Indestructibility, Strong Compactness, and Level by Level Equivalence *†

Arthur W. Apter‡
Department of Mathematics
Baruch College of CUNY
New York, New York 10010 USA
and
The CUNY Graduate Center, Mathematics
365 Fifth Avenue
New York, New York 10016 USA
http://faculty.baruch.cuny.edu/apter
awapter@alum.mit.edu

August 21, 2007
(revised April 20, 2009)

Abstract

We show the relative consistency of the existence of two strongly compact cardinals κ_1 and κ_2 which exhibit indestructibility properties for their strong compactness, together with level by level equivalence between strong compactness and supercompactness holding at all measurable cardinals except for κ_1. In the model constructed, κ_1's strong compactness is indestructible under arbitrary κ_1-directed closed forcing, κ_1 is a limit of measurable cardinals, κ_2's strong compactness is indestructible under κ_2-directed closed forcing which is also (κ_2, ∞)-distributive, and κ_2 is fully supercompact.

1 Introduction and Preliminaries

We begin by mentioning that we assume throughout familiarity with the large cardinal notions of measurability, strong compactness, and supercompactness. Readers are urged to consult [15] for

*2000 Mathematics Subject Classifications: 03E35, 03E55.
†Keywords: Supercompact cardinal, strongly compact cardinal, indestructibility, non-reflecting stationary set of ordinals, level by level equivalence between strong compactness and supercompactness.
‡The author's research was partially supported by PSC-CUNY grants and CUNY Collaborative Incentive grants. In addition, the author wishes to thank the referee for helpful comments, suggestions, and corrections which have been incorporated into the current version of the paper.

further details. In particular, we say that κ is supercompact (or strongly compact) up to a cardinal λ if κ is δ supercompact (or δ strongly compact) for every $\delta < \lambda$.

We continue with some key definitions. Suppose V is a model of ZFC in which for all regular cardinals $\kappa < \lambda$, κ is λ strongly compact iff κ is λ supercompact. Such a model will be said to witness level by level equivalence between strong compactness and supercompactness. We will also say that κ is a witness to level by level equivalence between strong compactness and supercompactness iff for every regular cardinal $\lambda > \kappa$, κ is λ strongly compact iff κ is λ supercompact. Models in which level by level equivalence between strong compactness and supercompactness holds nontrivially were first constructed by the author and Shelah in [9].

In [1], the author proved the following theorem.

Theorem 1 *Assume that $V \models$ ZFC + GCH satsifies the properties:*

1. *$\mathcal{K} \neq \emptyset$ is the class of supercompact cardinals.*
2. *Level by level equivalence between strong compactness and supercompactness holds for every measurable cardinal.*
3. *κ is the least supercompact cardinal.*

There is then a partial ordering $\mathbb{P} \in V$ such that $V^{\mathbb{P}} \models$ ZFC satisfies the properties:

1. *κ is the least strongly compact cardinal.*
2. *κ is a limit of measurable cardinals but is not supercompact.*
3. *$\mathcal{K} - \{\kappa\}$ is the class of supercompact cardinals.*
4. *Level by level equivalence between strong compactness and supercompactness holds for every measurable cardinal except for κ.*
5. *κ's strong compactness is indestructible under κ-directed closed forcing.*

Note that by construction, in the model for Theorem 1, κ is the only strongly compact cardinal exhibiting any sort of nontrivial indestructibility properties. Thus, one may wonder whether it is possible to have a model containing more than one strongly compact cardinal in which each strongly compact cardinal exhibits indestructibility properties for its strong compactness and level by level equivalence between strong compactness and supercompactness holds nontrivially at every measurable cardinal which is not strongly compact.

The purpose of this paper is to provide an affirmative answer to the above question. Specifically, we prove the following theorem.

Theorem 2 *Suppose that $V \models$ "ZFC + $\kappa_1 < \kappa_2$ are supercompact". There is then a model $\overline{V} \models$ ZFC satisfying the properties:*

1. *κ_1 is a non-supercompact strongly compact cardinal which is a limit of measurable cardinals.*

2. *κ_2 is supercompact.*

3. *No cardinal less than or equal to κ_1 is supercompact up to an inaccessible cardinal.*

4. *No cardinal is supercompact up to a measurable cardinal.*

5. *Level by level equivalence between strong compactness and supercompactness holds for every measurable cardinal except for κ_1.*

6. *κ_1 and κ_2 are the first two strongly compact cardinals.*

7. *κ_1's strong compactness is indestructible under arbitrary κ_1-directed closed forcing.*

8. *κ_2's strong compactness is indestructible under κ_2-directed closed forcing which is also (κ_2, ∞)-distributive.*

We take this opportunity to make two remarks concerning Theorem 2. Note that Theorem 5 of Apter-Hamkins [6] indicates that if κ is indestructibly supercompact (in Laver's sense of [17], i.e., κ's supercompactness is indestructible under arbitrary κ-directed closed forcing) and level by level equivalence between strong compactness and supercompactness holds, then no cardinal $\lambda > \kappa$ is 2^λ supercompact. This is in stark contrast to Theorem 2, which not only tells us, as Theorem 1 does, that level by level equivalence between strong compactness and supercompactness can hold nontrivially both below and above a cardinal κ_1 which is indestructibly strongly compact (in the sense of Apter-Gitik [4], i.e., κ_1's strong compactness is indestructible under arbitrary κ_1-directed closed forcing) but that there can be in addition a supercompact cardinal (namely κ_2) greater than κ_1 whose strong compactness is highly indestructible. Also, since in $\overline{V}$, it is the case that no cardinal is supercompact up to a measurable cardinal, κ_2 of necessity must be the only supercompact cardinal in $\overline{V}$, and $\overline{V}$ does not contain a measurable cardinal greater than κ_2.

We now very briefly give some preliminary information concerning notation and terminology. When forcing, $q \geq p$ means that q is stronger than p. For $\alpha < \beta$ ordinals, (α, β), $(\alpha, \beta]$, and $[\alpha, \beta)$ are as in standard interval notation. For κ a cardinal, $\mathbb{P}$ is κ-directed closed if every directed set of conditions of cardinality less than κ has an upper bound. $\mathbb{P}$ is κ-strategically closed if in the two person game in which the players construct an increasing sequence of conditions $\langle p_\alpha \mid \alpha \leq \kappa \rangle$, where player I plays odd stages and player II plays even and limit stages, player II has a strategy ensuring the game can always be continued. $\mathbb{P}$ is $\prec\kappa$-strategically closed if in the two person game in which the players construct an increasing sequence of conditions $\langle p_\alpha \mid \alpha < \kappa \rangle$, where player I plays odd stages and player II plays even and limit stages, player II has a strategy ensuring the game can always be continued. $\mathbb{P}$ is (κ, ∞)-distributive if given a sequence $\langle D_\alpha \mid \alpha < \kappa \rangle$ of dense open subsets of $\mathbb{P}$, $\bigcap_{\alpha < \kappa} D_\alpha$ is also a dense open subset of $\mathbb{P}$. Note that if $\mathbb{P}$ is (κ, ∞)-distributive, then forcing with $\mathbb{P}$ adds no new subsets of κ. If G is V-generic over $\mathbb{P}$, we will abuse notation slightly and use both $V[G]$ and $V^{\mathbb{P}}$ to indicate the universe obtained by forcing with $\mathbb{P}$. We will, from time to time, confuse terms with the sets they denote and write x when we actually mean $\dot{x}$ or $\check{x}$, especially when x is in V or is a variant of G.

We recall for the benefit of readers the definition given by Hamkins in Section 3 of [14] of the lottery sum of a collection of partial orderings. If $\mathfrak{A}$ is a collection of partial orderings, then the lottery sum is the partial ordering $\oplus\mathfrak{A} = \{\langle \mathbb{P}, p \rangle \mid \mathbb{P} \in \mathfrak{A} \text{ and } p \in \mathbb{P}\} \cup \{0\}$, ordered with 0 below everything and $\langle \mathbb{P}, p\rangle \leq \langle \mathbb{P}', p'\rangle$ iff $\mathbb{P} = \mathbb{P}'$ and $p \leq p'$. Intuitively, if G is V-generic over $\oplus\mathfrak{A}$, then G first selects an element of $\mathfrak{A}$ (or as Hamkins says in [14], "holds a lottery among the posets in $\mathfrak{A}$") and then forces with it.[1]

A corollary of Hamkins' work on gap forcing found in [12] and [13] will be employed in the proof of Theorem 2. We therefore state as a separate theorem what is relevant for this paper, along with some associated terminology, quoting from [12] and [13] when appropriate. Suppose $\mathbb{P}$ is a partial ordering which can be written as $\mathbb{Q} * \dot{\mathbb{R}}$, where $|\mathbb{Q}| < \delta$, $\mathbb{Q}$ is nontrivial, and $\Vdash_{\mathbb{Q}}$ "$\dot{\mathbb{R}}$ is δ-strategically closed". In Hamkins' terminology of [12] and [13], $\mathbb{P}$ admits a gap at δ. In Hamkins' terminology of [12] and [13], $\mathbb{P}$ is mild with respect to a cardinal κ iff every set of ordinals x in $V^{\mathbb{P}}$ of size less than κ has a "nice" name τ in V of size less than κ, i.e., there is a set y in V, $|y| < \kappa$, such that any ordinal forced by a condition in $\mathbb{P}$ to be in τ is an element of y. Also, as in the terminology of [12], [13], and elsewhere, an embedding $j : \overline{V} \to \overline{M}$ is amenable to $\overline{V}$ when $j \restriction A \in \overline{V}$ for any $A \in \overline{V}$. The specific corollary of Hamkins' work from [12] and [13] we will be using is then the following.

Theorem 3 (Hamkins) *Suppose that $V[G]$ is a generic extension obtained by forcing that admits a gap at some regular $\delta < \kappa$. Suppose further that $j : V[G] \to M[j(G)]$ is an embedding with critical point κ for which $M[j(G)] \subseteq V[G]$ and $M[j(G)]^{\delta} \subseteq M[j(G)]$ in $V[G]$. Then $M \subseteq V$; indeed, $M = V \cap M[j(G)]$. If the full embedding j is amenable to $V[G]$, then the restricted embedding $j \restriction V : V \to M$ is amenable to V. If j is definable from parameters (such as a measure or extender) in $V[G]$, then the restricted embedding $j \restriction V$ is definable from the names of those parameters in V. Finally, if $\mathbb{P}$ is mild with respect to κ and κ is λ strongly compact in $V[G]$ for any $\lambda \geq \kappa$, then κ is λ strongly compact in V.*

2 The Proof of Theorem 2

We turn now to the proof of Theorem 2.

[1] The terminology "lottery sum" is due to Hamkins, although the concept of the lottery sum of partial orderings has been around for quite some time and has been referred to at different junctures via the names "disjoint sum of partial orderings," "side-by-side forcing," and "choosing which partial ordering to force with generically."

Proof: Let $V \vDash$ "ZFC + $\kappa_1 < \kappa_2$ are supercompact". By first using the folklore result that GCH can be forced while preserving all ground model supercompact cardinals, then forcing as in [9], and then both taking the appropriate submodel and renaming cardinals if necessary, we slightly abuse notation and also assume in addition that $V \vDash$ ZFC + GCH satisfies the properties:

1. Level by level equivalence between strong compactness and supercompactness holds.
2. No cardinal greater than κ_1 is supercompact up to a measurable cardinal.
3. κ_1 and κ_2 are both the first two strongly compact and supercompact cardinals.

The partial ordering $\mathbb{P}^*$ of Theorem 1 may be defined with respect to κ_1 so as to have cardinality κ_1 and so that after forcing with $\mathbb{P}^*$, no cardinal less than or equal to κ_1 is supercompact up to an inaccessible cardinal. Consequently, if we now force with $\mathbb{P}^*$ and once again slightly abuse notation and denote the resulting generic extension by V, then by standard arguments and the work of [1] and Lévy-Solovay [18], we may assume that $V \vDash$ ZFC satisfies the properties:

1. κ_1 is the least strongly compact cardinal.
2. κ_1 is a limit of measurable cardinals but is not supercompact.
3. κ_1 is indestructibly strongly compact.
4. Level by level equivalence between strong compactness and supercompactness holds for every measurable cardinal except for κ_1.
5. GCH holds for every cardinal greater than or equal to κ_1.
6. κ_2 is both supercompact and the second strongly compact cardinal.
7. No cardinal greater than κ_1 is supercompact up to a measurable cardinal.
8. No cardinal less than or equal to κ_1 is supercompact up to an inaccessible cardinal.

The strategy in proving Theorem 2 will be to adjust in a suitable way our proof of Theorem 1 of [3]. More specifically, we will redefine the partial ordering used to prove the aforementioned theorem so as to add non-reflecting stationary sets of ordinals of high enough cofinality (namely κ_1) instead of Prikry sequences. This will destroy measurable cardinals witnessing failures of level by level equivalence between strong compactness and supercompactness, but will also allow us to preserve (unlike when a Prikry iteration is used) that κ_1 remains a non-supercompact indestructible strongly compact cardinal. It will also allow us to employ appropriately modified arguments from [3] in the proofs of Lemmas 2.1 – 2.3. Whereas the proofs of Lemmas 2.4 and 2.5 will not be very difficult, the proof of Lemma 2.6 will pose the greatest technical challenge. It is the heart of the argument, and requires the use of an ingenious method developed by Sargsyan. With Lemma 2.7, which shows that κ_1 and κ_2 are the first two strongly compact cardinals, the proof of Theorem 2 will be complete.

In accordance with this plan of attack, the partial ordering $\mathbb{P}$ used in the proof of Theorem 2 is a reverse Easton iteration of length κ_2, which we will index as $\langle\langle\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha\rangle \mid \alpha \in [\kappa_1, \kappa_2)\rangle$. It is a modification of the partial ordering used in [3]. Specifically, $\mathbb{P}_{\kappa_1}$ is the partial ordering for adding a Cohen subset of the least inaccessible cardinal greater than κ_1. The only nontrivial stages of forcing $\delta \in (\kappa_1, \kappa_2)$ occur at V-measurable cardinals. At such a stage δ, $\dot{\mathbb{Q}}_\delta$ has the form $\dot{\mathbb{Q}}^0_\delta * \dot{\mathbb{Q}}^1_\delta$, where for δ' the least measurable cardinal in V (and $V^{\mathbb{P}_\delta}$ as well) greater than δ, $\dot{\mathbb{Q}}^0_\delta$ is a term for the lottery sum of all δ-directed closed partial orderings which are also (δ, ∞)-distributive having rank less than δ'. If $\Vdash_{\mathbb{P}_\delta * \dot{\mathbb{Q}}^0_\delta}$ "Level by level equivalence between strong compactness and supercompactness fails at δ", then $\dot{\mathbb{Q}}^1_\delta$ is a term for the partial ordering adding a non-reflecting stationary set of ordinals of cofinality κ_1 to δ; otherwise, $\dot{\mathbb{Q}}^1_\delta$ is a term for trivial forcing. (Note that a precise definition of the partial ordering for adding a non-reflecting stationary set of ordinals of cofinality κ_1 to an inaccessible cardinal γ may found in, e.g., Section 1 of [9]. A property of this partial ordering we will use in what follows is that it is both κ_1-directed closed and $\prec\gamma$-strategically closed.)

Lemma 2.1 *Suppose $\delta \in (\kappa_1, \kappa_2)$ is measurable in V. Then $\Vdash_{\mathbb{P}_\delta}$ "Level by level equivalence between* *strong compactness and supercompactness holds at δ".*

Proof: We modify the proof of Lemma 2.1 of [3]. Since $V \models$ "GCH holds for every cardinal greater than or equal to κ_1", by standard arguments (see, e.g., the proof of Lemma 8.1 of [6]), $\Vdash_{\mathbb{P}_\delta}$ "δ is a measurable cardinal". We consequently assume inductively that for every measurable cardinal $\gamma < \delta$, $\gamma \in (\kappa_1, \kappa_2)$, $\Vdash_{\mathbb{P}_\gamma}$ "Level by level equivalence between strong compactness and supercompactness holds at γ".

Let $\lambda > \delta$ be a regular cardinal in $V^{\mathbb{P}_\delta}$ such that $\Vdash_{\mathbb{P}_\delta}$ "δ is λ strongly compact". Since by its definition, $\mathbb{P}_\delta$ is forcing equivalent to a partial ordering which admits a gap below δ and is mild with respect to δ, by Theorem 3, $V \models$ "δ is λ strongly compact". By our assumptions on V (including level by level equivalence between strong compactness and supercompactness for every measurable cardinal except for κ_1), $V \models$ "δ is λ supercompact and $\lambda < \delta'$ (so λ is non-measurable)". The proof of Lemma 2.1 will therefore be complete once we have shown that $\Vdash_{\mathbb{P}_\delta}$ "δ is λ supercompact".

To do this, fix $j : V \to M$ an elementary embedding witnessing the λ supercompactness of δ. We note that since $j(\delta) > \lambda > \delta > \kappa_1$, our inductive assumptions in V together with the fact that $V \models$ "GCH holds for every cardinal greater than or equal to κ_1" imply that in M, $\Vdash_{\mathbb{P}_\delta}$ "δ is a measurable cardinal and level by level equivalence between strong compactness and supercompactness holds at δ". Also, at stage δ in M in the definition of $j(\mathbb{P}_\delta)$, $(\dot{\mathbb{Q}}_\delta)^M = (\dot{\mathbb{Q}}^0_\delta)^M * (\dot{\mathbb{Q}}^1_\delta)^M$, where $(\dot{\mathbb{Q}}^0_\delta)^M$ is a term for the stage δ lottery sum performed in the definition of $j(\mathbb{P}_\delta)$ and $(\dot{\mathbb{Q}}^1_\delta)^M$ is a term for either trivial forcing or the forcing adding a non-reflecting stationary set of ordinals of cofinality $j(\kappa_1) = \kappa_1$ to δ. Thus, if we opt for trivial forcing in the stage δ lottery sum done in M in the definition of $j(\mathbb{P}_\delta)$, our inductive assumptions also allow us to take $(\dot{\mathbb{Q}}^1_\delta)^M$ as a term for trivial forcing. Consequently, above the appropriate condition in M, $j(\mathbb{P}_\delta)$ is forcing equivalent to $\mathbb{P}_\delta * \dot{\mathbb{P}}^{**}$, where $\dot{\mathbb{P}}^{**}$ is a term for a reverse Easton iteration of suitably closed partial orderings whose first nontrivial stage takes place well beyond λ. Since $V \models$ "GCH holds for every cardinal greater than or equal to κ_1", we may once again employ the arguments used in the proof of Lemma 8.1 of [6] to show that $\Vdash_{\mathbb{P}_\delta}$ "δ is λ supercompact". This completes the proof of Lemma 2.1.

$\square$

Lemma 2.2 *If $\lambda > \delta > \kappa_1$ are regular cardinals and $V \models$ "δ is λ supercompact", then $\Vdash_{\mathbb{P}_\delta}$ "δ is λ supercompact".*

Proof: We follow the proof of Lemma 2.2 of [3]. Since $V \models$ "δ is λ supercompact", let $j : V \to M$ be an elementary embedding witnessing this fact. Since $\lambda > \delta$, by GCH in V for cardinals greater than or equal to κ_1, $M \models$ "δ is measurable". Therefore, by Lemma 2.1 applied in M, $\Vdash_{\mathbb{P}_\delta}$ "δ is a measurable cardinal and level by level equivalence between strong compactness and supercompactness holds at δ". Consequently, the argument given in the last paragraph of the proof of Lemma 2.1 now applies to show that in V, $\Vdash_{\mathbb{P}_\delta}$ "δ is λ supercompact". This completes the proof of Lemma 2.2.

□

Since in Lemma 2.2, δ and λ can be arbitrary cardinals greater than κ_1, as in [3], it immediately follows that in $V^{\mathbb{P}}$, κ_2 is supercompact.

Lemma 2.3 $V^{\mathbb{P}} \models$ "*Level by level equivalence between strong compactness and supercompactness holds for every measurable cardinal $\delta \in (\kappa_1, \kappa_2)$".*

Proof: Suppose $\delta, \lambda \in (\kappa_1, \kappa_2)$ and $V^{\mathbb{P}} \models$ "$\lambda > \delta$ is a regular cardinal and δ is λ strongly compact". As in Lemma 2.1, by Theorem 3, the definition of $\mathbb{P}$, and level by level equivalence between strong compactness and supercompactness in V for every measurable cardinal except for κ_1, we have that $V \models$ "δ is λ supercompact and $\lambda < \delta'$ (so λ is non-measurable)". Thus, if we write $\mathbb{P} = \mathbb{P}_\delta * \dot{\mathbb{Q}}^0_\delta * \dot{\mathbb{Q}}^1_\delta * \dot{\mathbb{S}} = \mathbb{R} * \dot{\mathbb{S}}$, the definition of $\mathbb{P}$ tells us that $\Vdash_{\mathbb{R}}$ "Forcing with $\dot{\mathbb{S}}$ adds no bounded subsets of δ'". Since $V^{\mathbb{P}} = V^{\mathbb{R} * \dot{\mathbb{S}}} \models$ "δ is λ strongly compact", we may hence infer that $\Vdash_{\mathbb{R}}$ "δ is λ strongly compact" and that $\dot{\mathbb{Q}}^1_\delta$ is a term for trivial forcing (since otherwise, $V^{\mathbb{P}} \models$ "δ contains a non-reflecting stationary subset of ordinals of cofinality κ_1 and δ is weakly compact", a contradiction). As a consequence, another appeal to the definition of $\mathbb{P}$ indicates that not only is it the case that $\Vdash_{\mathbb{P}_\delta * \dot{\mathbb{Q}}^0_\delta}$ "δ is λ strongly compact", but it is additionally true that $\Vdash_{\mathbb{P}_\delta * \dot{\mathbb{Q}}^0_\delta}$ "δ is λ supercompact" as well. As we now know that $\mathbb{P}$ is forcing equivalent to a partial ordering of the form $\mathbb{P}_\delta * \dot{\mathbb{Q}}^0_\delta * \dot{\mathbb{T}}$ where $\Vdash_{\mathbb{P}_\delta * \dot{\mathbb{Q}}^0_\delta}$ "Forcing with $\dot{\mathbb{T}}$ adds no bounded subsets of δ'", $V^{\mathbb{P}} \models$ "δ is λ

supercompact". This completes the proof of Lemma 2.3.

□

Lemma 2.4 $V^{\mathbb{P}} \models$ "*No cardinal is supercompact up to a measurable cardinal". In fact,* $V^{\mathbb{P}} \models$ "*No cardinal less than or equal to* κ_1 *is supercompact up to an inaccessible cardinal".*

Proof: Since $V \models$ "No cardinal is supercompact up to a measurable cardinal" and $\mathbb{P}$ admits a sufficiently small gap above κ_1, by Theorem 3, $V^{\mathbb{P}} \models$ "No cardinal greater than κ_1 is supercompact up to a measurable cardinal". Since the first nontrivial stage of forcing in $\mathbb{P}$ occurs at the least V-inaccessible cardinal greater than κ_1 and $V \models$ "No cardinal less than or equal to κ_1 is supercompact up to an inaccessible cardinal", $V^{\mathbb{P}} \models$ "No cardinal less than or equal to κ_1 is supercompact up to an inaccessible cardinal". This completes the proof of Lemma 2.4.

□

Lemma 2.5 $V^{\mathbb{P}} \models$ "*Level by level equivalence between strong compactness and supercompactness holds for every measurable cardinal except for* κ_1".

Proof: Since $V^{\mathbb{P}} \models$ "κ_2 is supercompact", by Lemma 2.4, $V^{\mathbb{P}} \models$ "There are no measurable cardinals greater than κ_2". Thus, by Lemma 2.3, $V^{\mathbb{P}} \models$ "Level by level equivalence between strong compactness and supercompactness holds for every measurable cardinal greater than κ_1". In addition, as forcing with $\mathbb{P}$ adds no bounded subsets of the least V-inaccessible cardinal greater than κ_1 and $V \models$ "Level by level equivalence between strong compactness and supercompactness holds for every measurable cardinal except for κ_1 and no cardinal less than or equal to κ_1 is supercompact up to an inaccessible cardinal", $V^{\mathbb{P}} \models$ "Level by level equivalence holds for every measurable cardinal less than κ_1 and κ_1 is not supercompact up to an inaccessible cardinal". Finally, because $V \models$ "$\mathbb{P}$ is κ_1-directed closed and κ_1 is indestructibly strongly compact", $V^{\mathbb{P}} \models$ "κ_1 is a non-supercompact indestructibly strongly compact cardinal". We therefore now infer that $V^{\mathbb{P}} \models$ "Level by level equivalence between strong compactness and supercompactness fails at κ_1". This completes the proof of Lemma 2.5.

$\square$

Lemma 2.6 $V^{\mathbb{P}} \models$ "*κ_2's strong compactness is indestructible under κ_2-directed closed forcing which is also (κ_2, ∞)-distributive*".

Proof: Suppose $\mathbb{Q} \in V^{\mathbb{P}}$ is such that $V^{\mathbb{P}} \models$ "$\mathbb{Q}$ is κ_2-directed closed and (κ_2, ∞)-distributive". Let $\dot{\mathbb{Q}}$ be a canonical term for $\mathbb{Q}$, and let $\lambda > \max(|\mathrm{TC}(\dot{\mathbb{Q}})|, 2^{\kappa_2})$ be an arbitrary regular cardinal. Take $j : V \to M$ as an elementary embedding witnessing the λ supercompactness of κ_2. By the choice of λ and the fact that $V \models$ "No cardinal greater than κ_2 is measurable", $M \models$ "κ_2 is measurable and there are no measurable cardinals in the interval $(\kappa_2, \lambda]$". Thus, $\mathbb{Q}$ is allowed in the stage κ_2 lottery sum held in $M^{\mathbb{P}}$ in the definition of $j(\mathbb{P})$. We therefore assume without loss of generality that we are forcing above a condition which picks $\mathbb{Q}$ in the stage κ_2 lottery sum held in $M^{\mathbb{P}}$ in the definition of $j(\mathbb{P})$. We will consequently in what follows slightly abuse notation and replace any term for the stage κ_2 lottery sum held in $M^{\mathbb{P}}$ in the definition of $j(\mathbb{P})$ with $\dot{\mathbb{Q}}$.

We consider below two cases.

Case 1: In M, $\Vdash_{\mathbb{P} * \dot{\mathbb{Q}}}$ "Level by level equivalence between strong compactness and supercompactness holds at κ_2". Under these circumstances, since $\dot{\mathbb{Q}}^1_{\kappa_2}$ is a term for trivial forcing (our slight abuse of notation allows us to consider the terms $\dot{\mathbb{Q}}$ and $\dot{\mathbb{Q}}^0_{\kappa_2}$ as being the same), in M, $j(\mathbb{P} * \dot{\mathbb{Q}})$ is forcing equivalent to $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}} * j(\dot{\mathbb{Q}})$, where the first nontrivial stage in the forcing denoted by $\dot{\mathbb{R}}$ takes place well after λ. Standard arguments, as given, e.g., in the proof of Lemma 8.3 of [6] now show that j lifts in $V^{\mathbb{P} * \dot{\mathbb{Q}}}$ to $j : V^{\mathbb{P} * \dot{\mathbb{Q}}} \to M^{j(\mathbb{P} * \dot{\mathbb{Q}})}$, i.e., $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ "κ_2 is λ supercompact". This completes our discussion of Case 1. $\square$

Case 2: In M, $\Vdash_{\mathbb{P} * \dot{\mathbb{Q}}}$ "Level by level equivalence between strong compactness and supercompactness fails at κ_2". Under these circumstances, the proof of Lemma 2.6 is virtually identical to the proof of Lemma 2.3 of Apter-Sargsyan [8]. Specifically, because λ has been chosen large enough, we may assume by taking a normal measure over κ_2 having trivial Mitchell rank that $k : M \to N$ is an elementary embedding witnessing the measurability of κ_2 definable in M such that $N \models$ "κ_2 is not measurable". It is the case that if $i : V \to N$ is an elementary embedding having critical point κ_2 and for any $x \subseteq N$ with $|x| \le \lambda$, there is some $y \in N$ such that $x \subseteq y$ and $N \vDash$ "$|y| < i(\kappa_2)$", then i witnesses the λ strong compactness of κ_2. Using this fact, it is easily verifiable that $i = k \circ j$ is an elementary embedding witnessing the λ strong compactness of κ_2. To complete our discussion of Case 2, we show that i lifts in $V^{\mathbb{P} * \dot{\mathbb{Q}}}$ to $i : V^{\mathbb{P} * \dot{\mathbb{Q}}} \to N^{i(\mathbb{P} * \dot{\mathbb{Q}})}$.

To do this, we use a modification of an argument originally due to Magidor, unpublished by him but found in, among other places, Lemma 8 of Apter-Hamkins [5]. The modification is due to Sargsyan. Note that throughout the course of the remainder of the proof of Lemma 2.6, we will refer readers to the construction given in Lemma 8 of [5] when relevant, and omit details already presented therein.

Let G_0 be V-generic over $\mathbb{P}$, and let H be $V[G_0]$-generic over $\mathbb{Q}$. Write $i(\mathbb{P}) = \mathbb{P} * \dot{\mathbb{P}}^1 * \dot{\mathbb{P}}^2 * \dot{\mathbb{P}}^3$, where $\dot{\mathbb{P}}^1$ is a term for the portion of the forcing defined from stage κ_2 to stage $k(\kappa_2)$, $\dot{\mathbb{P}}^2$ is a term for the forcing done at stage $k(\kappa_2)$, and $\dot{\mathbb{P}}^3$ is a term for the remainder of the forcing, i.e., the portion done after stage $k(\kappa_2)$. We will build in $V[G_0][H]$ generic objects for the different portions of $i(\mathbb{P})$.

We begin by constructing an $N[G_0]$-generic object G_1 for $\mathbb{P}^1$. The argument used is essentially the same as the one given in the construction of the generic object G_1 found in Lemma 8 of [5] (and will therefore be carried out in $M[G_0] \subseteq V[G_0] \subseteq V[G_0][H]$). Specifically, since $N \vDash$ "κ_2 is not measurable", only trivial forcing is done at stage κ_2 in N, which means that $\dot{\mathbb{P}}^1$ is forced to act nontrivially on ordinals in the open interval $(\kappa_2, k(\kappa_2))$. In addition, since GCH holds in N for cardinals greater than or equal to κ_1 (as it does in V and M), standard counting arguments show that $N[G_0] \vDash$ "$|\mathbb{P}^1| = k(\kappa_2)$ and $|\wp(\mathbb{P}^1)| = 2^{k(\kappa_2)} = k(\kappa_2^+)$". Consequently, since GCH for cardinals greater than or equal to κ_1 also yields that $M \vDash$ "$|k(\kappa_2^+)| = \kappa_2^+$", we may let $\langle D_\alpha \mid \alpha < \kappa_2^+ \rangle$ be an enumeration in $M[G_0]$ of the dense open subsets of $\mathbb{P}^1$ present in $N[G_0]$. We then build in $M[G_0]$ an $N[G_0]$-generic object G_1 for $\mathbb{P}^1$ by meeting in turn each member of $\langle D_\alpha \mid \alpha < \kappa_2^+ \rangle$, using the fact that $\mathbb{P}^1$ is $\prec\kappa_2^+$-strategically closed in $N[G_0]$ and $M[G_0]$. This strategic closure property of $\mathbb{P}_1$ follows from the fact that standard arguments show $N[G_0]$ remains κ_2-closed with respect to (both

$V[G_0]$ and) $M[G_0]$.

We next analyze the exact nature of $\dot{\mathbb{P}}^2$. By the definition of $\mathbb{P}$, the closure properties of M, and the fact that we are in Case 2, we may write $j(\mathbb{P} * \dot{\mathbb{Q}}) = \mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{Q}}' * \dot{\mathbb{R}} * j(\dot{\mathbb{Q}})$, where $\dot{\mathbb{Q}} * \dot{\mathbb{Q}}'$ is a term for the forcing taking place at stage κ_2 in M and $\dot{\mathbb{Q}}'$ is a term for the partial ordering which adds a non-reflecting stationary set of ordinals of cofinality κ_1 to κ_2. By elementarity, since $\dot{\mathbb{P}}^2$ is a term for the forcing which takes place at stage $k(\kappa_2)$ in N, we may write $\dot{\mathbb{P}}^2 = k(\dot{\mathbb{Q}}) * k(\dot{\mathbb{Q}}')$. We will construct in $M[G_0][H]$ generic objects for $k(\mathbb{Q})$ and $k(\mathbb{Q}')$.

For $k(\mathbb{Q})$, we use an argument containing ideas due to Woodin, also presented in Theorem 4.10 of [14], Lemma 4.2 of Apter [2], Lemma 3.4 of Apter-Sargsyan [7], and Lemma 2.3 of [8]. First, note that since N is given by an ultrapower, $N = \{k(h)(\kappa_2) \mid h : \kappa_2 \to M$ is a function in $M\}$. Further, since by the definition of G_1, $k''G_0 \subseteq G_0 * G_1$, k lifts in both $M[G_0]$ and $M[G_0][H]$ to $k : M[G_0] \to N[G_0][G_1]$. From these facts, we may now show that $k''H \subseteq k(\mathbb{Q})$ generates an $N[G_0][G_1]$-generic object G_2 over $k(\mathbb{Q})$. Specifically, given a dense open subset $D \subseteq k(\mathbb{Q})$, $D \in N[G_0][G_1]$, $D = i_{G_0 * G_1}(\dot{D})$ for some N-name $\dot{D} = k(\vec{D})(\kappa_2)$, where $\vec{D} = \langle D_\alpha \mid \alpha < \kappa_2 \rangle$ is a function in M. We may assume that every D_α is a dense open subset of $\mathbb{Q}$. Since $\mathbb{Q}$ is (κ_2, ∞)-distributive, it follows that $D' = \bigcap_{\alpha < \kappa_2} D_\alpha$ is also a dense open subset of $\mathbb{Q}$. As $k(D') \subseteq D$ and $H \cap D' \neq \emptyset$, $k''H \cap D \neq \emptyset$. Thus, $G_2 = \{p \in k(\mathbb{Q}) \mid \exists q \in k''H[q \geq p]\}$, which is definable in $M[G_0][H]$, is our desired $N[G_0][G_1]$-generic object over $k(\mathbb{Q})$. Then, since $k(\mathbb{Q}')$ is in $N[G_0][G_1][G_2]$ the partial ordering which adds a non-reflecting stationary set of ordinals of cofinality $k(\kappa_1)$ to $k(\kappa_2)$, we know that $N[G_0][G_1][G_2] \vDash$ "$|k(\mathbb{Q}')| = k(\kappa_2)$ and $|\wp(k(\mathbb{Q}'))| = 2^{k(\kappa_2)} = k(\kappa_2^+)$" Hence, since $N[G_0][G_1][G_2]$ remains κ_2-closed with respect to $M[G_0][H]$, which means $k(\mathbb{Q}')$ is $\prec\kappa_2^+$-strategically closed in $N[G_0][G_1][G_2]$ and $M[G_0][H]$, the same argument used in the construction of G_1 allows us to build in $M[G_0][H]$ an $N[G_0][G_1][G_2]$-generic object G_3 for $k(\mathbb{Q}')$.

We construct now (in $V[G_0][H]$) an $N[G_0][G_1][G_2][G_3]$-generic object for $\mathbb{P}^3$. We do this by combining the term forcing argument found in Lemma 8 of [5] with the argument for the creation of a "master condition" found in Lemma 2 of [4]. Specifically, we begin by showing the existence of a term $\tau \in M$ for a "master condition" for $j(\dot{\mathbb{Q}})$, i.e., we show the existence of a term $\tau \in M$

in the language of forcing with respect to $j(\mathbb{P})$ such that in M, $\Vdash_{j(\mathbb{P})}$ "$\tau \in j(\dot{\mathbb{Q}})$ extends every $j(\dot{q})$ for $\dot{q} \in \dot{H}$". We first note that since $\mathbb{P}$ is κ_2-c.c. in both V and M, as $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is κ_2-directed closed and $|\dot{\mathbb{Q}}| < \lambda$", the usual arguments show $M[G_0][H]$ remains λ-closed with respect to $V[G_0][H]$. This means $T = \{j(\dot{q}) \mid \exists r \in G_0[\langle r, q\rangle \in G_0 * H]\} \in M[G_0][H]$ has a name $\dot{T} \in M$ such that in M, $\Vdash_{j(\mathbb{P})}$ "$|\dot{T}| < \lambda < j(\kappa_2)$, any two elements of $\dot{T}$ are compatible, and $\dot{T}$ is a subset of a partial ordering (namely $j(\dot{\mathbb{Q}})$) which is $j(\kappa_2)$-directed closed". Thus, since $M^\lambda \subseteq M$, $\Vdash_{j(\mathbb{P})}$ "There is an upper bound for $\dot{T}$". A term τ for this upper bound is as desired.

We work for the time being in M. Consider the "term forcing" partial ordering $\mathbb{R}^*$ (see Foreman [11] for the first published account of term forcing or Cummings [10], Section 1.2.5, page 8 — the notion is originally due to Laver) associated with $\dot{\mathbb{R}} * j(\dot{\mathbb{Q}})$, i.e., $\sigma \in \mathbb{R}^*$ iff σ is a term in the forcing language with respect to $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{Q}}'$ and $\Vdash_{\mathbb{P}*\dot{\mathbb{Q}}*\dot{\mathbb{Q}}'}$ "$\sigma \in \dot{\mathbb{R}} * j(\dot{\mathbb{Q}})$", ordered by $\sigma_1 \geq \sigma_0$ iff $\Vdash_{\mathbb{P}*\dot{\mathbb{Q}}*\dot{\mathbb{Q}}'}$ "$\sigma_1 \geq \sigma_0$". Note that τ' defined as the term in the language of forcing with respect to $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{Q}}'$ composed of the tuple all of whose members are forced to be the trivial condition, with the exception of the last member, which is τ, is an element of $\mathbb{R}^*$.

Clearly, $\mathbb{R}^* \in M$. In addition, since $V \vDash$ "No cardinal greater than κ_2 is measurable", as in Case 1, $M \vDash$ "The first stage at which $\dot{\mathbb{R}} * j(\dot{\mathbb{Q}})$ is forced to do nontrivial forcing is greater than λ". Thus, $\Vdash_{\mathbb{P}*\dot{\mathbb{Q}}*\dot{\mathbb{Q}}'}$ "$\dot{\mathbb{R}} * j(\dot{\mathbb{Q}})$ is $\prec\lambda^+$-strategically closed", which, since $M^\lambda \subseteq M$, immediately implies that $\mathbb{R}^*$ itself is $\prec\lambda^+$-strategically closed in both V and M. Further, since $V^{\mathbb{P}} \vDash$ "$|\mathbb{Q}| < \lambda$", in M, $\Vdash_{\mathbb{P}*\dot{\mathbb{Q}}*\dot{\mathbb{Q}}'}$ "$|\dot{\mathbb{R}} * j(\dot{\mathbb{Q}})| < j(\lambda)$". Also, by GCH for cardinals greater than or equal to κ_1 in both V and M and the fact that j may be assumed to be given via an ultrapower embedding by a normal measure over $P_{\kappa_2}(\lambda)$, $|j(\lambda^+)| = |\{f \mid f : P_{\kappa_2}(\lambda) \to \lambda^+| = |[\lambda^+]^\lambda| = \lambda^+$ and $\Vdash_{\mathbb{P}*\dot{\mathbb{Q}}*\dot{\mathbb{Q}}'}$ "$|\wp(\dot{\mathbb{R}} * j(\dot{\mathbb{Q}}))| < 2^{j(\lambda)} = j(\lambda^+)$". Therefore, since as in the footnote given in the proof of Lemma 8 of [5], we may assume that $\mathbb{R}^*$ has cardinality less than $j(\lambda)$ in M, we may let $\langle D_\alpha \mid \alpha < \lambda^+\rangle \in V$ be an enumeration of the dense open subsets of $\mathbb{R}^*$ present in M. It is then possible using the $\prec\lambda^+$-strategic closure of $\mathbb{R}^*$ in V and the argument employed in the construction of G_1 to build in V an M-generic object G_4^* for $\mathbb{R}^*$ containing τ'.

Note now that since N is given by an ultrapower of M via a normal measure over κ_2, Fact 2 of Section 1.2.2 of [10] tells us that $k''G_4^*$ generates an N-generic object G_4^{**} over $k(\mathbb{R}^*)$ containing $k(\tau')$. By elementarity, $k(\mathbb{R}^*)$ is the term forcing in N defined with respect to $k(j(\mathbb{P})_{\kappa_2+1}) = \mathbb{P} * \dot{\mathbb{P}}^1 * \dot{\mathbb{P}}^2$. Therefore, since $i(\mathbb{P} * \dot{\mathbb{Q}}) = k(j(\mathbb{P} * \dot{\mathbb{Q}})) = \mathbb{P} * \dot{\mathbb{P}}^1 * \dot{\mathbb{P}}^2 * \dot{\mathbb{P}}^3$, G_4^{**} is N-generic over $k(\mathbb{R}^*)$, and $G_0 * G_1 * G_2 * G_3$ is $k(\mathbb{P} * \dot{\mathbb{Q}})$-generic over N, Fact 1 of Section 1.2.5 of [10] (see also [11]) tells us that for $G_4 = \{i_{G_0 * G_1 * G_2 * G_3}(\sigma) \mid \sigma \in G_4^{**}\}$, G_4 is $N[G_0][G_1][G_2][G_3]$-generic over $\mathbb{P}^3$. In addition, since the definition of τ tells us that in M, the statement "$\langle p, \dot{q}\rangle \in j(\mathbb{P} * \dot{\mathbb{Q}})$ implies that $\langle p, \dot{q}\rangle \Vdash_{j(\mathbb{P} * \dot{\mathbb{Q}})}$ '$\tau \geq \dot{q}$' " is true, by elementarity, in N, the statement "$\langle p, \dot{q}\rangle \in k(j(\mathbb{P} * \dot{\mathbb{Q}}))$ implies that $\langle p, \dot{q}\rangle \Vdash_{k(j(\mathbb{P} * \dot{\mathbb{Q}}))}$ '$k(\tau) \geq \dot{q}$' " is true. In other words, since $k \circ j = i$, in N, the statement "$\langle p, \dot{q}\rangle \in i(\mathbb{P} * \dot{\mathbb{Q}})$ implies that $\langle p, \dot{q}\rangle \Vdash_{i(\mathbb{P} * \dot{\mathbb{Q}})}$ '$k(\tau) \geq \dot{q}$' " is true. Thus, in N, $k(\tau)$ functions as a term for a "master condition" for $i(\dot{\mathbb{Q}})$, so since G_4^{**} contains $k(\tau')$, the construction of all of the above generic objects immediately yields that $i''(G_0 * H) \subseteq G_0 * G_1 * G_2 * G_3 * G_4$. This means that i lifts in $V^{\mathbb{P} * \dot{\mathbb{Q}}}$ to $i : V^{\mathbb{P} * \dot{\mathbb{Q}}} \to N^{i(\mathbb{P} * \dot{\mathbb{Q}})}$, i.e. $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ "κ_2 is λ strongly compact". This completes our discussion of Case 2. □

We now see that regardless if Case 1 or Case 2 holds, $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ "κ_2 is λ strongly compact". Since λ was arbitrary, this completes the proof of Lemma 2.6.

□

Lemma 2.7 *$V^{\mathbb{P}} \models$ "κ_1 and κ_2 are the first two strongly compact cardinals".*

Proof: It follows by Ketonen's characterization of strong compactness given in [16] that any cardinal which is strongly compact up to a strongly compact cardinal is itself strongly compact. Thus, since $V \models$ "κ_1 is the first strongly compact cardinal", it must be the case that for any $\delta < \kappa_1$, $V \models$ "δ is not strongly compact up to κ_1". In addition, as we observed in the proof of Lemma 2.5, $\mathbb{P}$ is κ_1-directed closed, and so cannot force a new degree of strong compactness for any $\delta < \kappa_1$. Therefore, since $V \models$ "κ_1 is both indestructibly strongly compact and the first strongly compact cardinal", $V^{\mathbb{P}} \models$ "κ_1 is both indestructibly strongly compact and the first strongly compact cardinal". Next, if $\delta \in (\kappa_1, \kappa_2)$ is such that $V^{\mathbb{P}} \models$ "δ is measurable", then because $\mathbb{P}$ admits a sufficiently small gap above κ_1, by Theorem 3, $V \models$ "δ is measurable". Hence, because by its definition, $\mathbb{P}$ is mild with respect to δ, Theorem 3 also tells us that forcing with $\mathbb{P}$ cannot create any new degrees of strong compactness of δ. As $V \vDash$ "No cardinal $\delta \in (\kappa_1, \kappa_2)$ is strongly compact", we may consequently infer that $V^{\mathbb{P}} \vDash$ "No cardinal $\delta \in (\kappa_1, \kappa_2)$ is strongly compact". Since we have already seen that $V^{\mathbb{P}} \vDash$ "κ_2 is supercompact", $V^{\mathbb{P}} \vDash$ "κ_2 is the second strongly compact cardinal". This completes the proof of Lemma 2.7.

$\square$

Since $V \vDash$ "κ_1 is a limit of measurable cardinals and $\mathbb{P}$ is κ_1-directed closed", $V^{\mathbb{P}} \vDash$ "κ_1 is a limit of measurable cardinals". Therefore, by taking $\overline{V} = V^{\mathbb{P}}$, Lemmas 2.1 - 2.7, their proofs, and the intervening remarks complete the proof of Theorem 2.

$\square$

Theorem 2 leaves open some interesting questions, which we pose in conclusion to this paper. In particular, is it possible to prove an analogue of Theorem 2 for two strongly compact cardinals in which the large cardinal structure of the model constructed has fewer restrictions? Is it possible to prove an analogue of Theorem 2 for two strongly compact cardinals in which κ_2's strong compactness is indestructible under arbitrary κ_2-directed closed forcing? Is it possible to prove an analogue of Theorem 2 for two strongly compact cardinals in which κ_2 is indestructibly supercompact? Is it possible to prove an analogue of Theorem 2 in which the model constructed contains more than two strongly compact cardinals?

References

[1] A. Apter, "Level by Level Equivalence and Strong Compactness", *Mathematical Logic Quarterly 50*, 2004, 51–64.

[2] A. Apter, "Some Remarks on Indestructibility and Hamkins' Lottery Preparation", *Archive for Mathematical Logic 42*, 2003, 717–735.

[3] A. Apter, "Supercompactness and Level by Level Equivalence are Compatible with Indestructibility for Strong Compactness", *Archive for Mathematical Logic 46*, 2007, 155–163.

[4] A. Apter, M. Gitik, "The Least Measurable can be Strongly Compact and Indestructible", *Journal of Symbolic Logic 63*, 1998, 1404–1412.

[5] A. Apter, J. D. Hamkins, "Exactly Controlling the Non-Supercompact Strongly Compact Cardinals", *Journal of Symbolic Logic 68*, 2003, 669–688.

[6] A. Apter, J. D. Hamkins, "Indestructibility and the Level-by-Level Agreement between Strong Compactness and Supercompactness", *Journal of Symbolic Logic 67*, 2002, 820–840.

[7] A. Apter, G. Sargsyan, "Identity Crises and Strong Compactness III: Woodin Cardinals", *Archive for Mathematical Logic 45*, 2006, 307–322.

[8] A. Apter, G. Sargsyan, "Universal Indestructibility for Degrees of Supercompactness and Strongly Compact Cardinals", *Archive for Mathematical Logic 47*, 2008, 133–142.

[9] A. Apter, S. Shelah, "On the Strong Equality between Supercompactness and Strong Compactness", *Transactions of the American Mathematical Society 349*, 1997, 103–128.

[10] J. Cummings, "A Model in which GCH Holds at Successors but Fails at Limits", *Transactions of the American Mathematical Society 329*, 1992, 1–39.

[11] M. Foreman, "More Saturated Ideals", in: *Cabal Seminar 79-81*, **Lecture Notes in Mathematics 1019**, Springer-Verlag, Berlin and New York, 1983, 1–27.

[12] J. D. Hamkins, "Gap Forcing", *Israel Journal of Mathematics 125*, 2001, 237–252.

[13] J. D. Hamkins, "Gap Forcing: Generalizing the Lévy-Solovay Theorem", *Bulletin of Symbolic Logic 5*, 1999, 264–272.

[14] J. D. Hamkins, "The Lottery Preparation", *Annals of Pure and Applied Logic 101*, 2000, 103–146.

[15] T. Jech, *Set Theory: The Third Millennium Edition, Revised and Expanded*, Springer-Verlag, Berlin and New York, 2003.

[16] J. Ketonen, "Strong Compactness and Other Cardinal Sins", *Annals of Mathematical Logic 5*, 1972, 47–76.

[17] R. Laver, "Making the Supercompactness of κ Indestructible under κ-Directed Closed Forcing", *Israel Journal of Mathematics 29*, 1978, 385–388.

[18] A. Lévy, R. Solovay, "Measurable Cardinals and the Continuum Hypothesis", *Israel Journal of Mathematics 5*, 1967, 234–248.