The Consistency of $V = \mathrm{HOD}$ with Level by Level Equivalence *†

Arthur W. Apter

Department of Mathematics

Baruch College of CUNY

New York, New York 10010 USA

and

The CUNY Graduate Center, Mathematics

365 Fifth Avenue

New York, New York 10016 USA

http://faculty.baruch.cuny.edu/aapter

awapter@alum.mit.edu

May 5, 2017

Abstract

We construct two models showing the relative consistency of $V = \mathrm{HOD}$ with the level by level equivalence between strong compactness and supercompactness. In the first model, various versions of the combinatorial principles $\square$ and $\diamondsuit$ hold. In the second model, the Ground Axiom (GA) holds.

1 Introduction and Preliminaries

In [4], the following theorem was proven.

Theorem 1 *Let* $V \vDash$ *"ZFC + GCH +* $\mathcal{K} \neq \emptyset$ *is the class of supercompact cardinals". There is then a partial ordering* $\mathbb{P} \subseteq V$ *such that* $V^{\mathbb{P}} \vDash$ *"ZFC + GCH +* $\mathcal{K}$ *is the class of supercompact cardinals + Level by level equivalence between strong compactness and supercompactness holds". In* $V^{\mathbb{P}}$, $\square_{\gamma}^{S}$ *holds for every infinite cardinal* γ, *where* $S = \mathrm{Safe}(\gamma)$. *In addition, in* $V^{\mathbb{P}}$, $\diamondsuit_{\mu}$ *holds*

---

for every μ which is inaccessible or the successor of a singular cardinal, and $\diamondsuit_\mu^+$ holds for every μ which is the successor of a regular cardinal.

Quoting from [4], in terminology used by Woodin, this theorem can be classified as an "inner model theorem proven via forcing." This is since the witnessing model satisfies pleasant properties one usually associates with an inner model, namely GCH and many instances of square and diamond, along with a property one might perhaps expect if a "nice" inner model containing supercompact cardinals ever were to be constructed (see [22] for a discussion of this topic), namely level by level equivalence between strong compactness and supercompactness. Theorem 1 can also be considered as a part of S. Friedman's "outer model program", as first described in [10].

The purpose of this paper is to prove two theorems in which we create via forcing models for the level by level equivalence between strong compactness and supercompactness satisfying both $V =$ HOD and additional inner model properties. In the first of these theorems, we prove a generalized version of Theorem 1 in which level by level equivalence between strong compactness and supercompactness and various versions of diamond and square are true. For the second of these theorems, we recall that *the Ground Axiom (GA)*, introduced in [11, 20], is the assertion that the set-theoretic universe V is not a forcing extension of any inner model $W \subseteq V$ by nontrivial set forcing $\mathbb{P} \in W$. Any canonical inner model, e.g., L, $L_{\mathcal{U}}$ for κ a measurable cardinal and $\mathcal{U}$ a normal measure over κ, $L[\vec{E}]$ for $\vec{E}$ a coherent sequence of extenders, etc., satisfies GA. In our second theorem, we construct a model in which both level by level equivalence between strong compactness and supercompactness and GA are true. Specifically, we will prove the following.

Theorem 2 *Let $V \vDash$ "ZFC + GCH + $\mathcal{K} \neq \emptyset$ is the class of supercompact cardinals". There is then a partial ordering $\mathbb{P} \subseteq V$ such that $V^{\mathbb{P}} \vDash$ "ZFC + GCH + $\mathcal{K}$ is the class of supercompact cardinals + Level by level equivalence between strong compactness and supercompactness holds + $V =$ HOD". In $V^{\mathbb{P}}$, $\square_\gamma^S$ holds for every infinite cardinal γ, where $S = \mathrm{Safe}(\gamma)$. In addition, in $V^{\mathbb{P}}$, $\diamondsuit_\mu$ holds for every regular cardinal μ, and $\diamondsuit_\mu^+$ holds for every μ such that $\mu = \lambda^+$, λ is regular, but λ is not the successor of a singular cardinal.*

Theorem 3 *Let $V \vDash$ "ZFC + GCH + $\mathcal{K} \neq \emptyset$ is the class of supercompact cardinals". Assume further that in V, there is a proper class of measurable cardinals. There is then a partial ordering $\mathbb{P} \subseteq V$ such that $V^{\mathbb{P}} \vDash$ "ZFC + GCH + $\mathcal{K}$ is the class of supercompact cardinals". In $V^{\mathbb{P}}$, there is a proper class of measurable cardinals. In addition, in $V^{\mathbb{P}}$, level by level equivalence between strong compactness and supercompactness, $V = \text{HOD}$, and GA all hold.*

We take this opportunity to make a few additional remarks concerning Theorems 2 and 3. Although the model witnessing the conclusions of Theorem 2 satisfies the same instances of the kind of square that Theorem 1 does, it does not satisfy exactly the same instances of the different versions of diamond. This is due to the coding partial ordering that we use to force $V = \text{HOD}$, which will decide generically where $\diamondsuit^*_\mu$ holds or fails whenever μ is the double successor of a singular cardinal. We will be able to infer that $\diamondsuit_\mu$ holds for every successor cardinal μ by Shelah's celebrated result of [21], since our forcing will preserve GCH. The coding oracle employed to force $V = \text{HOD}$ and GA in Theorem 3 requires a proper class of measurable cardinals, which is the reason for this additional hypothesis. Also, Theorem 3 is a generalization of [5, Theorem 4.2], where a model containing a proper class of supercompact cardinals in which GCH, $V = \text{HOD}$, and GA all hold is built via forcing.

We conclude Section 1 with a discussion of some preliminary material. Suppose V is a model of ZFC in which for all regular cardinals $\kappa < \lambda$, κ is λ strongly compact iff κ is λ supercompact, except possibly if κ is a measurable limit of cardinals δ which are λ supercompact. Such a universe will be said to witness *level by level equivalence between strong compactness and supercompactness*. For brevity, we will henceforth abbreviate this as just *level by level equivalence*. The exception is provided by a theorem of Menas [19], who showed that if κ is a measurable limit of cardinals δ which are λ strongly compact, then κ is λ strongly compact but need not be λ supercompact. Any model of ZFC with this property also witnesses the Kimchi-Magidor property [17] that the classes of strongly compact and supercompact cardinals coincide precisely, except at measurable limit points. Models in which GCH and level by level equivalence between strong compactness and supercompactness hold nontrivially were first constructed by Shelah and the author in [6].

We presume a basic knowledge of large cardinals and forcing. A good reference in this regard is [16]. When forcing, $q \geq p$ means that q *is stronger than* p. We will have some slight abuses of notation. In particular, when G is V-generic over $\mathbb{P}$, we take both $V[G]$ and $V^{\mathbb{P}}$ as being the generic extension of V by $\mathbb{P}$. We will also, from time to time, confuse terms with the sets they denote and write x when we actually mean $\dot{x}$ or $\check{x}$. For $\alpha < \beta$ ordinals, $[\alpha, \beta]$, $[\alpha, \beta)$, $(\alpha, \beta]$, and (α, β) are as in standard interval notation. For $\kappa < \lambda$ regular cardinals, $\mathrm{Coll}(\kappa, \lambda)$ is the usual Lévy collapse of all cardinals in the half-open interval $(\kappa, \lambda]$ to κ. For κ a regular cardinal and λ an ordinal, $\mathrm{Add}(\kappa, \lambda)$ is the standard partial ordering for adding λ many Cohen subsets of κ. The partial ordering $\mathbb{P}$ is κ-*directed closed* if every directed set of conditions of size less than κ has an upper bound.

We recall for the benefit of readers the definition given by Hamkins in [15, Section 3] of the lottery sum of a collection of partial orderings. If $\mathfrak{A}$ is a collection of partial orderings, then the *lottery sum* is the partial ordering $\oplus\mathfrak{A} = \{\langle \mathbb{P}, p \rangle \mid \mathbb{P} \in \mathfrak{A} \text{ and } p \in \mathbb{P}\} \cup \{0\}$, ordered with 0 below everything and $\langle \mathbb{P}, p \rangle \leq \langle \mathbb{P}', p' \rangle$ iff $\mathbb{P} = \mathbb{P}'$ and $p \leq p'$. Intuitively, if G is V-generic over $\oplus\mathfrak{A}$, then G first selects an element of $\mathfrak{A}$ (or as Hamkins says in [15], "holds a lottery among the posets in $\mathfrak{A}$") and then forces with it.[1]

A corollary of Hamkins' work of [12] on the approximation and cover properties (which is a generalization of his gap forcing results found in [13, 14]) will be employed in the proofs of Theorems 2 and 3. This corollary follows from [12, Theorems 3, 31, and Corollary 14]. We therefore state as a separate theorem what is relevant for this paper, along with some associated terminology, quoting from [13, 14, 12] when appropriate. Suppose $\mathbb{P}$ is a partial ordering which can be written as $\mathbb{Q} * \dot{\mathbb{R}}$, where $|\mathbb{Q}| \leq \delta$, $\mathbb{Q}$ is nontrivial, and $\Vdash_{\mathbb{Q}}$ "$\dot{\mathbb{R}}$ is δ^+-directed closed". In Hamkins' terminology of [12], $\mathbb{P}$ *admits a closure point at* δ. In Hamkins' terminology of [13, 14, 12], $\mathbb{P}$ is *mild with respect to a cardinal* κ iff every set of ordinals x in $V^{\mathbb{P}}$ of size below κ has a "nice" name τ in V of size below κ, i.e., there is a set y in V, $|y| < \kappa$, such that any ordinal forced by a condition in $\mathbb{P}$ to be in τ is an element of y. Also, as in the terminology of [13, 14, 12] and elsewhere, an embedding $j : V \to \overline{M}$

---

[1]The terminology "lottery sum" is due to Hamkins, although the concept of the lottery sum of partial orderings has been around for quite some time and has been referred to at different junctures via the names "disjoint sum of partial orderings," "side-by-side forcing," and "choosing which partial ordering to force with generically."

is *amenable to V* when $j \upharpoonright A \in V$ for any $A \in V$. The specific corollary of Hamkins' work from [12] we will be using is then the following.

Theorem 4 (Hamkins) *Suppose that $V[G]$ is a generic extension obtained by forcing with $\mathbb{P}$ that admits a closure point at some regular $\delta < \kappa$. Suppose further that $j : V[G] \rightarrow M[j(G)]$ is an elementary embedding with critical point κ for which $M[j(G)] \subseteq V[G]$ and $M[j(G)]^{\delta} \subseteq M[j(G)]$ in $V[G]$. Then $M \subseteq V$; indeed, $M = V \cap M[j(G)]$. If the full embedding j is amenable to $V[G]$, then the restricted embedding $j \upharpoonright V : V \rightarrow M$ is amenable to V. If j is definable from parameters (such as a measure or extender) in $V[G]$, then the restricted embedding $j \upharpoonright V$ is definable from the names of those parameters in V. Finally, if $\mathbb{P}$ is mild with respect to κ and κ is λ strongly compact in $V[G]$ for any $\lambda \geq \kappa$, then κ is λ strongly compact in V.*

It immediately follows from Theorem 4 that any cardinal κ which is λ supercompact in a generic extension obtained by forcing that admits a closure point below κ (such as at ω) must also be λ supercompact in the ground model. In particular, if $\overline{V}$ is a generic extension of V by a partial ordering admitting a closure point at ω in which each supercompact cardinal is preserved, the class of supercompact cardinals in $\overline{V}$ remains the same as in V. In addition, it follows from Theorem 4 that if $\mathbb{P}$ admits a closure point at ω, $\mathbb{P}$ is mild with respect to κ, and $V^{\mathbb{P}} \vDash$ "κ is λ strongly compact", then it is also true that $V \vDash$ "κ is λ strongly compact".

We end Section 1 by stating the definitions of the combinatorial notions we will be using. Readers may consult [4] for additional details, from which we will feel free to quote verbatim when appropriate. If κ is a regular uncountable cardinal, $\diamondsuit_{\kappa}$ is the principle stating that there exists a sequence of sets $\langle S_{\alpha} \mid \alpha < \kappa \rangle$ such that $S_{\alpha} \subseteq \alpha$, with the additional property that for every $X \subseteq \kappa$, $\{\alpha < \kappa \mid X \cap \alpha = S_{\alpha}\}$ is a stationary subset of κ. We recall that if λ is an uncountable cardinal

1. $\diamondsuit^{*}_{\lambda^{+}}$ is the assertion that there exists a sequence $\langle \mathcal{S}_{\alpha} \mid \alpha < \lambda^{+} \rangle$ such that

    (a) For every α, $\mathcal{S}_{\alpha}$ is a family of subsets of α with $|\mathcal{S}_{\alpha}| \leq \lambda$.

    (b) For every $X \subseteq \lambda^{+}$, there is $C \subseteq \lambda^{+}$ a club set such that $\forall \alpha \in C \; X \cap \alpha \in \mathcal{S}_{\alpha}$.

2. $\diamondsuit_{\lambda^+}^+$ is the assertion that there exists a sequence $\langle \mathcal{S}_\alpha \mid \alpha < \lambda^+ \rangle$ such that

    (a) For every α, $\mathcal{S}_\alpha$ is a family of subsets of α with $|\mathcal{S}_\alpha| \leq \lambda$.

    (b) For every $X \subseteq \lambda^+$, there is $C \subseteq \lambda^+$ a club set such $\forall \alpha \in C \ X \cap \alpha, C \cap \alpha \in \mathcal{S}_\alpha$.

We explicitly make the observation that trivially, by their definitions, $\diamondsuit_{\lambda^+}^+ \implies \diamondsuit_{\lambda^+}^*$. This of course means that if $\diamondsuit_{\lambda^+}^*$ fails, then so does $\diamondsuit_{\lambda^+}^+$. We will use this explicitly in our coding and in our proof of Lemma 2.3 showing that in our desired generic extension, $V = \mathrm{HOD}$.

We give a partial version of square, $\square_\kappa^S$ for κ an uncountable cardinal and $S \subseteq \kappa$ a set of regular cardinals, compatible with supercompact cardinals. As was mentioned in [4, Section 2.1], square sequences of this kind were first shown to be consistent with supercompactness by Foreman and Magidor [9, page 191], using techniques of Baumgartner. In the notation of Definition 1.1 found immediately below, they showed that $\square_{\kappa^{+\omega}}^{\{\kappa^{+n} \mid n < \omega\}}$ is consistent with κ being supercompact.

Given a set S of regular cardinals, we denote by $\mathrm{cof}(S)$ the class of ordinals α such that $\mathrm{cf}(\alpha) \in S$. We also let REG stand for the class of regular cardinals.

Definition 1.1 *Let κ be an infinite cardinal and let S be a set of regular cardinals which are less than or equal to κ. Then a $\square_\kappa^S$-sequence is a sequence $\langle C_\alpha \mid \alpha \in \kappa^+ \cap \mathrm{cof}(S) \rangle$ such that*

    1. C_α is club in α and $\mathrm{ot}(C_\alpha) \leq \kappa$.

    2. If $\beta \in \lim(C_\alpha) \cap \lim(C_{\alpha'})$ then $C_\alpha \cap \beta = C_{\alpha'} \cap \beta$.

    $\square_\kappa^S$ *holds if and only if there is a $\square_\kappa^S$-sequence.*

Definition 1.2 *For each infinite cardinal κ, a regular cardinal μ is safe for κ if and only if*

    1. $\mu \leq \kappa$.

    2. For every cardinal $\lambda \leq \kappa$, if λ is κ^+ supercompact then $\lambda \leq \mu$.

    $\mathrm{Safe}(\kappa)$ *is the set of safe regular cardinals for κ.*

We note that the safe set is a final segment of REG $\cap$ $(\kappa + 1)$, and that the safe set can only

be empty when κ is a singular limit of cardinals which are κ^+ supercompact. In addition, by the

remarks immediately following the statement of Theorem 4, Safe(γ) is upwards absolute to any

cardinal and cofinality preserving generic extension by a partial ordering admitting a closure point

at ω and preserving all regular instances of supercompactness.

2   The Proof of Theorem 2

We turn now to the proof of Theorem 2.

Proof: Suppose $V \vDash$ "ZFC + GCH + $\mathcal{K} \neq \emptyset$ is the class of supercompact cardinals". By Theorem

1, we may assume in addition that in V, level by level equivalence holds, and that $\square_\kappa^S$ holds for

every infinite cardinal κ, where $S = \mathrm{Safe}(\kappa)$.

We are in a position to define the partial ordering $\mathbb{P}$ used in the proof of Theorem 2. We pattern

our definition after the iteration found in [4, Section 3.2]. First, for λ a cardinal, let $\mathbb{Q}^\diamond(\lambda^+)$ be the

partial ordering of [4, page 71] which adds $\diamondsuit_{\lambda^+}^+$. For the exact definition, we refer readers to [4].

We note only that $\mathbb{Q}^\diamond(\lambda^+)$ is λ^+-directed closed and λ^{++}-c.c. $\mathbb{P} = \langle\langle \mathbb{P}_\delta, \dot{\mathbb{Q}}_\delta \rangle \mid \delta \in \mathrm{ORD}\rangle$ may now

be defined as the proper class Easton support iteration which begins by forcing with $\mathrm{Add}(\omega, 1)$

(i.e., $\mathbb{P}_0 = \{\emptyset\}$ and $\mathbb{Q}_0 = \mathrm{Add}(\omega, 1)$) so as to ensure a closure point at ω. We then let $\dot{\mathbb{Q}}_\delta$ be a term

for

1. $\mathrm{Add}(\delta, 1)$ for δ inaccessible.

2. The lottery sum of $\mathrm{Add}(\delta, \delta^+)$ and $\mathbb{Q}^\diamond(\delta)$ at cardinals δ where $\delta = \lambda^+$ for λ the successor of
   a singular cardinal (so in particular, $\lambda = \rho^+$ and $\delta = \rho^{++}$ where ρ is a singular cardinal).

3. The partial ordering $\mathbb{Q}^\diamond(\delta)$ at cardinals δ where $\delta = \lambda^+$ and λ is a regular cardinal which is
   not the successor of a singular cardinal (so in particular, either λ is inaccessible, or $\lambda = \rho^+$
   where ρ is a regular cardinal).

4. Trivial forcing if δ does not fall into any of the above three categories (so in particular, trivial
   forcing occurs if either δ is a singular cardinal or $\delta = \lambda^+$ and λ is a singular cardinal).

As in [4], we need to show that forcing with $\mathbb{P}$ preserves all regular instances of supercompactness. Standard arguments show that $V^{\mathbb{P}} \vDash$ ZFC and that forcing with $\mathbb{P}$ preserves all cardinals, cofinalities, and GCH. In addition, in analogy to what was done in [4], using the arguments of [1, Lemma 1.1] and [8, Theorem 12.2], it is easily verified that in $V^{\mathbb{P}}$, $\diamondsuit_\mu$ holds for every μ which is inaccessible, and $\diamondsuit_\mu^+$ holds at cardinals μ where $\mu = \lambda^+$ for λ a regular cardinal which is not the successor of a singular cardinal. Since GCH holds in $V^{\mathbb{P}}$, by Shelah's work of [21], $\diamondsuit_\mu$ holds in $V^{\mathbb{P}}$ for every successor cardinal μ and hence holds in $V^{\mathbb{P}}$ for every regular cardinal μ.

Lemma 2.1 *If $V \vDash$ "$\kappa < \lambda$ are such that κ is λ supercompact and λ is regular", then $V^{\mathbb{P}} \vDash$ "κ is λ supercompact".*

Proof: Let $\kappa < \lambda$ be as in the hypotheses of Lemma 2.1. If λ is either inaccessible or $\lambda = \delta^+$ where δ is a singular cardinal, then the argument that $V^{\mathbb{P}} \vDash$ "κ is λ supercompact" is exactly the same as in [4, Theorem 5, bottom of page 73 up to and including the first three paragraphs on page 74]. If $\lambda = \delta^+$ where δ is a regular cardinal which is not the successor of a singular cardinal, then the argument that $V^{\mathbb{P}} \vDash$ "κ is λ supercompact" is exactly the same as in [4, Theorem 5, fourth paragraph on page 74 through the end of the proof on page 76]. Suppose now that $\lambda = \delta^+$ where δ is the successor of a singular cardinal. Assume that $\mathbb{Q}_\lambda^\diamondsuit$ is chosen in the stage λ lottery held in the definition of $\mathbb{P}$, and that we are forcing above $p_0 \in \mathbb{P}$ which forces that this is indeed the case. The argument that $V^{\mathbb{P}} \vDash$ "κ is λ supercompact" is then once again exactly the same as in [4, Theorem 5, fourth paragraph on page 74 through the end of the proof on page 76]. If $\mathrm{Add}(\lambda, \lambda^+)$ is chosen in the stage λ lottery held in the definition of $\mathbb{P}$, then the argument that $V^{\mathbb{P}} \vDash$ "κ is λ supercompact" combines standard techniques with ideas originally due to Magidor [18]. Variants of Magidor's method are also given in [6, pages 119–120] as well as [2, Lemma 2.4] and are found other places in the literature as well. However, since the proof requires certain modifications from its original version, we present it completely below.

Getting specific, suppose $j : V \to M$ is an elementary embedding witnessing the λ supercompactness of κ which is generated by a supercompact ultrafilter over $P_\kappa(\lambda)$. Assume we are forcing above $p_1 \in \mathbb{P}$ which forces $\mathrm{Add}(\lambda, \lambda^+)$ to be chosen in the stage λ lottery held in the definition of

$\mathbb{P}$. This allows us (with a slight abuse of notation) to write $\mathbb{P} = \mathbb{P}_\kappa * \dot{\mathbb{P}}(\kappa, \lambda) * \dot{\mathrm{Add}}(\lambda, \lambda^+) * \dot{\mathbb{Q}}$, where $\dot{\mathbb{P}}(\kappa, \lambda)$ is a term for the portion of $\mathbb{P}$ acting on ordinals in the half-open interval $[\kappa, \lambda)$ and $\dot{\mathbb{Q}}$ is a term for the portion of $\mathbb{P}$ acting on ordinals above λ. Let G_0 be V-generic over $\mathbb{P}_\kappa$, G_1 be $V[G_0]$-generic over $\mathbb{P}(\kappa, \lambda)$, and G_3 be $V[G_0][G_1]$-generic over $\mathrm{Add}(\lambda, \lambda^+)$. Write $j(\mathbb{P}_\kappa * \dot{\mathbb{P}}(\kappa, \lambda) * \dot{\mathrm{Add}}(\lambda, \lambda^+)) = \mathbb{P}_\kappa * \dot{\mathbb{P}}(\kappa, \lambda) * \dot{\mathrm{Add}}(\lambda, \lambda^+)) * \dot{\mathbb{R}} * \dot{\mathbb{P}}(j(\kappa), j(\lambda)) * \dot{\mathrm{Add}}(j(\lambda), j(\lambda^+))$, where $\dot{\mathbb{R}}$ is a term for the portion of $j(\mathbb{P})$ acting on ordinals in the open interval $(\lambda, j(\kappa))$. We construct in $V[G_0][G_1][G_2]$ an $M[G_0][G_1][G_2]$-generic object G_3 over $\mathbb{R}$, an $M[G_0][G_1][G_2][G_3]$-generic object G_4 over $\mathbb{P}(j(\kappa), j(\lambda))$, and an $M[G_0][G_1][G_2][G_3][G_4]$-generic object G_5 over $(\mathrm{Add}(j(\lambda), j(\lambda^+))^{M[G_0][G_1][G_2][G_3][G_4]}$ such that $j''(G_0 * G_1 * G_2) \subseteq G_0 * G_1 * G_2 * G_3 * G_4 * G_5$. This allows us to lift $j : V \to M$ in $V[G_0][G_1][G_2]$ to the elementary embedding $j : V[G_0][G_1][G_2] \to M[G_0][G_1][G_2][G_3][G_4][G_5]$ witnessing the λ supercompactness of κ in $V[G_0][G_1][G_2]$. Since $\Vdash_{\mathbb{P}_\kappa * \dot{\mathbb{P}}(\kappa, \lambda) * \mathrm{Add}(\lambda, \lambda^+)}$ "$\dot{\mathbb{Q}}$ is λ^+-directed closed", this will suffice to show that $V^{\mathbb{P}} \vDash$ "κ is λ supercompact".

We begin by observing that since $\mathbb{P}_\kappa * \dot{\mathbb{P}}(\kappa, \lambda) * \dot{\mathrm{Add}}(\lambda, \lambda^+)$ is λ^+-c.c., $M[G_0][G_1][G_2]$ remains λ-closed with respect to $V[G_0][G_1][G_2]$. In V, since M is given via an ultrapower by a normal measure over $P_\kappa(\lambda)$, $|j(\kappa)|$ and $|j(\kappa^+)|$ may be calculated as $|\{f \mid f : P_\kappa(\lambda) \to \kappa\}| = |\{f \mid f : \lambda \to \kappa\}| = 2^\lambda = \lambda^+$ and $|\{f \mid f : P_\kappa(\lambda) \to \kappa^+\}| = |\{f \mid f : \lambda \to \kappa^+\}| = 2^\lambda = \lambda^+$ respectively. Also, by elementarity, since $V \vDash$ "$2^\kappa = \kappa^+$", $M \vDash$ "$2^{j(\kappa)} = (j(\kappa))^+ = j(\kappa^+)$". In addition, since $M[G_0][G_1][G_2] \vDash$ "$\mathbb{R}$ is an Easton support iteration of length $j(\kappa)$", $M[G_0][G_1][G_2] \vDash$ "$|\mathbb{R}| = j(\kappa)$ and $2^{j(\kappa)} = (j(\kappa))^+ = j(\kappa^+)$". This means the number of dense open subsets of $\mathbb{R}$ present in $M[G_0][G_1][G_2]$ is $j(\kappa^+)$. Further, as $M[G_0][G_1][G_2] \vDash$ "$\mathbb{R}$ is λ^+-directed closed" and $M[G_0][G_1][G_2]$ is λ-closed with respect to $V[G_0][G_1][G_2]$, $\mathbb{R}$ is λ^+-directed closed in $V[G_0][G_1][G_2]$ as well. Since λ^+ is preserved from V to $V[G_0][G_1][G_2]$, we may let $\langle D_\beta \mid \beta < \lambda^+\rangle \in V[G_0][G_1][G_2]$ enumerate the dense open subsets of $\mathbb{R}$ present in $M[G_0][G_1][G_2]$. We may now use the fact that $\mathbb{R}$ is λ^+-directed closed in $V[G_0][G_1][G_2]$ to meet each D_β and thereby construct in $V[G_0][G_1][G_2]$ an $M[G_0][G_1][G_2]$-generic object G_3 over $\mathbb{R}$. Our construction guarantees that $j''G_0 \subseteq G_0 * G_1 * G_2 * G_3$, so j lifts in $V[G_0][G_1][G_2]$ to $j : V[G_0] \to M[G_0][G_1][G_2][G_3]$.

By the fact that $\mathbb{R}$ is λ^+-directed closed in $M[G_0][G_1][G_2]$, $M[G_0][G_1][G_2][G_3]$ remains λ-

closed with respect to $V[G_0][G_1][G_2][G_3] = V[G_0][G_1][G_2]$. Therefore, since $V[G_0] \vDash$ "$|\mathbb{P}(\kappa, \lambda)| < \lambda$", $M[G_0][G_1][G_2][G_3] \vDash$ "$|\mathbb{P}(j(\kappa), j(\lambda))| < j(\lambda)$" and $j''G_1 \in M[G_0][G_1][G_2][G_3]$. Also, by its definition, $M[G_0][G_1][G_2][G_3] \vDash$ "$\mathbb{P}(j(\kappa), j(\lambda))$ is $j(\kappa)$-directed closed", so since $j(\kappa) > \lambda^+$, $M[G_0][G_1][G_2][G_3] \vDash$ "$\mathbb{P}(j(\kappa), j(\lambda))$ is λ^+-directed closed" as well. Since it is once again the case that $V[G_0][G_1][G_2] \vDash$ "$\mathbb{P}(j(\kappa), j(\lambda))$ is λ^+-directed closed", this means we can let q^* be a master condition for $j''G_1$ and use the construction mentioned in the previous paragraph to build in $V[G_0][G_1][G_2]$ an $M[G_0][G_1][G_2][G_3]$-generic object G_4 for $\mathbb{P}(j(\kappa), j(\lambda))$ such that $q^* \in G_4$. Our construction guarantees that $j''(G_0 * G_1) \subseteq G_0 * G_1 * G_2 * G_3 * G_4$, so we may lift j to $j : V[G_0][G_1] \to M[G_0][G_1][G_2][G_3][G_4]$. Once again, because $M[G_0][G_1][G_2][G_3] \vDash$ "$\mathbb{P}(j(\kappa), j(\lambda))$ is λ^+-directed closed", $M[G_0][G_1][G_2][G_3][G_4]$ remains λ-closed with respect to $V[G_0][G_1][G_2][G_3][G_4] = V[G_0][G_1][G_2]$.

We now use Magidor's methods mentioned above to construct in $V[G_0][G_1][G_2]$ an $M[G_0][G_1][G_2][G_3][G_4]$-generic object G_5 over $(\mathrm{Add}(j(\lambda), j(\lambda^+))^{M[G_0][G_1][G_2][G_3][G_4]}$ such that $j''(G_0 * G_1 * G_2) \subseteq G_0 * G_1 * G_2 * G_3 * G_4 * G_5$. We follow the presentation given in the proof of [2, Lemma 2.4]. For $\zeta \in (\lambda, \lambda^+)$ and $p \in (\mathrm{Add}(\lambda, \lambda^+))^{V[G_0][G_1]}$, let $p \upharpoonright \zeta = \{\langle\langle \rho, \sigma \rangle, \eta \rangle \in p \mid \sigma < \zeta\}$ and $G_2 \upharpoonright \zeta = \{p \upharpoonright \zeta \mid p \in G_2\}$. Clearly, $V[G_0][G_1][G_2] \vDash$ "$|G_2 \upharpoonright \zeta| \le \lambda$ for all $\zeta \in (\lambda, \lambda^+)$". Thus, since $\mathrm{Add}(j(\lambda), j(\lambda^+))^{M[G_0][G_1][G_2][G_3][G_4]}$ is $j(\lambda)$-directed closed and $j(\lambda) > \lambda^+$, $q_\zeta = \bigcup\{j(p) \mid p \in G_2 \upharpoonright \zeta\}$ is well-defined and is an element of $(\mathrm{Add}(j(\lambda), j(\lambda^+)))^{M[G_0][G_1][G_2][G_3][G_4]}$. Further, if $\langle \rho, \sigma \rangle \in \mathrm{dom}(q_\zeta) - \mathrm{dom}(\bigcup_{\beta < \zeta} q_\beta)$ ($\bigcup_{\beta < \zeta} q_\beta$ is well-defined by closure), then $\sigma \in [\bigcup_{\beta < \zeta} j(\beta), j(\zeta))$. To see this, assume to the contrary that $\sigma < \bigcup_{\beta < \zeta} j(\beta)$. Let β be minimal such that $\sigma < j(\beta)$. It must thus be the case that for some $p \in G_2 \upharpoonright \zeta$, $\langle \rho, \sigma \rangle \in \mathrm{dom}(j(p))$. Since by elementarity and the definitions of $G_2 \upharpoonright \beta$ and $G_2 \upharpoonright \zeta$, for $p \upharpoonright \beta = q \in G_2 \upharpoonright \beta$, $j(q) = j(p) \upharpoonright j(\beta) = j(p \upharpoonright \beta)$, it must be the case that $\langle \rho, \sigma \rangle \in \mathrm{dom}(j(q))$. This means $\langle \rho, \sigma \rangle \in \mathrm{dom}(q_\beta)$, a contradiction.

Since GCH is preserved to $M[G_0][G_1][G_2][G_3][G_4]$, an application of [16, Lemma 15.4, page 227] shows that $M[G_0][G_1][G_2][G_3][G_4] \vDash$ "$\mathrm{Add}(j(\lambda), j(\lambda^+))$ is $j(\lambda^+)$-c.c. and has $j(\lambda^+)$ many maximal antichains". This means that if $\mathcal{A} \in M[G_0][G_1][G_2][G_3][G_4]$ is a maximal antichain of $\mathrm{Add}(j(\lambda), j(\lambda^+))$, $\mathcal{A} \subseteq \mathrm{Add}(j(\lambda), \beta)$ for some $\beta \in (j(\lambda), j(\lambda^+))$. Thus, since $V \vDash$ "$|j(\lambda^+)| =$

$|\{f \mid f : P_\kappa(\lambda) \to \lambda^+\}| = |\{f \mid f : \lambda \to \lambda^+\}| = |[\lambda^+]^\lambda| = \lambda^{+"}$, we can let $\langle \mathcal{A}_\zeta \mid \zeta \in (\lambda, \lambda^+) \rangle \in V[G_0][G_1][G_2]$ be an enumeration of all of the maximal antichains of $\mathrm{Add}(j(\lambda), j(\lambda^+))$ present in $M[G_0][G_1][G_2][G_3][G_4]$.

We define now an increasing sequence $\langle r_\zeta \mid \zeta \in (\lambda, \lambda^+) \rangle$ of elements of $\mathrm{Add}(j(\lambda), j(\lambda^+))$ such that $\forall \zeta \in (\lambda, \lambda^+)[r_\zeta \geq q_\zeta$ and $r_\zeta \in \mathrm{Add}(j(\lambda), j(\zeta))]$ and such that $\forall \mathcal{A} \in \langle \mathcal{A}_\zeta \mid \zeta \in (\lambda, \lambda^+) \rangle \exists \beta \in (\lambda, \lambda^+) \exists r \in \mathcal{A}[r_\beta \geq r]$. Assuming we have such a sequence, $G_5 = \{p \in \mathrm{Add}(j(\lambda), j(\lambda^+)) \mid \exists r \in \langle r_\zeta \mid \zeta \in (\lambda, \lambda^+) \rangle[r \geq p]\}$ is an $M[G_0][G_1][G_2][G_3][G_4]$-generic object over $(\mathrm{Add}(j(\lambda), j(\lambda^+)))^{M[G_0][G_1][G_2][G_3][G_4]}$. To define $\langle r_\zeta \mid \zeta \in (\lambda, \lambda^+) \rangle$, if ζ is a limit, we let $r_\zeta = \bigcup_{\beta \in (\lambda, \zeta)} r_\beta$. By the facts $\langle r_\beta \mid \beta \in (\lambda, \zeta) \rangle$ is (strictly) increasing and $M[G_0][G_1][G_2][G_3][G_4]$ is λ-closed with respect to $V[G_0][G_1][G_2]$, this definition is valid. Assuming now r_ζ has been defined and we wish to define $r_{\zeta+1}$, let $\langle \mathcal{B}_\beta \mid \beta < \eta \leq \lambda \rangle$ be the subsequence of $\langle \mathcal{A}_\beta \mid \beta \leq \zeta + 1 \rangle$ containing each antichain $\mathcal{A}$ such that $\mathcal{A} \subseteq \mathrm{Add}(j(\lambda), j(\zeta + 1))$. Because $q_\zeta, r_\zeta \in \mathrm{Add}(j(\lambda), j(\zeta))$, $q_{\zeta+1} \in \mathrm{Add}(j(\lambda), j(\zeta + 1))$, and $j(\zeta) < j(\zeta + 1)$, the condition $r'_{\zeta+1} = r_\zeta \cup q_{\zeta+1}$ is well-defined. This is since by our earlier observations, any new elements of $\mathrm{dom}(q_{\zeta+1})$ won't be present in either $\mathrm{dom}(q_\zeta)$ or $\mathrm{dom}(r_\zeta)$. We can thus, using the fact $M[G_0][G_1][G_2][G_3][G_4]$ is λ-closed with respect to $V[G_0][G_1][G_2]$, define by induction an increasing sequence $\langle s_\beta \mid \beta < \eta \rangle$ such that $s_0 \geq r'_{\zeta+1}$, $s_\rho = \bigcup_{\beta < \rho} s_\beta$ if ρ is a limit ordinal, and $s_{\beta+1} \geq s_\beta$ is such that $s_{\beta+1}$ extends some element of $\mathcal{B}_\beta$. The just mentioned closure fact implies $r_{\zeta+1} = \bigcup_{\beta < \eta} s_\beta$ is a well-defined condition.

In order to show that G_5 is $M[G_0][G_1][G_2][G_3][G_4]$-generic over $\mathrm{Add}(j(\lambda), j(\lambda^+))$, we must show that $\forall \mathcal{A} \in \langle \mathcal{A}_\zeta \mid \zeta \in (\lambda, \lambda^+) \rangle \exists \beta \in (\lambda, \lambda^+) \exists r \in \mathcal{A}[r_\beta \geq r]$. To do this, we first note that $\langle j(\zeta) \mid \zeta < \lambda^+ \rangle$ is unbounded in $j(\lambda^+)$. To see this, if $\beta < j(\lambda^+)$ is an ordinal, then for some $f : P_\kappa(\lambda) \to M$ representing β, we can assume that for $p \in P_\kappa(\lambda)$, $f(p) < \lambda^+$. Thus, by the regularity of λ^+ in V and the fact that $V \vDash "|P_\kappa(\lambda)| = \lambda"$, $\beta_0 = \bigcup_{\rho < \lambda} f(\rho) < \lambda^+$, and $j(\beta_0) > \beta$. This means by our earlier remarks that if $\mathcal{A} \in \langle \mathcal{A}_\zeta \mid \zeta < \lambda^+ \rangle$, $\mathcal{A} = \mathcal{A}_\rho$, then we can let $\beta \in (\lambda, \lambda^+)$ be such that $\mathcal{A} \subseteq \mathrm{Add}(j(\lambda), j(\beta))$. By construction, for $\eta > \max(\beta, \rho)$, there is some $r \in \mathcal{A}$ such that $r_\eta \geq r$. And, as any $p \in \mathrm{Add}(\lambda, \lambda^+)$ is such that for some $\zeta \in (\lambda, \lambda^+)$, $p = p \upharpoonright \zeta$, G_5 is such that if $p \in G_2$, $j(p) \in G_5$. This means that $j''(G_0 * G_1 * G_2) \subseteq G_0 * G_1 * G_2 * G_3 * G_4 * G_5$, thereby

completing the proof of Lemma 2.1.

$\square$

Lemma 2.1 immediately implies that all V-supercompact cardinals are preserved to $V^{\mathbb{P}}$. By its definition, we may write $\mathbb{P} = \mathbb{P}' * \dot{\mathbb{P}}''$, where $|\mathbb{P}'| = \omega$, $\mathbb{P}' = \mathrm{Add}(\omega, 1)$ is nontrivial, and $\Vdash_{\mathbb{P}'}$ "$\dot{\mathbb{P}}''$ is $\aleph_1$-directed closed". Therefore, by our remarks in the paragraph immediately following Theorem 4, $\mathcal{K}$ remains the class of supercompact cardinals in $V^{\mathbb{P}}$. Further, by the upwards absoluteness of any form of square in a cardinal preserving generic extension (see the discussion given in the proof of [1, Theorem 1]) and our remarks in the last paragraph of Section 1, in $V^{\mathbb{P}}$, $\square_\gamma^S$ holds for every infinite cardinal γ, where $S = \mathrm{Safe}(\gamma)$.

Lemma 2.2 $V^{\mathbb{P}} \vDash$ "*Level by level equivalence holds*".

Proof: We mimic the proofs of [1, Lemma 1.3] and [4, Lemma 4.1]. Suppose $V^{\mathbb{P}} \vDash$ "$\kappa < \lambda$ are regular cardinals such that κ is λ strongly compact and κ isn't a measurable limit of cardinals δ which are λ supercompact". By Lemma 2.1, any cardinal δ such that δ is λ supercompact in V remains λ supercompact in $V^{\mathbb{P}}$. We may therefore infer that $V \vDash$ "$\kappa < \lambda$ are regular cardinals such that κ isn't a measurable limit of cardinals δ which are λ supercompact".

By the definition of $\mathbb{P}$, it is easily seen that $\mathbb{P}$ is mild with respect to κ. Hence, by the factorization of $\mathbb{P}$ given above and Theorem 4, $V \vDash$ "κ is λ strongly compact". Consequently, by level by level equivalence between strong compactness and supercompactness in V, $V \vDash$ "κ is λ supercompact", so another application of Lemma 2.1 yields that $V^{\mathbb{P}} \vDash$ "κ is λ supercompact". This completes the proof of Lemma 2.2.

$\square$

Lemma 2.3 $V^{\mathbb{P}} \vDash$ "$V = \mathrm{HOD}$".

Proof: We use ideas given by Brooke-Taylor in [7, Theorem 9] (see also the proof of [5, Lemma 4.3]). In particular, to show that $V^{\mathbb{P}} \vDash$ "$V = \mathrm{HOD}$", it will suffice to show that every set of ordinals

in $V^{\mathbb{P}}$ is ordinal definable using a coding oracle given by where $\diamondsuit^*_{\kappa^{++}}$ holds or fails for κ a singular cardinal.

To do this, suppose $p \in \mathbb{P}$, $\dot{x}$, and α are such that $p \Vdash$ "$\dot{x} \subseteq \alpha$" and that α is a limit ordinal. By the definition of $\mathbb{P}$, there must exist some ordinal β such that $p \Vdash$ "$\dot{x} \in V^{\mathbb{P}_\beta}$". Also, since $\mathbb{P}$ is an Easton support proper class length iteration, there must be some ordinal γ such that support$(p) \subseteq \gamma$. This allows us to write $p = \langle p_\sigma \mid \sigma < \gamma \rangle$. Let $\delta > \max(\alpha, \beta, \gamma)$ be fixed but arbitrary, with $\langle \delta_\sigma \mid \sigma < \alpha \rangle$ the first α many singular cardinals greater than δ. Define $\rho = \sup(\{\delta_\sigma \mid \sigma < \alpha\})$. Take $q > p$, $q = \langle q_\sigma \mid \sigma < \rho \rangle$ as the condition such that $q_\sigma = p_\sigma$ for $\sigma < \gamma$. For $\sigma \geq \gamma$, $\sigma < \rho$, q_σ is the trivial condition, except if $\zeta < \alpha$ and $\sigma = \delta_\zeta^{++}$. At such a σ, q_σ is defined as the term such that $\Vdash_{\mathbb{P}_\sigma}$ "$q_\sigma \in \dot{\mathbb{Q}}_\sigma$ forces that $\dot{\mathbb{Q}}^{\diamondsuit}(\sigma)$ is chosen in the stage σ lottery if $\zeta \in \dot{x}$, but $\mathrm{Add}(\sigma, \sigma^+)$ is chosen in the stage σ lottery if $\zeta \notin \dot{x}$". The definition of $\mathbb{P}$ tells us that in $V^{\mathbb{P}}$, $\diamondsuit^+_\sigma$, and hence also $\diamondsuit^*_\sigma$, both hold if $\mathbb{Q}^{\diamondsuit}(\sigma)$ is chosen in the stage σ lottery. On the other hand, the arguments found in [7, paragraph immediately preceding Section 3, pages 644–645] combined with the closure properties of $\mathbb{P}$ tell us that $\diamondsuit^*_\sigma$ fails if $\mathrm{Add}(\sigma, \sigma^+)$ is chosen in the stage σ lottery. This means that as in the proofs of [7, Theorem 9] and [5, Lemma 4.3], the proper class of conditions forcing that the set x is coded using the coding oracle mentioned in the first paragraph of the proof of this lemma is dense in $\mathbb{P}$. Since x is an arbitrary set of ordinals, again as in the proofs of [7, Theorem 9] and [5, Lemma 4.3], $V^{\mathbb{P}} \vDash$ "$V = \mathrm{HOD}$". This completes the proof of Lemma 2.3.

$\square$

Lemmas 2.1 – 2.3, the paragraph immediately preceding the proof of Lemma 2.1, and the intervening remarks complete the proof of Theorem 2.

$\square$

Turning now to the proof of Theorem 3, suppose $V \vDash$ "ZFC + GCH + $\mathcal{K} \neq \emptyset$ is the class of supercompact cardinals". Assume in addition that in V, there is a proper class of measurable cardinals. By [3, Theorem 1], we may further assume that V satisfies level by level equivalence and that every measurable cardinal κ in V carries $2^{2^{\kappa}} = \kappa^{++}$ many normal measures (which is of

course the maximal number of normal measures a V-measurable cardinal can carry).

We pattern the definition of the partial ordering used in the proof of Theorem 3 after the definition of the forcing conditions $\mathbb{P}^0$ used in the proof of [3, Theorem 2]. Specifically, let $\langle \delta_\alpha \mid \alpha \in \mathrm{Ord} \rangle$ enumerate all V-measurable cardinals which are not themselves limits of V-measurable cardinals. By our assumption that V contains a proper class of measurable cardinals, such an enumeration is possible. The partial ordering $\mathbb{P} = \langle \langle \mathbb{P}_\delta, \dot{\mathbb{Q}}_\delta \rangle \mid \delta \in \mathrm{Ord} \rangle$ may now be defined as the proper class Easton support iteration which begins by forcing with $\mathrm{Add}(\omega, 1)$ so as once again to ensure a closure point at ω. $\dot{\mathbb{Q}}_\delta$ is then a term for trivial forcing, except if δ is such that $\delta = \sigma^+$ where σ is a V-measurable cardinal which is not a limit of V-measurable cardinals. In this case, $\dot{\mathbb{Q}}_\delta$ is a term for the lottery sum of trivial forcing and $\mathrm{Coll}(\delta^+, \delta^{++})$.

As in the proof of Theorem 2, standard arguments show $V^{\mathbb{P}} \vDash \mathrm{ZFC}$ and that forcing with $\mathbb{P}$ preserves GCH. In addition, the arguments in the proof of [5, Theorem 4.2, pages 369–371] show that $V^{\mathbb{P}} \vDash$ "$\mathcal{K}$ is the class of supercompact cardinals + There is a proper class of measurable cardinals + $V = \mathrm{HOD} + \mathrm{GA}$". Thus, the proof of Theorem 3 is completed by the following lemma.

Lemma 2.4 $V^{\mathbb{P}} \vDash$ "*Level by level equivalence holds*".

Proof: To prove Lemma 2.4, we will show that if $\kappa < \lambda$ are such that $V^{\mathbb{P}} \vDash$ "κ is λ strongly compact and λ is regular", then $V^{\mathbb{P}} \vDash$ "Either κ is λ supercompact, or κ is a measurable limit of cardinals δ which are λ supercompact". Towards this end, suppose $\kappa < \lambda$ are as just stated. Write $\mathbb{P} = \mathbb{P}' * \dot{\mathbb{P}}''$, where $|\mathbb{P}'| = \omega$, $\mathbb{P}' = \mathrm{Add}(\omega, 1)$ is nontrivial, and $\Vdash_{\mathbb{P}'}$ "$\dot{\mathbb{P}}''$ is $\aleph_1$-directed closed". This factorization, combined with $\mathbb{P}$'s definition, yield that $\mathbb{P}$ admits a closure point at ω and is mild with respect to κ. Therefore, by Theorem 4, $V \vDash$ "κ is λ strongly compact".

We consider now two cases.

Case 1: λ is a trivial stage of forcing, i.e., either $\lambda = \sigma^+$, where σ is in V a measurable cardinal which is not a limit of measurable cardinals and trivial forcing is chosen in the stage σ^+ lottery held in the definition of $\mathbb{P}$, or $\lambda \neq \sigma^+$, where σ is in V a measurable cardinal which is not a limit of measurable cardinals. Because level by level equivalence holds in V, we may consequently

14

infer that $V \vDash$ "Either κ is λ supercompact, or κ is a measurable limit of cardinals δ which are λ supercompact". We will prove that $V^{\mathbb{P}} \vDash$ "Either κ is λ supercompact, or κ is a measurable limit of cardinals δ which are λ supercompact". This means we can now let $\delta \leq \kappa$ be such that $V \vDash$ "δ is λ supercompact". We will show that $V^{\mathbb{P}} \vDash$ "δ is λ supercompact". It will then follow that $V^{\mathbb{P}} \vDash$ "Either κ is λ supercompact, or κ is a measurable limit of cardinals δ which are λ supercompact".

Towards this end, write $\mathbb{P} = \mathbb{P}_\delta * \dot{\mathbb{P}}(\delta, \lambda) * \dot{\mathbb{P}}^\lambda$, where $\dot{\mathbb{P}}(\delta, \lambda)$ is a term for the portion of $\mathbb{P}$ such that $\mathbb{P}_\delta * \dot{\mathbb{P}}(\delta, \lambda) = \mathbb{P}_{\lambda+1}$ and $\dot{\mathbb{P}}^\lambda$ is a term for the rest of $\mathbb{P}$. It is then the case that $\Vdash_{\mathbb{P}_\delta * \dot{\mathbb{P}}(\delta,\lambda)}$ "$\dot{\mathbb{P}}^\lambda$ is λ^+-directed closed". Thus, to prove that $V^{\mathbb{P}} \vDash$ "δ is λ supercompact", it will suffice to show that $V^{\mathbb{P}_\delta * \dot{\mathbb{P}}(\delta,\lambda)} \vDash$ "δ is λ supercompact".

To do this, let G be V-generic over $\mathbb{P}_\delta * \dot{\mathbb{P}}(\delta, \lambda)$, with $G = G_0 * G_1$ the corresponding factorization of G. With a slight abuse of notation, by assuming we are forcing above the appropriate condition and using the fact that GCH is preserved to $V^{\mathbb{P}_\alpha}$ for any ordinal α, the definition of $\mathbb{P}$ immediately allows us to infer that $V[G_0] \vDash$ "$|\mathbb{P}(\delta, \lambda)| \leq \lambda$". Let $j : V \to M$ be an elementary embedding witnessing the λ supercompactness of δ which is generated by a supercompact ultrafilter over $P_\delta(\lambda)$. Write $j(\mathbb{P}_\delta * \dot{\mathbb{P}}(\delta, \lambda)) = \mathbb{P}_\delta * \dot{\mathbb{P}}(\delta, \lambda) * \dot{\mathbb{R}} * j(\dot{\mathbb{P}}(\delta, \lambda)) = \mathbb{P}_\delta * \dot{\mathbb{P}}(\delta, \lambda) * \dot{\mathbb{R}} * \dot{\mathbb{P}}(j(\delta), j(\lambda))$. This means we may use the arguments given in the proof of Lemma 2.1 in the construction of the generic objects G_3 and G_4 to build in $V[G_0][G_1]$ an $M[G_0][G_1]$-generic object G_2 for $\mathbb{R}$ and an $M[G_0][G_1][G_2]$-generic object G_3^* for $\mathbb{P}(j(\delta), j(\lambda))$ and lift in $V[G_0][G_1]$ the embedding j to $j : V[G_0][G_1] \to M[G_0][G_1][G_2][G_3^*]$. The lifted j is then a witness to the λ supercompactness of δ in $V[G_0][G_1]$.

Case 2: λ is a nontrivial stage of forcing, i.e., $\lambda = \sigma^+$, where σ is in V a measurable cardinal which is not a limit of measurable cardinals and $\mathrm{Coll}(\sigma^+, \sigma^{++})$ is chosen in the stage σ^+ lottery held in the definition of $\mathbb{P}$. Then because $V^{\mathbb{P}} \vDash$ "$|(\sigma^{++})^V| = \sigma^+ = (\sigma^+)^V$ and κ is $\lambda = \sigma^+$ strongly compact", we know that $V^{\mathbb{P}} \vDash$ "κ is $(\sigma^{++})^V$ strongly compact" as well. Thus, by the remarks in the paragraph immediately following Theorem 4, $V \vDash$ "κ is $\sigma^{++} = \lambda^+$ strongly compact". Hence, by level by level equivalence, $V \vDash$ "Either κ is λ^+ supercompact, or κ is a measurable limit of cardinals δ which are

λ^+ supercompact". As in Case 1, let $\delta \leq \kappa$ be such that $V \vDash$ "δ is λ^+ supercompact". We adopt also the same notation as in Case 1. In analogy to Case 1, once again by slightly abusing notation, assuming that we are forcing above the appropriate condition, and using the facts that GCH is preserved to $V^{\mathbb{P}_\alpha}$ for any ordinal α and $(\lambda^+)^V = (\lambda^+)^{V[G_0]}$, it consequently follows by the definition of $\mathbb{P}$ that $V[G_0] \vDash$ "$|\mathbb{P}(\delta, \lambda^+)| = \lambda^+$". We may then repeat the argument of Case 1 to infer that $V[G_0][G_1] \vDash$ "δ is $(\lambda^+)^V$ supercompact". As $V[G_0][G_1] \vDash$ "$|(\lambda^+)^V| = \sigma^+ = \lambda$" and $\Vdash_{\mathbb{P}_\delta * \dot{\mathbb{P}}(\delta,\lambda^+)}$ "$\dot{\mathbb{P}}^\lambda$ is λ'-directed closed for λ' the least inaccessible cardinal above λ", $V^{\mathbb{P}} \vDash$ "Either κ is λ supercompact, or κ is a measurable limit of cardinals δ which are λ supercompact".

Cases 1 and 2 show that regardless if λ is a trivial stage of forcing, if $V^{\mathbb{P}} \vDash$ "κ is λ strongly compact", then $V^{\mathbb{P}} \vDash$ "Either κ is λ supercompact, or κ is a measurable limit of cardinals δ which are λ supercompact". Since this means that $V^{\mathbb{P}} \vDash$ "Level by level equivalence holds", the proof of Lemma 2.4 is now complete.

$\square$

Lemma 2.4 and the paragraph immediately preceding Lemma 2.4 complete the proof of Theorem 3.

$\square$

We conclude this paper by asking what other combinatorial properties can consistently hold in a model containing supercompact cardinals which satisfies both level by level equivalence and $V = \mathrm{HOD}$. In particular, can such a model satisfy the same combinatorial properties as the model of Theorem 1? Can a model be constructed which satisfies both the combinatorial properties of Theorem 2 (or suitable generalizations thereof) and GA? Can the assumption of a proper class of measurable cardinals be removed from the proof of Theorem 3? If these sorts of models were to be constructed via forcing, one would most likely have to employ different kinds of coding oracles from the ones used above.

References

[1] A. Apter, "Diamond, Square, and Level by Level Equivalence", *Archive for Mathematical Logic 44*, 2005, 387–395.

[2] A. Apter, "Expanding κ's Power Set in its Ultrapowers", *Radovi Matematicki 10*, 2001, 149–156.

[3] A. Apter, "Level by Level Equivalence and the Number of Normal Measures over $P_\kappa(\lambda)$", *Fundamenta Mathematicae 194*, 2007, 253–265.

[4] A. Apter, J. Cummings, "An L-like Model Containing Very Large Cardinals", *Archive for Mathematical Logic 47*, 2008, 65–78.

[5] A. Apter, Sh. Friedman, "Coding into HOD via Normal Measures with Some Applications", *Mathematical Logic Quarterly 57*, 2011, 366–372.

[6] A. Apter, S. Shelah, "On the Strong Equality between Supercompactness and Strong Compactness", *Transactions of the American Mathematical Society 349*, 1997, 103–128.

[7] A. Brooke-Taylor, "Large Cardinals and Definable Well-Orders on the Universe", *Journal of Symbolic Logic 74*, 2009, 641–654.

[8] J. Cummings, M. Foreman, M. Magidor, "Squares, Scales, and Stationary Reflection", *Journal of Mathematical Logic 1*, 2001, 35–98.

[9] M. Foreman, M. Magidor, "A Very Weak Square Principle", *Journal of Symbolic Logic 62*, 1997, 175–196.

[10] S. D. Friedman, "Large Cardinals and L-like Universes", in: *Set Theory: Recent Trends and Applications*, Alessandro Andretta, editor, **Quaderni di Matematica**, volume 17, 2005, 93–110.

[11] G. Fuchs, J. D. Hamkins, J. Reitz, "Set-Theoretic Geology", *Annals of Pure and Applied Logic 166*, 2015, 464–501.

[12] J. D. Hamkins, "Extensions with the Approximation and Cover Properties have No New Large Cardinals", *Fundamenta Mathematicae 180*, 2003, 257–277.

[13] J. D. Hamkins, "Gap Forcing", *Israel Journal of Mathematics 125*, 2001, 237–252.

[14] J. D. Hamkins, "Gap Forcing: Generalizing the Lévy-Solovay Theorem", *Bulletin of Symbolic Logic 5*, 1999, 264–272.

[15] J. D. Hamkins, "The Lottery Preparation", *Annals of Pure and Applied Logic 101*, 2000, 103–146.

[16] T. Jech, *Set Theory: The Third Millennium Edition, Revised and Expanded*, Springer-Verlag, Berlin and New York, 2003.

[17] Y. Kimchi, M. Magidor, "The Independence between the Concepts of Compactness and Supercompactness", circulated manuscript.

[18] M. Magidor, "On the Existence of Nonregular Ultrafilters and the Cardinality of Ultrapowers", *Transactions of the American Mathematical Society 249*, 1979, 97–111.

[19] T. Menas, "On Strong Compactness and Supercompactness", *Annals of Mathematical Logic 7*, 1974, 327–359.

[20] J. Reitz, "The Ground Axiom", *Journal of Symbolic Logic 72*, 2007, 1299–1317.

[21] S. Shelah, "Diamonds", *Proceedings of the American Mathematical Society 138*, 2010, 2151–2161.

[22] W. H. Woodin, "In Search of Ultimate-L: The 19th Midrasha Mathematicae Lectures", *Bulletin of Symbolic Logic 23*, 2017, 1–109.