A Note on Indestructibility and Strong Compactness *†

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Abstract

If $\kappa < \lambda$ are such that κ is both supercompact and indestructible under κ-directed closed forcing which is also (κ^+, ∞)-distributive and λ is 2^λ supercompact, then by [3, Theorem 5], $\{\delta < \kappa \mid \delta$ is δ^+ strongly compact yet δ isn't δ^+ supercompact$\}$ must be unbounded in κ. We show that the large cardinal hypothesis on λ is necessary by constructing a model containing a supercompact cardinal κ in which no cardinal $\delta > \kappa$ is $2^\delta = \delta^+$ supercompact, κ's supercompactness is indestructible under κ-directed closed forcing which is also (κ^+, ∞)-distributive, and for every measurable cardinal δ, δ is δ^+ strongly compact iff δ is δ^+ supercompact.

1 Introduction and Preliminaries

In [3], it was shown (see Theorem 5) that if $\kappa < \lambda$ are such that κ is indestructibly supercompact and λ is 2^λ supercompact, then $\{\delta < \kappa \mid \delta$ is δ^+ strongly compact yet δ isn't δ^+ supercompact$\}$ must be unbounded in κ. The only use of indestructibility in this proof is that κ remains supercompact after forcing with the partial ordering which first (if necessary) makes $2^\lambda = \lambda^+$ and $2^{\lambda^+} = \lambda^{++}$ and then does a reverse Easton iteration of length λ which adds a non-reflecting stationary set of

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ordinals of cofinality κ to each measurable cardinal in a final segment of the open interval (κ, λ). Thus, we actually have the following result.

Theorem 1 *Suppose $\kappa^+ \le \gamma < \lambda$ are such that κ is supercompact, κ's supercompactness is indestructible under κ-directed closed forcing which is also (γ, ∞)-distributive, and λ is 2^λ supercompact. Then $A = \{\delta < \kappa \mid \delta$ is δ^+ strongly compact yet δ isn't δ^+ supercompact$\}$ is unbounded in κ.*

The purpose of this note is to show that the large cardinal hypothesis on λ in Theorem 1 is necessary. Specifically, we prove the following theorem.

Theorem 2 *Suppose $V \models$ "ZFC + GCH + κ is supercompact + No cardinal $\delta > \kappa$ is $2^\delta = \delta^+$ supercompact + For every cardinal δ, δ is δ^+ strongly compact iff δ is δ^+ supercompact". There is then a partial ordering $\mathbb{P} \in V$ such that $V^{\mathbb{P}} \models$ "ZFC + κ is supercompact + No cardinal $\delta > \kappa$ is $2^\delta = \delta^+$ supercompact". In $V^{\mathbb{P}}$, κ's supercompactness is indestructible under κ-directed closed forcing which is also (κ^+, ∞)-distributive. Further, in $V^{\mathbb{P}}$, δ is δ^+ strongly compact iff δ is δ^+ supercompact.*

The existence of models V satisfying the hypotheses of Theorem 2 (and much more) was first shown in [4]. By a result of Menas [12], $V \models$ "No cardinal $\delta < \kappa$ is both measurable and a limit of cardinals γ which are either δ^+ strongly compact or δ^+ supercompact", since if δ is the least such cardinal, then $V \models$ "δ is δ^+ strongly compact but not δ^+ supercompact". Hence, there must of necessity be some restrictions on the large cardinal structure of V below κ.

We conclude Section 1 with a very brief discussion of some preliminary material. We presume a basic knowledge of large cardinals and forcing. A good reference in this regard is [8]. We also mention that the partial ordering $\mathbb{P}$ is *κ-directed closed* if for every directed set D of conditions of size less than κ, there is a condition in $\mathbb{P}$ extending each member of D. $\mathbb{P}$ is *(κ, ∞)-distributive* if the intersection of κ many dense-open subsets of $\mathbb{P}$ is dense open. It therefore follows that forcing with any partial ordering $\mathbb{P}$ which is both κ-directed closed and (κ^+, ∞)-distributive preserves either the κ^+ strong compactness or κ^+ supercompactness of κ, since forcing with $\mathbb{P}$ preserves $P_\kappa(\kappa^+)$.

We abuse notation slightly and take $V^{\mathbb{P}}$ as being the generic extension of V by $\mathbb{P}$. An *indestructibly supercompact cardinal* is one as first given by Laver in [10], i.e., κ is indestructibly supercompact if κ's supercompactness is preserved in any generic extension via a κ-directed closed partial ordering. For δ any ordinal, δ' is the least cardinal $\gamma > \delta$ such that $V \vDash$ "γ is γ^+ supercompact".

A corollary of Hamkins' work on gap forcing found in [6, 7] will be employed in the proof of Theorem 2. We therefore state as a separate theorem what is relevant for this paper, along with some associated terminology, quoting from [6, 7] when appropriate. Suppose $\mathbb{P}$ is a partial ordering which can be written as $\mathbb{Q} * \dot{\mathbb{R}}$, where $|\mathbb{Q}| < \delta$, $\mathbb{Q}$ is nontrivial, and $\Vdash_{\mathbb{Q}}$ "$\dot{\mathbb{R}}$ is δ^+-directed closed". In Hamkins' terminology of [6, 7], $\mathbb{P}$ *admits a gap at* δ. In Hamkins' terminology of [6, 7], $\mathbb{P}$ is *mild with respect to a cardinal* κ iff every set of ordinals x in $V^{\mathbb{P}}$ of size below κ has a "nice" name τ in V of size below κ, i.e., there is a set y in V, $|y| < \kappa$, such that any ordinal forced by a condition in $\mathbb{P}$ to be in τ is an element of y. Also, as in the terminology of [6, 7] and elsewhere, an embedding $j : \overline{V} \to \overline{M}$ is *amenable to* $\overline{V}$ when $j \restriction A \in \overline{V}$ for any $A \in \overline{V}$. The specific corollary of Hamkins' work from [6, 7] we will be using is then the following.

Theorem 3 (Hamkins) *Suppose that $V[G]$ is a generic extension obtained by forcing with $\mathbb{P}$ that admits a gap at some regular $\delta < \kappa$. Suppose further that $j : V[G] \to M[j(G)]$ is an embedding with critical point κ for which $M[j(G)] \subseteq V[G]$ and $M[j(G)]^{\delta} \subseteq M[j(G)]$ in $V[G]$. Then $M \subseteq V$; indeed, $M = V \cap M[j(G)]$. If the full embedding j is amenable to $V[G]$, then the restricted embedding $j \restriction V : V \to M$ is amenable to V. If j is definable from parameters (such as a measure or extender) in $V[G]$, then the restricted embedding $j \restriction V$ is definable from the names of those parameters in V. Finally, if $\mathbb{P}$ is mild with respect to κ and κ is λ strongly compact in $V[G]$ for any $\lambda \geq \kappa$, then κ is λ strongly compact in V.*

2 The Proof of Theorem 2

We turn now to the proof of Theorem 2.

Proof: Suppose $V \vDash$ "ZFC + GCH + κ is supercompact + No cardinal $\delta > \kappa$ is $2^{\delta} = \delta^+$

supercompact + For every cardinal δ, δ is δ^+ strongly compact iff δ is δ^+ supercompact". Let f be a Laver function [10] for κ, i.e., $f : \kappa \to V_\kappa$ is such that for every $x \in V$ and every $\lambda \geq |\mathrm{TC}(x)|$, there is an elementary embedding $j : V \to M$ generated by a supercompact ultrafilter over $P_\kappa(\lambda)$ such that $j(f)(\kappa) = x$. The partial ordering $\mathbb{P}$ which is used to establish Theorem 2 is the reverse Easton iteration of length κ which begins by adding a Cohen subset of ω and then (possibly) does nontrivial forcing only at those cardinals $\delta < \kappa$ which are at least δ^+ supercompact in V. At such a stage δ, if $f(\delta) = \dot{\mathbb{Q}}$ and $\Vdash_{\mathbb{P}_\delta}$ "$\dot{\mathbb{Q}}$ is a δ-directed closed, (δ^+, ∞)-distributive partial ordering having rank below δ'", then $\mathbb{P}_{\delta+1} = \mathbb{P}_\delta * \dot{\mathbb{Q}}$. If this is not the case, then $\mathbb{P}_{\delta+1} = \mathbb{P}_\delta * \dot{\mathbb{Q}}$, where $\dot{\mathbb{Q}}$ is a term for trivial forcing.

Lemma 2.1 $V^{\mathbb{P}} \models$ *"κ's supercompactness is indestructible under κ-directed closed forcing which is also (κ^+, ∞)-distributive".*

Proof: We follow the proof of [2, Lemma 2.1]. Let $\mathbb{Q} \in V^{\mathbb{P}}$ be such that $V^{\mathbb{P}} \models$ "$\mathbb{Q}$ is κ-directed closed and (κ^+, ∞)-distributive". Take $\dot{\mathbb{Q}}$ as a term for $\mathbb{Q}$ such that $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is κ-directed closed and (κ^+, ∞)-distributive". Suppose $\lambda \geq \max(\kappa^{++}, |\mathrm{TC}(\dot{\mathbb{Q}})|)$ is an arbitrary cardinal, and let $\gamma = 2^{|[\lambda]^{<\kappa}|}$. Take $j : V \to M$ as an elementary embedding witnessing the γ supercompactness of κ generated by a supercompact ultrafilter over $P_\kappa(\gamma)$ such that $j(f)(\kappa) = \dot{\mathbb{Q}}$. Since $V \models$ "No cardinal δ above κ is $2^\delta = \delta^+$ supercompact", $\gamma \geq 2^{[\kappa^+]^{<\kappa}}$, and $M^\gamma \subseteq M$, $M \models$ "κ is $2^\kappa = \kappa^+$ supercompact and no cardinal δ in the half-open interval $(\kappa, \gamma]$ is $2^\delta = \delta^+$ supercompact". Hence, the definition of $\mathbb{P}$ implies that $j(\mathbb{P} * \dot{\mathbb{Q}}) = \mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}} * j(\dot{\mathbb{Q}})$, where the first stage at which $\dot{\mathbb{R}}$ is forced to do nontrivial forcing is well above γ. Laver's original argument from [10] now applies and shows $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ "κ is λ supercompact". (Simply let $G_0 * G_1 * G_2$ be V-generic over $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$, lift j in $V[G_0][G_1][G_2]$ to $j : V[G_0] \to M[G_0][G_1][G_2]$, take a master condition p for $j''G_1$ and a $V[G_0][G_1][G_2]$-generic object G_3 over $j(\mathbb{Q})$ containing p, lift j again in $V[G_0][G_1][G_2][G_3]$ to $j : V[G_0][G_1] \to M[G_0][G_1][G_2][G_3]$, and show by the γ^+-directed closure of $\mathbb{R} * j(\dot{\mathbb{Q}})$ that the supercompactness measure over $(P_\kappa(\lambda))^{V[G_0][G_1]}$ generated by j is actually a member of $V[G_0][G_1]$.) As λ and $\mathbb{Q}$ were arbitrary, this completes the proof of Lemma 2.1.

□

Since trivial forcing is both κ-directed closed and (κ^+, ∞)-distributive, Lemma 2.1 implies that $V^{\mathbb{P}} \models$ "κ is supercompact". Also, because $\mathbb{P}$ may be defined so that $|\mathbb{P}| = \kappa$, standard arguments in tandem with the results of [11] show that $V^{\mathbb{P}} \models$ "No cardinal $\delta > \kappa$ is either $2^\delta = \delta^+$ strongly compact or supercompact".

Lemma 2.2 *If $V \models$ "δ is δ^+ supercompact", then $V^{\mathbb{P}} \models$ "δ is δ^+ supercompact".*

Proof: Suppose $V \models$ "δ is δ^+ supercompact". As $V \models$ "No cardinal $\delta > \kappa$ is $2^\delta = \delta^+$ supercompact" and $V^{\mathbb{P}} \models$ "κ is supercompact", we may assume that $\delta < \kappa$.

Write $\mathbb{P} = \mathbb{P}_\delta * \dot{\mathbb{P}}^\delta$. Since by the definition of $\mathbb{P}$, $\Vdash_{\mathbb{P}_\delta}$ "$\dot{\mathbb{P}}^\delta$ is both δ-directed closed and (δ^+, ∞)-distributive", to show $V^{\mathbb{P}} = V^{\mathbb{P}_\delta * \dot{\mathbb{P}}^\delta} \models$ "δ is δ^+ supercompact", it suffices to show that $V^{\mathbb{P}_\delta} \models$ "δ is δ^+ supercompact". To do this, we consider the following two cases.

Case 1: $|\mathbb{P}_\delta| < \delta$. If this occurs, then by the results of [11], $V^{\mathbb{P}_\delta} \models$ "δ is δ^+ supercompact". $\square$

Case 2: $|\mathbb{P}_\delta| \geq \delta$. In this situation, by the definition of $\mathbb{P}$, $|\mathbb{P}_\gamma| < \delta$ for every $\gamma < \delta$, and δ is a limit of cardinals γ which are γ^+ supercompact. Hence, $|\mathbb{P}_\delta| = \delta$. Let $j : V \to M$ be an elementary embedding witnessing the δ^+ supercompactness of δ generated by a supercompact ultrafilter over $P_\delta(\delta^+)$ such that $M \models$ "δ isn't δ^+ supercompact". We may now infer that only trivial forcing is done at stage δ in M in the definition of $j(\mathbb{P}_\delta)$. It then follows that $j(\mathbb{P}_\delta) = \mathbb{P}_\delta * \dot{\mathbb{Q}}$, where the first stage at which $\dot{\mathbb{Q}}$ is forced to do nontrivial forcing is well above δ^+. A standard diagonalization argument (see, e.g., the proof of [3, Lemma 8.1]) now shows that $V^{\mathbb{P}_\delta} \models$ "δ is δ^+ supercompact". $\square$

Cases 1 and 2 complete the proof of Lemma 2.2.

$\square$

Lemma 2.3 *$V^{\mathbb{P}} \models$ "δ is δ^+ strongly compact iff δ is δ^+ supercompact".*

Proof: Suppose $V^{\mathbb{P}} \models$ "δ is δ^+ strongly compact". By Lemma 2.2 and our remarks above, we may assume without loss of generality that $\delta < \kappa$ and $V \models$ "δ isn't δ^+ supercompact". Let $\gamma = \sup(\{\alpha < \delta \mid \alpha \text{ is } \alpha^+ \text{ supercompact}\})$, and write $\mathbb{P} = \mathbb{P}_\gamma * \dot{\mathbb{Q}}$. By the definition of $\mathbb{P}$, $\Vdash_{\mathbb{P}_\gamma}$ "$\dot{\mathbb{Q}}$

is both δ'-directed closed and $((\delta')^+, \infty)$-distributive" (from which it follows that $\Vdash_{\mathbb{P}_\gamma}$ "$\dot{\mathbb{Q}}$ is both δ-directed closed and (δ^+, ∞)-distributive"). Consequently, $V^{\mathbb{P}_\gamma} \vDash$ "δ is δ^+ strongly compact". Further, by its definition, $\mathbb{P}_\gamma$ admits a gap at $\aleph_1$.

If $|\mathbb{P}_\gamma| < \delta$, then by the results of [11], $V \vDash$ "δ is δ^+ strongly compact". Hence, by our hypotheses on V, $V \vDash$ "δ is δ^+ supercompact", which is contradictory to our assumptions. If $|\mathbb{P}_\gamma| \geq \delta$, then we first assume that $\mathbb{P}_\gamma$ is mild with respect to δ. Under these circumstances, by Theorem 3, $V \vDash$ "δ is δ^+ strongly compact", which means we reach the same contradiction as when $|\mathbb{P}_\gamma| < \delta$. Thus, we may assume without loss of generality that $\mathbb{P}_\gamma$ isn't mild with respect to δ.

We consider now the following two cases. Our argument is analogous to the one given in the proof of [1, Lemma 2.3].

Case 1: $(\delta^+)^V < (\delta^+)^{V^{\mathbb{P}_\gamma}}$. If this is the situation, then as δ is measurable and hence a cardinal in $V^{\mathbb{P}_\gamma}$, $V^{\mathbb{P}_\gamma} \vDash$ "$|(\delta^+)^V| = \delta$". Therefore, since for any ordinal ρ having cardinality δ, δ is measurable iff δ is ρ strongly compact iff δ is ρ supercompact, $V^{\mathbb{P}_\gamma} \vDash$ "δ is $(\delta^+)^V$ supercompact". By Theorem 3, $V \vDash$ "δ is $(\delta^+)^V = \delta^+$ supercompact", an immediate contradiction. $\square$

Case 2: $(\delta^+)^V = (\delta^+)^{V^{\mathbb{P}_\gamma}}$. To handle when this occurs, we use an idea due to Hamkins, which has also appeared in [5] in a more general context (as well as in this context in [1, Lemma 2.3]). Hamkins' argument is as follows. Let G be V-generic over $\mathbb{P}_\gamma$, and let $j : V[G] \to M[j(G)]$ be an elementary embedding witnessing the δ^+ strong compactness of δ generated by a δ-additive, fine ultrafilter over $P_\delta(\delta^+)$ present in $V[G]$. As $M[j(G)]^\delta \subseteq M[j(G)]$, by Theorem 3, the embedding $j^* = j \restriction V : V \to M$ is definable in V. Note that j and j^* agree on the ordinals. Since j is a δ^+ strong compactness embedding in $V[G]$, there is some $X \subseteq j(\delta^+)$, $X \in M[j(G)]$ with $j''\delta^+ \subseteq X$ and $M[j(G)] \vDash$ "$|X| < j(\delta^+)$". Therefore, since δ^+ is regular in $V[G]$, $j(\delta^+)$ is regular in $M[j(G)]$, so we can find an $\alpha < j(\delta^+)$ with $\alpha > \sup(X) \geq \sup(j''\delta^+)$. This means that if $x \subseteq \delta^+$ is such that $x \subseteq \beta < \delta^+$, $j(\alpha) \notin j(x) \subseteq j(\beta)$. But then, $\mathcal{U} = \{x \subseteq \delta^+ \mid \alpha \in j^*(x)\}$ defines in V a δ-additive, uniform ultrafilter over δ^+ which gives measure 1 to sets having size δ^+. By a theorem of Ketonen [9], δ is δ^+ strongly compact in V. Again by our hypotheses on V, $V \vDash$ "δ is δ^+ supercompact", a contradiction. $\square$

Thus, assuming $V^{\mathbb{P}} \models$ "δ is δ^+ strongly compact" leads to the conclusion that $V \models$ "δ is δ^+ supercompact". Since this contradicts our initial assumptions, the proof of Lemma 2.3 is now complete.

$\square$

Lemmas 2.1 – 2.3 and the intervening remarks complete the proof of Theorem 2.

$\square$

We take this opportunity to observe that our preceding work actually shows that if $V^{\mathbb{P}} \models$ "$\mathbb{Q}$ is both κ-directed closed and (κ^+, ∞)-distributive", then $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ "δ is δ^+ strongly compact iff δ is δ^+ supercompact". This easily follows for $\delta \leq \kappa$, since any forcing which is both κ-directed closed and (κ^+, ∞)-distributive will preserve the conclusions of Lemma 2.3. For $\delta > \kappa$, the arguments of Lemma 2.3 with $\mathbb{P} * \dot{\mathbb{Q}}$ replacing $\mathbb{P}_\gamma$ show that if $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ "δ is δ^+ strongly compact", then $V \models$ "δ is δ^+ supercompact". This, of course, contradicts our initial hypotheses on V. Thus, we may in fact infer that $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ "No cardinal $\delta > \kappa$ is δ^+ strongly compact".

The methods we have used still leave open some interesting questions, with which we conclude this note. Specifically, is it possible to prove an analogue of Theorem 2 in which κ is (fully) indestructibly supercompact? Is it possible to prove an analogue of Theorem 2 in which, e.g., for every cardinal δ, δ is δ^{++} strongly compact iff δ is δ^{++} supercompact? Hamkins' idea of [5] used in the proof of Lemma 2.3 does not yet seem to generalize to the situation where δ is γ strongly compact but $\gamma \geq \delta^{++}$. Finally, in a question first posed in [3], is it possible to construct a model containing an indestructibly supercompact cardinal κ in which for every pair of regular cardinals $\delta < \gamma$, δ is γ strongly compact iff δ is γ supercompact? As Theorem 1 indicates, an answer to this final question would take place in a model with some restrictions on its large cardinal structure.

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