Indestructibility, Strong Compactness, and Level by Level Equivalence ∗†

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Abstract

We show the relative consistency of the existence of two strongly compact cardinals \(\kappa_1\) and \(\kappa_2\) which exhibit indestructibility properties for their strong compactness, together with level by level equivalence between strong compactness and supercompactness holding at all measurable cardinals except for \(\kappa_1\). In the model constructed, \(\kappa_1\)’s strong compactness is indestructible under arbitrary \(\kappa_1\)-directed closed forcing, \(\kappa_1\) is a limit of measurable cardinals, \(\kappa_2\)’s strong compactness is indestructible under \(\kappa_2\)-directed closed forcing which is also \((\kappa_2,\infty)\)-distributive, and \(\kappa_2\) is fully supercompact.

1 Introduction and Preliminaries

We begin by mentioning that we assume throughout familiarity with the large cardinal notions of measurability, strong compactness, and supercompactness. Readers are urged to consult [15] for

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further details. In particular, we say that $\kappa$ is supercompact (or strongly compact) up to a cardinal $\lambda$ if $\kappa$ is $\delta$ supercompact (or $\delta$ strongly compact) for every $\delta < \lambda$.

We continue with some key definitions. Suppose $V$ is a model of ZFC in which for all regular cardinals $\kappa < \lambda$, $\kappa$ is $\lambda$ strongly compact iff $\kappa$ is $\lambda$ supercompact. Such a model will be said to witness level by level equivalence between strong compactness and supercompactness. We will also say that $\kappa$ is a witness to level by level equivalence between strong compactness and supercompactness iff for every regular cardinal $\lambda > \kappa$, $\kappa$ is $\lambda$ strongly compact iff $\kappa$ is $\lambda$ supercompact. Models in which level by level equivalence between strong compactness and supercompactness holds nontrivially were first constructed by the author and Shelah in [9].

In [1], the author proved the following theorem.

**Theorem 1** Assume that $V \models ZFC + GCH$ satisfies the properties:

1. $\mathcal{K} \neq \emptyset$ is the class of supercompact cardinals.

2. Level by level equivalence between strong compactness and supercompactness holds for every measurable cardinal.

3. $\kappa$ is the least supercompact cardinal.

There is then a partial ordering $\mathbb{P} \in V$ such that $V^\mathbb{P} \models ZFC$ satisfies the properties:

1. $\kappa$ is the least strongly compact cardinal.

2. $\kappa$ is a limit of measurable cardinals but is not supercompact.

3. $\mathcal{K} - \{\kappa\}$ is the class of supercompact cardinals.

4. Level by level equivalence between strong compactness and supercompactness holds for every measurable cardinal except for $\kappa$.

5. $\kappa$’s strong compactness is indestructible under $\kappa$-directed closed forcing.
Note that by construction, in the model for Theorem 1, \( \kappa \) is the only strongly compact cardinal exhibiting any sort of nontrivial indestructibility properties. Thus, one may wonder whether it is possible to have a model containing more than one strongly compact cardinal in which each strongly compact cardinal exhibits indestructibility properties for its strong compactness and level by level equivalence between strong compactness and supercompactness holds nontrivially at every measurable cardinal which is not strongly compact.

The purpose of this paper is to provide an affirmative answer to the above question. Specifically, we prove the following theorem.

**Theorem 2** Suppose that \( V \models \text{"ZFC + } \kappa_1 < \kappa_2 \text{ are supercompact"} \). There is then a model \( \bar{V} \models \text{ZFC} \) satisfying the properties:

1. \( \kappa_1 \) is a non-supercompact strongly compact cardinal which is a limit of measurable cardinals.

2. \( \kappa_2 \) is supercompact.

3. No cardinal less than or equal to \( \kappa_1 \) is supercompact up to an inaccessible cardinal.

4. No cardinal is supercompact up to a measurable cardinal.

5. Level by level equivalence between strong compactness and supercompactness holds for every measurable cardinal except for \( \kappa_1 \).

6. \( \kappa_1 \) and \( \kappa_2 \) are the first two strongly compact cardinals.

7. \( \kappa_1 \)'s strong compactness is indestructible under arbitrary \( \kappa_1 \)-directed closed forcing.

8. \( \kappa_2 \)'s strong compactness is indestructible under \( \kappa_2 \)-directed closed forcing which is also \( (\kappa_2, \infty) \)-distributive.

We take this opportunity to make two remarks concerning Theorem 2. Note that Theorem 5 of Apter-Hamkins [6] indicates that if \( \kappa \) is indestructibly supercompact (in Laver’s sense of [17], i.e., \( \kappa \)'s supercompactness is indestructible under arbitrary \( \kappa \)-directed closed forcing) and level by
level equivalence between strong compactness and supercompactness holds, then no cardinal $\lambda > \kappa$ is $2^\lambda$ supercompact. This is in stark contrast to Theorem 2, which not only tells us, as Theorem 1 does, that level by level equivalence between strong compactness and supercompactness can hold nontrivially both below and above a cardinal $\kappa_1$ which is indestructibly strongly compact (in the sense of Apter-Gitik [4], i.e., $\kappa_1$’s strong compactness is indestructible under arbitrary $\kappa_1$-directed closed forcing) but that there can be in addition a supercompact cardinal (namely $\kappa_2$) greater than $\kappa_1$ whose strong compactness is highly indestructible. Also, since in $\mathcal{V}$, it is the case that no cardinal is supercompact up to a measurable cardinal, $\kappa_2$ of necessity must be the only supercompact cardinal in $\mathcal{V}$, and $\mathcal{V}$ does not contain a measurable cardinal greater than $\kappa_2$.

We now very briefly give some preliminary information concerning notation and terminology. When forcing, $q \geq p$ means that $q$ is stronger than $p$. For $\alpha < \beta$ ordinals, $(\alpha, \beta)$, $[\alpha, \beta]$, and $[\alpha, \beta)$ are as in standard interval notation. For $\kappa$ a cardinal, $\mathbb{P}$ is $\kappa$-directed closed if every directed set of conditions of cardinality less than $\kappa$ has an upper bound. $\mathbb{P}$ is $\kappa$-strategically closed if in the two person game in which the players construct an increasing sequence of conditions $\langle p_\alpha \mid \alpha \leq \kappa \rangle$, where player I plays odd stages and player II plays even and limit stages, player II has a strategy ensuring the game can always be continued. $\mathbb{P}$ is $\prec \kappa$-strategically closed if in the two person game in which the players construct an increasing sequence of conditions $\langle p_\alpha \mid \alpha < \kappa \rangle$, where player I plays odd stages and player II plays even and limit stages, player II has a strategy ensuring the game can always be continued. $\mathbb{P}$ is $(\kappa, \infty)$-distributive if given a sequence $\langle D_\alpha \mid \alpha < \kappa \rangle$ of dense open subsets of $\mathbb{P}$, $\bigcap_{\alpha < \kappa} D_\alpha$ is also a dense open subset of $\mathbb{P}$. Note that if $\mathbb{P}$ is $(\kappa, \infty)$-distributive, then forcing with $\mathbb{P}$ adds no new subsets of $\kappa$. If $G$ is $\mathcal{V}$-generic over $\mathbb{P}$, we will abuse notation slightly and use both $\mathcal{V}[G]$ and $\mathcal{V}^\mathbb{P}$ to indicate the universe obtained by forcing with $\mathbb{P}$. We will, from time to time, confuse terms with the sets they denote and write $x$ when we actually mean $\dot{x}$ or $\check{x}$, especially when $x$ is in $\mathcal{V}$ or is a variant of $G$.

We recall for the benefit of readers the definition given by Hamkins in Section 3 of [14] of the lottery sum of a collection of partial orderings. If $\mathcal{A}$ is a collection of partial orderings, then the lottery sum is the partial ordering $\oplus \mathcal{A} = \{ \langle \mathbb{P}, p \rangle \mid \mathbb{P} \in \mathcal{A} \text{ and } p \in \mathbb{P} \} \cup \{0\}$, ordered with 0 below
everything and \( \langle P, p \rangle \leq \langle P', p' \rangle \) iff \( P = P' \) and \( p \leq p' \). Intuitively, if \( G \) is \( V \)-generic over \( \oplus \mathcal{A} \), then \( G \) first selects an element of \( \mathcal{A} \) (or as Hamkins says in [14], “holds a lottery among the posets in \( \mathcal{A} \)”) and then forces with it.¹

A corollary of Hamkins’ work on gap forcing found in [12] and [13] will be employed in the proof of Theorem 2. We therefore state as a separate theorem what is relevant for this paper, along with some associated terminology, quoting from [12] and [13] when appropriate. Suppose \( P \) is a partial ordering which can be written as \( Q \ast \check{R} \), where \( |Q| < \delta \), \( Q \) is nontrivial, and \( \models \check{R} \) “\( \check{R} \) is \( \delta \)-strategically closed”. In Hamkins’ terminology of [12] and [13], \( P \) admits a gap at \( \delta \). In Hamkins’ terminology of [12] and [13], \( P \) is mild with respect to a cardinal \( \kappa \) iff every set of ordinals \( x \) in \( V^P \) of size less than \( \kappa \) has a “nice” name \( \tau \) in \( V \) of size less than \( \kappa \), i.e., there is a set \( y \) in \( V \), \( |y| < \kappa \), such that any ordinal forced by a condition in \( P \) to be in \( \tau \) is an element of \( y \). Also, as in the terminology of [12], [13], and elsewhere, an embedding \( j : V \to M \) is amenable to \( V \) when \( j \upharpoonright A \in V \) for any \( A \in V \). The specific corollary of Hamkins’ work from [12] and [13] we will be using is then the following.

**Theorem 3 (Hamkins)** Suppose that \( V[G] \) is a generic extension obtained by forcing that admits a gap at some regular \( \delta < \kappa \). Suppose further that \( j : V[G] \to M[j(G)] \) is an embedding with critical point \( \kappa \) for which \( M[j(G)] \subseteq V[G] \) and \( M[j(G)]^\delta \subseteq M[j(G)] \) in \( V[G] \). Then \( M \subseteq V \); indeed, \( M = V \cap M[j(G)] \). If the full embedding \( j \) is amenable to \( V[G] \), then the restricted embedding \( j \upharpoonright V : V \to M \) is amenable to \( V \). If \( j \) is definable from parameters (such as a measure or extender) in \( V[G] \), then the restricted embedding \( j \upharpoonright V \) is definable from the names of those parameters in \( V \). Finally, if \( P \) is mild with respect to \( \kappa \) and \( \kappa \) is \( \lambda \) strongly compact in \( V[G] \) for any \( \lambda \geq \kappa \), then \( \kappa \) is \( \lambda \) strongly compact in \( V \).

## 2 The Proof of Theorem 2

We turn now to the proof of Theorem 2.

¹The terminology “lottery sum” is due to Hamkins, although the concept of the lottery sum of partial orderings has been around for quite some time and has been referred to at different junctures via the names “disjoint sum of partial orderings,” “side-by-side forcing,” and “choosing which partial ordering to force with generically.”
**Proof:** Let $V \models \text{"ZFC + } \kappa_1 < \kappa_2 \text{ are supercompact"}$. By first using the folklore result that GCH can be forced while preserving all ground model supercompact cardinals, then forcing as in [9], and then both taking the appropriate submodel and renaming cardinals if necessary, we slightly abuse notation and also assume in addition that $V \models \text{ZFC + GCH}$ satisfies the properties:

1. Level by level equivalence between strong compactness and supercompactness holds.
2. No cardinal greater than $\kappa_1$ is supercompact up to a measurable cardinal.
3. $\kappa_1$ and $\kappa_2$ are both the first two strongly compact and supercompact cardinals.

The partial ordering $\mathbb{P}^*$ of Theorem 1 may be defined with respect to $\kappa_1$ so as to have cardinality $\kappa_1$ and so that after forcing with $\mathbb{P}^*$, no cardinal less than or equal to $\kappa_1$ is supercompact up to an inaccessible cardinal. Consequently, if we now force with $\mathbb{P}^*$ and once again slightly abuse notation and denote the resulting generic extension by $V$, then by standard arguments and the work of [1] and Lévy-Solovay [18], we may assume that $V \models \text{ZFC}$ satisfies the properties:

1. $\kappa_1$ is the least strongly compact cardinal.
2. $\kappa_1$ is a limit of measurable cardinals but is not supercompact.
3. $\kappa_1$ is indestructibly strongly compact.
4. Level by level equivalence between strong compactness and supercompactness holds for every measurable cardinal except for $\kappa_1$.
5. GCH holds for every cardinal greater than or equal to $\kappa_1$.
6. $\kappa_2$ is both supercompact and the second strongly compact cardinal.
7. No cardinal greater than $\kappa_1$ is supercompact up to a measurable cardinal.
8. No cardinal less than or equal to $\kappa_1$ is supercompact up to an inaccessible cardinal.
The strategy in proving Theorem 2 will be to adjust in a suitable way our proof of Theorem 1 of [3]. More specifically, we will redefine the partial ordering used to prove the aforementioned theorem so as to add non-reflecting stationary sets of ordinals of high enough cofinality (namely \(\kappa_1\)) instead of Prikry sequences. This will destroy measurable cardinals witnessing failures of level by level equivalence between strong compactness and supercompactness, but will also allow us to preserve (unlike when a Prikry iteration is used) that \(\kappa_1\) remains a non-supercompact indestructible strongly compact cardinal. It will also allow us to employ appropriately modified arguments from [3] in the proofs of Lemmas 2.1 – 2.3. Whereas the proofs of Lemmas 2.4 and 2.5 will not be very difficult, the proof of Lemma 2.6 will pose the greatest technical challenge. It is the heart of the argument, and requires the use of an ingenious method developed by Sargsyan. With Lemma 2.7, which shows that \(\kappa_1\) and \(\kappa_2\) are the first two strongly compact cardinals, the proof of Theorem 2 will be complete.

In accordance with this plan of attack, the partial ordering \(\mathbb{P}\) used in the proof of Theorem 2 is a reverse Easton iteration of length \(\kappa_2\), which we will index as \(\langle \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle \mid \alpha \in [\kappa_1, \kappa_2) \rangle\). It is a modification of the partial ordering used in [3]. Specifically, \(\mathbb{P}_{\kappa_1}\) is the partial ordering for adding a Cohen subset of the least inaccessible cardinal greater than \(\kappa_1\). The only nontrivial stages of forcing \(\delta \in (\kappa_1, \kappa_2)\) occur at \(V\)-measurable cardinals. At such a stage \(\delta\), \(\dot{\mathbb{Q}}_\delta\) has the form \(\dot{\mathbb{Q}}^0_\delta \ast \dot{\mathbb{Q}}^1_\delta\), where for \(\delta'\) the least measurable cardinal in \(V\) (and \(V^{\mathbb{P}_\delta}\) as well) greater than \(\delta\), \(\dot{\mathbb{Q}}^0_\delta\) is a term for the lottery sum of all \(\delta\)-directed closed partial orderings which are also \((\delta, \infty)\)-distributive having rank less than \(\delta'\). If \(\Vdash_{\mathbb{P}_\delta * \dot{\mathbb{Q}}^0_\delta} \text{"Level by level equivalence between strong compactness and supercompactness fails at } \delta\)\”, then \(\dot{\mathbb{Q}}^1_\delta\) is a term for the partial ordering adding a non-reflecting stationary set of ordinals of cofinality \(\kappa_1\) to \(\delta\); otherwise, \(\dot{\mathbb{Q}}^1_\delta\) is a term for trivial forcing. (Note that a precise definition of the partial ordering for adding a non-reflecting stationary set of ordinals of cofinality \(\kappa_1\) to an inaccessible cardinal \(\gamma\) may found in, e.g., Section 1 of [9]. A property of this partial ordering we will use in what follows is that it is both \(\kappa_1\)-directed closed and \(\prec \gamma\)-strategically closed.)

**Lemma 2.1** Suppose \(\delta \in (\kappa_1, \kappa_2)\) is measurable in \(V\). Then \(\Vdash_{\mathbb{P}_\delta} \text{"Level by level equivalence between } \)
**strong compactness and supercompactness holds at \( \delta \).**

**Proof:** We modify the proof of Lemma 2.1 of [3]. Since \( V \models \text{“GCH holds for every cardinal greater than or equal to } \kappa_1 \text{”} \), by standard arguments (see, e.g., the proof of Lemma 8.1 of [6]), \( \Vdash_{P_\delta} \text{“} \delta \text{ is a measurable cardinal”} \). We consequently assume inductively that for every measurable cardinal \( \gamma < \delta, \gamma \in (\kappa_1, \kappa_2) \), \( \Vdash \text{“} \text{Level by level equivalence between strong compactness and supercompactness holds at } \gamma \text{”} \).

Let \( \lambda > \delta \) be a regular cardinal in \( V^{P_\delta} \) such that \( \Vdash_{P_\delta} \text{“} \delta \text{ is } \lambda \text{ strongly compact”} \). Since by its definition, \( P_\delta \) is forcing equivalent to a partial ordering which admits a gap below \( \delta \) and is mild with respect to \( \delta \), by Theorem 3, \( V \models \text{“} \delta \text{ is } \lambda \text{ strongly compact”} \). By our assumptions on \( V \) (including level by level equivalence between strong compactness and supercompactness for every measurable cardinal except for \( \kappa_1 \)), \( V \models \text{“} \delta \text{ is } \lambda \text{ supercompact and } \lambda < \delta' \text{ (so } \lambda \text{ is non-measurable)”} \). The proof of Lemma 2.1 will therefore be complete once we have shown that \( \Vdash_{P_\delta} \text{“} \delta \text{ is } \lambda \text{ supercompact”} \).

To do this, fix \( j : V \rightarrow M \) an elementary embedding witnessing the \( \lambda \) supercompactness of \( \delta \). We note that since \( j(\delta) > \lambda > \delta > \kappa_1 \), our inductive assumptions in \( V \) together with the fact that \( V \models \text{“GCH holds for every cardinal greater than or equal to } \kappa_1 \text{”} \) imply that in \( M, \Vdash_{P_\delta} \text{“} \delta \text{ is a measurable cardinal and level by level equivalence between strong compactness and supercompactness holds at } \delta \text{”} \). Also, at stage \( \delta \) in \( M \) in the definition of \( j(P_\delta) \), \( (\hat{Q}_\delta)^M = (\hat{Q}_\delta^0)^M * (\hat{Q}_\delta^1)^M \), where \( (\hat{Q}_\delta^0)^M \) is a term for the stage \( \delta \) lottery sum performed in the definition of \( j(P_\delta) \) and \( (\hat{Q}_\delta^1)^M \) is a term for either trivial forcing or the forcing adding a non-reflecting stationary set of ordinals of cofinality \( j(\kappa_1) = \kappa_1 \) to \( \delta \). Thus, if we opt for trivial forcing in the stage \( \delta \) lottery sum done in \( M \) in the definition of \( j(P_\delta) \), our inductive assumptions also allow us to take \( (\hat{Q}_\delta^1)^M \) as a term for trivial forcing. Consequently, above the appropriate condition in \( M \), \( j(P_\delta) \) is forcing equivalent to \( P_\delta * \hat{P}^{**} \), where \( \hat{P}^{**} \) is a term for a reverse Easton iteration of suitably closed partial orderings whose first nontrivial stage takes place well beyond \( \lambda \). Since \( V \models \text{“GCH holds for every cardinal greater than or equal to } \kappa_1 \text{”} \), we may once again employ the arguments used in the proof of Lemma 8.1 of [6] to show that \( \Vdash_{P_\delta} \text{“} \delta \text{ is } \lambda \text{ supercompact”} \). This completes the proof of Lemma 2.1.
Lemma 2.2 If $\lambda > \delta > \kappa_1$ are regular cardinals and $V \models \text{“} \delta \text{ is } \lambda \text{ supercompact} \text{”}$, then $\Vdash_{\mathcal{P}_\delta} \text{“} \delta \text{ is } \lambda \text{ supercompact} \text{”}$.

Proof: We follow the proof of Lemma 2.2 of [3]. Since $V \models \text{“} \delta \text{ is } \lambda \text{ supercompact} \text{”}$, let $j : V \rightarrow M$ be an elementary embedding witnessing this fact. Since $\lambda > \delta$, by GCH in $V$ for cardinals greater than or equal to $\kappa_1$, $M \models \text{“} \delta \text{ is measurable} \text{”}$. Therefore, by Lemma 2.1 applied in $M$, $\Vdash_{\mathcal{P}_\delta} \text{“} \delta \text{ is a measurable cardinal and level by level equivalence between strong compactness and supercompactness holds at } \delta \text{”}$. Consequently, the argument given in the last paragraph of the proof of Lemma 2.1 now applies to show that in $V$, $\Vdash_{\mathcal{P}_\delta} \text{“} \delta \text{ is } \lambda \text{ supercompact} \text{”}$. This completes the proof of Lemma 2.2. 

□

Since in Lemma 2.2, $\delta$ and $\lambda$ can be arbitrary cardinals greater than $\kappa_1$, as in [3], it immediately follows that in $V^P$, $\kappa_2$ is supercompact.

Lemma 2.3 $V^P \models \text{“} \text{Level by level equivalence between strong compactness and supercompactness holds for every measurable cardinal } \delta \in (\kappa_1, \kappa_2) \text{”} \text{"}.

Proof: Suppose $\delta, \lambda \in (\kappa_1, \kappa_2)$ and $V^P \models \text{“} \lambda > \delta \text{ is a regular cardinal and } \delta \text{ is } \lambda \text{ strongly compact} \text{”}$. As in Lemma 2.1, by Theorem 3, the definition of $\mathcal{P}$, and level by level equivalence between strong compactness and supercompactness in $V$ for every measurable cardinal except for $\kappa_1$, we have that $V \models \text{“} \delta \text{ is } \lambda \text{ supercompact and } \lambda < \delta' \text{ (so } \lambda \text{ is non-measurable)} \text{”}$. Thus, if we write $\mathcal{P} = \mathcal{P}_\delta \ast \dot{\mathcal{Q}}^0_\delta \ast \dot{\mathcal{Q}}^1_\delta \ast \dot{\mathcal{S}} = \mathbb{R} \ast \dot{\mathcal{S}}$, the definition of $\mathcal{P}$ tells us that $\Vdash_{\mathbb{R}} \text{“} \text{Forcing with } \dot{\mathcal{S}} \text{ adds no bounded subsets of } \delta' \text{”}$. Since $V^P = V^{\mathbb{R} \ast \dot{\mathcal{S}}} \models \text{“} \delta \text{ is } \lambda \text{ strongly compact} \text{”}$, we may hence infer that $\Vdash_{\mathbb{R}} \text{“} \delta \text{ is } \lambda \text{ strongly compact} \text{”}$ and that $\dot{\mathcal{Q}}^1_\delta$ is a term for trivial forcing (since otherwise, $V^P \models \text{“} \delta \text{ contains a non-reflecting stationary subset of ordinals of cofinality } \kappa_1 \text{ and } \delta \text{ is weakly compact} \text{”}$, a contradiction). As a consequence, another appeal to the definition of $\mathcal{P}$ indicates that not only is it the case that $\Vdash_{\mathcal{P}_\delta \ast \dot{\mathcal{Q}}^0_\delta} \text{“} \delta \text{ is } \lambda \text{ strongly compact} \text{”}$, but it is additionally true that $\Vdash_{\mathcal{P}_\delta \ast \dot{\mathcal{Q}}^0_\delta} \text{“} \delta \text{ is } \lambda \text{ supercompact} \text{”} \text{ as well. As we now know that } \mathcal{P} \text{ is forcing equivalent to a partial ordering of the form } \mathcal{P}_\delta \ast \dot{\mathcal{Q}}^0_\delta \ast \dot{T} \text{ where } \Vdash_{\mathcal{P}_\delta \ast \dot{\mathcal{Q}}^0_\delta} \text{“} \text{Forcing with } \dot{T} \text{ adds no bounded subsets of } \delta' \text{”, } V^P \models \text{“} \delta \text{ is } \lambda \text{ supercompact} \text{”}.$
supercompact”. This completes the proof of Lemma 2.3.

Lemma 2.4 \( V^P \models \text{“No cardinal is supercompact up to a measurable cardinal”}. \) In fact, \( V^P \models \text{“No cardinal less than or equal to } \kappa_1 \text{ is supercompact up to an inaccessible cardinal”}. \)

Proof: Since \( V \models \text{“No cardinal is supercompact up to a measurable cardinal”} \) and \( P \) admits a sufficiently small gap above \( \kappa_1 \), by Theorem 3, \( V^P \models \text{“No cardinal greater than } \kappa_1 \text{ is supercompact up to a measurable cardinal”}. \) Since the first nontrivial stage of forcing in \( P \) occurs at the least \( V \)-inaccessible cardinal greater than \( \kappa_1 \) and \( V \models \text{“No cardinal less than or equal to } \kappa_1 \text{ is supercompact up to an inaccessible cardinal”}, \( V^P \models \text{“No cardinal less than or equal to } \kappa_1 \text{ is supercompact up to an inaccessible cardinal”}. \) This completes the proof of Lemma 2.4.

Lemma 2.5 \( V^P \models \text{“Level by level equivalence between strong compactness and supercompactness holds for every measurable cardinal except for } \kappa_1 \text{”}. \)

Proof: Since \( V^P \models \text{“} \kappa_2 \text{ is supercompact}” \), by Lemma 2.4, \( V^P \models \text{“There are no measurable cardinals greater than } \kappa_2 \text{”}. \) Thus, by Lemma 2.3, \( V^P \models \text{“Level by level equivalence between strong compactness and supercompactness holds for every measurable cardinal greater than } \kappa_1 \text{”}. \) In addition, as forcing with \( P \) adds no bounded subsets of the least \( V \)-inaccessible cardinal greater than \( \kappa_1 \) and \( V \models \text{“Level by level equivalence between strong compactness and supercompactness holds for every measurable cardinal except for } \kappa_1 \text{ and no cardinal less than or equal to } \kappa_1 \text{ is supercompact up to an inaccessible cardinal”}, \( V^P \models \text{“Level by level equivalence holds for every measurable cardinal less than } \kappa_1 \text{ and } \kappa_1 \text{ is not supercompact up to an inaccessible cardinal”}. \) Finally, because \( V \models \text{“} P \text{ is } \kappa_1 \text{-directed closed and } \kappa_1 \text{ is indestructibly strongly compact}” \), \( V^P \models \text{“} \kappa_1 \text{ is a non-supercompact indestructibly strongly compact cardinal}” \). We therefore now infer that \( V^P \models \text{“Level by level equivalence between strong compactness and supercompactness fails at } \kappa_1 \text{”}. \) This completes the proof.
Lemma 2.6 \( V^\mathbb{P} \models \) “\( \kappa_2 \)'s strong compactness is indestructible under \( \kappa_2 \)-directed closed forcing which is also \( (\kappa_2, \infty) \)-distributive”.

**Proof:** Suppose \( \mathbb{Q} \in V^\mathbb{P} \) is such that \( V^\mathbb{P} \models \) “\( \mathbb{Q} \) is \( \kappa_2 \)-directed closed and \( (\kappa_2, \infty) \)-distributive”. Let \( \dot{\mathbb{Q}} \) be a canonical term for \( \mathbb{Q} \), and let \( \lambda > \max(|\text{TC}(\dot{\mathbb{Q}})|, 2^{\kappa_2}) \) be an arbitrary regular cardinal. Take \( j: V \to M \) as an elementary embedding witnessing the \( \lambda \) supercompactness of \( \kappa_2 \). By the choice of \( \lambda \) and the fact that \( V \models \) “No cardinal greater than \( \kappa_2 \) is measurable”, \( M \models \) “\( \kappa_2 \) is measurable and there are no measurable cardinals in the interval \( (\kappa_2, \lambda] \)”.

Thus, \( \mathbb{Q} \) is allowed in the stage \( \kappa_2 \) lottery sum held in \( M^\mathbb{P} \) in the definition of \( j(\mathbb{P}) \). We therefore assume without loss of generality that we are forcing above a condition which picks \( \mathbb{Q} \) in the stage \( \kappa_2 \) lottery sum held in \( M^\mathbb{P} \) in the definition of \( j(\mathbb{P}) \). We will consequently in what follows slightly abuse notation and replace any term for the stage \( \kappa_2 \) lottery sum held in \( M^\mathbb{P} \) in the definition of \( j(\mathbb{P}) \) with \( \dot{\mathbb{Q}} \).

We consider below two cases.

**Case 1:** In \( M, \models_{\mathbb{P}^\ast\dot{\mathbb{Q}}} \) “Level by level equivalence between strong compactness and supercompactness holds at \( \kappa_2 \)”.

Under these circumstances, since \( \dot{\mathbb{Q}}^1_{\kappa_2} \) is a term for trivial forcing (our slight abuse of notation allows us to consider the terms \( \dot{\mathbb{Q}} \) and \( \dot{\mathbb{Q}}^0_{\kappa_2} \) as being the same), in \( M, j(\mathbb{P}^\ast\dot{\mathbb{Q}}) \) is forcing equivalent to \( \mathbb{P}^\ast\dot{\mathbb{Q}}^\ast\dot{\mathbb{R}}^\ast j(\mathbb{Q}) \), where the first nontrivial stage in the forcing denoted by \( \dot{\mathbb{R}} \) takes place well after \( \lambda \). Standard arguments, as given, e.g., in the proof of Lemma 8.3 of [6] now show that \( j \) lifts in \( V^{\mathbb{P}^\ast\dot{\mathbb{Q}}} \) to \( j : V^{\mathbb{P}^\ast\dot{\mathbb{Q}}} \to M^{j(\mathbb{P}^\ast\dot{\mathbb{Q}})} \), i.e., \( V^{\mathbb{P}^\ast\dot{\mathbb{Q}}} \models \) “\( \kappa_2 \) is \( \lambda \) supercompact”. This completes our discussion of Case 1.

**Case 2:** In \( M, \models_{\mathbb{P}^\ast\dot{\mathbb{Q}}} \) “Level by level equivalence between strong compactness and supercompactness fails at \( \kappa_2 \)”.

Under these circumstances, the proof of Lemma 2.6 is virtually identical to the proof of Lemma 2.3 of Apter-Sargsyan [8]. Specifically, because \( \lambda \) has been chosen large enough, we may assume by taking a normal measure over \( \kappa_2 \) having trivial Mitchell rank that \( k : M \to N \) is an elementary embedding witnessing the measurability of \( \kappa_2 \) definable in \( M \) such that \( N \models \) “\( \kappa_2 \) is not
measurable”. It is the case that if \( i : V \to N \) is an elementary embedding having critical point \( \kappa_2 \) and for any \( x \subseteq N \) with \( |x| \leq \lambda \), there is some \( y \in N \) such that \( x \subseteq y \) and \( N \models "|y| < i(\kappa_2)" \), then \( i \) witnesses the \( \lambda \) strong compactness of \( \kappa_2 \). Using this fact, it is easily verifiable that \( i = k \circ j \) is an elementary embedding witnessing the \( \lambda \) strong compactness of \( \kappa_2 \). To complete our discussion of Case 2, we show that \( i \) lifts in \( V^{\mathcal{P}\hat{Q}} \) to \( i : V^{\mathcal{P}\hat{Q}} \to N^{i(\mathcal{P}\hat{Q})} \).

To do this, we use a modification of an argument originally due to Magidor, unpublished by him but found in, among other places, Lemma 8 of Apter-Hamkins [5]. The modification is due to Sargsyan. Note that throughout the course of the remainder of the proof of Lemma 2.6, we will refer readers to the construction given in Lemma 8 of [5] when relevant, and omit details already presented therein.

Let \( G_0 \) be \( V \)-generic over \( \mathcal{P} \), and let \( H \) be \( V[G_0] \)-generic over \( \mathcal{Q} \). Write \( i(\mathcal{P}) = \mathcal{P} \ast \hat{\mathcal{P}}_1 \ast \hat{\mathcal{P}}_2 \ast \hat{\mathcal{P}}_3 \), where \( \hat{\mathcal{P}}_1 \) is a term for the portion of the forcing defined from stage \( \kappa_2 \) to stage \( k(\kappa_2) \), \( \hat{\mathcal{P}}_2 \) is a term for the forcing done at stage \( k(\kappa_2) \), and \( \hat{\mathcal{P}}_3 \) is a term for the remainder of the forcing, i.e., the portion done after stage \( k(\kappa_2) \). We will build in \( V[G_0][H] \) generic objects for the different portions of \( i(\mathcal{P}) \).

We begin by constructing an \( N[G_0] \)-generic object \( G_1 \) for \( \mathcal{P}^1 \). The argument used is essentially the same as the one given in the construction of the generic object \( G_1 \) found in Lemma 8 of [5] (and will therefore be carried out in \( M[G_0] \subseteq V[G_0] \subseteq V[G_0][H] \)). Specifically, since \( N \models "\kappa_2 \) is not measurable”

\( N[G_0][G_0] \) remains \( \kappa_2 \)-closed with respect to (both
V[G_0] and) M[G_0].

We next analyze the exact nature of $\mathbb{P}^2$. By the definition of $\mathbb{P}$, the closure properties of $M$, and the fact that we are in Case 2, we may write $j(P \ast Q) = P \ast Q \ast Q' \ast R \ast j(Q)$, where $Q \ast Q'$ is a term for the forcing taking place at stage $\kappa_2$ in $M$ and $Q'$ is a term for the partial ordering which adds a non-reflecting stationary set of ordinals of cofinality $\kappa_1$ to $\kappa_2$. By elementarity, since $\mathbb{P}^2$ is a term for the forcing which takes place at stage $k(\kappa_2)$ in $N$, we may write $\mathbb{P}^2 = k(Q) \ast k(Q')$. We will construct in $M[G_0][H]$ generic objects for $k(Q)$ and $k(Q')$.

For $k(Q)$, we use an argument containing ideas due to Woodin, also presented in Theorem 4.10 of [14], Lemma 4.2 of Apter [2], Lemma 3.4 of Apter-Sargsyan [7], and Lemma 2.3 of [8]. First, note that since $N$ is given by an ultrapower, $N = \{k(h)(\kappa_2) \mid h : \kappa_2 \to M \text{ is a function in } M\}$. Further, since by the definition of $G_1$, $k''G_0 \subseteq G_0 \ast G_1$, $k$ lifts in both $M[G_0]$ and $M[G_0][H]$ to $k : M[G_0] \to N[G_0][G_1]$. From these facts, we may now show that $k''H \subseteq k(Q)$ generates an $N[G_0][G_1]$-generic object $G_2$ over $k(Q)$. Specifically, given a dense open subset $D \subseteq k(Q)$, $D \in N[G_0][G_1]$, $D = i_{G_0 \ast G_1}(D')$ for some $N$-name $D = k(D)(\kappa_2)$, where $D' = \langle D_\alpha \mid \alpha < \kappa_2 \rangle$ is a function in $M$. We may assume that every $D_\alpha$ is a dense open subset of $Q$. Since $Q$ is $(\kappa_2, \infty)$-distributive, it follows that $D' = \bigcap_{\alpha < \kappa_2} D_\alpha$ is also a dense open subset of $Q$. As $k(D') \subseteq D$ and $H \cap D' \neq \emptyset$, $k''H \cap D \neq \emptyset$. Thus, $G_2 = \{ p \in k(Q) \mid \exists q \in k''H[q \geq p] \}$, which is definable in $M[G_0][H]$, is our desired $N[G_0][G_1]$-generic object over $k(Q)$. Then, since $k(Q')$ is in $N[G_0][G_1][G_2]$ the partial ordering which adds a non-reflecting stationary set of ordinals of cofinality $k(\kappa_1)$ to $k(\kappa_2)$, we know that $N[G_0][G_1][G_2] \models "|k(Q')| = k(\kappa_2) \text{ and } |\psi(k(Q'))| = 2^{k(\kappa_2)} = k(\kappa_2^+)"$ Hence, since $N[G_0][G_1][G_2]$ remains $\kappa_2$-closed with respect to $M[G_0][H]$, which means $k(Q')$ is $<\kappa_2^+$-strategically closed in $N[G_0][G_1][G_2]$ and $M[G_0][H]$, the same argument used in the construction of $G_1$ allows us to build in $M[G_0][H]$ an $N[G_0][G_1][G_2]$-generic object $G_3$ for $k(Q')$.

We construct now (in $V[G_0][H]$) an $N[G_0][G_1][G_2][G_3]$-generic object for $\mathbb{P}^3$. We do this by combining the term forcing argument found in Lemma 8 of [5] with the argument for the creation of a “master condition” found in Lemma 2 of [4]. Specifically, we begin by showing the existence of a term $\tau \in M$ for a “master condition” for $j(Q)$, i.e., we show the existence of a term $\tau \in M$
in the language of forcing with respect to \( j(\mathbb{P}) \) such that in \( M \), \( \models_{j(\mathbb{P})} \) \( \tau \in j(\check{Q}) \) extends every \( j(\check{q}) \) for \( \check{q} \in \check{H} \). We first note that since \( \mathbb{P} \) is \( \kappa_2 \)-c.c. in both \( V \) and \( M \), as \( \models_{\mathbb{P}} \) \( \check{Q} \) is \( \kappa_2 \)-directed closed and \( |\check{Q}| < \lambda \), the usual arguments show \( M[G_0][H] \) remains \( \lambda \)-closed with respect to \( V[G_0][H] \). This means \( T = \{ j(\check{q}) \mid \exists r \in G_0(\langle r, q \rangle \in G_0 \ast H) \} \in M[G_0][H] \) has a name \( \check{T} \in M \) such that in \( M \), \( \models_{j(\mathbb{P})} \) \( |\check{T}| < \lambda < j(\kappa_2) \), any two elements of \( \check{T} \) are compatible, and \( \check{T} \) is a subset of a partial ordering (namely \( j(\check{Q}) \)) which is \( j(\kappa_2) \)-directed closed”. Thus, since \( M^\lambda \subseteq M \), \( \models_{j(\mathbb{P})} \) “There is an upper bound for \( T \”). A term \( \tau \) for this upper bound is as desired.

We work for the time being in \( M \). Consider the “term forcing” partial ordering \( \mathbb{R}^* \) (see Foreman [11] for the first published account of term forcing or Cummings [10], Section 1.2.5, page 8 — the notion is originally due to Laver) associated with \( \check{R} \ast j(\check{Q}) \), i.e., \( \sigma \in \mathbb{R}^* \) iff \( \sigma \) is a term in the forcing language with respect to \( \mathbb{P} \ast \check{Q} \ast \check{Q}' \) and \( \models_{\mathbb{P} \ast \check{Q} \ast \check{Q}'} \) \( \sigma \in \check{R} \ast j(\check{Q}) \), ordered by \( \sigma_1 \geq \sigma_0 \) iff \( \models_{\mathbb{P} \ast \check{Q} \ast \check{Q}'} \) \( \sigma_1 \geq \sigma_0 \). Note that \( \tau' \) defined as the term in the language of forcing with respect to \( \mathbb{P} \ast \check{Q} \ast \check{Q}' \) composed of the tuple all of whose members are forced to be the trivial condition, with the exception of the last member, which is \( \tau \), is an element of \( \mathbb{R}^* \).

Clearly, \( \mathbb{R}^* \in M \). In addition, since \( V \models \) “No cardinal greater than \( \kappa_2 \) is measurable”, as in Case 1, \( M \models \) “The first stage at which \( \check{R} \ast j(\check{Q}) \) is forced to do nontrivial forcing is greater than \( \lambda \)”. Thus, \( \models_{\mathbb{P} \ast \check{Q} \ast \check{Q}'} \) “\( \check{R} \ast j(\check{Q}) \) is \( \prec \lambda^+ \)-strategically closed”, which, since \( M^\lambda \subseteq M \), immediately implies that \( \mathbb{R}^* \) itself is \( \prec \lambda^+ \)-strategically closed in both \( V \) and \( M \). Further, since \( V^\mathbb{P} \models \) “\( |\check{Q}| < \lambda \)”, in \( M \), \( \models_{\mathbb{P} \ast \check{Q} \ast \check{Q}'} \) “\( |\check{R} \ast j(\check{Q})| < j(\lambda) \)”. Also, by GCH for cardinals greater than or equal to \( \kappa_1 \) in both \( V \) and \( M \) and the fact that \( j \) may be assumed to be given via an ultrapower embedding by a normal measure over \( P_{\kappa_2}(\lambda) \), \( |j(\lambda^+)| = |\{ f \mid f : P_{\kappa_2}(\lambda) \rightarrow \lambda^+ \} = |[\lambda^+]^\lambda| = \lambda^+ \) and \( \models_{\mathbb{P} \ast \check{Q} \ast \check{Q}'} \) “\( |\check{R} \ast j(\check{Q})| < 2^{j(\lambda)} = j(\lambda^+) \)”. Therefore, since as in the footnote given in the proof of Lemma 8 of [5], we may assume that \( \mathbb{R}^* \) has cardinality less than \( j(\lambda) \) in \( M \), we may let \( \langle D_\alpha \mid \alpha < \lambda^+ \rangle \in V \) be an enumeration of the dense open subsets of \( \mathbb{R}^* \) present in \( M \). It is then possible using the \( \prec \lambda^+ \)-strategic closure of \( \mathbb{R}^* \) in \( V \) and the argument employed in the construction of \( G_1 \) to build in \( V \) an \( M \)-generic object \( G_4^* \) for \( \mathbb{R}^* \) containing \( \tau' \).

Note now that since \( N \) is given by an ultrapower of \( M \) via a normal measure over \( \kappa_2 \), Fact 2
of Section 1.2.2 of [10] tells us that \( k' G_4^* \) generates an \( N \)-generic object \( G_4^{**} \) over \( k(\mathbb{R}^*) \) containing \( k(\tau') \). By elementarity, \( k(\mathbb{R}^*) \) is the term forcing in \( N \) defined with respect to \( k(j(\mathbb{P})_{\kappa_2+1}) = \mathbb{P} \ast \bar{\mathbb{P}}^1 \ast \bar{\mathbb{P}}^2 \). Therefore, since \( i(\mathbb{P} \ast \bar{\mathbb{Q}}) = k(j(\mathbb{P} \ast \bar{\mathbb{Q}})) = \mathbb{P} \ast \bar{\mathbb{P}}^1 \ast \bar{\mathbb{P}}^2 \ast \bar{\mathbb{P}}^3 \), \( G_4^{**} \) is \( N \)-generic over \( k(\mathbb{R}^*) \), and \( G_0 \ast G_1 \ast G_2 \ast G_3 \) is \( k(\mathbb{P} \ast \bar{\mathbb{Q}}) \)-generic over \( N \), Fact 1 of Section 1.2.5 of [10] (see also [11]) tells us that for \( G_4 = \{ i_{G_0 \ast G_1 \ast G_2 \ast G_3}(\sigma) \mid \sigma \in G_4^{**} \} \), \( G_4 \) is \( N[G_0][G_1][G_2][G_3] \)-generic over \( \mathbb{P}^3 \). In addition, since the definition of \( \tau \) tells us that in \( M \), the statement \( \langle p, \dot{q} \rangle \in j(\mathbb{P} \ast \bar{\mathbb{Q}}) \) implies that \( \langle p, \dot{q} \rangle \models \bar{\mathcal{M}}_{j(\mathbb{P} \ast \bar{\mathbb{Q}})} ' \tau \geq \dot{q}' '' \) is true, by elementarity, in \( N \), the statement \( \langle p, \dot{q} \rangle \in k(j(\mathbb{P} \ast \bar{\mathbb{Q}})) \) implies that \( \langle p, \dot{q} \rangle \models k(\tau) \geq \dot{q}'' \) is true. In other words, since \( k \circ j = i \), in \( N \), the statement \( \langle p, \dot{q} \rangle \in i(\mathbb{P} \ast \bar{\mathbb{Q}}) \) implies that \( \langle p, \dot{q} \rangle \models \bar{\mathcal{M}}_i(\mathbb{P} \ast \bar{\mathbb{Q}}) ' k(\tau) \geq \dot{q}''' \) is true. Thus, in \( N \), \( k(\tau) \) functions as a term for a “master condition” for \( i(\bar{\mathbb{Q}}) \), so since \( G_4^{**} \) contains \( k(\tau') \), the construction of all of the above generic objects immediately yields that \( i''(G_0 \ast H) \subseteq G_0 \ast G_1 \ast G_2 \ast G_3 \ast G_4 \). This means that \( i \) lifts in \( \mathbb{V}^\mathbb{P} \) to \( i : \mathbb{V}^\mathbb{P} \ast \bar{\mathbb{Q}} \rightarrow N[i(\mathbb{P} \ast \bar{\mathbb{Q}})] \), i.e. \( \mathbb{V}^\mathbb{P} \ast \bar{\mathbb{Q}} \models "\kappa_2 \text{ is } \lambda \text{ strongly compact}" \). This completes our discussion of Case 2. \[ \square \]

We now see that regardless if Case 1 or Case 2 holds, \( \mathbb{V}^\mathbb{P} \ast \bar{\mathbb{Q}} \models "\kappa_2 \text{ is } \lambda \text{ strongly compact}" \). Since \( \lambda \) was arbitrary, this completes the proof of Lemma 2.6. \[ \square \]

**Lemma 2.7** \( \mathbb{V}^\mathbb{P} \models "\kappa_1 \text{ and } \kappa_2 \text{ are the first two strongly compact cardinals}".  

**Proof:** It follows by Ketonen’s characterization of strong compactness given in [16] that any cardinal which is strongly compact up to a strongly compact cardinal is itself strongly compact. Thus, since \( \mathbb{V} \models "\kappa_1 \text{ is the first strongly compact cardinal}" \), it must be the case that for any \( \delta < \kappa_1 \), \( \mathbb{V} \models "\delta \text{ is not strongly compact up to } \kappa_1" \). In addition, as we observed in the proof of Lemma 2.5, \( \mathbb{P} \) is \( \kappa_1 \)-directed closed, and so cannot force a new degree of strong compactness for any \( \delta < \kappa_1 \). Therefore, since \( \mathbb{V} \models "\kappa_1 \text{ is both indestructibly strongly compact and the first strongly compact cardinal}" \), \( \mathbb{V}^\mathbb{P} \models "\kappa_1 \text{ is both indestructibly strongly compact and the first strongly compact cardinal}" \). Next, if \( \delta \in (\kappa_1, \kappa_2) \) is such that \( \mathbb{V}^\mathbb{P} \models "\delta \text{ is measurable}" \), then because \( \mathbb{P} \) admits a sufficiently small gap above \( \kappa_1 \), by Theorem 3, \( \mathbb{V} \models "\delta \text{ is measurable}" \). Hence, because by its
definition, \(P\) is mild with respect to \(\delta\), Theorem 3 also tells us that forcing with \(P\) cannot create any new degrees of strong compactness of \(\delta\). As \(V \vDash \text{“No cardinal } \delta \in (\kappa_1, \kappa_2) \text{ is strongly compact”}\), we may consequently infer that \(V^P \vDash \text{“No cardinal } \delta \in (\kappa_1, \kappa_2) \text{ is strongly compact”}\). Since we have already seen that \(V^P \vDash \text{“}\kappa_2 \text{ is supercompact”}\), \(V^P \vDash \text{“}\kappa_2 \text{ is the second strongly compact cardinal”}\). This completes the proof of Lemma 2.7.

\[\square\]

Since \(V \vDash \text{“}\kappa_1 \text{ is a limit of measurable cardinals and } P \text{ is } \kappa_1\text{-directed closed”}\), \(V^P \vDash \text{“}\kappa_1 \text{ is a limit of measurable cardinals”}\). Therefore, by taking \(V = V^P\), Lemmas 2.1 - 2.7, their proofs, and the intervening remarks complete the proof of Theorem 2.

\[\square\]

Theorem 2 leaves open some interesting questions, which we pose in conclusion to this paper. In particular, is it possible to prove an analogue of Theorem 2 for two strongly compact cardinals in which the large cardinal structure of the model constructed has fewer restrictions? Is it possible to prove an analogue of Theorem 2 for two strongly compact cardinals in which \(\kappa_2\)’s strong compactness is indestructible under arbitrary \(\kappa_2\text{-directed closed forcing}\)? Is it possible to prove an analogue of Theorem 2 for two strongly compact cardinals in which \(\kappa_2\) is indestructibly supercompact? Is it possible to prove an analogue of Theorem 2 in which the model constructed contains more than two strongly compact cardinals?

References


