On Tall Cardinals and Some Related Generalizations *†

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Abstract

We continue the study of tall cardinals and related notions begun by Hamkins in [11] and answer three of his questions posed in that paper.

1 Introduction and preliminaries

We begin with the following definitions.

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Definition 1.1 (Hamkins [11]) *Suppose κ is a cardinal and $\lambda \geq \kappa$ is an arbitrary ordinal. κ is* λ tall *if there is an elementary embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \lambda$ and $M^\kappa \subseteq M$. κ is* tall *if κ is λ tall for every ordinal λ. κ is* strongly tall *if for every ordinal $\lambda \geq \kappa$, there is an elementary embedding witnessing that κ is λ tall which is generated by a κ-complete measure on some set.*

The first part of the next definition (i.e., *tall with bounded closure*) is due to Hamkins and is found in [11, Section 5].

Definition 1.2 *κ is* tall with bounded closure *if κ is not a tall cardinal, yet there is a cardinal δ with $\omega \leq \delta < \kappa$ such that for all ordinals $\lambda \geq \kappa$, there is an elementary embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \lambda$ and $M^\delta \subseteq M$. Let $\omega \leq \delta < \kappa$ be a fixed cardinal. κ is* tall with bounded closure δ *if for every ordinal $\lambda \geq \kappa$, there is an elementary embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \lambda$, $M^\delta \subseteq M$, and $M^{\delta^+} \not\subseteq M$.*

In [11], Hamkins made a systematic study of tall cardinals and some related notions and posed the following questions.

1. (implicit to [11, Section 4]) Is it possible to construct a model containing infinitely many tall cardinals in which the measurable and tall cardinals coincide precisely?

2. ([11, Question 5.5]) Is it possible to construct a model containing a tall cardinal with bounded closure?

3. ([11, Question 2.12]) Are strong tallness and strong compactness equivalent? Are they equiconsistent?

The purpose of this paper is to answer Questions 1 and 2 affirmatively and Question 3 negatively. Specifically, we prove the following eight theorems, along with a corollary to one of them. Theorem 1 addresses Question 1. Theorems 2 – 7 address Question 2. Theorem 8 addresses Question 3.

Theorem 1 *Suppose $V \models$ "ZFC + GCH + κ is supercompact + No cardinal $\lambda > \kappa$ is measurable". There is then a partial ordering $\mathbb{P} \subseteq V$ such that $V^{\mathbb{P}} \models$ "ZFC + κ is supercompact + No cardinal $\lambda > \kappa$ is measurable + For every $\delta < \kappa$, δ is measurable iff δ is tall".*

Theorem 2 *Suppose κ is a strong cardinal. Then κ is a tall cardinal having bounded closure ω.*

Theorem 3 *Suppose $V = \mathcal{K}$ and $V \models$ "ZFC + κ is strong + $\eta > \kappa$ is such that $o(\eta) = \omega_1$ + No cardinal $\delta > \eta$ is measurable". There is then a partial ordering $\mathbb{P} \in V$ such that $V^{\mathbb{P}} \models$ "κ is a tall cardinal having bounded closure ω + There are neither any tall cardinals nor any tall cardinals having bounded closure δ for $\omega_1 \leq \delta < \kappa$".*

Theorem 4 *The following conditions are equivalent:*

1. *There is an elementary embedding $j : V \to M$ with critical point κ such that $(j(\kappa))^\omega \subseteq M$ yet $(j(\kappa))^{\omega_1} \not\subseteq M$.*
2. *There exists a Rudin-Keisler increasing sequence of ultrafilters over κ having length ω_1.*
3. *There exists an elementary embedding $j : V \to M$ such that $M^\omega \subseteq M$ and a sequence $\langle \eta_\alpha \mid \alpha < \omega_1 \rangle$ of ordinals below $j(\kappa)$ such that for every $\alpha < \omega_1$ and for every $f : [\kappa]^\alpha \to \kappa$, it is the case that $\eta_\alpha \neq j(f)(\langle \eta_\beta \mid \beta < \alpha \rangle)$.*

Theorem 5 *Suppose that there is no sharp for a strong cardinal (i.e., that o pistol does not exist). If there is an elementary embedding $j : V \to M$ with critical point κ such that $(j(\kappa))^\omega \subseteq M$ yet $(j(\kappa))^{\omega_1} \not\subseteq M$, then either $o(\kappa) \geq \omega_1$ in $\mathcal{K}$, or κ is measurable in $\mathcal{K}$ and $\{\nu < \kappa \mid o^{\mathcal{K}}(\nu) \geq \omega_1\}$ is unbounded in κ.*

Theorem 6 *Suppose that $o(\kappa) \geq \omega_1$ in $\mathcal{K}$. Then there is a generic extension V of $\mathcal{K}$ with an elementary embedding $j : V \to M$ having critical point κ such that $M^\omega \subseteq M$ yet $(j(\kappa))^{\omega_1} \not\subseteq M$.*

Theorem 7 *Suppose that κ is a measurable cardinal in $\mathcal{K}$ and $\{\nu < \kappa \mid o^{\mathcal{K}}(\nu) \geq \omega_1\}$ is unbounded in κ. Then there is a generic extension V of $\mathcal{K}$ and an elementary embedding $j : V \to M$ having critical point κ such that $M^\omega \subseteq M$ yet $(j(\kappa))^{\omega_1} \not\subseteq M$.*

Theorem 8 *The following theories are equiconsistent:*

a) ZFC + There is a strong cardinal and a proper class of measurable cardinals.

b) ZFC + There is a strongly tall cardinal.

We take this opportunity to make a few remarks concerning Theorems 1 – 8. Theorem 1 provides a positive answer to Question 1, since in $(V_\kappa)^{V^{\mathbb{P}}}$, there is a proper class of tall cardinals, and the tall and measurable cardinals precisely coincide. As we will show, however, the use of a supercompact cardinal is unnecessary in order to construct a model witnessing a positive answer to Question 1. We prove Theorem 1 in this form, though, because we feel it is of independent interest to show that the tall and measurable cardinals can coincide precisely below a supercompact cardinal. In addition, Question 1 is a direct analogue of a famous question concerning strongly compact and measurable cardinals, which we will discuss at greater length in Section 2 after the proof of Theorem 1. Theorem 2 shows that any strong cardinal is in fact tall with bounded closure ω. Theorem 3 provides a positive answer to Question 2, and Theorems 3 – 7 address the consistency strength of the existence of a tall cardinal κ exhibiting tallness with closure bounded below κ. Theorem 8 exactly pins down the consistency strength of the existence of a strongly tall cardinal and shows that it is much weaker than the consistency strength of a strongly compact cardinal.

Before beginning the proofs of our theorems, we briefly mention some preliminary information and terminology. Essentially, our notation and terminology are standard, and when this is not the case, this will be clearly noted. For $\alpha < \beta$ ordinals, $[\alpha, \beta]$, $[\alpha, \beta)$, $(\alpha, \beta]$, and (α, β) are as in the usual interval notation. If $\kappa \geq \omega$ is a regular cardinal and λ is an arbitrary ordinal, then $\mathrm{Add}(\kappa, \lambda)$ is the standard partial ordering for adding λ Cohen subsets of κ.

When forcing, $q \geq p$ will mean that *q is stronger than p*. If G is V-generic over $\mathbb{P}$, we will abuse notation slightly and use both $V[G]$ and $V^{\mathbb{P}}$ to indicate the universe obtained by forcing with $\mathbb{P}$. If $x \in V[G]$, then $\dot{x}$ will be a term in V for x. We may, from time to time, confuse terms with the sets they denote and write x when we actually mean $\dot{x}$ or $\check{x}$, especially when x is some variant of the generic set G, or x is in the ground model V. The abuse of notation mentioned above will be compounded by writing $x \in V^{\mathbb{P}}$ instead of $\dot{x} \in V^{\mathbb{P}}$. Any term for trivial forcing will always be taken as a term for the partial ordering $\{\emptyset\}$. If φ is a formula in the forcing language with respect to $\mathbb{P}$ and $p \in \mathbb{P}$, then $p \parallel \varphi$ means that p *decides* φ.

From time to time within the course of our discussion, we will refer to partial orderings $\mathbb{P}$ as being *Easton support iterations of Prikry type forcings.* By this we will mean an Easton support iteration as first given by the second author in [5], to which we refer readers for a discussion of the basic properties of and terminology associated with such an iteration.

As in [9], we will say that the partial ordering $\mathbb{P}$ is *κ^+-weakly closed and satisfies the Prikry property* if it meets the following criteria.

1. $\mathbb{P}$ has two partial orderings $\leq$ and $\leq^*$ with $\leq^* \subseteq \leq$.

2. For every $p \in \mathbb{P}$ and every statement φ in the forcing language with respect to $\mathbb{P}$, there is some $q \in \mathbb{P}$ such that $p \leq^* q$ and $q \parallel \varphi$.

3. The partial ordering $\leq^*$ is κ-closed, i.e., there is an upper bound for every increasing chain of conditions having length κ.

Key to the proof of Theorem 1 is the following result due to the second author and Shelah. It is a corollary of the work of [9, Section 2].

Theorem 9 *Suppose $V \models$ "ZFC + GCH + $\delta < \kappa$ are such that δ is a regular cardinal and κ is a strong cardinal". There is then a δ^+-weakly closed partial ordering $\mathbb{I}(\delta, \kappa)$ satisfying the Prikry property having cardinality κ such that $V^{\mathbb{I}(\delta,\kappa)} \models$ "κ is a strong cardinal whose strongness is indestructible under κ^+-weakly closed partial orderings satisfying the Prikry property".*

We mention that we are assuming some familiarity with the large cardinal notions of measurability, measurable cardinals of high Mitchell order, tallness, hypermeasurability, strongness, strong compactness, and supercompactness. Interested readers may consult [13], [17], [18], or [21]. In addition, we are assuming some familiarity with basic inner model and core model theory, as presented in [22] and [19]. In particular, $\mathcal{K}$ will always denote the core model. Finally, we are assuming some familiarity with the Rudin-Keisler ordering on ultrafilters, for which we refer readers to [6].

2 Models where the measurable and tall cardinals coincide precisely

We begin with the proof of Theorem 1, which we restate for the convenience of readers.

Theorem 1 *Suppose $V \models$ "ZFC + GCH + κ is supercompact + No cardinal $\lambda > \kappa$ is measurable". There is then a partial ordering $\mathbb{P} \subseteq V$ such that $V^{\mathbb{P}} \models$ "ZFC + κ is supercompact + No cardinal $\lambda > \kappa$ is measurable + For every $\delta < \kappa$, δ is measurable iff δ is tall".*

Proof: We start with the key fact that a Magidor iteration [16] of Prikry forcing preserves the tallness of a strong cardinal.

Lemma 2.1 *Suppose $\delta < \kappa$ and $V \models$ "κ is a strong cardinal". Let $\mathbb{P}(\delta, \kappa)$ be the Magidor iteration of Prikry forcing which adds a Prikry sequence to every measurable cardinal in the open interval (δ, κ). Then $V^{\mathbb{P}(\delta,\kappa)} \models$ "κ is a tall cardinal".*

Proof: Let $\lambda > \kappa$ be an arbitrary strong limit cardinal of cofinality at least κ. By the proof of [11, Theorem 4.1], we may take $j : V \to M$ to be an elementary embedding witnessing the λ tallness of κ generated by the (κ, λ)-extender $\mathcal{E} = \langle E_a \mid a \in [\lambda]^{<\omega} \rangle$ such that $M \models$ "κ is not measurable". Since $M^\kappa \subseteq M$, it is the case that $\mathcal{E}$ is κ^+-*directed*, i.e., for each κ sequence $\langle E_i \mid i < \kappa \rangle$ of measures from $\mathcal{E}$, there is some $E \in \mathcal{E}$ such that for each $i < \kappa$, $E_i <_{\mathrm{RK}} E$ (so E projects onto E_i as in the Rudin-Keisler ordering). To see this, let $j_i : \mathrm{Ult}(V, E_i) \to M$ be the canonical elementary embedding of $\mathrm{Ult}(V, E_i)$ into M, and let $\tau_i = j_i([\mathrm{id}]_{E_i})$. Because $M^\kappa \subseteq M$, $\tau =_{\mathrm{df}} \langle \tau_i \mid i < \kappa \rangle \in M$. Consequently, there must be some $E \in \mathcal{E}$ such that for some $\sigma \in \mathrm{Ult}(V, E)$ and $j_E : \mathrm{Ult}(V, E) \to M$ the canonical elementary embedding, $\tau = j_E(\sigma)$. However, this just means that for every $i < \kappa$, $E_i <_{\mathrm{RK}} E$.

For each $E \in \mathcal{E}$, $E = E_b$ for $b \in [\lambda]^{<\omega}$, let $k_E : V \to \mathrm{Ult}(V, E) = M_E$ be the canonical elementary embedding. Note that since the canonical elementary embedding $\ell_E : M_E \to M$ is such that $\mathrm{cp}(\ell_E) > \kappa$ and $M \models$ "κ is not measurable", $M_E \models$ "κ is not measurable" as well. Therefore, if we consider now E^* defined in $V^{\mathbb{P}(\delta,\kappa)}$ by $p \Vdash$ "$\dot{x} \in \dot{E}^*$" iff there is $q \in k_E(\mathbb{P}(\delta, \kappa))$, $q \geq k_E(p)$

such that $|k_E(p) - q| = 0$ (where $| \quad |$ is the distance function from [16]), $k_E(p) \restriction \kappa = q \restriction \kappa = p$, and $q \Vdash$ "$b \in k_E(\dot{x})$", then because $M_E \models$ "κ is not measurable", the arguments of [16, Theorem 2.5] show that E^* is well-defined and is a κ-additive ultrafilter extending E. It is routine (although tedious) to verify that $\mathcal{E}^* = \langle E^*_a \mid a \in [\lambda]^{<\omega} \rangle \in V^{\mathbb{P}(\delta,\kappa)}$ is hence a (κ, λ)-extender extending $\mathcal{E}$ which is κ^+-directed. (Note that κ^+-directedness follows because by its definition, projection maps in the sense of the Rudin-Keisler ordering between members of $\mathcal{E}^*$ remain projection maps in the same sense.) To show that in fact $\mathcal{E}^*$ witnesses that $V^{\mathbb{P}(\delta,\kappa)} \models$ "κ is λ tall", it suffices to show that for $M_* = \mathrm{Ult}(V^{\mathbb{P}(\delta,\kappa)}, \mathcal{E}^*)$, $M_*^\kappa \subseteq M_*$.

To see this, suppose $\langle a_i \mid i < \kappa \rangle \in V^{\mathbb{P}(\delta,\kappa)}$ is a κ sequence of members of M_*. There must be $E^*_i \in \mathcal{E}^*$ and $a'_i \in M_i = \mathrm{Ult}(V^{\mathbb{P}(\delta,\kappa)}, E^*_i)$ such that for $\ell^*_i : M_i \to M_*$ the canonical elementary embedding, $\ell^*_i(a'_i) = a_i$. Since $\mathcal{E}^*$ is κ^+-directed, let $E^* \in \mathcal{E}^*$ be such that for each $i < \kappa$, $E^*_i <_{\mathrm{RK}} E^*$. Consider $M_{E^*} = \mathrm{Ult}(V^{\mathbb{P}(\delta,\kappa)}, E^*)$, with $j_{E^*} : M_{E^*} \to M_*$ the canonical elementary embedding. Note that there is a canonical elementary embedding $j^*_i : M_i \to M_{E^*}$ generated by the projection of E^* to E^*_i and that $j_{E^*} \circ j^*_i : M_i \to M_*$ is an elementary embedding. Also, since M_{E^*} is the ultrapower via a measure, it is κ-closed with respect to $V^{\mathbb{P}(\delta,\kappa)}$. Therefore, $a'' = \langle a''_i \mid i < \kappa \rangle \in M_{E^*}$, where for $i < \kappa$, $a''_i = j^*_i(a'_i)$. However, $j_{E^*}(a'') = \langle a_i \mid i < \kappa \rangle \in M_*$, so $M_*^\kappa \subseteq M_*$. Thus, $V^{\mathbb{P}(\delta,\kappa)} \models$ "κ is λ tall". Since λ was arbitrary, this completes the proof of Lemma 2.1.

□

Since the proof of Theorem 1 requires that we force over a ground model V satisfying certain indestructibility properties for strongness, we next show that this is possible in the following lemma.

Lemma 2.2 *Suppose $V \models$ "ZFC + GCH + κ is supercompact + No cardinal $\eta > \kappa$ is measurable". Let $\mathcal{C} = \{\delta < \kappa \mid \delta$ is a strong cardinal which is not a limit of strong cardinals$\}$". There is then a partial ordering $\mathbb{I} \in V$ such that $V^{\mathbb{I}} \models$ "ZFC + κ is supercompact + No cardinal $\eta > \kappa$ is measurable + For every $\delta \in \mathcal{C}$, δ is a strong cardinal whose strongness is indestructible under δ^+-weakly closed partial orderings satisfying the Prikry property".*

Proof: Let $\langle \delta_\alpha \mid \alpha < \kappa \rangle$ enumerate in increasing order the members of $\mathcal{C}$. For every $\alpha < \kappa$, let $\gamma_\alpha = (\sup_{\beta<\alpha} \delta_\beta)^+$, where $\gamma_0 = \omega$. $\mathbb{I} = \langle \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle \mid \alpha < \kappa \rangle$ is defined as the Easton support iteration of Prikry type forcings of length κ such that $\mathbb{P}_0 = \{\emptyset\}$. For every $\alpha < \kappa$, $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$, where $\dot{\mathbb{Q}}_\alpha$ is a term for the partial ordering $\mathbb{I}(\gamma_\alpha, \delta_\alpha)$ of Theorem 9 as defined in $V^{\mathbb{P}_\alpha}$. Note that this definition makes sense, since inductively, it is the case that $|\mathbb{P}_\alpha| < \delta_\alpha$. By the Hamkins-Woodin results [12], $V^{\mathbb{P}_\alpha} \vDash$ "δ_α is a strong cardinal", meaning that $\mathbb{P}_{\alpha+1}$ may be correctly defined. Note also that the only strong cardinals on which $\mathbb{I}$ acts nontrivially are those strong cardinals which are not limits of strong cardinals in V. In other words, if $V \vDash$ "δ is a strong cardinal which is a limit of strong cardinals", then $\mathbb{I}$ acts trivially on δ.

By its definition, $V^{\mathbb{P}_{\alpha+1}} \vDash$ "δ_α is a strong cardinal whose strongness is indestructible under δ_α^+-weakly closed partial orderings satisfying the Prikry property". Factor $\mathbb{I}$ as $\mathbb{I} = \mathbb{P}_{\alpha+1} * \dot{\mathbb{P}}^{\alpha+1}$. Since also by its definition, $\Vdash_{\mathbb{P}_{\alpha+1}}$ "$\dot{\mathbb{P}}^{\alpha+1}$ is δ_α^+-weakly closed and satisfies the Prikry property", $V^{\mathbb{P}_{\alpha+1} * \dot{\mathbb{P}}^{\alpha+1}} = V^{\mathbb{I}} \vDash$ "δ_α is a strong cardinal whose strongness is indestructible under δ_α^+-weakly closed partial orderings satisfying the Prikry property". Consequently, since α was arbitrary, $V^{\mathbb{I}} \vDash$ "For every $\delta \in \mathcal{C}$, δ is a strong cardinal whose strongness is indestructible under δ^+-weakly closed partial orderings satisfying the Prikry property".

To show that $V^{\mathbb{I}} \vDash$ "κ is supercompact", we follow the proof of [1, Lemma 2.1]. Let $\lambda \geq \kappa^+ = 2^\kappa$ be any regular cardinal. Take $j : V \to M$ as an elementary embedding witnessing the λ supercompactness of κ. By [4, Lemma 2.1], in M, κ is a limit of strong cardinals. In addition, since $V \vDash$ "No cardinal $\eta > \kappa$ is measurable", $M \vDash$ "No cardinal $\eta \in (\kappa, \lambda]$ is measurable". Hence, in M, $j(\mathbb{I})$ is forcing equivalent to $\mathbb{I} * \dot{\mathbb{Q}}$, where the first nontrivial stage in $\dot{\mathbb{Q}}$ takes place well after λ.

We may now apply the argument of [5, Lemma 1.5]. Specifically, let G be V-generic over $\mathbb{I}$. Since GCH in V implies that $V \vDash$ "$2^\lambda = \lambda^+$", we may let $\langle \dot{x}_\alpha \mid \alpha < \lambda^+ \rangle$ be an enumeration in V of all of the canonical $\mathbb{I}$-names of subsets of $P_\kappa(\lambda)$. Because $\mathbb{I}$ is κ-c.c. and $M^\lambda \subseteq M$, $M[G]^\lambda \subseteq M[G]$. By [5, Lemmas 1.4 and 1.2], we may therefore define an increasing sequence $\langle p_\alpha \mid \alpha < \lambda^+ \rangle$ of elements of $j(\mathbb{I})/G$ such that if $\alpha < \beta < \lambda^+$, p_β is an Easton extension of p_α,[1] every initial segment of the sequence is in $M[G]$, and for every $\alpha < \lambda^+$, $p_{\alpha+1} \parallel$ "$\langle j(\beta) \mid \beta < \lambda \rangle \in j(\dot{x}_\alpha)$". The remainder

[1]Roughly speaking, this means that p_β extends p_α as in a usual reverse Easton iteration, except that at coordinates at which, e.g., Prikry forcing or some variant or generalization thereof occurs in p_α, measure 1 sets are shrunk and stems are not extended. For a more precise definition, readers are urged to consult [5].

of the argument of [5, Lemma 1.5] remains valid and shows that a supercompact ultrafilter $\mathcal{U}$ over $(P_\kappa(\lambda))^{V[G]}$ may be defined in $V[G]$ by $x \in \mathcal{U}$ iff $x \subseteq (P_\kappa(\lambda))^{V[G]}$ and for some $\alpha < \lambda^+$ and some $\mathbb{I}$-name $\dot{x}$ of x, in $M[G]$, $p_\alpha \Vdash_{j(\mathbb{I})/G}$ "$\langle j(\beta) \mid \beta < \lambda \rangle \in j(\dot{x})$". (The fact that $j''G = G$ tells us $\mathcal{U}$ is well-defined.) Thus, $\Vdash_{\mathbb{I}}$ "κ is λ supercompact". Since λ was arbitrary, $V^{\mathbb{I}} \models$ "κ is supercompact". Finally, since $\mathbb{I}$ may be defined so that $|\mathbb{I}| = \kappa$, $V^{\mathbb{I}} \models$ "No cardinal $\eta > \kappa$ is measurable". This completes the proof of Lemma 2.2.

$\square$

We assume now that our ground model, which with an abuse of notation we relabel as V, has the properties of the model $V^{\mathbb{I}}$ constructed in Lemma 2.2. Given this, and adopting the notation of Lemma 2.1, let $\mathbb{P}(\gamma_\alpha, \delta_\alpha)$ for every $\alpha < \kappa$ be the Magidor iteration of Prikry forcing from [16] which adds a Prikry sequence to every measurable cardinal in the open interval $(\gamma_\alpha, \delta_\alpha)$. The partial ordering $\mathbb{P} = \langle \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle \mid \alpha < \kappa \rangle$ with which we force is defined as the Easton support iteration of Prikry type forcings of length κ such that $\mathbb{P}_0 = \{\emptyset\}$. For every $\alpha < \kappa$, $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$, where $\dot{\mathbb{Q}}_\alpha$ is a term for the partial ordering $\mathbb{P}(\gamma_\alpha, \delta_\alpha)$ of Lemma 2.1 as defined in V (and *not as defined in* $V^{\mathbb{P}_\alpha}$).

To see that this makes sense, i.e., that $V^{\mathbb{P}_\alpha} \models$ "$\mathbb{P}(\gamma_\alpha, \delta_\alpha)$ as defined in V is γ_α^+-weakly closed and satisfies the Prikry property", we note that by their definitions, the cardinality of $\mathbb{P}_\alpha$ is less than the least measurable cardinal in the open interval $(\gamma_\alpha, \delta_\alpha)$. Consequently, by the results of [14], for any $\delta \in (\gamma_\alpha, \delta_\alpha)$, δ is measurable in V iff δ is measurable in $V^{\mathbb{P}_\alpha}$, and every normal measure μ^* over δ in $V^{\mathbb{P}_\alpha}$ has the form $\{x \subseteq \delta \mid \exists y \in \mu[y \subseteq x]\}$, where $\mu \in V$ is some normal measure over δ. This means that informally, every normal measure μ^* used in the Magidor iteration of Prikry forcing $\mathbb{P}^*(\gamma_\alpha, \delta_\alpha)$ as defined in $V^{\mathbb{P}_\alpha}$ which adds a Prikry sequence to every measurable cardinal in the open interval $(\gamma_\alpha, \delta_\alpha)$ may be replaced by its ground model counterpart μ. More formally, let $p = \langle \langle s_\beta, A_\beta \rangle \mid \beta < \delta_\alpha \rangle \in \mathbb{P}^*(\gamma_\alpha, \delta_\alpha)$, where s_β is a finite sequence of ordinals and $\langle A_\beta \mid \beta < \delta_\alpha \rangle$ is a sequence of terms for measure 1 sets which are forced to be members of the appropriate normal measure. We proceed inductively. Let $\mathbb{R}_\beta = \mathbb{P}^*(\gamma_\alpha, \delta_\alpha) \restriction \beta$, i.e., $\mathbb{R}_\beta$ is the Magidor iteration defined up to stage β in $V^{\mathbb{P}_\alpha}$. By the work of [16], for some normal measure $\mu^*_\beta \in V^{\mathbb{P}_\alpha * \dot{\mathbb{R}}_\beta}$, $\Vdash_{\mathbb{P}_\alpha * \dot{\mathbb{R}}_\beta}$ "$A_\beta \in \mu^*_\beta$". By the results of [14], there must exist some normal measure

$\mu_\beta \in V^{\mathbb{R}_\beta}$ and some term B_β such that $\Vdash_{\mathbb{P}_\alpha}$ "$B_\beta \in \mu_\beta$ and $B_\beta \subseteq A_\beta$". By replacing each A_β with B_β, we inductively define a condition $q = \langle\langle s_\beta, B_\beta\rangle \mid \beta < \delta_\alpha\rangle \in \mathbb{P}(\gamma_\alpha, \delta_\alpha) \subseteq \mathbb{P}^*(\gamma_\alpha, \delta_\alpha)$ such that $q \geq_{\mathbb{P}^*(\gamma_\alpha,\delta_\alpha)} p$. Thus, $\mathbb{P}(\gamma_\alpha, \delta_\alpha)$ is dense in $\mathbb{P}^*(\gamma_\alpha, \delta_\alpha)$, a partial ordering which is γ_α^+-weakly closed and satisfies the Prikry property in $V^{\mathbb{P}_\alpha}$. It therefore immediately follows that $V^{\mathbb{P}_\alpha} \models$ "$\mathbb{P}(\gamma_\alpha, \delta_\alpha)$ as defined in V is γ_α^+-weakly closed and satisfies the Prikry property".

Lemma 2.3 $V^{\mathbb{P}} \models$ "*Any $\delta \in \mathcal{C}$ is a tall cardinal*".

Proof: Suppose $\delta \in \mathcal{C}$. It is then the case that for some $\alpha < \kappa$, $\delta = \delta_\alpha$. Because each component of $\mathbb{P}$ is an element of V, it is possible to write $\mathbb{P} = (\prod_{\beta<\alpha} \mathbb{P}(\gamma_\beta, \delta_\beta)) \times \mathbb{P}(\gamma_\alpha, \delta_\alpha) \times (\prod_{\beta>\alpha} \mathbb{P}(\gamma_\beta, \delta_\beta)) = \mathbb{P}^0 \times \mathbb{P}^1 \times \mathbb{P}^2$, where $\mathbb{P}^0, \mathbb{P}^1, \mathbb{P}^2 \in V$ and the ordering on $\mathbb{P}^0$ and $\mathbb{P}^2$ is the one used in an Easton support iteration of Prikry type forcings. Since by our observations above, $\mathbb{P}^2$ is in fact δ^+-weakly closed and satisfies the Prikry property, by the indestructibility properties of V, $V^{\mathbb{P}^2} \models$ "δ is a strong cardinal". Further, by the fact $\mathbb{P}^2$ is δ^+-weakly closed, the definition of $\mathbb{P}^1 = \mathbb{P}(\gamma_\alpha, \delta_\alpha)$ as the Magidor iteration of Prikry forcing adding a Prikry sequence to each measurable cardinal in the open interval $(\gamma_\alpha, \delta_\alpha)$ is the same in both V and $V^{\mathbb{P}^2}$. By Lemma 2.1, this means that $V^{\mathbb{P}^2 \times \mathbb{P}^1} \models$ "δ is a tall cardinal". Finally, because $|\mathbb{P}^0| < \delta$, by [11, Theorem 2.13], $V^{\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^0} = V^{\mathbb{P}} \models$ "δ is a tall cardinal". This completes the proof of Lemma 2.3.

$\square$

Lemma 2.4 $V^{\mathbb{P}} \models$ "*Any measurable cardinal is either a member of $\mathcal{C}$ or a limit of members of $\mathcal{C}$*".

Proof: Since $|\mathbb{P}| = \kappa$ and $V \models$ "No cardinal $\eta > \kappa$ is measurable", by the results of [14], $V^{\mathbb{P}} \models$ "No cardinal $\eta > \kappa$ is measurable" as well. We may thus assume that $\delta < \kappa$ and $V^{\mathbb{P}} \models$ "δ is measurable", since $V^{\mathbb{P}} \models$ "κ is supercompact and a limit of members of $\mathcal{C}$". If in addition $V^{\mathbb{P}} \models$ "δ is neither a member of $\mathcal{C}$ nor a limit of members of $\mathcal{C}$", then let $\alpha < \kappa$ be such that α is least with $\delta_\alpha > \delta$. Because in both V and $V^{\mathbb{P}}$, δ is not a limit of members of $\mathcal{C}$, it must be the case that $\delta \in (\gamma_\alpha, \delta_\alpha)$.

As in the proof of Lemma 2.3, write $\mathbb{P} = (\prod_{\beta<\alpha} \mathbb{P}(\gamma_\beta, \delta_\beta)) \times \mathbb{P}(\gamma_\alpha, \delta_\alpha) \times (\prod_{\beta>\alpha} \mathbb{P}(\gamma_\beta, \delta_\beta)) = \mathbb{P}^0 \times \mathbb{P}^1 \times \mathbb{P}^2$. The work of [16] shows that $V^{\mathbb{P}^1} \models$ "There are no measurable cardinals in the open interval $(\gamma_\alpha, \delta_\alpha)$", i.e., that $V^{\mathbb{P}(\gamma_\alpha, \delta_\alpha)} \vDash$ "There are no measurable cardinals in the open interval $(\gamma_\alpha, \delta_\alpha)$". Since as we observed in the proof of Lemma 2.3, $\mathbb{P}^1$ retains its properties in $V^{\mathbb{P}^2}$, it is also the case that $V^{\mathbb{P}^2 \times \mathbb{P}^1} \vDash$ "There are no measurable cardinals in the open interval $(\gamma_\alpha, \delta_\alpha)$". Because $|\mathbb{P}^0| < \delta$, the results of [14] again imply that $V^{\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^0} = V^{\mathbb{P}} \vDash$ "There are no measurable cardinals in the open interval $(\gamma_\alpha, \delta_\alpha)$". This contradiction completes the proof of Lemma 2.4.

□

By [11, Corollary 2.7], any measurable limit of tall cardinals is also a tall cardinal. In addition, the same argument as found in the proof of Lemma 2.2 shows that $V^{\mathbb{P}} \vDash$ "κ is supercompact". These facts, together with Lemmas 2.1 – 2.4, complete the proof of Theorem 1.

□

As we mentioned when making our introductory comments in Section 1, it is completely unnecessary to use a supercompact cardinal in order to construct a model in which the tall and measurable cardinals precisely coincide. An inaccessible limit of strong cardinals is more than enough for this purpose. To see this, suppose κ is an inaccessible limit of strong cardinals instead of a supercompact cardinal. Suppose further that the partial orderings $\mathbb{I}$ and $\mathbb{P}$ of Theorem 1 are both defined as they were in our original proof, i.e., as Easton support iterations of length κ. By [2, Lemma 0.6], in $V^{\mathbb{I} * \dot{\mathbb{P}}}$, κ remains inaccessible. Thus, the proofs we gave above show that in $(V_\kappa)^{V^{\mathbb{I} * \dot{\mathbb{P}}}}$, there is a proper class of tall cardinals, and the tall and measurable cardinals precisely coincide.

As we also mentioned in our introductory comments in Section 1, a famous question (essentially due to Magidor) asks whether it is possible to construct a model of ZFC containing infinitely many strongly compact cardinals in which the measurable and strongly compact cardinals precisely coincide. To date, this question remains open, and has defied every effort to obtain a positive answer. We were able to prove Theorem 1 because the work of [9] shows that it is possible to do Prikry forcing above a strong cardinal while preserving strongness. However, as is fairly well known (see, e.g., [16, Section 4] and [3, Lemma 3.1]), adding a Prikry sequence above a strongly compact cardinal destroys strong compactness. Thus, the methods of this paper cannot be used to provide a positive answer to Magidor's question.

3 Tall cardinals with bounded degrees of closure

Having completed the proof of Theorem 1, we turn now to the proofs of Theorems 2 – 7. We begin with the proof of Theorem 2, which we again restate.

Theorem 2 *Suppose κ is a strong cardinal. Then κ is a tall cardinal having bounded closure ω.*

Proof: Suppose $V \models$ "ZFC + κ is a strong cardinal". Let $\lambda > \kappa$ be a strong limit cardinal. Let $j : V \to M$ be an elementary embedding such that $M \supseteq H(\lambda^{+\omega_1})$ which is generated by a (κ, δ)-extender $\mathcal{E}$ for the appropriate strong limit cardinal $\delta > \lambda$. Consider $\mathcal{E}' = \mathcal{E} \restriction \lambda^{+\omega_1}$, with $j' : V \to M'$ the elementary embedding generated by $\mathcal{E}'$ and $k : M' \to M$ the canonical elementary embedding. Since $k \circ j' = j$, $j(\kappa) > \lambda$, and $\mathrm{cp}(k) > \lambda$, $j'(\kappa) > \lambda$. Because $\mathcal{E}' \notin M'$, M' is not ω_1 closed. To see this, let $\alpha < \omega_1$, and define $\mathcal{E}_\alpha = \mathcal{E} \restriction \lambda^{+\alpha}$. Let $\beta < \omega_1$ be such that $\mathcal{E}_\alpha \in H(\lambda^{+\beta})$. Note that $\mathcal{E}_\alpha \in M'$, since $H(\lambda^{+\beta}) \subseteq M'$ and $\mathcal{E}_\alpha \in H(\lambda^{+\beta})$. Hence, if $(M')^{\omega_1} \subseteq M'$, then $\langle \mathcal{E}_\gamma \mid \gamma < \omega_1 \rangle \in M'$, thereby allowing us to recover $\mathcal{E}'$ within M'.

To prove that M' is ω closed, it suffices to show that every countable set of generators of $\mathcal{E}'$ is a member of M'. To see this, let $\langle x_n \mid n < \omega \rangle$ be a countable sequence of elements of M'. Then there are functions $\langle f_n \mid n < \omega \rangle$ and a sequence $\langle a_n \mid n < \omega \rangle$ of generators of $\mathcal{E}'$ such that $x_n = j(f_n)(a_n)$. Let a be a generator of $\mathcal{E}'$ coding $\langle a_n \mid n < \omega \rangle$, with $\pi_n(a) = a_n$. By hypothesis, $a \in M'$. Since $\langle j(f_n) \mid n < \omega \rangle \in M'$ and $\langle \pi_n \mid n < \omega \rangle$ is definable in M', $\langle x_n \mid n < \omega \rangle \in M'$ as well.

The proof of Theorem 2 will thus be complete once we have established that every countable set of generators of $\mathcal{E}'$ is a member of M'. Consequently, let a be such a set, and let $\tau < \omega_1$ be such that $a \in H(\lambda^{+\tau})$. For $\mathcal{E}'' = \mathcal{E}' \restriction \lambda^{+\tau+1}$ and $M'' = \mathrm{Ult}(V, \mathcal{E}'')$, it is then the case that $H(\lambda^{+\tau}) \subseteq M''$, from which it immediately follows that $a \in M''$. Since for $i : M'' \to M'$ the canonical elementary embedding, $i(a) = a$, we have that $i(a) = a \in M'$. This completes the proof of Theorem 2.

$\square$

For the convenience of readers, we also restate Theorem 3 before giving its proof.

Theorem 3 *Suppose $V = \mathcal{K}$ and $V \models$ "ZFC + κ is strong + $\eta > \kappa$ is such that $o(\eta) = \omega_1$ + No cardinal $\delta > \eta$ is measurable". There is then a partial ordering $\mathbb{P} \in V$ such that $V^{\mathbb{P}} \models$ "κ is a* *tall cardinal having bounded closure ω + There are neither any tall cardinals nor any tall cardinals having bounded closure δ for $\omega_1 \le \delta < \kappa$".*

Proof: Suppose $V = \mathcal{K}$ and $V \models$ "ZFC + κ is strong + $\eta > \kappa$ is such that $o(\eta) = \omega_1$ + No cardinal $\delta > \eta$ is measurable". We define an Easton support iteration of Prikry type forcings $\mathbb{P} = \langle\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha\rangle \mid \alpha \le \eta\rangle$ of length η as follows. For $\alpha < \eta$, $\dot{\mathbb{Q}}_\alpha$ is a term for trivial forcing, unless α is a measurable cardinal in V such that $o(\alpha) < \omega_1$. If this is the case, then $\dot{\mathbb{Q}}_\alpha$ is a term for the forcing of [5] which adds a Magidor sequence (see [15]) of order type $\omega^{o(\alpha)}$ to α. For $\alpha = \eta$, $\dot{\mathbb{Q}}_\alpha$ is a term for the Prikry type forcing from [5] which changes the cofinality of η to ω_1 without adding over $V^{\mathbb{P}_\eta}$ any new bounded subsets of η. This partial ordering can be defined in $V^{\mathbb{P}_\eta}$ so as to be η^+-c.c. and have cardinality η^+. Consequently, forcing with $\dot{\mathbb{Q}}_\eta$ over $V^{\mathbb{P}_\eta}$ adds no new countable sets of ordinals. This is since if $A \in V^{\mathbb{P}_\eta * \dot{\mathbb{Q}}_\eta} = V^{\mathbb{P}}$ were a new countable set of ordinals, then because $V^{\mathbb{P}_\eta} \models$ "$\mathbb{Q}_\eta$ is η^+-c.c.", there is a set $B \in V$ such that $B \supseteq A$ and $|B| = \eta$. However, since no new countable subsets are added to B, this is impossible.

Let G be V-generic over $\mathbb{P}$, with $G \restriction \eta = G_\eta$. We claim that $V' = V[G]$ is our desired model in which κ is a tall cardinal having closure ω but not closure ω_1 and in which there are no tall cardinals having closure ω_1.

We begin by showing that $V' \models$ "κ is not a tall cardinal having closure ω_1". If this is not true, then choose some $\lambda > \eta$, and let $j' : V' \to M'$ be such that $\text{cp}(j') = \kappa$, $j'(\kappa) > \lambda$, and $(M')^{\omega_1} \subseteq M'$. Consider $j = j' \restriction \mathcal{K}$. Note that j is given as an iterated ultrapower of $\mathcal{K}$ using extenders at and above κ (see [20] and [22]). Then η is regular in $(\mathcal{K})^{M'}$, since it is regular in $\mathcal{K}$ and $(\mathcal{K})^{M'}$ is an iterated ultrapower of $\mathcal{K}$ by its extenders.

By elementarity, M' is a generic extension of $j(\mathcal{K}) = (\mathcal{K})^{M'}$ by $j(\mathbb{P})$. In addition, by its definition, forcing with $\mathbb{P}$ does not change the cofinality of any cardinal below κ to ω_1. Hence, by elementarity, forcing with $j(\mathbb{P})$ does not change the cofinality of any cardinal below $j(\kappa)$ to ω_1 in M'. However, since $V' \models$ "$\text{cof}(\eta) = \omega_1$" and $(M')^{\omega_1} \subseteq M'$, $M' \models$ "$\text{cof}(\eta) = \omega_1$" as well. This immediately contradicts that $j(\kappa) > \lambda > \eta$.

To show that $V' \models$ "κ is a tall cardinal having closure ω", let $\lambda > \eta$ be a regular cardinal. Let $\mathcal{E}$ be a (κ, λ)-extender, with $k : V \to M$ the corresponding elementary embedding. By the arguments of [9], in $V[G_\eta]$, $\mathcal{E}$ extends to a (κ, λ)-extender $\mathcal{E}^*$. Let $k^* : V[G_\eta] \to M^*$ be the corresponding elementary embedding. Since no new ω sequences and no new subsets of κ are added to $V[G_\eta]$ after forcing with $\mathbb{Q}_\eta$, $\mathcal{E}^*$ remains an extender in V' with well-founded ultrapower. Let $k' : V' \to M'$ be the corresponding elementary embedding. Then $k' \restriction V[G_\eta] = k^*$, since no new subsets of κ were added to $V[G_\eta]$ after forcing with $\mathbb{Q}_\eta$. Hence, $k'(\kappa) > \lambda$. Also, since we may assume that $M^\kappa \subseteq M$, this is preserved to $V[G_\eta]$, i.e., $(M^*)^\kappa \subseteq M^*$. As $V[G_\eta]$ and V' have the same countable sets of ordinals, in V', $(M')^\omega \subseteq M'$. Consequently, because λ was an arbitrary regular cardinal, $V' \models$ "κ is a tall cardinal having closure ω".

It remains to show that $V' \models$ "There are no tall cardinals having closure ω_1". Because $\mathbb{P}$ may be defined so that $|\mathbb{P}| = \eta^+$, by the results of [14] and the fact that $V = \mathcal{K}$ and $\mathcal{K}$ contains no measurable cardinals above η, $V' \models$ "There are no measurable cardinals greater than η". It thus suffices to show that $V' \models$ "No $\delta \in (\kappa, \eta)$ is a tall cardinal having closure ω_1". To see this, suppose to the contrary that $V' \models$ "$\delta \in (\kappa, \eta)$ is a tall cardinal having closure ω_1". Take $i : V' \to N$ with $i(\delta) > \eta$ and $N^{\omega_1} \subseteq N$. As above (see [20] and [22]), $i^* = i \restriction \mathcal{K}$ is given as an iterated ultrapower of $\mathcal{K}$ using extenders at and above δ. In addition, because $\mathcal{K} \models$ "η is regular" and $i^*(\mathcal{K})$ is an inner model of $\mathcal{K}$, $i^*(\mathcal{K}) \models$ "η is regular" as well. Since $V' \models$ "$\mathrm{cof}(\eta) = \omega_1$", N must be a generic extension of $i^*(\mathcal{K})$, and $N^{\omega_1} \subseteq N$ in V', this means that $N \models$ "$\mathrm{cof}(\eta) = \omega_1$". However, as $\eta < i(\delta)$, by reflection, it follows that in V', unboundedly many $\mathcal{K}$-regular cardinals below δ have their cofinalities changed to ω_1. By the definition of $\mathbb{P}$, this is impossible. This contradiction completes the proof of Theorem 3.

□

Theorems 2 and 3 raise the question of classifying the consistency strength of the existence of embeddings witnessing a bounded degree of closure, which we address now. We deal here with ω_1, but the same arguments actually apply to any regular $\delta \leq \kappa$. We begin with Theorem 4, which gives the equivalence of three conditions for the existence of such elementary embeddings.

Theorem 4 *The following conditions are equivalent:*

1. *There is an elementary embedding $j : V \to M$ with critical point κ such that $(j(\kappa))^\omega \subseteq M$ yet $(j(\kappa))^{\omega_1} \not\subseteq M$.*

2. *There exists a Rudin-Keisler increasing sequence of ultrafilters over κ having length ω_1.*

3. *There exists an elementary embedding $j : V \to M$ such that $M^\omega \subseteq M$ and a sequence $\langle \eta_\alpha \mid \alpha < \omega_1 \rangle$ of ordinals below $j(\kappa)$ such that for every $\alpha < \omega_1$ and for every $f : [\kappa]^\alpha \to \kappa$, it is the case that $\eta_\alpha \neq j(f)(\langle \eta_\beta \mid \beta < \alpha \rangle)$.*

Proof: To show that (1) $\Longrightarrow$ (2), let $a \subseteq j(\kappa)$, $|a| = \aleph_1$ be such that $a \notin M$. Let $\langle \eta_\alpha \mid \alpha < \omega_1 \rangle$ be an increasing enumeration of a. Define a κ-complete ultrafilter U_α over V_κ by

$$x \in U_\alpha \text{ iff } \langle \eta_\beta \mid \beta < \alpha \rangle \in j(x).$$

Clearly, if $\gamma < \alpha$, then $U_\gamma \leq_{\mathrm{RK}} U_\alpha$. We claim that for every $\gamma < \omega_1$, there is some α, $\gamma < \alpha < \omega_1$ such that $U_\gamma <_{\mathrm{RK}} U_\alpha$ (i.e., the inequality is strict). To see this, suppose otherwise. Then there is $\gamma < \omega_1$ such that for every α with $\gamma < \alpha < \omega_1$, we have that $U_\gamma =_{\mathrm{RK}} U_\alpha$. For every α with $\gamma < \alpha < \omega_1$, we fix a function $f_\alpha : [\kappa]^\gamma \to [\kappa]^\alpha$ witnessing $U_\gamma =_{\mathrm{RK}} U_\alpha$. We have $j(f_\alpha)(\langle \eta_\beta \mid \beta < \gamma \rangle) = \langle \eta_\beta \mid \beta < \alpha \rangle$. But $j(\langle f_\alpha \mid \gamma < \alpha < \omega_1 \rangle) = \langle j(f_\alpha) \mid \gamma < \alpha < \omega_1 \rangle$, so $\langle j(f_\alpha) \mid \gamma < \alpha < \omega_1 \rangle \in M$. Since $\langle \eta_\beta \mid \beta < \gamma \rangle$ is a countable sequence of ordinals below $j(\kappa)$, $\langle \eta_\beta \mid \beta < \gamma \rangle \in M$. Consequently, $\langle j(f_\alpha)(\langle \eta_\beta \mid \beta < \gamma \rangle) \mid \gamma < \alpha < \omega_1 \rangle \in M$, from which it follows that $\langle \eta_\alpha \mid \alpha < \omega_1 \rangle \in M$. This contradicts that $a \notin M$, thereby proving (1) $\Longrightarrow$ (2).

To show that (2) $\Longrightarrow$ (1), let $\langle U_\alpha \mid \alpha < \omega_1 \rangle$ be a Rudin-Keisler increasing sequence of ultrafilters over κ. For $\alpha < \omega_1$, denote by $j_\alpha : V \to M_\alpha$ the ultrapower embedding generated by U_α, and for $\alpha < \beta < \omega_1$, denote by $j_{\alpha,\beta} : M_\alpha \to M_\beta$ the elementary embedding generated by a projection of U_β to U_α. Let $\langle \langle M, i_\alpha \rangle \mid \alpha < \omega_1 \rangle$ be the direct limit of the system $\langle \langle M_\alpha, j_{\alpha,\beta} \rangle \mid \alpha < \beta < \omega_1 \rangle$, where $i_\alpha : M_\alpha \to M$. It is then the case that M and the limit embedding $j : V \to M$ are as desired. To see this, we first note that $M^\omega \subseteq M$. This follows since if $x \subseteq M$ is countable, then for some $\alpha < \omega_1$, x has a preimage in M_α. However, $(M_\alpha)^\kappa \subseteq M_\alpha$, since M_α is the ultrapower by an ultrafilter over κ.

We now define $a \subseteq j(\kappa)$ with $|a| = \aleph_1$ such that $a \notin M$ by $a = \{i_\alpha([\mathrm{id}]_{U_\alpha}) \mid \alpha < \omega_1\}$. To see that a is as desired, assume to the contrary that $a \in M$. It must then be true that for some $\beta < \omega_1$ and some $b \in M_\beta$, $i_\beta(b) = a$. But then for every $\alpha \geq \beta$, U_α must be Rudin-Keisler equivalent to U_β, which is impossible. Since each $i_\alpha([\mathrm{id}]_{U_\alpha}) < j(\kappa)$, this completes the proof of (2) $\Longrightarrow$ (1).

To show that (2) $\Longrightarrow$ (3), we use the previous construction. The set a just defined is as desired, since $\langle U_\alpha \mid \alpha < \omega_1 \rangle$ is a strictly increasing Rudin-Keisler sequence of ultrafilters.

Finally, to show that (3) $\Longrightarrow$ (2), we use $\langle \eta_\alpha \mid \alpha < \omega_1 \rangle$ to define the U_α s as in (1) $\Longrightarrow$ (2) and argue as in (1) $\Longrightarrow$ (2). Even if $a \in M$, the argument remains valid. This completes the proof of Theorem 4.

□

We remark that in general, $(j(\kappa))^\omega \subseteq M$ does not imply that $M^\omega \subseteq M$. To see this, suppose $\kappa < \lambda$ are both measurable cardinals. We construct $j : V \to M$ by first taking an ultrapower via a measure over κ, and then taking an iterated ultrapower ω many times by a measure over λ. It will then be the case that $(j(\kappa))^\omega \subseteq M$ but $(j(\lambda))^\omega \not\subseteq M$. In addition, an argument using the work of [9] shows that it is impossible to replace the condition of Theorem 4(3) with $\eta_\alpha \neq j(f)(\xi_1, \ldots, \xi_n)$ whenever $n < \omega$, $\xi_1, \ldots, \xi_n < \eta_\alpha$, and $f : [\kappa]^n \to \kappa$.

Corollary 3.1 *Suppose that κ is a $\mathcal{P}^\lambda(\kappa)$ hypermeasurable cardinal for $\lambda \geq 2$ of cofinality different from ω. Then there is an elementary embedding $j : V \to M$ having critical point κ such that $(j(\kappa))^\omega \subseteq M$ yet $(j(\kappa))^{\omega_1} \not\subseteq M$. Moreover, the embedding may be constructed so that $\mathcal{P}^\lambda(\kappa) \subseteq M$.*

Proof: Let $i : V \to N$ be an elementary embedding witnessing the $\mathcal{P}^\lambda(\kappa)$ hypermeasurability of κ. Since $\lambda \geq 2$, i clearly witnesses the $\mathcal{P}^2(\kappa)$ hypermeasurability of κ as well. Hence, $\mathcal{P}^2(\kappa) \subseteq N$, from which it follows that $i(\kappa) > (2^\kappa)^+$.

We define by induction an increasing sequence $\langle \eta_\alpha \mid \alpha < \omega_1 \rangle$ of ordinals below $(2^\kappa)^+$ satisfying the assumptions of Theorem 4(3) as follows. Begin by setting $\eta_0 = \kappa$. Assume now that $\langle \eta_\beta \mid \beta < \alpha \rangle$ has been defined. To define η_α, we first note that $\langle \eta_\beta \mid \beta < \alpha \rangle \in N$, since $\mathcal{P}^2(\kappa) \subseteq N$. Let

$$U_\alpha = \{x \subseteq [\kappa]^\alpha \mid \langle \eta_\beta \mid \beta < \alpha \rangle \in i(x)\},$$

with $i_\alpha : V \to N_\alpha$ the corresponding ultrapower embedding. Note that $2^\kappa < i_\alpha(\kappa) < (2^\kappa)^+$. Consider $k_\alpha : N_\alpha \to N$ defined by setting $k_\alpha([f]_{U_\alpha}) = i(f)(\langle \eta_\beta \mid \beta < \alpha\rangle)$. Let $\langle A_\xi \mid \xi < (2^\kappa)^{N_\alpha}\rangle \in N_\alpha$ list all subsets of κ. Then since $k_\alpha \restriction \kappa + 1$ is the identity, we will have $k_\alpha(A_\xi) = A_\xi$. But then $k_\alpha(\langle A_\xi \mid \xi < (2^\kappa)^{N_\alpha}\rangle) = \langle A_\xi \mid \xi < (2^\kappa)^{N_\alpha}\rangle$, since $\mathcal{P}(\kappa) \subseteq N_\alpha$. In particular, $(2^\kappa)^{N_\alpha} \geq 2^\kappa$ and $k_\alpha \restriction 2^\kappa + 1$ is the identity. Thus, $\mathrm{cp}(k_\alpha) = ((2^\kappa)^+)^{N_\alpha}$. We now set $\eta_\alpha = ((2^\kappa)^+)^{N_\alpha}$, thereby completeing our construction.

By (3) $\implies$ (2) of Theorem 4, we can let $\langle U^*_\alpha \mid \alpha < \omega_1\rangle$ be a strictly increasing Rudin-Keisler sequence of ultrafilters over κ. Denote by $\rho_{\beta,\alpha}$ the projection from U^*_β onto U^*_α, whenever $\alpha \leq \beta < \omega_1$. Then, for every countable sequence a of elements of $\mathcal{P}^\lambda(\kappa)$, define

$$E_a = \{x \subseteq V_\kappa \mid a \in i(x)\}.$$

This definition makes sense, since by our assumption that $\mathrm{cof}(\lambda) \neq \omega$, we may also assume that $(\mathcal{P}^\lambda(\kappa))^\omega \subseteq \mathcal{P}^\lambda(\kappa)$. If a is a subsequence of b, denote by $\pi_{\beta,a}$ the projection of E_b onto E_a. Let $\mathcal{E} = \langle\langle E_a, \pi_{b,a} \mid a, b \in [\mathcal{P}^\lambda(\kappa)]^{\aleph_0}, a$ is a subsequence of $b\rangle$ be the corresponding extender, with $i_\mathcal{E} : V \to N_\mathcal{E}$ the associated embedding. Then because i witnesses the $\mathcal{P}^\lambda(\kappa)$ hypermeasurability of κ, $i_\mathcal{E}$ does as well. In particular, $N_\mathcal{E} \supseteq \mathcal{P}^\lambda(\kappa)$ and $(N_\mathcal{E})^\omega \subseteq N_\mathcal{E}$.

Consider now $\langle E_a \times U^*_\alpha \mid a \in [\mathcal{P}^\lambda(\kappa)]^{\aleph_0}, \alpha < \omega_1\rangle$ with projections $\langle\langle \pi_{b,a}, \rho_{b,a}\rangle \mid a$ is a subsequence of b, $\alpha \leq \beta < \omega_1\rangle$. It is a directed system whose limit model M will be as desired. To see this, use $i_\mathcal{E}(\langle U^*_\alpha \mid \alpha < \omega_1\rangle)$ over $N_\mathcal{E}$ as in (2) $\implies$ (1) of Theorem 4 to obtain M. Because $\mathcal{P}^\lambda(\kappa) \subseteq N_\mathcal{E}$, $i_\mathcal{E}(\kappa) > |\mathcal{P}^\lambda(\kappa)|$. Since the embedding generated by $\langle U^*_\alpha \mid \alpha < \omega_1\rangle$ over V has critical point κ, the embedding generated by $i_\mathcal{E}(\langle U^*_\alpha \mid \alpha < \omega_1\rangle)$ from $N_\mathcal{E}$ to M has critical point $i_\mathcal{E}(\kappa)$. Consequently, $\mathcal{P}^\lambda(\kappa) \subseteq M$. This completes the proof of Corollary 3.1.

□

We turn our attention now to addressing the strength of the existence of an elementary embedding $j : V \to M$ with critical point κ such that $(j(\kappa))^\omega \subseteq M$ yet $(j(\kappa))^{\omega_1} \not\subseteq M$. We will prove three theorems in this regard, beginning with the following.

Theorem 5 *Suppose that there is no sharp for a strong cardinal (i.e., that o pistol does not exist).*

If there is an elementary embedding $j : V \to M$ with critical point κ such that $(j(\kappa))^\omega \subseteq M$ yet $(j(\kappa))^{\omega_1} \nsubseteq M$, then either $o(\kappa) \geq \omega_1$ in $\mathcal{K}$, or κ is measurable in $\mathcal{K}$ and $\{\nu < \kappa \mid o^{\mathcal{K}}(\nu) \geq \omega_1\}$ is unbounded in κ.

Proof: Suppose otherwise, i.e., that κ is a measurable cardinal in $\mathcal{K}$, $o(\kappa) < \omega_1$ in $\mathcal{K}$, and $\{\nu < \kappa \mid o^{\mathcal{K}}(\nu) \geq \omega_1\}$ is bounded in κ. Consider $j^* = j \restriction \mathcal{K}$. Then $j^* : \mathcal{K} \to (\mathcal{K})^M$ is an iterated ultrapower of $\mathcal{K}$. Hence, each $x \in (\mathcal{K})^M$ is of the form $j^*(f)(\kappa_{\alpha_1}, \ldots, \kappa_{\alpha_n})$ for some $f : [\kappa]^n \to \mathcal{K}$ in $\mathcal{K}$ and some critical points $\kappa_{\alpha_1}, \ldots, \kappa_{\alpha_n}$ of the iteration. Note that by our assumptions, only measures are involved in the iteration up to $j^*(\kappa)$. We ignore the iteration above $j^*(\kappa)$, if any, since it is irrelevant for the arguments below. Then it is possible to find an increasing sequence of critical points (generators of j^*) not in M which has all initial segments in M. Consider the least possible δ which is the limit of such a sequence.

Let θ be a generator of j^*. Then at some stage during the iteration, θ was a measurable cardinal, and a measure over θ was applied. We may assume that always the smaller measures in the Mitchell order are applied before the larger ones. Denote by $\text{meas}(\theta)$ the final image of θ during the continuation of the iteration to the final model $(\mathcal{K})^M$. Then $\text{meas}(\theta)$ is a measurable cardinal in $(\mathcal{K})^M$, and $\text{meas}(\theta) \leq j^*(\kappa)$. There are $f_\theta : [\kappa]^{n_\theta} \to \kappa + 1$ in $\mathcal{K}$ and a sequence of generators $\rho_{1,\theta} < \cdots < \rho_{n_\theta,\theta} < \theta$ such that $j^*(f)(\rho_{1,\theta}, \ldots, \rho_{n_\theta,\theta}) = \text{meas}(\theta)$.

Lemma 3.2 *Let $\langle \rho_\beta \mid \beta < \alpha \rangle$ be a sequence of generators corresponding to the same measurable cardinal λ, i.e., for every $\beta, \gamma < \alpha$, $\text{meas}(\rho_\beta) = \text{meas}(\rho_\gamma)$. Then $\alpha < \omega_1$.*

Proof: By our assumptions, the measurable cardinals of order at least ω_1 are bounded below κ. Hence, there are only countably many normal measures over κ. Consequently, if $\alpha \geq \omega_1$, the same measure was used during the iteration uncountably many times. However, since $M^\omega \subseteq M$, it is impossible to use the same measure even $\omega + 1$ many times. This completes the proof of Lemma 3.2.

□

Lemma 3.3 $|j^*(\kappa)|^{\mathcal{K}} = (\kappa^+)^{\mathcal{K}} = \kappa^+$.

Proof: Suppose that $|j^*(\kappa)|^{\mathcal{K}} \geq (\kappa^{++})^{\mathcal{K}}$. Since by our assumptions, there are only countably many normal measures over κ in $\mathcal{K}$, some of these measures must be used more than ω many times in the iteration. This is impossible, however, since $(j(\kappa))^\omega \subseteq M$. This completes the proof of Lemma 3.3.

$\square$

It now immediately follows that δ must be singular in $\mathcal{K}$ with cofinality less than κ. This is since $\kappa \leq \delta \leq j^*(\kappa)$, by Lemma 3.3, $|j^*(\kappa)|^{\mathcal{K}} = (\kappa^+)^{\mathcal{K}} = \kappa^+$, and both κ and κ^+ remain regular in V.

Suppose that δ is a regular cardinal in $(\mathcal{K})^M$.

Lemma 3.4 *There is a continuous, increasing sequence $\langle \kappa_\alpha \mid \alpha < \omega_1 \rangle$ of critical points of the iteration such that:*

1. $\delta = \bigcup_{\alpha<\omega_1} \kappa_\alpha$.
2. *For every $f : \kappa \to \kappa$ such that $f \in \mathcal{K}$, and for every $\alpha < \omega_1$, it is the case that $(j^*(f)''\kappa_\alpha) \cap [\kappa_\alpha, \delta) = \emptyset$.*

Proof: Suppose otherwise. Let $\langle \tau_\alpha \mid \alpha < \omega_1 \rangle$ be a continuous, increasing sequence of critical points of the iteration having limit δ. Consider the set

$$S = \{\alpha < \omega_1 \mid \exists g : \kappa \to \kappa, g \in \mathcal{K} \text{ such that } (j^*(g)''\tau_\alpha) \cap [\tau_\alpha, \delta) \neq \emptyset\}.$$

Then S is stationary, for if not, pick a club $C \subseteq \omega_1$ such that $C \cap S = \emptyset$. Consider $\langle \tau_\alpha \mid \alpha \in C \rangle$. This sequence satisfies clause (2) above, so there will be a club $C' \subseteq C$ where $\langle \tau_\alpha \mid \alpha \in C' \rangle$ satisfies clause (1) as well, contrary to our assumptions. Therefore, by applying Fodor's Theorem to the function $f(\alpha) =$ The least $\beta < \alpha$ with $(j^*(g)''\tau_\beta) \cap [\tau_\alpha, \delta) = \emptyset\}$ for some $g : \kappa \to \kappa$, $g \in \mathcal{K}$, there are $\alpha^* < \omega_1$ and a stationary $S^* \subseteq S$ such that for every $\alpha \in S^*$, there exists $g_\alpha : \kappa \to \kappa$, $g_\alpha \in \mathcal{K}$ such that $(j^*(g_\alpha)''\tau_{\alpha^*}) \cap [\tau_\alpha, \delta) \neq \emptyset$.

We now argue that there is a set $E \in \mathcal{K}$ consisting of functions from κ to κ such that $|E|^{\mathcal{K}} < \kappa$ and $E \supseteq \{g_\alpha \mid \alpha < \omega_1\}$. This follows from the following

Claim 3.5 *Let $A \subseteq \kappa^+$, $|A| < \kappa$. Then there is $B \in \mathcal{K}$, $|B|^{\mathcal{K}} < \kappa$ such that $B \supseteq A$.*

Proof: There is $\eta < \kappa^+$, $\eta \supseteq A$. Let $t_\eta \in \mathcal{K}$ be a bijection between κ and η. Consider $x = t_\eta^{-1\prime\prime} A$. Then $x \subseteq \kappa$ and $|x| < \kappa$. Hence, there is $\xi < \kappa$ such that $x \subseteq \xi$. The set $B = t_\eta'' \xi$ is as desired.

□

Consider next $j^*(E) = j^{*\prime\prime} E$. This set is in $(\mathcal{K})^M$. But now the set

$$x = \{\sup((j^*(g)'' \tau_{\alpha^*}) \cap \delta) \mid g \in j^*(E)\}$$

is in $(\mathcal{K})^M$. In addition, x is unbounded in δ and has cardinality less than κ in $(\mathcal{K})^M$. But this means that δ is singular in $(\mathcal{K})^M$, contrary to our assumptions. This completes the proof of Lemma 3.4.

□

It is now possible to infer that there must be some $\alpha_0 < \omega_1$ such that the measures originating from κ_{α_0} are used unboundedly often below δ. This is since by Lemma 3.4, $\alpha \leq \beta < \omega_1$ implies that $\text{meas}(\kappa_\alpha) \geq \text{meas}(\kappa_\beta) \geq \delta$. This follows because the fact that κ_α is a critical point of the iteration j^* implies that $\text{meas}(\kappa_\alpha) = \min\{j^*(f)''\kappa_\alpha \mid f : \kappa \to \kappa, f \in \mathcal{K}\}$. For any $\alpha < \omega_1$, $\kappa_\alpha < \delta$, and by clause (2) of Lemma 3.4, $\delta \leq \text{meas}(\kappa_\alpha)$. Therefore, the sequence $\langle \text{meas}(\kappa_\alpha) \mid \alpha < \omega_1 \rangle$ is non-increasing. This means that $\text{meas}(\kappa_\alpha)$ should stabilize, i.e., there are $\alpha_0 < \omega_1$ and $\mu \geq \delta$ such that for every α with $\alpha_0 \leq \alpha < \omega_1$, $\mu = \text{meas}(\kappa_\alpha)$. However, by our assumptions, there are only countably many normal measures over κ_{α_0}, which means that one of them should be used $\aleph_1$ many times in the iteration. By Lemma 3.2, this is impossible. Thus, δ cannot be regular in $(\mathcal{K})^M$.

Suppose now that δ is singular in $(\mathcal{K})^M$. Then $\text{cof}(\delta) < \kappa$ in $(\mathcal{K})^M$.[2] Hence, clearly, δ is not a generator and $\delta < j^*(\kappa)$. This means we can pick a function $g_\delta : [\kappa]^{n_\delta} \to \kappa$ in $\mathcal{K}$ and a sequence of generators $\xi_{1,\delta} < \cdots < \xi_{n_\delta,\delta} < \delta$ such that $j^*(g_\delta)(\xi_{1,\delta}, \ldots, \xi_{n_\delta,\delta}) = \delta$. Let $(\text{cof}(\delta))^{(\mathcal{K})^M} = \epsilon$. Fix a closed, cofinal sequence $\langle \sigma_i \mid i < \epsilon \rangle \in (\mathcal{K})^M$ for δ. Let $\langle \eta_\tau \mid \tau < \omega_1 \rangle$ be an increasing sequence

[2]Note that $(\text{cof}(\delta))^{(\mathcal{K})^M} = (\text{cof}(\delta))^{\mathcal{K}} \geq \text{cof}(\delta) = (\text{cof}(\delta))^M = \omega_1$, since otherwise, say if $\text{cof}(\delta) < \epsilon = (\text{cof}(\delta))^M$, then necessarily, ϵ must have cofinality $\text{cof}(\delta) = \omega_1$. However, $M \models$ "ϵ is regular" and $\epsilon < \kappa$. This means that $j(\epsilon) = \epsilon$, a contradiction.

of generators unbounded in δ such that $\langle \eta_\tau \mid \tau < \omega_1 \rangle \notin M$ and $\eta_0 > \xi_{n_\delta,\delta}$. We would like to use Mitchell's Covering Lemma [19, Theorem 4.19, page 1566] to cover inside M either the sequence $\langle \eta_\tau \mid \tau < \omega_1 \rangle$ or a final segment $s' = \langle \eta_\tau \mid \gamma < \tau < \omega_1 \rangle$ of this sequence by a set z of size below κ. Then $\mathcal{P}(z) = \mathcal{P}(z)^M$, so $s' \in M$. However, because $\eta_\tau < j(\kappa)$ for $\tau < \omega_1$ and $(j(\kappa))^\omega \subseteq M$, $s = \langle \eta_\tau \mid \tau \leq \gamma \rangle \in M$. Thus, $s^\frown s' = \langle \eta_\tau \mid \tau < \omega_1 \rangle \in M$, a contradiction.

If the sequence $\langle \text{meas}(\eta_\tau) \mid \tau < \omega_1 \rangle \in M$ and moreover, there is a set $A \in (\mathcal{K})^M$ of size $\epsilon' < \kappa$ in $(\mathcal{K})^M$ which covers $\langle \eta_\tau \mid \tau < \omega_1 \rangle$, then working inside M, for sufficiently large θ, we pick an elementary submodel $N \prec H_\theta$ such that $N^\omega \subseteq N$, $|N| = \epsilon' + 2^{\aleph_0}$, and $\delta, \langle \sigma_i \mid i < \epsilon \rangle, \langle \text{meas}(\eta_\tau) \mid \tau < \omega_1 \rangle, A \in N$. Then by Mitchell's Covering Lemma, there are $\zeta < \delta$, $h \in (\mathcal{K})^M$, and a system of indiscernibles C such that $N \cap \delta \subseteq h[\zeta; C]$. Moreover, for every limit $i < \epsilon$ with $\zeta < \sigma_i$, we have that all but boundedly many indiscernibles for measures in $\sigma_i \cap A$ are in C. Now, using a regressive function, we will obtain that all but boundedly many indiscernibles for measures in A are in C. In particular, a final segment of $\langle \eta_\tau \mid \tau < \omega_1 \rangle \in C$, and we are done.

In general, we need not have $\langle \text{meas}(\eta_\tau) \mid \tau < \omega_1 \rangle \in M$. We can compensate for this by working a little harder. Specifically, we define a tree T and begin by putting $\langle \eta_\tau \mid \tau < \omega_1 \rangle$ at the first level of T. Set

$$\text{Succ}_T(\langle \eta_\tau \rangle) = \{\rho_{\eta_\tau,k} \mid k < n_{\eta_\tau}\}$$

for every $\eta_\tau \neq \kappa$. If $\eta_\tau = \kappa$, then set $\text{Succ}_T(\langle \eta_\tau \rangle) = \emptyset$. The next level (and all further levels as well) are defined similarly. Thus, if $\rho_{\eta_\tau,k} = \kappa$, then set $\text{Succ}(\langle \eta_\tau, \rho_{\eta_\tau,k} \rangle) = \emptyset$. Otherwise set

$$\text{Succ}_T(\langle \eta_\tau, \rho_{\eta_\tau,k} \rangle) = \{\rho_{\rho_{\eta_\tau,k}m} \mid m < n_{\eta_{\rho_\tau,k}}\}.$$

The tree T will be well founded, since ordinals along its branches are decreasing.

Consider the set of nodes $S = \{\rho \mid \exists t \in T[t^\frown \rho \in T$ and all immediate successors (and consequently all successors) correspond to $j^*(\kappa)\}$. Note that $|S| \leq |T| = \omega_1$. In addition, by the definition of T, generators corresponding to $j^*(\kappa)$ (which are just measures that started originally from κ) appear only at terminal nodes (κ), or possibly at nodes one step before terminal ones. Further, $x = \{\text{meas}(\rho) \mid \rho \in S\} \in M$. To see this, observe that each $\text{meas}(\rho)$ is of the form $j^*(f_\rho)(\xi_{1,\rho}, \ldots, \xi_{n_\rho,\rho})$, with $\xi_{1,\rho}, \ldots, \xi_{n_\rho,\rho}$ being generators for $j^*(\kappa)$. We have already shown that the number of generators for $j^*(\kappa)$ is at most countable. Note that the total number of functions f_ρ which are used in T has size at most ω_1 (and the total number of functions relevant for x is at most countable). If we let $\langle t_i \mid i < \omega_1 \rangle$ be an enumeration of all of these functions in V, then $j(\langle t_i \mid i < \omega_1 \rangle) = \langle j^*(t_i) \mid i < \omega_1 \rangle \in M$. Hence, by using the generators for x and the sequence $\langle j^*(t_i) \mid i < \omega_1 \rangle$, we may now infer that $x \in M$.

Let us now cover $\langle j^*(t_i) \mid i < \omega_1 \rangle$ by a set of size less than κ in $(\mathcal{K})^M$. We will do this by covering $\langle t_i \mid i < \omega_1 \rangle$ by a set in $\mathcal{K}$ of cardinality less than κ and then applying j^* to our covering set. In particular, we argue in V as follows: $2^\kappa = \kappa^+$ in $\mathcal{K}$ (and in V as well since $o(\kappa) < \omega_1$). This means that we can code in $\mathcal{K}$ functions by ordinals less than κ^+. Thus, there is $\gamma < \kappa^*$ such that the codes for $\langle t_i \mid i < \omega_1 \rangle$ are all below γ. Pick a bijection $h : \gamma \to \kappa$ in $\mathcal{K}$. There is $\gamma^* < \kappa$ such that the images of the codes of the t_i s are below γ^*. Then $h''\gamma^*$ will be the desired covering of the set of the codes of the t_i s.

Now, back in M, let $B = j^*(h''\gamma^*)$. For sufficiently large θ, we pick an elementary submodel $N \prec H_\theta$ such that $N^\omega \subseteq N$, $|N| = \gamma^* + 2^{\aleph_0}$, and $\delta, \kappa, j^*(\kappa), B \in N$. Then as before, by Mitchell's Covering Lemma, there are $\zeta < \delta$, $h \in (\mathcal{K})^M$, and a system of indiscernibles C such that $N \cap \delta \subseteq h[\zeta; C]$. Moreover, for every limit $i < \epsilon$ with $\zeta < \sigma_i$, we have that all but boundedly many indiscernibles for measures in $\sigma_i \cap N$ are in C. Now, using a regressive function, we will obtain that all but boundedly many indiscernibles for measures in N are in C. In particular, again as before, a final segment of $\langle \eta_\tau \mid \tau < \omega_1 \rangle \in C$, and we are done. This completes the proof of Theorem 5.

□

Theorem 6 *Suppose that $o(\kappa) \geq \omega_1$ in $\mathcal{K}$. Then there is a generic extension V of $\mathcal{K}$ with an elementary embedding $j : V \to M$ having critical point κ such that $M^\omega \subseteq M$ yet $(j(\kappa))^{\omega_1} \not\subseteq M$.*

Proof: By the Theorem of [6], assuming that $o(\kappa) \geq \omega_1$ in $\mathcal{K}$, it is possible to force over $\mathcal{K}$ to obtain a strictly increasing Rudin-Keisler sequence of ultrafilters over κ having length ω_1. Theorem 6 then follows by the proof of Theorem 4, (2) $\Longrightarrow$ (1).

□

Theorem 7 *Suppose that κ is a measurable cardinal in $\mathcal{K}$ and $\{\nu < \kappa \mid o^{\mathcal{K}}(\nu) \geq \omega_1\}$ is unbounded in κ. Then there is a generic extension V of $\mathcal{K}$ and an elementary embedding $j : V \to M$ having critical point κ such that $M^\omega \subseteq M$ yet $(j(\kappa))^{\omega_1} \not\subseteq M$.*

Proof: Fix a normal measure U over κ. For each $\nu < \kappa$, let ν^* be the least cardinal above ν with $o(\nu) = \omega_1$. Let $\vec{W}(\nu^*) = \langle W(\nu^*, \xi) \mid \xi < \omega_1 \rangle$ witness that $o(\nu^*) = \omega_1$, i.e., $\vec{W}(\nu^*)$ is an increasing sequence in the Mitchell ordering $\triangleleft$ [18] of normal measures over κ.

We now turn $\vec{W}(\nu^*)$ into a Rudin-Keisler increasing sequence of ultrafilters. Let $\langle\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle \mid \alpha < \kappa \rangle$ be an Easton support iteration of Prikry type forcings of length κ, where for every $\alpha < \kappa$, $\dot{\mathbb{Q}}_\alpha$ is a term for the forcing of [6, Section 2] (see also [7]) which adds either a Prikry or Magidor sequence to every measurable cardinal $\gamma \in (\alpha, \alpha^*)$. Note that for all such γ, $o(\gamma) < \omega_1$ by the definition of α^*. This extends $\vec{W}(\alpha^*) = \langle W(\alpha^*, \xi) \mid \xi < \omega_1 \rangle$ into a Rudin-Keisler increasing commutative sequence $\vec{W}'(\alpha^*) = \langle W'(\alpha^*, \xi) \mid \xi < \omega_1 \rangle$ of α^* complete ultrafilters over α^*.

Let G be $\mathcal{K}$-generic over $\mathbb{P} = \mathbb{P}_\kappa$. We claim that in $V = \mathcal{K}[G]$, there is an elementary embedding $j : V \to M$ having critical point κ such that $M^\omega \subseteq M$ yet $(j(\kappa))^{\omega_1} \not\subseteq M$. To see this, fix some $\xi < \omega_1$. Define an ultrafilter U_ξ over κ^2 in $\mathcal{K}$ by

$$x \in U_\xi \text{ iff } \{\nu < \kappa \mid \{\zeta < \nu^* \mid (\nu, \zeta) \in x\} \in W(\nu^*, \xi)\} \in U.$$

The ultrapower by U_ξ is the ultrapower by U followed by the ultrapower by $(j_U(\vec{W}))(\kappa^*, \xi)$. Let $j_\xi : \mathcal{K} \to M_\xi$ be the corresponding elementary embedding. Then we can write $j_\xi = j_{(j_U(\vec{W}))(\kappa^*, \xi)} \circ j_U$ and obtain a commutative system of embeddings.

Consider now what happens in V. By using the argument found in the proof of Lemma 2.2 for the construction of the supercompact ultrafilter $\mathcal{U}$, we may extend the ultrafilter U_ξ of $\mathcal{K}$ to an ultrafilter U'_ξ of V by constructing an increasing sequence of conditions successively deciding the statements "$(\kappa, \xi) \in j_\xi(\dot{x})$" for all suitable canonical names $\dot{x}$. Because $o(\zeta) = \xi$, by the definition of $\mathbb{P}$, for a typical (ν, ζ), a Magidor sequence of order type ω^ξ was added to ζ. Also, by elementarity, in the ultrapower by U'_ξ, the same thing is true. Thus, let $j_{U'_\xi} : \mathcal{K} \to M'_\xi \simeq V^{\kappa^2}/U'_\xi$. Let $[\mathrm{id}]_{U'_\xi} = \langle \kappa, \tilde{\xi} \rangle$. Then M'_ξ has a Magidor sequence of order type ω^ξ for $\tilde{\xi}$ over its ground model $\mathcal{K}^{M'_\xi}$.

Let $\rho < \xi$. Set $\sigma_{\xi,\rho}(\nu, \zeta) = (\nu, \zeta_\rho)$, where ζ_ρ is the ρ^{th} member of the Magidor sequence added to ζ. Note that by their definitions, $\sigma_{\xi,\rho}$ will project the extension U'_ξ of U_ξ to the extension U'_ρ of U_ρ. Consequently,

$$\langle \langle U'_\xi \mid \xi < \omega_1 \rangle, \langle \sigma_{\xi,\rho} \mid \rho \leq \xi < \omega_1 \rangle \rangle$$

forms a Rudin-Keisler commutative sequence. We check that it is strictly increasing. By Theorem 4, (2) $\Longrightarrow$ (1), this will suffice to prove Theorem 7.

To do this, we suppose otherwise. Then there are $\rho < \xi < \omega_1$ such that $U'_\rho =_{\text{RK}} U'_\xi$. Let $f : \kappa^2 \to \kappa^2$ be a witnessing isomorphism. Then in the ultrapower by U'_ξ we will have $j_{U'_\xi}(f)(\kappa, \tilde{\xi_\rho}) = (\kappa, \tilde{\xi})$ since $U'_\rho = \{x \subseteq \kappa^2 \mid \langle \kappa, \tilde{\xi_\rho} \rangle \in j_{U'_\xi}(x)\}$ because of the projection map $\sigma_{\xi,\rho}$.

By the next claim (Claim 3.6), we will be able to assume that f is the identity in the first coordinate and is strictly increasing in the second coordinate once the first one has been fixed, i.e., if $\tau < \tau' < \nu^*$ and $f(\nu, \tau) = (\alpha, \beta)$, $f(\nu, \tau') = (\alpha', \beta')$, then $\nu = \alpha = \alpha'$ and $\tau < \beta < \beta'$.

Claim 3.6 *There is $f' : \kappa^2 \to \kappa^2$ such that*

1. $[f']_{U'_\xi} = [f]_{U'_\xi}$.
2. *For every inaccessible $\nu < \kappa$ and $\tau < \tau' < \nu^*$, if $f'(\nu, \tau) = (\alpha, \beta)$ and $f'(\nu, \tau') = (\alpha', \beta')$, then $\nu = \alpha = \alpha'$ and $\tau < \beta < \beta'$.*

Proof: Without loss of generality, we assume that for every inaccessible ν and every $\tau < \nu^*$, it is the case that $f(\nu, \tau) < \nu^*$. Therefore, for any inaccessible cardinal $\nu < \kappa$, we may define in V the set $C_\nu = \{\tau < \nu^* \mid \text{For all } \sigma < \tau, \text{ the second coordinate of } f(\nu, \sigma) \text{ is less than } \tau\}$, which is a club subset of ν^*. Note that the forcing above ν^* does not add subsets to ν^*, nothing is done over ν^* itself, and $\mathbb{P}_{\nu^*}$ satisfies ν^*-c.c. Hence, there is a club $E_\nu \in \mathcal{K}$, $E_\nu \subseteq C_\nu$. Consequently, by normality, we have that $E_\nu \in W(\nu^*, \theta)$ for every $\theta < \omega_1$. It then follows that E_ν and C_ν will each be in $W'(\nu^*, \theta)$. Thus, for every $\theta < \omega_1$, the set $x = \{(\nu, \tau) \in \kappa^2 \mid \tau \in E_\nu\} \in U_\theta$. However, U'_θ extends U_θ, so in particular, $x \in U'_\xi$. This means that $f \restriction x$ is as desired. This completes the proof of Claim 3.6.

□

For every inaccessible cardinal $\nu < \kappa$ and every $\tau \in [\nu, \nu^*)$, set $f_\nu(\tau) =$ The second coordinate of $f(\nu, \tau)$. Then $j_{U'_\xi}(f)_\kappa(\tilde{\xi_\rho}) = \tilde{\xi}$. Pick a set $A_\nu \in W(\nu^*, \rho) - W(\nu^*, \xi)$. Let $g_\nu = f_\nu \restriction A_\nu$. Note that each g_ν is strictly increasing. Also, in the ultrapower by U'_ξ, $g_\kappa(\tilde{\xi_\rho}) = \tilde{\xi}$.

For $\nu < \kappa$ an inaccessible cardinal, define $h_\nu \in \mathcal{K}$ by $h_\nu(\tau) = \{\mu \mid \exists p \in \mathbb{P}_{\nu^*}[p \Vdash \text{“}g_\nu(\tau) = \mu\text{”}]\}$. By its definition, $h_\nu : A_\nu \to \mathcal{P}(\nu^*)$, and for every $\tau \in A_\nu$, $g_\nu(\tau) \in h_\nu(\tau)$ and $\min(h_\nu(\tau)) > \tau$.

Claim 3.7 *There is $B_\nu \in W(\nu^*, \xi)$ such that $B_\nu \cap \bigcup \text{rng}(h_\nu) = \emptyset$.*

Proof: If not, then $\bigcup \text{rng}(h_\nu) \in W(\nu^*, \xi)$. Consequently, $\nu^* \in j_{W(\nu^*,\xi)}(\bigcup \text{rng}(h_\nu))$. So there is $\tau \in j_{W(\nu^*,\xi)}(A_\nu)$ such that $\nu^* \in (j_{W(\nu^*,\xi)}(h_\nu))(\tau)$. But $\min((j_{W(\nu^*,\xi)}(h_\nu))(\tau)) > \tau$, so $\nu^* > \tau$. Then $(j_{W(\nu^*,\xi)}(h_\nu))(\tau) = h_\nu(\tau)$. This is since ν^* is the critical point of the embedding $j_{W(\nu^*,\xi)}$ and $|h_\nu(\tau)| < \nu^*$. (This last fact follows because as we have already observed, the forcing above ν^* does not add subsets to ν^*, nothing is done over ν^* itself, and $\mathbb{P}_{\nu^*}$ satisfies ν^*-c.c.) But $h_\nu(\tau) \subseteq \nu^*$, so $\nu^* \notin h_\nu(\tau)$. This contradiction completes the proof of Claim 3.7.

$\square$

We now look at what happens at κ in the ultrapower by U'_ξ. It is the case that $\tilde{\xi} \in B_\kappa$. To see this, let $z = \{(\nu, \zeta) \mid \zeta \in B_\nu\}$. We have that $z \in U_\xi \subseteq U'_\xi$. Hence $(\kappa, \tilde{\xi}) \in j_{U'_\xi}(z)$, so $\tilde{\xi} \in B_\kappa$. Then $\tilde{\xi} = g_\kappa(\tilde{\xi_\rho}) \in h_\kappa(\tilde{\xi_\rho})$ and $B_\kappa \cap \bigcup(\text{rng}(h_\kappa)) = \emptyset$. This is impossible. This completes the proof of Theorem 7.

$\square$

We conclude Section 3 by noting that it is possible to prove Theorem 6 by forcing over an arbitrary model V^* of ZFC in which κ has a coherent sequence of measures of length at least ω_1. In addition, it is possible to prove Theorem 7 by forcing over an arbitrary model V^* of ZFC in which $\{\nu < \kappa \mid \text{There is a coherent sequence of measures over } \nu \text{ of length at least } \omega_1\}$ is unbounded in κ. In order to minimize the technical details involved, however, we force over $\mathcal{K}$ instead.

4 The consistency strength of strongly tall cardinals

Recall that κ is *strongly tall* if for every ordinal $\lambda \geq \kappa$, there is an elementary embedding witnessing that κ is λ tall which is generated by a κ-complete measure on some set. We address the consistency strength of strongly tall cardinals with the following theorem.

Theorem 8 *The following theories are equiconsistent:*

a) ZFC + There is a strong cardinal and a proper class of measurable cardinals.

b) ZFC + There is a strongly tall cardinal.

Proof: We begin with the proof of Theorem 8(a). Suppose $V \models$ "ZFC + κ is strong + There is a proper class of measurable cardinals". Assume without loss of generality that $V \models$ GCH as well. Fix a proper class $\langle \lambda_\alpha \mid \alpha \in \mathrm{Ord} \rangle$ satisfying the following properties.

1. $\lambda_0 > \kappa$.

2. $\alpha < \beta$ implies that $\lambda_\alpha < \lambda_\beta$.

3. If α is a limit ordinal, then $\lambda_\alpha = \bigcup_{\beta<\alpha} \lambda_\beta$.

4. For every α, $\lambda_{\alpha+1}$ is a measurable cardinal.

We now define the partial ordering $\mathbb{P}$ used in the proof of Theorem 8(a). Let α be an ordinal. Let $\mathbb{Q}_\alpha$ be the reverse Easton iteration of length $\lambda_{\alpha+1}$ which does trivial forcing except at regular cardinals in the half-open interval $(\lambda_\alpha, \lambda_{\alpha+1}]$. At such a stage δ, the forcing used is $\mathrm{Add}(\delta, \lambda_\alpha)$. $\mathbb{P}$ is taken as the Easton support product $\prod_{\alpha\in\mathrm{Ord}} \mathbb{Q}_\alpha$.

Let G be V-generic over $\mathbb{P}$. Standard arguments show that $V[G] \models$ ZFC. The proof of Theorem 8(a) will be complete once we have shown that $V[G] \models$ "κ is strongly tall". Towards this end, let $\lambda \geq \kappa$ be a regular cardinal. Choose α such that $\lambda_\alpha > \lambda$ and $\mathrm{cof}(\alpha) >> \lambda$.

Work for the time being in V. Fix a (κ, λ)-extender $\mathcal{E}$, with $j : V \to M$ the corresponding ultrapower embedding. It is then the case that $\mathrm{cp}(j) = \kappa$, $\lambda_\alpha > j(\kappa) > \lambda$, and $M^\kappa \subseteq M$. For every $a \in [\lambda]^{\leq\kappa}$, set $U_a = \{x \subseteq V_\kappa \mid a \in j(x)\}$. Let $j_a : V \to M_a$ be the corresponding ultrapower embedding. If a is a subsequence of b, then we denote by $\pi_{b,a}$ the obviously defined projection of U_b onto U_a and let $k_{a,b} : M_a \to M_b$ be the corresponding elementary embedding between the ultrapowers. Then

$$\langle\langle M_a \mid a \in [\lambda]^{\leq\kappa}\rangle, \langle k_{a,b} \mid a, b \in [\lambda]^{\leq\kappa}, a \text{ is a subsequence of } b\rangle\rangle$$

is a κ^+-directed system having limit $\langle M, \langle j_a \mid a \in [\lambda]^{\leq\kappa}\rangle\rangle$.

Let $W \in V$ be a normal ultrafilter over $\lambda_{\alpha+1}$, with $i : V \to N$ the corresponding ultrapower embedding. For every $a \in [\lambda]^{\leq\kappa}$, let $W_a = U_a \times W$, with $i_a : V \to N_a$ the corresponding ultrapower embedding. Note that i_a may be obtained either by first applying U_a and then $j_a(W)$ (which is actually W) or by first applying W and then U_a.

Define a κ^+-directed system

$$\langle\langle N_a \mid a \in [\lambda]^{\leq\kappa}\rangle, \langle \ell_{a,b} \mid a, b \in [\lambda]^{\leq\kappa}, a \text{ is a subsequence of } b\rangle\rangle$$

in the obvious manner. Let $\langle N_{\mathcal{E}}, \langle i_a \mid a \in [\lambda]^{\leq\kappa}\rangle\rangle$ be its limit, with $i_{\mathcal{E}} : V \to N_{\mathcal{E}}$ the corresponding embedding. Note that $N_{\mathcal{E}}$ may be viewed as the ultrapower by $\mathcal{E} \times W$ or as the iterated ultrapower by $\mathcal{E}$ and then by $j(W) = W$ or as the ultrapower by first applying W and then $\mathcal{E}$.

Consider now $V[G]$. Let W' be an extension of W in $V[G]$ and $i' : V[G] \to N[G']$, $i' \supseteq i$ the corresponding ultrapower embedding.

Write $\mathbb{P} = \mathbb{P}_{<\alpha} \times \mathbb{P}_\alpha \times \mathbb{P}_{>\alpha}$, with $\mathbb{P}_{<\alpha} = \prod_{\beta<\alpha} \mathbb{Q}_\beta$, $\mathbb{P}_\alpha = \mathbb{Q}_\alpha$, and $\mathbb{P}_{>\alpha} = \prod_{\beta>\alpha} \mathbb{Q}_\beta$. Since $\mathbb{P}$ is defined as a product forcing, the order of the products just given can be changed. In addition, $i(\mathbb{P}_{<\alpha}) = \mathbb{P}_{<\alpha}$, and W is not affected by $\mathbb{P}_{>\alpha}$ because of its closure. Let $G_{<\alpha} = G \restriction \mathbb{P}_{<\alpha}$, $G_\alpha = G \restriction \mathbb{P}_\alpha$, and $G_{>\alpha} = G \restriction \mathbb{P}_{>\alpha}$. Then $G' \restriction \mathbb{P}_{<\alpha} = G_{<\alpha}$, $G' \restriction \mathbb{P}_\alpha = G_\alpha$, and $G' \restriction \mathbb{P}_{>\alpha}$ is generated by $i''G_{>\alpha}$. Denote by $G^{>\alpha}$ the part of G' in the interval $(\lambda_{\alpha+1}, i(\lambda_{\alpha+1})]$. Let $\langle f_\xi \mid \xi < \lambda_\alpha\rangle$ be the Cohen functions from $\lambda_{\alpha+1}$ to $\lambda_{\alpha+1}$ added by G_α over $\lambda_{\alpha+1}$, with $\langle f'_\xi \mid \xi < i(\lambda_\alpha)\rangle$ the corresponding functions from $i(\lambda_{\alpha+1})$ to $i(\lambda_{\alpha+1})$ added by G'.

Consider now $i_{\mathcal{E}} : V \to N_{\mathcal{E}}$. We extend it to $i'_{\mathcal{E}} : V[G] \to N_{\mathcal{E}}[G_{\mathcal{E}}]$ as follows. We first generate $G^{\leq\alpha}_{\mathcal{E}}$, the part of $G_{\mathcal{E}}$ below λ_α, by $j''G_{<\alpha}$. We then use $G^{>\alpha}$ to generate the part of $G_{\mathcal{E}}$ in the interval $(\lambda_{\alpha+1}, i(\lambda_{\alpha+1})]$. Finally, the part of $G_{\mathcal{E}}$ above $i(\lambda_{\alpha+1})$ is generated by $i''_{\mathcal{E}}G_{>\alpha}$.

For every $a \in [\lambda]^{\leq\kappa}$, let $W'_a = \{x \subseteq V_\kappa \times \lambda_{\alpha+1} \mid (a, \lambda_{\alpha+1}) \in i'_{\mathcal{E}}(x)\}$ be the extension of W_a in $V[G]$ and $i'_a : V[G] \to N_a[G'_a]$, $i'_a \supseteq i_a$ the corresponding ultrapower embedding. Then the κ^+-directed system

$$\langle \langle N_a \mid a \in [\lambda]^{\leq\kappa} \rangle, \langle \ell_{a,b} \mid a, b \in [\lambda]^{\leq\kappa}, a \text{ is a subsequence of } b \rangle \rangle$$

extends in the obvious fashion to the κ^+-directed system

$$\langle \langle N_a[G'_a] \mid a \in [\lambda]^{\leq\kappa} \rangle, \langle \ell'_{a,b} \mid a, b \in [\lambda]^{\leq\kappa}, a \text{ is a subsequence of } b \rangle \rangle$$

with limit $\langle N_{\mathcal{E}}[G_{\mathcal{E}}], \langle i'_a \mid a \in [\lambda]^{\leq\kappa} \rangle \rangle$ and $i'_{\mathcal{E}} : V[G] \to N_{\mathcal{E}}[G_{\mathcal{E}}]$ the corresponding embedding.

Fix $a \in [\lambda]^{\leq\kappa}$. Let the Cohen functions added by G'_a over $i_a(\lambda_{\alpha+1})$ corresponding to $\langle f_\xi \mid \xi < \lambda_\alpha \rangle$ be denoted by $\langle f'_{a,\xi} \mid \xi < i_a(\lambda_\alpha) \rangle$.

We come now to the crucial point of the construction. For every $\zeta \in a$, let ζ_a be the ordinal represented in M_a by the coordinate ζ, i.e., $j_a(\zeta_a) = \zeta$. We change the value of $f'_{a,i_a(\zeta)}(\lambda_{\alpha+1})$ to ζ_a and let $f''_{a,i_a(\zeta)}$ be the resulting function. Since we have changed only one value, $f''_{a,i_a(\zeta)}$ remains Cohen generic. Note that the number of changes made is at most κ, which is small relative to $\lambda_{\alpha+1}$. Consequently, after all of the changes have been made to $G_{\mathcal{E}}$, the resulting set $G''_{\mathcal{E}}$ remains $N_a[G_{\mathcal{E}} \restriction i_a(\lambda_{\alpha+1})]$-generic. Let $i''_a : V[G] \to N_a[G''_a]$ be the corresponding embedding. Note that since the generic set has been changed, $i''_a \neq i'_a$. Regardless, we have a κ^+-directed system

$$\langle \langle N_a[G''_a] \mid a \in [\lambda]^{\leq\kappa} \rangle, \langle \ell''_{a,b} \mid a, b \in [\lambda]^{\leq\kappa}, a \text{ is a subsequence of } b \rangle \rangle$$

with limit $\langle N_{\mathcal{E}}[G''_{\mathcal{E}}], \langle i''_a \mid a \in [\lambda]^{\leq\kappa} \rangle \rangle$ and $i''_{\mathcal{E}} : V[G] \to N_{\mathcal{E}}[G''_{\mathcal{E}}]$ the corresponding embedding. Since $G''_{\mathcal{E}} \in N_{\mathcal{E}}[G_{\mathcal{E}}]$, $N_{\mathcal{E}}[G''_{\mathcal{E}}] = N_{\mathcal{E}}[G_{\mathcal{E}}]$. Using κ^+-directedness, it follows that $(N_{\mathcal{E}}[G_{\mathcal{E}}])^\kappa \subseteq N_{\mathcal{E}}[G_{\mathcal{E}}]$.

By using an appropriate coding of $[\lambda]^{\leq\kappa}$ in V, any ultrafilter of the form U_a for $a \in [\lambda]^{\leq\kappa}$ may be replaced by an ultrafilter of the form $U_{\{\zeta\}}$ for some $\zeta < \lambda$. Consequently, any system defined using $\langle U_a \mid a \in [\lambda]^{\leq\kappa} \rangle$ may be replaced by a system defined from this coding using only $\langle U_{\{\zeta\}} \mid \zeta < \lambda \rangle$, i.e., the two systems will have the same direct limit.

Let $W^* = \{z \subseteq \lambda_{\alpha+1} \mid \lambda_{\alpha+1} \in i''_{\mathcal{E}}(z)\}$. By its definition, and using the fact mentioned in the preceding paragraph, W^* extends W and projects onto W_a for every $a \in [\lambda]^{\leq\kappa}$. Let $i^* : V[G] \to N^*$

be the corresponding elementary embedding and $k^* : N^* \to N_{\mathcal{E}}[G''_{\mathcal{E}}]$ be the standard embedding which forms a commutative diagram, i.e., $k^*([g]_{W^*}) = i''_{\mathcal{E}}(g)(\lambda_{\alpha+1})$. Because $i''_{\mathcal{E}}$ is definable in $V[G]$, i^* is definable in $V[G]$ as well.

The next claim is used to finish the proof of Theorem 8(a).

Claim 4.1 *k^* is a map onto $N_{\mathcal{E}}[G''_{\mathcal{E}}]$ and hence $N^* = N_{\mathcal{E}}[G''_{\mathcal{E}}]$.*

Proof: Note that every element of $N_{\mathcal{E}}[G''_{\mathcal{E}}]$ is of the form $i''_{\mathcal{E}}(h)(\zeta, \lambda_{\alpha+1})$ for some $h : \kappa \times \lambda_{\alpha+1} \to V[G]$ and $\zeta < \lambda$.

Fix $\zeta < \lambda$ and consider f_ζ. We have

$$k^*([f_\zeta]_{W^*}) = (i''_{\mathcal{E}}(f_\zeta))(\lambda_{\alpha+1}) = \zeta$$

because of the change we made to the value of the Cohen function. Then for any $h : \kappa \times \lambda_{\alpha+1} \to V[G]$,

$$i''_{\mathcal{E}}(h)(\zeta, \lambda_{\alpha+1}) = i''_{\mathcal{E}}(h)((i''_{\mathcal{E}}(f_\zeta))(\lambda_{\alpha+1}), \lambda_{\alpha+1}) = i''_{\mathcal{E}}(t)(\lambda_{\alpha+1}) = k^*([t]_{W^*}),$$

where $t(\rho) = h(f_\zeta(\rho), \rho)$ for every $\rho < \lambda_{\alpha+1}$.

$\square$

Because $W^* \supseteq W$ and W^* projects onto U_a for every $a \in [\lambda]^{\leq\kappa}$, $i^*(\kappa) > \lambda_{\alpha+1}$. Since $N^* = N_{\mathcal{E}}[G''_{\mathcal{E}}]$, $N_{\mathcal{E}}[G''_{\mathcal{E}}] = N_{\mathcal{E}}[G_{\mathcal{E}}]$, and $(N_{\mathcal{E}}[G_{\mathcal{E}}])^\kappa \subseteq N_{\mathcal{E}}[G_{\mathcal{E}}]$, i^* maps $V[G]$ into a κ-closed inner model. Consequently, because for any ordinal α, $\lambda_{\alpha+1}$ is a measurable cardinal in V, $V[G] \models$ "κ is strongly tall". This completes the proof of Theorem 8(a).

$\square$

Having completed the proof of Theorem 8(a), we turn now to the proof of Theorem 8(b). Suppose κ is a strongly tall cardinal and that there is no inner model with two strong cardinals. We show this implies that there are arbitrarily large measurable cardinals in $\mathcal{K}$ (which of course can be assumed to contain one strong cardinal).

Suppose $\theta > \kappa$. Let $\lambda >> \theta$ be a strong limit cardinal. Let U be a κ-complete uniform ultrafilter on some cardinal δ with corresponding elementary embedding $j : V \to M$ such that $j(\kappa) > \lambda$. Then $\delta \geq \theta$, and uniformity and κ-completeness together imply that $\mathrm{cof}(\delta) \geq \kappa$.

Let $\mathbb{P}_U$ be Prikry tree forcing defined with respect to U (see [7] for the exact definition). Force with $\mathbb{P}_U$ over V. Then as with ordinary Prikry forcing, V and $V^{\mathbb{P}_U}$ have the same bounded subsets of κ, and δ has cofinality ω in $V^{\mathbb{P}_U}$. Therefore, if δ was regular in V, work of Schindler [20] shows that δ is measurable in $\mathcal{K}$.

Suppose now that $V \vDash$ "$\mathrm{cof}(\delta) = \eta < \delta$". Let $\langle \delta_n \mid n < \omega \rangle$ be the cofinal ω sequence added by $\mathbb{P}_U$. Observe that there is no set $x \in V$ of cardinality less than δ in V covering $\{\delta_n \mid n < \omega\}$. To see this, suppose otherwise. Without loss of generality, we can assume that $x \subseteq \delta$. By the uniformity of U, it is the case that $x \notin U$. This, however, implies that a final segment of the δ_n s will be in the compliment of x, an immediate contradiction. Hence, by applying covering arguments to Schindler's core model [20], for every $\tau < \delta$, there is a measurable cardinal in $\mathcal{K}$ above τ. In particular, there is a measurable cardinal in $\mathcal{K}$ above θ. This completes the proofs of both Theorem 8(b) and Theorem 8.

□

5 Concluding remarks

We conclude by posing some questions and making some related comments. These are as follows:

1. Is it possible to obtain a model of ZFC in which the first ω strongly compact and measurable cardinals precisely coincide? More generally, is it possible to obtain a model of ZFC in which there are infinitely many (including possibly even proper class many) strongly compact cardinals, and the measurable and strongly compact cardinals precisely coincide?

2. Is $V = \mathcal{K}$ really needed in the hypotheses of Theorem 3, or is it possible to construct a model with a tall cardinal having bounded closure ω in which there are neither any tall cardinals nor any tall cardinals having bounded closure δ for $\omega_1 \leq \delta < \kappa$ by forcing over an arbitrary model V of ZFC satisfying the current assumptions?

3. Is the existence of $\eta > \kappa$ with $o(\eta) = \omega_1$ really needed in order to construct a model with a tall cardinal having bounded closure ω in which there are neither any tall cardinals nor any

tall cardinals having bounded closure δ for $\omega_1 \le \delta < \kappa$? In particular, are the theories "ZFC + There is a tall cardinal with bounded closure ω" and "ZFC + There is a tall cardinal with bounded closure" equiconsistent? Are the theories "ZFC + There is a strong cardinal" and "ZFC + There is a tall cardinal with bounded closure" equiconsistent?

4. Suppose that there is no $\eta > \kappa$ with $o(\eta) = \omega_1$ in $\mathcal{K}$. Is it possible to have an elementary embedding $j : V \to M$ such that $\mathrm{cp}(j) = \kappa$, $j(\kappa) \ge \kappa^{++}$, and $M^\omega \subseteq M$, yet for no elementary embedding $j' : V \to M'$ with $\mathrm{cp}(j') = \kappa$ and $j'(\kappa) \ge \kappa^{++}$ is it the case that $(M')^{\omega_1} \subseteq M'$?

5. The same question as Question 4, except that we require in addition that $\kappa^{++} = (\kappa^{++})^M$.

Question 1 and its generalized version are variants of Magidor's question posed at the end of Section 2. Also, in Theorem 3, it is possible to eliminate the assumption of no measurable cardinals above η.

Note that assuming the existence in $\mathcal{K}$ of $\eta > \kappa$ with $o(\eta) = \omega_1$, it is possible first to use the construction given in the proof of Theorem 3 and then force with $\mathrm{Add}(\omega, 1) * \dot{\mathrm{C}}\mathrm{oll}(\kappa^+, <\lambda)$. Here, $\lambda \ge \eta^+$ is a fixed regular cardinal, and $\mathrm{Coll}(\kappa^+, <\lambda)$ is the standard Lévy collapse which makes $\lambda = \kappa^{++}$. By the results of [14], the model obtained after forcing with $\mathrm{Add}(\omega, 1)$ also witnesses the conclusions of Theorem 3. Since forcing with $\mathrm{Coll}(\kappa^+, <\lambda)$ will add no new subsets of κ, the relevant extender is not affected, and in our final model V, there is an elementary embedding $j : V \to M$ such that $\mathrm{cp}(j) = \kappa$, $j(\kappa) \ge \kappa^{++}$, $M^\omega \subseteq M$, and $M^{\omega_1} \not\subseteq M$. By Hamkins' results of [10], because $\mathrm{Add}(\omega, 1) * \dot{\mathrm{C}}\mathrm{oll}(\kappa^+, <\lambda)$ "admits a closure point at ω" (see [10] for a definition of this terminology), there is no elementary embedding $j' : V \to M'$ such that $\mathrm{cp}(j') = \kappa$, $j'(\kappa) \ge \kappa^{++}$, and $(M')^{\omega_1} \subseteq M'$. With a little more work (i.e., by using a preparatory forcing similar to the one given in [8]), it is possible to ensure also that $\kappa^{++} = (\kappa^{++})^M$.

It is unclear at all, however, whether an assumption beyond $o(\kappa) = \kappa^{++}$ is really needed. In fact, this prompts us to ask the related question

6. Suppose $o(\kappa) = \kappa^{++}$. Is it possible to force in $V^{\mathbb{P}}$ an ω-directed but not ω_1-directed sequence $\langle\langle U_\alpha, \pi_{\alpha,\beta}\rangle \mid \alpha \le \beta < \kappa^{++}\rangle$ of ultrafilters over κ such that there is no κ-directed sequence of

ultrafilters of length κ^{++} in $V^{\mathbb{P}}$?

We end by conjecturing that it is possible and that methods from [8] may be relevant.

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