ALL UNCOUNTABLE CARDINALS IN THE GITIK MODEL ARE ALMOST RAMSEY AND CARRY ROWBOTTOM FILTERS

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Abstract. Using the analysis developed in our earlier paper [ADK14], we show that every uncountable cardinal in Gitik's model of [Git80] in which all uncountable cardinals are singular is almost Ramsey and is also a Rowbottom cardinal carrying a Rowbottom filter. We assume that the model of [Git80] is constructed from a proper class of strongly compact cardinals, each of which is a limit of measurable cardinals. Our work consequently reduces the best previously known upper bound in consistency strength for the theory ZF + "All uncountable cardinals are singular" + "Every uncountable cardinal is both almost Ramsey and a Rowbottom cardinal carrying a Rowbottom filter".

1. Introduction

In this paper, we analyse the large cardinal properties that each uncountable cardinal exhibits in Gitik's model of [Git80] in which all uncountable cardinals are singular. Specifically, we will prove the following theorem.

Theorem 1.1. *Let $V \models \mathsf{ZFC}$ + "There is a proper class of strongly compact cardinals, each of which is a limit of measurable cardinals" + "Every limit of strongly compact cardinals is singular". There is then a proper class partial ordering $\mathbb{P} \subseteq V$ and a symmetric model $V(G)$ of the theory ZF + "All uncountable cardinals are singular" + "Every uncountable cardinal is both almost Ramsey and a Rowbottom cardinal carrying a Rowbottom filter".*

Theorem 1.1 generalises earlier work found in our previous paper [ADK14]. We will take as a convention that the word *cardinal* refers to the well-ordered cardinals, i.e., to the alephs.

We note that our extra assumption (unused by Gitik in [Git80]) that there is a proper class of strongly compact cardinals, each of which is a limit of measurable cardinals, is an entirely reasonable one. It follows, e.g., if there is a proper class of supercompact cardinals, since any supercompact cardinal is automatically both strongly compact and a limit of measurable cardinals. It is also a theorem of Kimchi and Magidor, unpublished by them but with a proof appearing in [Apt98], that it is consistent, relative to a proper class of supercompact cardinals, for the proper classes of supercompact and strongly compact cardinals to coincide precisely, except at measurable limit points. In fact, although Magidor's work of [Mag76] shows that it is consistent, relative to the existence of a strongly compact cardinal,

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for the least strongly compact and least measurable cardinal to coincide, it is not known whether it is even relatively consistent for the first ω strongly compact and measurable cardinals to coincide. (It is a theorem of Magidor, unpublished by him but appearing in [AC00], that for any fixed $n \in \omega$, it is consistent relative to the existence of n supercompact cardinals for the first n strongly compact and measurable cardinals to coincide.) It is therefore conceivable that it is a theorem of ZFC that if there is a proper class of strongly compact cardinals, then there is also a proper class of strongly compact cardinals, each of which is a limit of measurable cardinals.

We would also like to take this opportunity to point out that Theorem 1.1 dramatically reduces the best previously known upper bound in consistency strength for ZF + "All uncountable cardinals are singular" + "Every uncountable cardinal is both almost Ramsey and a Rowbottom cardinal carrying a Rowbottom filter". The work of [Apt85] and [Apt92] shows that the consistency of this theory can be established using hypotheses strictly between the consistency strength of ZFC + "There is a supercompact limit of supercompact cardinals" and ZFC + "There is an almost huge cardinal". These theories are both considerably stronger than ZFC + "There is a proper class of strongly compact cardinals, each of which is a limit of measurable cardinals". (The consistency of ZF + $\neg \mathsf{AC}_\omega$ + "Every successor cardinal is regular" + "All uncountable cardinals are almost Ramsey" had previously been shown in [AK08] to follow from the consistency of ZFC + "There are cardinals $\kappa < \lambda$ such that κ is 2^λ supercompact and λ is the least regular almost Ramsey cardinal greater than κ".)

We conclude the Introduction by recalling that the cardinal κ is *almost Ramsey* if $\forall \alpha < \kappa[\kappa \to (\alpha)_2^{<\omega}]$, i.e., if given $\alpha < \kappa$ and $f : [\kappa]^{<\omega} \to 2$, there is a *homogeneous set* $X \subseteq \kappa$ having order-type α (so X is such that $|F''[X]^n| = 1$ for every $n < \omega$). The cardinal κ is *Rowbottom* if $\forall \lambda < \kappa[\kappa \to [\kappa]^{<\omega}_{\lambda,\omega}]$, i.e., if given $\lambda < \kappa$ and $f : [\kappa]^{<\omega} \to \lambda$, there is a *homogeneous set* $X \subseteq \kappa$, $|X| = \kappa$ such that $|f''[X]^{<\omega}| \leq \omega$. If κ carries a filter $\mathcal{F}$ such that for any $f : [\kappa]^{<\omega} \to \lambda$, there is a set $X \in \mathcal{F}$ which is homogeneous for $\mathcal{F}$, then $\mathcal{F}$ is called a *Rowbottom filter.*

2. The Gitik construction

For our construction, we follow to a large extent the presentation given in our earlier paper [ADK14]. We will frequently quote verbatim from that paper. Specifically, we assume knowledge of forcing as presented in [Kun80, Ch. VII], [Jec03, Ch. 14], and of symmetric submodels as presented in [Jec03, Ch.15]. By a result of Felgner [Fel71], we also assume that our ground model V satisfies the global Axiom of Choice (global AC). We will use global AC when choosing the relevant ultrafilters in the discussion to be given below about type 1 and type 2 regular cardinals.

We start with an increasing sequence of cardinals,

$$\langle \kappa_\xi \ ; \ \xi \in \mathsf{Ord} \rangle,$$

such that for every $\xi \in \mathsf{Ord}$, κ_ξ is both strongly compact and a limit of measurable cardinals, and such that the sequence has no regular limits.

We will construct a model in which all uncountable cardinals are singular. We will do this by slightly modifying and adapting Gitik's construction of [Git80], whose notation and terminology we freely use.

Our construction is a finite support product of Prikry-like forcings which are interweaved in order to prove a Prikry-like lemma for that part of the forcing.

Call Reg the class of regular cardinals in V. For convenience call $\omega =: \kappa_{-1}$. For an $\alpha \in \mathsf{Reg}$ we define a $\mathsf{cf}'\alpha$ to distinguish between the following categories.

(type 1) This occurs if there is a largest $\kappa_\xi \leq \alpha$ (i.e., $\alpha \in [\kappa_\xi, \kappa_{\xi+1})$). We then define $\mathsf{cf}'\alpha := \alpha$.

If $\alpha = \kappa_\xi$ and $\xi \neq -1$, then let Φ_{κ_ξ} be a measure for κ_ξ (which need not be normal). If $\alpha = \omega$ then let Φ_ω be any uniform ultrafilter over ω.

If $\alpha > \kappa_\xi$ is inaccessible, then let H_α be a κ_ξ-complete fine ultrafilter over $\mathcal{P}_{\kappa_\xi}(\alpha)$ and let $h_\alpha : \mathcal{P}_{\kappa_\xi}(\alpha) \to \alpha$ be a bijection. Define

$$\Phi_\alpha := \{X \subseteq \alpha \ ; \ h_\alpha^{-1}``X \in H_\alpha\}.$$

This is a uniform κ_ξ-complete ultrafilter over α.

If $\alpha > \kappa_\xi$ is not inaccessible, then let Φ_α be any κ_ξ-complete uniform ultrafilter over α.

(type 2) This occurs if there is no largest strongly compact $\leq \alpha$. We then let β be the largest (singular) limit of strongly compacts $\leq \alpha$. Define $\mathsf{cf}'\alpha := \mathsf{cf}\beta$. Let

$$\langle \kappa_\nu^\alpha \ ; \ \nu < \mathsf{cf}'\alpha \rangle$$

be a fixed ascending sequence of strongly compacts $\geq \mathsf{cf}'\alpha$ such that $\beta = \bigcup\{\kappa_\nu^\alpha \ ; \ \nu < \mathsf{cf}'\alpha\}$.

If α is inaccessible, then for each $\nu < \mathsf{cf}'\alpha$, let $H_{\alpha,\nu}$ be a fine ultrafilter over $\mathcal{P}_{\kappa_\nu^\alpha}(\alpha)$ and $h_{\alpha,\nu} : \mathcal{P}_{\kappa_\nu}(\alpha) \to \alpha$ a bijection. Define

$$\Phi_{\alpha,\nu} := \{X \subseteq \alpha \ ; \ h_\alpha^{-1}``X \in H_{\alpha,\nu}\}.$$

Again, $\Phi_{\alpha,\nu}$ is a κ_ν^α-complete uniform ultrafilter over α.

If α is not inaccessible, then for each $\nu < \mathsf{cf}'\alpha$, let $\Phi_{\alpha,\nu}$ be any κ_ν^α-complete uniform ultrafilter over α.

This $\mathsf{cf}'\alpha$ will be used when we want to organise the choice of ultrafilters for the type 2 cardinals.

We use the fine ultrafilters H_α and $H_{\alpha,\nu}$ to make sure that in the end only the strongly compacts and their singular limits remain cardinals in our final symmetric model $V(G)$. For type 1 ordinals we will do some tree-Prikry like forcings to singularise in cofinality ω. Type 1 cardinals in the open intervals $(\kappa_\xi, \kappa_{\xi+1})$ will be collapsed to κ_ξ because enough of these forcings will be isomorphic to strongly compact Prikry forcings (or "fake" strongly compact Prikry forcings in the case of $\xi = -1$). This is why we use fine ultrafilters for these cardinals.

To singularise type 2 ordinals Gitik used a technique he credits in [Git80] to Magidor, a Prikry-type forcing that relies on the countable cofinal sequence $\vec{c}$ that we build for $\mathsf{cf}'\alpha$ to pick a countable sequence of ultrafilters $\langle \Phi_{\vec{c}(n)} \ ; \ n \in \omega \rangle$. To show that the type 2 cardinals are collapsed, we use again the fine ultrafilters.

As usual with Prikry-type forcings, one has to prove a Prikry-like lemma (see [Git80, Lemma 5.1]). For the arguments one requires the forcing conditions to grow nicely.

These conditions can be viewed as trees. These trees will grow from "left to right" in order to ensure that a type 2 cardinal α will have the necessary information from the Prikry sequence[1] at stage $\mathsf{cf}'\alpha$. Let us take a look at the definition of the stems of the Prikry sequences to be added.

[1]For the rest of this paper, we will abuse terminology by using phrases like "Prikry sequence" and "Prikry forcing" when referring to our Prikry-like forcing notions.

Definition 2.1. For a set $t \subseteq \mathsf{Reg} \times \omega \times \mathsf{Ord}$ we define the sets

$$\begin{aligned}\mathsf{dom}(t) :=& \{\alpha \in \mathsf{Reg} \ ; \ \exists m \in \omega \exists \gamma \in \mathsf{Ord}((\alpha, m, \gamma) \in t)\}, \text{ and}\\ \mathsf{dom}^2(t) :=& \{(\alpha, m) \in \mathsf{Reg} \times \omega \ ; \ \exists \gamma \in \mathsf{Ord}((\alpha, m, \gamma) \in t)\}.\end{aligned}$$

Let P_1 be the class of all finite subsets t of $\mathsf{Reg} \times \omega \times \mathsf{Ord}$, such that for every $\alpha \in \mathsf{dom}(t)$, $t(\alpha) := \{(m, \gamma) \ ; \ (\alpha, m, \gamma) \in t\}$ is an injective function from some finite subset of ω into α.

To add a Prikry sequence to a type 2 cardinal α, we want to have some information on the Prikry sequence of the cardinal $\mathsf{cf}'\alpha$. We also want to make sure that these stems are appropriately ordered for the induction in the proof of the aforementioned Prikry-like lemma. So we define the following.

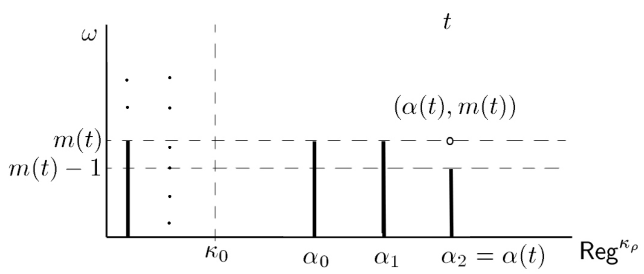
Definition 2.2. Let P_2 be the class of all $t \in P_1$ such that the following hold.

(1) For every $\alpha \in \mathsf{dom}(t)$, $\mathsf{cf}'\alpha \in \mathsf{dom}(t)$ and $\mathsf{dom}(t(\mathsf{cf}'\alpha)) \supseteq \mathsf{dom}(t(\alpha))$.
(2) If $\{\alpha_0, \dots, \alpha_{n-1}\}$ is an increasing enumeration of $\mathsf{dom}(t) \setminus \kappa_0$, then there are $m, j \in \omega$, such that $m \geq 1$, $j \leq n-1$ with the properties that
   - for every $k < j$ we have that $\mathsf{dom}(t(\alpha_k)) = m + 1$ and
   - for every $k \in \{j, \dots, n-1\}$ we have that $\mathsf{dom}(t(\alpha_k)) = m$.

   These m and α_j are unique for t and are denoted by $m(t) := m$ and $\alpha(t) := \alpha_j$.
(3) If $\mathsf{cf}'\alpha(t) < \alpha(t)$, then $m(t) \in \mathsf{dom}(t(\mathsf{cf}'\alpha(t)))$.

We may think of the point $(\alpha(t), m(t))$ as the point we have to fill in next, in order to extend t.

Let us call elements of P_2 stems. In the following image we can see roughly what a stem t with a domain $\{\alpha_0, \alpha_1, \alpha_2\}$ above κ_0 looks like.



In order to add Prikry sequences, we will use the ultrafilters and define the partial ordering with which we will force.

Definition 2.3. Let P_3 be the class of pairs (t, T) such that

(1) $t \in P_2$,
(2) $T \subseteq P_2$ is a set (and not a proper class),
(3) $t \in T$,
(4) for every $t' \in T$ we have $t' \supseteq t$ or $t' \subseteq t$, and $\mathsf{dom}(t') = \mathsf{dom}(t)$,
(5) for every $t' \in T$, if $t' = r \cup \{(\alpha(r), m(r), \beta)\}$ then $t'^{-} := r \in T$, i.e., T is tree-like,
(6) for every $t' \in T$ with $t' \supseteq t$, if $\alpha(t')$ is of type 1 (i.e., $\mathsf{cf}'(\alpha(t')) = \alpha(t')$) then
$$\mathsf{Suc}_T(t') := \{\beta \ ; \ t' \cup \{(\alpha(t'), m(t'), \beta)\} \in T\} \in \Phi_{\alpha(t')}, \text{ and}$$
(7) for every $t' \in T$ with $t' \supseteq t$, if $\alpha(t')$ is of type 2 (i.e., $\mathsf{cf}'\alpha(t') < \alpha(t')$) and $m(t') \in \mathsf{dom}(t'(\mathsf{cf}'(\alpha(t'))))$ then
$$\mathsf{Suc}_T(t') := \{\beta \ ; \ t' \cup \{(\alpha(t'), m(t'), \beta)\} \in T\} \in \Phi_{\alpha(t'), t'(\mathsf{cf}'\alpha(t'))(m(t'))}.$$

For a (t,T) in P_3 and a subset $x \subseteq \mathsf{Reg}$ we write $T \restriction x$ for $\{t' \restriction x \ ; \ t' \in T\}$, where $t' \restriction x = \{(\alpha, m, \gamma) \in t' \ ; \ \alpha \in x\}$.

We call t the trunk of (t,T).

This P_3 is the forcing we are going to use. It is partially ordered by

$$(t,T) \leq (s,S) :\iff \mathsf{dom}(t) \supseteq \mathsf{dom}(s) \text{ and } T \restriction \mathsf{dom}(s) \subseteq S.$$

A full generic extension via this P_3 adds too many subsets of ordinals and makes every infinite ordinal in V countable. By restricting to sets of ordinals which can be approximated with finite domains we ensure that the former strongly compacts are still cardinals in the symmetric model to be constructed, and the power sets stay small.

To build a symmetric model we need an automorphism group of the complete Boolean algebra $B = B(P_3)$ that is induced by P_3, as in [Jec03, Corollary 14.12].[2] We start by considering $\mathcal{G}$ to be the group of permutations of $\mathsf{Reg} \times \omega \times \mathsf{Ord}$ whose elements a satisfy the following properties.

- For every $\alpha \in \mathsf{Reg}$ there is a permutation a_α of α that moves only finitely many elements of α, and is such that for each $n \in \omega$ and each $\beta \in \alpha$,
$$a(\alpha, n, \beta) = (\alpha, n, a_\alpha(\beta)).$$
The finite subset of α that a_α moves, we denote by $\mathsf{supp}(a_\alpha)$, which stands for "support of a_α".
- For only finitely many $\alpha \in \mathsf{Reg}$ is a_α not the identity. This finite subset of Reg we denote by $\mathsf{dom}(a)$.

We extend $\mathcal{G}$ to P_3 as follows. For $a \in \mathcal{G}$ and $(t,T) \in P_3$, define

$$a(t,T) := (a``t, \{a``t' \ ; \ t' \in T\}),$$

where $a``t := \{(\alpha, n, a_\alpha(\beta)) \ ; \ (\alpha, n, \beta) \in t\}$.

Unfortunately, in general $a(t,T)$ is not a member of P_3 because of the branching condition at type 2 cardinals. In particular, it is possible that for some $\alpha \in \mathsf{dom}(t)$ of type 2, and some $t' \in T$ with $\alpha = \alpha(t')$, we have that $a_{\mathsf{cf}'\alpha}(t'(\mathsf{cf}'\alpha)(m(t'))) = \gamma \neq t'(\mathsf{cf}'\alpha)(m(t'))$, and even though we had before $\mathsf{Suc}_T(t') \in \Phi_{\alpha, t'(\mathsf{cf}'\alpha)(m(t'))}$, it is not true that $\mathsf{Suc}_T(t') \in \Phi_{\alpha,\gamma}$.

To overcome this problem, for an $a \in \mathcal{G}$, define $P^a \subseteq P_3$ as follows.

(t,T) is in P^a iff the following hold:

(1) $\mathsf{dom}(t) \supseteq \mathsf{dom}(a)$,
(2) for every $\alpha \in \mathsf{dom}(t)$ we have that $\mathsf{dom}(t(\alpha)) = \mathsf{dom}(t(\mathsf{cf}'\alpha))$, and
(3) for every $\alpha \in \mathsf{dom}(t)$, we have that
$$\mathsf{rng}(t(\alpha)) \supseteq \{\beta \in \mathsf{supp}(a_\alpha) \ ; \ \exists q \in T(\beta \in \mathsf{rng}(q(\alpha)))\}.$$

The equality in (2) ensures that there will be no severe mixup in the requirements for membership in ultrafilters of the form $\Phi_{\alpha,\gamma}$. In (3) we require that the stem of each condition already contains all the ordinals that the a_α could move. This will prevent any trouble with membership in the ultrafilters. One may think that this requirement should be $\mathsf{supp}(a_\alpha) \subseteq \mathsf{rng}(t(\alpha))$ but this is not the case; note that there might be some γ in a_α which doesn't appear in the range of any $q \in T$.

Now, we have that $a : P^a \to P^a$ is an automorphism. Also, as mentioned in [Git80, page 68], for every $a \in \mathcal{G}$ the class P^a is a dense subclass of P_3. Therefore,

[2]Note that even though P_3 is a proper class partial ordering, it is still possible to define both $B = B(P_3)$ and a basic theory of symmetric models.

a can be extended to a unique automorphism of the complete Boolean algebra B. We denote the automorphism of B with the same letter, and also by $\mathcal{G}$ the automorphism group of B that consists of these extended automorphisms. By [Jec03, (14.36)], every automorphism a of B induces an automorphism of the Boolean valued model V^B.

Proceeding to the definition of our symmetric model, for every $e \subseteq \mathsf{Reg}$ define

$$E_e := \{(t,T) \in P_3 \ ; \ \mathsf{dom}(t) \subseteq e\},$$

$$I := \{E_e \ ; \ e \subset \mathsf{Reg} \text{ is finite and closed under } \mathsf{cf}'\},$$

$$\mathsf{fix}E_e := \{a \in \mathcal{G} \ ; \ \forall \alpha \in e(a_\alpha \text{ is the identity on } \alpha)\},$$

and let $\mathcal{F}$ be the normal filter (see [Jec03, (15.34)]) over $\mathcal{G}$ that is generated by

$$\{\mathsf{fix}E_e \ ; \ E_e \in I\}.$$

For each $\dot{x}$ in the Boolean valued model V^B, define its symmetry group

$$\mathsf{sym}(\dot{x}) := \{a \in \mathcal{G} \ ; \ a(\dot{x}) = \dot{x}\}.$$

A name $\dot{x}$ is called symmetric iff its symmetry group is in the filter $\mathcal{F}$. The class of hereditarily symmetric names HS is defined by recursion on the rank of the name, i.e.,

$$\mathsf{HS} := \{\dot{x} \in V^B \ ; \ \forall \dot{y} \in \mathsf{tc}_{\mathsf{dom}}(\dot{x})(\mathsf{sym}(\dot{y}) \in \mathcal{F})\},$$

where $\mathsf{tc}_{\mathsf{dom}}(\dot{x})$ is defined as the union of all x_n, which are defined recursively by $x_0 := \{\dot{x}\}$ and $x_{n+1} := \bigcup\{\mathsf{dom}(\dot{y}) \ ; \ \dot{y} \in x_n\}$.

We will say that an $E_e \in I$ supports a name $\dot{x} \in \mathsf{HS}$ if $\mathsf{fix}E_e \subseteq \mathsf{sym}(\dot{x})$.

For some V-generic ultrafilter G on B we define the symmetric model

$$V(G) := \{\dot{x}^G \ ; \dot{x} \in \mathsf{HS}\}.$$

By [Jec03, Lemma 15.51], $V \subseteq V(G) \subseteq V[G]$. By [Jec03, Lemma 15.51] and the work of [Git80], $V(G) \models \mathsf{ZF}$ + "All uncountable cardinals are singular".

For each $(t,T) \in P_3$ and each $E_e \in I$, define

$$(t,T) \restriction^* E_e = (t \restriction e, \{t' \restriction e \ ; \ t' \in T\}).$$

Let $\langle \delta_\alpha \ ; \ \alpha \in \mathsf{Ord}\rangle$ be the continuous, increasing enumeration of $\langle \kappa_\alpha \ ; \ \alpha \in \mathsf{Ord}\rangle$, together with the limit points of this sequence. By the proof of [ADK14, Theorem 2.5] (including the work of the appendix), the following key fact is true.

Lemma 2.4. *If $X \in V(G)$ is a set of ordinals, $X \subseteq \delta_\alpha$ for some ordinal α, then there is an $E_e \in I$ with $e \subseteq \delta_{\alpha+1}$ such that $X \in V[G \restriction^* E_e]$, where $G \restriction^* E_e := \{(t,T) \restriction^* E_e \ ; \ (t,T) \in G\}$.*

We are now in a position to state and prove the lemma used to establish Theorem 1.1. Specifically, we have the following.

Lemma 2.5. *In $V(G)$, every uncountable cardinal κ is both almost Ramsey and is also a Rowbottom cardinal carrying a Rowbottom filter.*

Proof. Suppose $V(G) \models$ "$\kappa > \omega$ is a cardinal". By [ADK14, Corollary 2.9], $\kappa = \delta_\alpha$ for some ordinal α.

To show that $V(G) \models$ "κ is almost Ramsey", let $f : [\kappa]^{<\omega} \to 2$, $f \in V(G)$. Since f can be coded by a subset of κ, by Lemma 2.4, $f \in V[G \restriction^* E_e]$ for some $e \subseteq \delta_{\alpha+1}$. Again by the proof of [ADK14, Theorem 2.5] (including the work of the appendix), we can write $V[G \restriction^* E_e] = V[H_0][H_1]$, where H_0 is V-generic over a partial ordering $\mathbb{E}$ such that $|\mathbb{E}| < \delta_\alpha$, and H_1 is $V[H_0]$-generic over a partial ordering $\mathbb{Q}$ which adds no bounded subsets of δ_α.

Note now that in V, κ is a limit of both (non-measurable) Ramsey and measurable cardinals. This is since either κ is strongly compact or a limit of strongly compacts. By the Lévy-Solovay results [LS67], in $V[H_0]$, κ is a limit of (non-measurable) Ramsey and measurable cardinals as well. Further, because forcing over $V[H_0]$ with $\mathbb{Q}$ adds no bounded subsets of $\kappa = \delta_\alpha$, in $V[H_0][H_1] = V[G \restriction^* E_e]$, κ is also a limit of Ramsey cardinals. Consequently, by [AK08, Proposition 1], κ is almost Ramsey in $V[G \restriction^* E_e]$. This means that for every $\beta < \kappa$, there is a set $X_\beta \in V[G \restriction^* E_e]$ which is homogeneous for f and has order type at least β. As $V[G \restriction^* E_e] \subseteq V(G)$, $X_\beta \in V(G)$. Since f was arbitrary, it therefore follows that $V(G) \models$ "κ is almost Ramsey".

To show that $V(G) \models$ "κ is a Rowbottom cardinal carrying a Rowbottom filter", we first note that there is some $E_{e_0} \in I$ such that $V[G \restriction^* E_{e_0}] \models$ "$\mathsf{cf}(\kappa) = \omega$ and $\kappa = \sup(\langle \zeta_i \ ; \ i < \omega \rangle)$, where each ζ_i is measurable". To see this, observe that because (by hypothesis) $V \models$ "κ is a limit of measurable cardinals" and $V(G) \models$ "All uncountable cardinals are singular", by Lemma 2.4, there is some $e_0 \subseteq \delta_{\alpha+1}$ and $E_{e_0} \in I$ such that $V[G \restriction^* E_{e_0}] \models$ "$\mathsf{cf}(\kappa) = \omega$ and $\kappa = \sup(\langle \zeta_i \ ; \ i < \omega \rangle)$, where each ζ_i is measurable in V". However, as in the preceding paragraph, we can write $V[G \restriction^* E_{e_0}] = V[H_0][H_1]$, where H_0 is V-generic over a partial ordering $\mathbb{E}$ such that $|\mathbb{E}| < \delta_\alpha = \kappa$, and H_1 is $V[H_0]$-generic over a partial ordering $\mathbb{Q}$ which adds no bounded subsets of δ_α. By the results of [LS67], it is still the case that $V[H_0] \models$ "κ is a limit of measurable cardinals". Because forcing with $\mathbb{Q}$ adds no bounded subsets of κ, $V[H_0][H_1] = V[G \restriction^* E_{e_0}]$ is as desired. Consequently, for the remainder of the proof of Lemma 2.5, we fix E_{e_0}, $\langle \zeta_i \ ; \ i < \omega \rangle \in V[G \restriction^* E_{e_0}]$, and $\langle \mu_i \ ; \ i < \omega \rangle \in V[G \restriction^* E_{e_0}]$ such that $V[G \restriction^* E_{e_0}] \models$ "$\kappa = \sup(\langle \zeta_i \ ; \ i < \omega \rangle)$, where each ζ_i is a measurable cardinal, and each μ_i is a normal measure over ζ_i". We also define $\mathcal{F} \in V[G \restriction^* E_{e_0}]$ by $\mathcal{F} = \{X \subseteq \kappa \ ; \ \exists n < \omega \forall i \geq n [X \cap \zeta_i \in \mu_i]\}$. Clearly, $\mathcal{F}$ generates a filter (in any model of ZF in which it is a member).

Now, let $f : [\kappa]^{<\omega} \to \lambda$, $f \in V(G)$, with $\omega \leq \lambda < \kappa$ a cardinal in $V(G)$. As before, since f can be coded by a subset of κ, by Lemma 2.4, $f \in V[G \restriction^* E_e]$ for some $e \subseteq \delta_{\alpha+1}$. Without loss of generality, by coding if necessary, we may assume in addition that $V[G \restriction^* E_e] \supseteq V[G \restriction^* E_{e_0}]$. As in the preceding paragraph, $V[G \restriction^* E_e] = V[H^*_0][H^*_1]$, where H^*_0 is V-generic over a partial ordering having cardinality less than κ. Therefore, by the results of [LS67], we may further assume that in $V[G \restriction^* E_e]$, a final segment $\mathbb{F}$ of $\langle \zeta_i \ ; \ i < \omega \rangle$ is composed of measurable cardinals, and that for any i such that $\zeta_i \in \mathbb{F}$, μ'_i defined in $V[G \restriction^* E_e]$ by $\mu'_i = \{X \subseteq \zeta_i \ ; \ \exists Y \in \mu_i [Y \subseteq X]\}$ is a normal measure over ζ_i. Let n_0 be least such that $\zeta_{n_0} \in \mathbb{F}$. By a theorem of Prikry (see [Kan03, Theorem 8.7]), $\mathcal{F}^* = \{X \subseteq \kappa \ ; \ \exists n \geq n_0 \forall i \geq n [X \cap \zeta_i \in \mu'_i]\} \in V[G \restriction^* E_e]$ is such that for some $Z^* \in \mathcal{F}^*$, Z^* is homogeneous for f. By the definitions of $\mathcal{F}$ and $\mathcal{F}^*$ and the fact that every μ'_i measure 1 set contains a μ_i measure 1 set for $\zeta_i \in \mathbb{F}$, it then immediately follows that for some $Z \in \mathcal{F}$, $Z \subseteq Z^*$, Z is homogeneous for f. Thus, $\mathcal{F} \in V[G \restriction^* E_{e_0}] \subseteq V(G)$ generates a Rowbottom filter for κ in $V(G)$. This completes the proof of both Lemma 2.5 and Theorem 1.1. qed

We conclude by asking whether it is possible to remove the additional assumption that every strongly compact cardinal is a limit of measurable cardinals. We conjecture that this is indeed possible, although with a fair bit of work.

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