_2 \supseteq \check{s} \wedge \alpha(\check{r}'_2) > \mathsf{cf}'\alpha(\check{r}'_2) < \kappa_\xi.$$

With the same arguments as for (6), there is some $X \in V$ such that $(t, T) \uparrow (r'_1) \Vdash \check{X} \in \check{\Phi}_{\alpha(\check{r}'_2), \Gamma(\mathsf{cf}'\alpha(\check{r}'_2))(m(\check{r}'_2))}$ and for every $\check{\beta} \in V^{\mathbb{E}}$ we have that

$$(t, T) \uparrow (r'_1) \Vdash \check{\beta} \in \check{X} \leftrightarrow \check{r}'_2 \cup \{(\alpha(\check{r}'_2), m(\check{r}'_2), \check{\beta})\} \in \bar{\sigma}.$$

But since $r'_1 \cup r'_2 = r' \in P_2$, we have that $(t, T) \uparrow (r'_1)$ decides the value of $\Gamma(\mathsf{cf}'\alpha(\check{r}'_2))(m(\check{r}'_2))$ to be $\gamma := r'_1(\mathsf{cf}'\alpha(r'_2))(m(r'_2))$. So $(t, T) \uparrow (r'_1) \Vdash \check{X} \in \check{\Phi}_{\mathsf{cf}'\alpha(\check{r}'_2), \gamma}$. Since $X \in V$ and $\Phi_{\mathsf{cf}'\alpha(\check{r}'_2), \gamma} \in V$ we have that $X \in \Phi_{\mathsf{cf}'\alpha(\check{r}'_2), \gamma}$. So take an arbitrary $\beta \in X$. Then $(t, T) \uparrow (r'_1) \Vdash (r'_2 \cup \{(\alpha(r'_2), m(r'_2), \beta)\})^\vee \in \bar{\sigma}$ which by the definition of R means that $r' \cup \{(\alpha(r'), m(r'), \beta)\} \in R$ and consequently $X \subseteq \mathsf{Suc}_R(r') \in \Phi_{\alpha(r'), r'(\mathsf{cf}'\alpha(r'))(m(r'))}$.

Similarly for the other cases where $\alpha(r') > \kappa_\xi$ and $\mathsf{cf}'\alpha(r') \geq \kappa_\xi$, and $\alpha(r') < \kappa_\xi$.

To conclude Subclaim 2 we want to show that the last condition for membership in J is fulfilled, i.e., if $r' \in R$, $\alpha \geq \kappa_\xi$, and $n < \omega$ are such that $(\alpha, n) \in \mathsf{dom}^2(r') \setminus \mathsf{dom}^2(t)$ and $\mathsf{cf}'\alpha < \alpha$, then $\Phi_{\alpha, r'(\mathsf{cf}'\alpha)(n)}$ is κ_ξ-complete. Let $q \subseteq r'$ be such that for some $q' \in R$, $q = q' \cup \{(\alpha, n, \beta)\}$, $\alpha(q') = \alpha$, and $m(q') = n$. Note that $r'(\mathsf{cf}'\alpha)(n) = q'(\mathsf{cf}'\alpha)(n)$.

If $\mathsf{cf}'\alpha \geq \kappa_\xi$ then clearly $\Phi_{\alpha,q'(\mathsf{cf}'\alpha)(n)}$ is κ_ξ-complete.

If $\mathsf{cf}'\alpha < \kappa_\xi$ then note that $q' \restriction (e \cap \kappa_\xi) \in T$ and $(t,T) \in \mathbb{E}$, i.e., for some $(s,S) \in J$, $(s,S) \restriction^* E_{e\cap\kappa_\xi} = (t,T)$. So $q'(\mathsf{cf}'\alpha)(n)$ must be high enough for the ultrafilter $\Phi_{\alpha,q'(\mathsf{cf}'\alpha)(n)}$ to be κ_ξ-complete. qed Subclaim 2

Lastly, we want to show that $i((r,R)) \leq_{\mathbb{E}*\dot{\mathbb{Q}}} ((t,T),\sigma)$. Let $i((r,R))) = ((r_1,R_1),\rho)$, and $\rho = (\check{r}'_2, \bar{\rho})\check{}$. By the definition of R and of i we immediately get that $(r_1,R_1) \leq_{\mathbb{E}} (t,T)$. So it remains to show that $(r_1,R_1) \Vdash \bar{\rho} \subseteq \bar{\sigma}$, i.e.,

$$(r_1,R_1) \Vdash \forall\pi(\pi \in \bar{\rho} \to \pi \in \bar{\sigma}),$$

or equivalently that

$$\forall\pi \in V^{\mathbb{E}}\forall(b,B) \leq (r_1,R_1)\exists(b',B') \leq (b,B)((b',B') \Vdash \neg\pi \in \bar{\rho} \text{ or } (b',B') \Vdash \pi \in \bar{\sigma}).$$

So let $\pi \in V^{\mathbb{E}}$ and $(b,B) \leq (r_1,R_1)$ be arbitrary. There is some $(b',B') \leq (b,B)$ that decides "$\pi \in \bar{\rho}$". Assume that $(b',B') \nVdash \neg\pi \in \bar{\rho}$. Then $(b',B') \Vdash \pi \in \bar{\rho}$. By the definition of $\bar{\rho}$, there must be some $r' \in R$ such that $(b',B') \Vdash \pi = (r' \restriction (e \setminus \kappa_\xi))^\vee$ and (b',B') is compatible with $(r_1,R_1) \uparrow (r' \restriction (e \cap \kappa_\xi))$. Call $r'_1 := r' \restriction (e \cap \kappa_\xi)$, $r'_2 := r' \restriction (e \setminus \kappa_\xi)$, and let (b'',B'') be stronger than both (b',B') and $(r_1,R_1) \uparrow (r'_1)$.

By Subclaim 1, $(t,T) \uparrow (r'_1) \Vdash \check{r}'_2 \in \bar{\sigma}$. Then $(r_1,R_1) \uparrow (r'_1) \Vdash \check{r}'_2 \in \bar{\sigma}$ so $(b'',B'') \Vdash \check{r}'_2 \in \bar{\sigma}$. qed Claim 2

So we have shown that $\mathbb{Q}$ can indeed be seen as the top part of E_e, cut in κ_ξ. qed

Lemma A.3. *(The Prikry lemma for $\mathbb{Q}$) In $V[G^*]$, let $\tau_1,\ldots,\tau_k$ be $\mathbb{Q}$-names, and ϕ be a formula with k free variables. Then for every forcing condition $(s,S) \in \mathbb{Q}$ there is a stronger condition $(s,W) \in \mathbb{Q}$ which decides $\phi(\tau_1,\ldots,\tau_n)$.*

This proof is almost identical to Gitik's Prikry style lemma [Git80, Lemma 5.1].

Proof. Work in $V[G^*]$. Let $(s,S) \in \mathbb{Q}$.

Let $r \in S$. If $\alpha(r)$ is of type 1 then call

$$\Phi_r := \Phi_{\alpha(r)}.$$

If $\alpha(r)$ is of type 2 and $\mathsf{cf}'\alpha(r) \geq \kappa_\xi$ then call

$$\Phi_r := \Phi_{\alpha(r),r(\mathsf{cf}'\alpha(r))(m(r))}.$$

If $\alpha(r)$ is of type 2 and $\mathsf{cf}'\alpha(r) < \kappa_\xi$ then let $\gamma \in \mathsf{cf}'\alpha$ be such that $\bigcup G^*(\mathsf{cf}'\alpha(r))(m(r)) = \gamma$, and call

$$\Phi_r := \Phi_{\alpha(r),\gamma}.$$

For all $r \in P_2 \cap (e \setminus \kappa_\xi) \times \omega \times \kappa_\rho$ define

$$\bar{\Phi}_r := \{X \subseteq \alpha(r)\ ;\ \exists Y \in \Phi_r(Y \subseteq X)\}.$$

For each $r \in S$, the ultrafilter Φ_r is at least κ_ξ complete. Since $|\mathbb{E}| < \kappa_\xi$, we can use arguments from the Lévy-Solovay theorem [LS67] to get that in $V[G^*]$, each $\bar{\Phi}_r$ is at least κ_ξ-complete as well. Also define the following sets.

$$\begin{aligned}
S_0 &:= \{r \in S\ ;\ r \supseteq s \text{ and } \exists R \subseteq S((r,R) \in \mathbb{Q} \text{ and } (r,R) \Vdash_{\mathbb{Q}} \phi(\tau_1,\ldots,\tau_k))\}\\
S_1 &:= \{r \in S\ ;\ r \supseteq s \text{ and } \exists R \subseteq S((r,R) \in \mathbb{Q} \text{ and } (r,R) \Vdash_{\mathbb{Q}} \neg\phi(\tau_1,\ldots,\tau_k))\}\\
S_2 &:= \{r \in S\ ;\ r \supseteq s \text{ and } \forall R \subseteq S(\text{if } (r,R) \in \mathbb{Q} \text{ then } (r,R) \text{ does not decide}\\
&\qquad \phi(\tau_1,\ldots,\tau_k))\}.
\end{aligned}$$

Clearly, $S = S_0 \cup S_1 \cup S_2$. Let $e \setminus \kappa_\xi := \{\alpha_0, \ldots, \alpha_{n-1}\}$. We will now enumerate the set $((e \setminus \kappa_\xi) \times \omega) \setminus \mathsf{dom}^2(s)$ from left to right and upwards, by a function x that is recursively defined as follows. First,

$$x(0) := (\alpha(s), m(s)).$$

Now let $x(i) = (\alpha_j, m)$ for some $\alpha_j \in e \setminus \kappa_\xi$ and $m \in \omega$. If $j < n-1$ then let

$$x(i+1) := (\alpha_{j+1}, m),$$

and if $j = n-1$ then

$$x(i+1) := (\alpha_0, m+1).$$

For every $i \in \omega$ define

$$\tilde{F}_i := \{r \in S \ ; \ \mathsf{dom}^2(r) \setminus \mathsf{dom}^2(s) = x``(i+1)\}.$$

Now for $i \leq j$ we will define recursively on $j - i$ a set of functions $F_{i,j} : \tilde{F}_i \to 3$. For $\ell < 3$ and $r \in \tilde{F}_i$ let

$$F_{i,i}(r) = \ell :\iff r \in S_\ell.$$

For $i < j$ let $F_{i,j}(r) = \ell :\iff$ the set

$$\{\beta \ ; \ r \cup \{(\alpha(r), m(r), \beta)\} \in S \text{ and } F_{i+1,j}(r \cup \{(\alpha(r), m(r), \beta)\}) = \ell\}$$

is in the ultrafilter Φ_r. Define recursively on $i < \omega$ a subset $\tilde{F}'_i \subset \tilde{F}_i$. Using the definition and the ω-completeness of the ultrafilter $\bar{\Phi}_s$, we find (in V), a set $\tilde{F}'_0 \subseteq \tilde{F}_0$ which is homogeneous for all functions in the set $\{F_{0,j} \ ; \ j < \omega\}$ and which is such that the set

$$\{\beta \ ; \ s \cup \{(\alpha(s), m(s), \beta)\} \in \tilde{F}'_0\}$$

is in Φ_s. By *homogeneous* here we mean that for all $t_1, t_2 \in \tilde{F}'_0$ and for all $0 \leq j < \omega$, $F_{0,j}(t_1) = F_{0,j}(t_2)$. For $i > 0$ we take

$$\tilde{F}'_i := \{r \in \tilde{F}_i \ ; \ r^- \in \tilde{F}'_{i-1} \text{ and } \forall i \leq j (F_{i,j}(r) = F_{i-1,j}(r^-))\},$$

where r^- is defined as in Definition 2.3(5). By the induction hypothesis, it follows that $\tilde{F}'_i$ is homogeneous for all functions in the set $\{F_{i,j} \ ; \ i \leq j < \omega\}$. The definition of the functions $F_{i,j}$ implies that for every $r \in \tilde{F}'_{i-1}$,

$$\{\beta \ ; \ r \cup \{(\alpha(r), m(r), \beta)\} \in \tilde{F}'_i\} \in \Phi_r. \tag{2}$$

Define the set

$$\tilde{F} := \{s\} \cup \bigcup \{\tilde{F}'_i \ ; \ i < \omega\}.$$

Claim. If $s_1, s_2 \in \tilde{F}$, $(s_1, A_1), (s_2, A_2) \in \mathbb{Q}$, and $(s_1, A_1), (s_2, A_2) \leq_{\mathbb{Q}} (s, S)$, then it is impossible to have that $(s_1, A_1) \Vdash_{\mathbb{Q}} \phi(\tau_1, \ldots, \tau_k)$ and $(s_2, A_2) \Vdash_{\mathbb{Q}} \neg\phi(\tau_1, \ldots, \tau_k)$.

Proof of claim. We have that for some $i_1, i_2 < \omega$ and for every $j = 1, 2$,

$$\mathsf{dom}^2(s_j) = \mathsf{dom}^2(s) \cup \{x(\ell) \ ; \ \ell < i_j\}.$$

Without loss of generality we may assume that $i_1 \leq i_2$. If $i_1 < i_2$ then we can increase the $\mathsf{dom}^2(s_1)$, one step at a time until we get $i_1 = i_2$. We have that the set

$$E := \{\beta < \alpha(s_1) \ ; \ s_1 \cup \{(\alpha(s_1), m(s_1), \beta)\} \in \tilde{F}'_{i_1}\} \text{ is in } \Phi_{s_1}$$

and since (s_1, A_1) is a condition in $\mathbb{Q}$ we have that also the set

$$E' := \{\beta < \alpha(s_1) \ ; \ s_1 \cup \{(\alpha(s_1), m(s_1), \beta)\} \in A_1\} \text{ is in } \Phi_{s_1}.$$

Let $\beta \in E \cap E'$, let $\bar{s}_1 := s_1 \cup \{(\alpha(s_1), m(s_1), \beta)\}$, and let $\bar{A}_1 := \{t \in A_1 \ ; \ t \supseteq \bar{s}_1\}$. Then we have that $\bar{s}_1 \in \tilde{F}'_{i_1} \subseteq \tilde{F}$ and that $(\bar{s}_1, \bar{A}_1) \leq_{\mathbb{Q}} (s_1, A_1)$. Therefore, $(\bar{s}_1, \bar{A}_1) \Vdash_{\mathbb{Q}} \phi(\tau_1, \ldots, \tau_n)$, and

$$\mathsf{dom}^2(\bar{s}_1) = \mathsf{dom}^2(s) \cup \{x(k) \ ; \ k < i_1 + 1\}.$$

This way we keep increasing i_1 until we get $i_1 = i_2$. Denote $i_2 - i_1$ by i. If $i = 0$ then $s_1 = s_2 = r$, therefore (s_1, A_1) and (s_2, A_2) are compatible which is a contradiction. If $i > 0$ then $s_1, s_2 \in \tilde{F}'_{i-1}$. Because $(s_1, A_1) \leq_{\mathbb{Q}} (s, S)$ and $(s_1, A_1) \Vdash_{\mathbb{Q}} \phi(\tau_1, \ldots, \tau_n)$, we have that $s_1 \in S_0$, thus $F_{i-1,i-1}(s_1) = 0$. Similarly we get that $F_{i-1,i-1}(s_2) = 1$ which contradicts the homogeneity of $\tilde{F}'_{i-1}$ for $F_{i-1,i-1}$. qed Claim

So to finish the proof of this lemma, we first show that $(s, \tilde{F})$ is indeed a condition in $\mathbb{Q}$. It suffices to show that for every $s' \in \tilde{F}$, the set $\mathsf{Suc}_{\tilde{F}}(s')$ of successors of s' in $\tilde{F}$ is in the ultrafilter $\Phi_{s'}$. We have that

$$\begin{aligned}\mathsf{Suc}_{\tilde{F}}(s') =& \{\beta < \alpha(s') \ ; \ s' \cup \{(\alpha(s'), m(s'), \beta)\} \in \tilde{F}\} \\ =& \{\beta < \alpha(s') \ ; \ s' \cup \{(\alpha(s'), m(s'), \beta)\} \in \tilde{F}_{i+1}, \}\end{aligned}$$

where $i \in \omega$ is such that $s' \in \tilde{F}_i$. By (2) we get that $\mathsf{Suc}_{\tilde{F}}(s') \in \Phi_{s'}$.

Finally, let $(s_1, A_1) \in \mathbb{Q}$ be any condition that decides $\phi(\tau_1, \ldots, \tau_n)$ and extends $(s, \tilde{F})$. Without loss of generality assume that $(s_1, A_1) \Vdash_{\mathbb{Q}} \phi(\tau_1, \ldots, \tau_n)$. By the definition of $\mathbb{Q}$ we can assume that $\mathsf{dom}(s_1) = e \setminus \kappa_\xi$, and hence $s_1 \in \tilde{F}$ and $A_1 \subseteq \tilde{F}$. Suppose that $(s, \tilde{F})$ does not decide $\phi(\tau_1, \ldots, \tau_n)$, then there must be some $(s_2, A_2) \leq_{\mathbb{Q}} (s, \tilde{F})$ such that $(s_2, A_2) \Vdash_{\mathbb{Q}} \neg\phi(\tau_1, \ldots, \tau_n)$. But this contradicts the Claim. Therefore $(s, \tilde{F})$ decides $\phi(\tau_1, \ldots, \tau_n)$. qed

Lemma A.4. *In $V[G^*]$, the partial order $(\mathbb{Q}, \leq_{\mathbb{Q}})$ does not add bounded subsets to κ_ξ.*

Proof. We work in $V[G^*]$. Consider the relation $\leq^*_{\mathbb{Q}} \subseteq \leq_{\mathbb{Q}}$, defined as $(s, S) \leq^*_{\mathbb{Q}} (r, R)$ iff $s = r$ and $S \subseteq R$.

Claim. $(\mathbb{Q}, \leq^*_{\mathbb{Q}})$ is κ_ξ-closed.

Proof of claim. Let $\gamma < \kappa_\xi$ and let $\langle (s_\zeta, S_\zeta) \ ; \ \zeta < \gamma \rangle$ be a $\leq^*_{\mathbb{Q}}$-descending sequence of elements in $\mathbb{Q}$. Since $\mathbb{E}$ is small forcing with respect to κ_ξ, we will use the Lévy-Solovay theorem [LS67] to get that all ultrafilters involved in the definition of $\mathbb{Q}$ can be extended to κ_ξ-complete ultrafilters in $V[G^*]$.

Let $s = s_\zeta$ for all $\zeta < \gamma$ and define a set $S \subseteq \bigcap_{\zeta < \gamma} S_\zeta$ inductively (above s) as follows. If $s' \subseteq s$ then $s' \in S$. Assume that $s \subseteq s' \in S$. Since for every $\zeta < \gamma$, $s' \in S_\zeta$, there is a set Φ, that is a κ_ξ-complete ultrafilter over $\alpha(s')$ in V, and such that for every $\zeta < \gamma$,

$$\mathsf{Suc}_{S_\zeta}(s') := \{\beta \ ; \ s' \cup \{(\alpha(s'), m(s'), \beta)\} \in S_\zeta\} \in \Phi.$$

In particular, we choose these Φ to fit with the definition of P_3, i.e., if $\alpha(s')$ is of type 1 then $\Phi = \Phi_{\alpha(s')}$, if $\alpha(s')$ is of type 2 and $\mathsf{cf}'\alpha(s') \geq \kappa_\xi$ then $\Phi = \Phi_{\alpha(s'), s'(\mathsf{cf}'\alpha(s'))(m(s'))}$, and if $\alpha(s')$ is of type 2 and $\mathsf{cf}'\alpha(s') < \kappa_\xi$ then $\Phi = \Phi_{\alpha(s'), \bigcup G^*(\mathsf{cf}'\alpha(s'))(m(s'))}$. By the Lévy-Solovay theorem [LS67],

$$\Phi^* := \{X \subseteq \alpha(s') \ ; \ \exists Y \subseteq X (Y \in \Phi)\}$$

is a κ_ξ-complete ultrafilter over $\alpha(s')$. So $\bigcap_{\zeta < \gamma} \mathsf{Suc}_{S_\zeta}(s') \in \Phi^*$ and consequently there is some $Y_{s'} \subseteq \mathsf{Suc}_{S_\zeta}(s')$ such that $Y_{s'} \in \Phi$. We define then

$$s' \cup \{(\alpha(s'), m(s'), \beta)\} \in S :\Longleftrightarrow \beta \in Y_{s'}.$$

Clearly $(s,S) \in \mathbb{Q}$ and for every $\zeta < \gamma$, $(s,S) \leq^*_{\mathbb{Q}} (s_\zeta, S_\zeta)$. qed Claim

Now to show that $(\mathbb{Q}, \leq_{\mathbb{Q}})$ does not add new bounded subsets to κ_ξ, we proceed as usual with Prikry type forcings. Let $(s,S) \in \mathbb{Q}$, let τ be a $\mathbb{Q}$-name, $\gamma < \kappa_\xi$, and assume that $(s,S) \Vdash_{\mathbb{Q}} \tau \subseteq \check{\gamma}$. Using the Prikry Lemma for $\mathbb{Q}$ (by Lemma A.3) and that $(\mathbb{Q}, \leq^*_{\mathbb{Q}})$ is κ_ξ-closed (by the claim), we get a $\leq^*_{\mathbb{Q}}$-decreasing sequence $\langle (s, S_\zeta) \; ; \; \zeta < \gamma \rangle$ of conditions in $\mathbb{Q}$ such that for each $\zeta < \gamma$, (s, S_ζ) decides the statement "$\check{\zeta} \in \tau$". Again by using the claim we get a condition (s, S') stronger than all (s, S_ζ), therefore one such that

$$(s,S') \Vdash \tau = \check{x}, \text{ where } x = \{\zeta \in \gamma \; ; \; (s,S') \Vdash \check{\zeta} \in \tau\}.$$

qed

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