Indestructibility, HOD, and the Ground Axiom *†

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Abstract

Let φ_1 stand for the statement $V = \mathrm{HOD}$ and φ_2 stand for the Ground Axiom. Suppose T_i for $i = 1, \ldots, 4$ are the theories "ZFC + φ_1 + φ_2", "ZFC + $\neg\varphi_1$ + φ_2", "ZFC + φ_1 + $\neg\varphi_2$", and "ZFC + $\neg\varphi_1$ + $\neg\varphi_2$" respectively. We show that if κ is indestructibly supercompact and $\lambda > \kappa$ is inaccessible, then for $i = 1, \ldots, 4$, $A_i =_{\mathrm{df}} \{\delta < \kappa \mid \delta$ is an inaccessible cardinal which is not a limit of inaccessible cardinals and $V_\delta \vDash T_i\}$ must be unbounded in κ. The large cardinal hypothesis on λ is necessary, as we further demonstrate by constructing via forcing four models in which $A_i = \emptyset$ for $i = 1, \ldots, 4$. In each of these models, there is an indestructibly supercompact cardinal κ, and no cardinal $\delta > \kappa$ is inaccessible. We show it is also the case that if κ is indestructibly supercompact, then $V_\kappa \vDash T_1$, so by reflection, $B_1 =_{\mathrm{df}} \{\delta < \kappa \mid \delta$ is an inaccessible limit of inaccessible cardinals and $V_\delta \vDash T_1\}$ is unbounded in κ. Consequently, it is not possible to construct a model in which κ is indestructibly supercompact and $B_1 = \emptyset$. On the other hand, assuming κ is supercompact and no cardinal $\delta > \kappa$ is inaccessible, we demonstrate that it is possible to construct a model in which κ is indestructibly supercompact and for every inaccessible cardinal $\delta < \kappa$, $V_\delta \vDash T_1$. It is thus not possible to prove in ZFC that $B_i =_{\mathrm{df}} \{\delta < \kappa \mid \delta$ is an inaccessible limit of inaccessible cardinals and $V_\delta \vDash T_i\}$ for $i = 2, \ldots, 4$ is unbounded in κ if κ is indestructibly supercompact.

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We start with a very brief discussion of some preliminary material. We presume a basic knowledge of large cardinals and forcing. If κ is a cardinal, the partial ordering $\mathbb{P}$ is *κ-directed closed* if for every directed set D of conditions of size less than κ, there is a condition in $\mathbb{P}$ extending each member of D. When G is V-generic over $\mathbb{P}$, we abuse notation slightly and take both $V[G]$ and $V^{\mathbb{P}}$ as being the generic extension of V by $\mathbb{P}$.

We continue with some key definitions. As in [14], the cardinal κ is *indestructibly supercompact* if κ's supercompactness is preserved after forcing with a κ-directed closed partial ordering. The *Ground Axiom (GA)* is the assertion that the universe of sets V is not a generic extension of any inner model $W \subseteq V$ via some nontrivial (set) partial ordering $\mathbb{P} \in W$. GA was formulated by Hamkins and Reitz and studied by Reitz [15, 16] and Hamkins, Reitz, and Woodin [10]. Although GA is *prima facie* a second order statement, as Reitz has shown in [15, 16], it is actually first-order expressible. In addition, as was shown in [8, 5], if Paul Corazza's *Wholeness Axiom (WA)* (first introduced in [7]) is consistent, then it is consistent with GA. Since Corazza showed in [7] that WA is consistent relative to the existence of an I_3 cardinal and also showed in [7] that WA implies the existence of a cardinal κ which is super-n-huge for every $n \in \omega$, we know that GA is relatively consistent with some fairly large cardinals.

It is a very interesting fact that the large cardinal structure of the universe above either a supercompact or strong cardinal κ with suitable indestructibility properties can affect the large cardinal structure below κ. On the other hand, these effects can be mitigated if the universe contains relatively few large cardinals. These sorts of occurrences have been studied in [6, 1, 2, 3, 4].

The purpose of this paper is to continue investigating this phenomenon, but in the context of models of ZFC in which $V = \mathrm{HOD}$ can be either true or false and GA can be either true or false. Specifically, we prove four theorems, taking as our notation throughout that φ_1 stands for the statement $V = \mathrm{HOD}$, φ_2 stands for the Ground Axiom, T_i for $i = 1, \ldots, 4$ are the theories "ZFC + φ_1 + φ_2", "ZFC + $\neg\varphi_1$ + φ_2", "ZFC + φ_1 + $\neg\varphi_2$", and "ZFC + $\neg\varphi_1$ + $\neg\varphi_2$" respectively, and for $i = 1, \ldots, 4$, $A_i =_{\mathrm{df}} \{\delta < \kappa \mid \delta$ is an inaccessible cardinal which is not a limit of inaccessible cardinals and $V_\delta \models T_i\}$. We begin with the following theorem.

Theorem 1 *Suppose $\lambda > \kappa$ is inaccessible and κ is indestructibly supercompact. Then for each $i = 1, \ldots, 4$, A_i is unbounded in κ.*

Proof: Let V be our ground model. Suppose δ is any cardinal and ρ is the least inaccessible cardinal greater than δ. We describe four δ-directed closed partial orderings $\mathbb{P}^*_i$ for $i = 1, \ldots, 4$ such that $V^{\mathbb{P}^*_i} \models$ "$V_\rho \models T_i$". These partial orderings are as follows:

1. $\mathbb{P}^*_1$ is the partial ordering of [16, Theorem 11] (see also [15, 10]) as defined in V_ρ using a coding based on regular cardinals in the open interval (δ, ρ) such that $V_\rho^{\mathbb{P}^*_1} \models T_1$. For the exact definition of $\mathbb{P}^*_1$, we refer readers to [16]. We do note, however, that the work of [15, 16] shows that this coding may be done in a way such that $\mathbb{P}^*_1$ is δ-directed closed and forcing with $\mathbb{P}^*_1$ preserves the inaccessibility of ρ (which of course means that forcing with $\mathbb{P}^*_1$ preserves the fact that ρ is the least inaccessible cardinal greater than δ). The work of [15, 16] additionally shows that this coding may be done so that $V_\rho^{\mathbb{P}^*_1}$ is a model for the *Continuum Coding Axion (CCA)* of [15, 16, 10], which says that for every ordinal α and every $x \subseteq \alpha$, there is some ordinal θ such that $\beta \in x$ iff for every $\beta < \alpha$, $2^{\aleph_{\theta+\beta+1}} = \aleph_{\theta+\beta+2}$.[1]

2. $\mathbb{P}^*_2$ is the partial ordering of [10, Theorem 2] as defined in V_ρ using a coding based on regular cardinals in the open interval (δ, ρ) such that $V_\rho^{\mathbb{P}^*_2} \models T_2$. For the exact definition of $\mathbb{P}^*_2$, we refer readers to [10]. The work of [10] in conjunction with the work of [15, 16] once again show that this may be done in a way such that $\mathbb{P}^*_2$ is δ-directed closed and forcing with $\mathbb{P}^*_2$ preserves the inaccessibility of ρ.

3. $\mathbb{P}^*_3$ is the partial ordering of [16, Theorem 18] (see also [15]) as defined in V_ρ using a coding based on regular cardinals in the open interval (δ, ρ) such that $V_\rho^{\mathbb{P}^*_3} \models T_3$. For the exact definition of $\mathbb{P}^*_3$, we once more refer readers to [16]. The work of [16] again shows that this may be done in a way such that $\mathbb{P}^*_3$ is δ-directed closed and forcing with $\mathbb{P}^*_3$ preserves the inaccessibility of ρ.

[1]As pointed out by the referee, and as mentioned in [15, 16], the CCA implies a strengthening of itself in which there are unboundedly many θ which can be used to code the set of ordinals x. It is this stronger version that is used to infer GA in [15, 16]. Also, CCA clearly implies $V = \mathrm{HOD}$.

4. Fix an arbitrary regular cardinal $\gamma \in (\delta, \rho)$. $\mathbb{P}^*_4$ is then the partial ordering adding a Cohen subset of γ. Clearly, $\mathbb{P}^*_4$ is δ-directed closed. By the defintion of GA, $V_\rho^{\mathbb{P}^*_4} \models \neg$GA, and since $\mathbb{P}^*_4$ is *almost homogeneous* (i.e., for any $p, q \in \mathbb{P}^*_4$, there is an automorphism π of $\mathbb{P}^*_4$ such that $\pi(p)$ is compatible with q), as in [10, Theorem 1] (see also [13, pages 244–245]), $V_\rho^{\mathbb{P}^*_4} \models V \neq \text{HOD}$. Thus, since forcing with $\mathbb{P}^*_4$ preserves ρ's inaccessibility, $V_\rho^{\mathbb{P}^*_4} \models T_4$.

Having completed our description of the $\mathbb{P}^*_i$, we now follow the proof of [1, Theorem 2]. Suppose $\lambda > \kappa$ is inaccessible and κ is indestructibly supercompact. Without loss of generality, assume that λ is the least inaccessible cardinal above κ. Let i for $i = 1, \ldots, 4$ be fixed but arbitrary. Force with one of the partial orderings $\mathbb{P}^*_i$ as defined over the open interval (κ, λ). After this forcing, which is κ-directed closed, λ remains the least inaccessible cardinal above κ. In particular, after the forcing, λ is an inaccessible cardinal which is not a limit of inaccessible cardinals. Further, by the definition of $\mathbb{P}^*_i$, $V^{\mathbb{P}^*_i} \models$ "$V_\lambda \models T_i$". Since κ is suitably indestructible, by reflection, $A_i =_{\text{df}} \{\delta < \kappa \mid \delta$ is an inaccessible cardinal which is not a limit of inaccessible cardinals and $V_\delta \models T_i\}$ is unbounded in κ after the forcing has been performed. Once more, we infer by the fact $\mathbb{P}^*_i$ is κ-directed closed that A_i is unbounded in κ in the ground model.

$\square$

That the assumption of an inaccessible cardinal λ above the supercompact cardinal κ is necessary is shown by our next theorem.

Theorem 2 *Suppose $V \models$ "ZFC + κ is supercompact + No cardinal $\delta > \kappa$ is inaccessible". Then for each $i = 1, \ldots, 4$, there is a partial ordering $\mathbb{P} \in V$ such that $V^{\mathbb{P}} \models$ "ZFC + κ is indestructibly supercompact + No cardinal $\delta > \kappa$ is inaccessible + $A_i = \emptyset$".*

Proof: Suppose $V \models$ "ZFC + κ is supercompact + No cardinal $\delta > \kappa$ is inaccessible". Without loss of generality, by first doing a preliminary forcing if necessary, we assume in addition that $V \models$ GCH.

Assume i for $i = 1, \ldots, 4$ is given but arbitrary. Fix $k \neq i$, $k = 1, \ldots, 4$. Let $\langle \delta_j \mid j < \kappa \rangle$ be the continuous, increasing enumeration of $\{\omega\} \cup \{\delta < \kappa \mid \delta$ is either an inaccessible cardinal or a limit of inaccessible cardinals}. Let f be a Laver function [14] for κ, i.e., $f : \kappa \to V_\kappa$ is such that for every $x \in V$ and every $\lambda \geq |\mathrm{TC}(x)|$, there is an elementary embedding $j : V \to M$ generated by a supercompact ultrafilter over $P_\kappa(\lambda)$ such that $j(f)(\kappa) = x$. We define now a length κ reverse Easton iteration $\mathbb{P} = \langle\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha\rangle \mid \alpha < \kappa\rangle$ by four cases as follows, taking as an inductive hypothesis that $\Vdash_{\mathbb{P}_\alpha}$ "$\delta_{\alpha+1}$ is inaccessible" (so $\Vdash_{\mathbb{P}_\alpha}$ "$\delta_{\alpha+1}$ is the least inaccessible cardinal greater than δ_α"):

1. $\mathbb{P}_0 = \{\emptyset\}$.

2. If δ_α is not an inaccessible limit of inaccessible cardinals, then $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$, where $\dot{\mathbb{Q}}_\alpha$ is a term for the partial ordering $\mathbb{P}^*_k$ of Theorem 1 defined using ordinals in the open interval $(\delta_\alpha, \delta_{\alpha+1})$, so that $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ is (at least) δ_α-directed closed".

3. If δ_α is an inaccessible limit of inaccessible cardinals and $f(\delta_\alpha) = \langle \dot{\mathbb{Q}}, \delta\rangle$ where $\delta \in (\delta_\alpha, \delta_{\alpha+1})$ and $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}$ is δ_α-directed closed and has cardinality less than $\delta_{\alpha+1}$", let γ' be the least (singular) strong limit cardinal greater than $\max(|\mathrm{TC}(\dot{\mathbb{Q}})|, \delta)$. Then $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}} * \dot{\mathbb{Q}}' = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$, where $\dot{\mathbb{Q}}'$ is a term for the partial ordering $\mathbb{P}^*_k$ of Theorem 1 defined using ordinals in the open interval $(\gamma', \delta_{\alpha+1})$, so that $\Vdash_{\mathbb{P}_\alpha * \dot{\mathbb{Q}}}$ "$\dot{\mathbb{Q}}'$ is (at least) γ'-directed closed".

4. If δ_α is an inaccessible limit of inaccessible cardinals and Case 3 does not hold, then $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$, where $\dot{\mathbb{Q}}_\alpha$ is a term for the partial ordering $\mathbb{P}^*_k$ of Theorem 1 defined using ordinals in the open interval $(\delta_\alpha, \delta_{\alpha+1})$, so that $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ is (at least) δ_α-directed closed".

An easy induction shows that for any $\alpha < \kappa$, $|\mathbb{P}_\alpha| < \delta_{\alpha+1}$. From this, it follows that the inductive hypothesis holds and $\mathbb{P}$ is well-defined, i.e., that $V^{\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha} = V^{\mathbb{P}_{\alpha+1}} \models$ "$\delta_{\alpha+1}$ is inaccessible and $V_{\delta_{\alpha+1}} \models T_k$".

Lemma 1.1 $V^{\mathbb{P}} \models$ "*κ is indestructibly supercompact*".

Proof: We follow the proof of [1, Lemma 2.1]. Let $\mathbb{Q} \in V^{\mathbb{P}}$ be such that $V^{\mathbb{P}} \models$ "$\mathbb{Q}$ is κ-directed closed". Take $\dot{\mathbb{Q}}$ as a term for $\mathbb{Q}$ such that $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is κ-directed closed". Suppose $\lambda \geq |\mathrm{TC}(\dot{\mathbb{Q}})|$ is an arbitrary cardinal, and let $\gamma = 2^{|[\lambda]^{<\kappa}|}$. Take $j : V \to M$ as an elementary embedding witnessing the γ supercompactness of κ generated by a supercompact ultrafilter over $P_\kappa(\gamma)$ such that $j(f)(\kappa) = \langle \dot{\mathbb{Q}}, \gamma \rangle$. Since $V \models$ "No cardinal $\delta > \kappa$ is inaccessible" and $M^\gamma \subseteq M$, the definition of $\mathbb{P}$ implies that $j(\mathbb{P} * \dot{\mathbb{Q}}) = \mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}} * j(\dot{\mathbb{Q}})$, where the first stage at which $\dot{\mathbb{R}}$ is forced to do nontrivial forcing is well above γ. Laver's original argument from [14] now applies and shows that $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ "κ is λ supercompact". (Simply let $G_0 * G_1 * G_2$ be V-generic over $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$, lift j in $V[G_0][G_1][G_2]$ to $j : V[G_0] \to M[G_0][G_1][G_2]$, take a master condition p for $j''G_1$ and a $V[G_0][G_1][G_2]$-generic object G_3 over $j(\mathbb{Q})$ containing p, lift j again in $V[G_0][G_1][G_2][G_3]$ to $j : V[G_0][G_1] \to M[G_0][G_1][G_2][G_3]$, and show by the γ^+-directed closure of $\mathbb{R} * j(\dot{\mathbb{Q}})$ that the supercompactness measure over $(P_\kappa(\lambda))^{V[G_0][G_1]}$ generated by j is actually a member of $V[G_0][G_1]$.) As λ and $\mathbb{Q}$ were arbitrary, this completes the proof of Lemma 1.1.

$\square$

Lemma 1.2 *In $V^{\mathbb{P}}$, $A_k = \{\delta < \kappa \mid \delta$ is an inaccessible cardinal which is not a limit of inaccessible cardinals$\}$.*

Proof: For any $\delta < \kappa$ such that $V \models$ "δ is an inaccessible cardinal which is not a limit of inaccesssible cardinals", let $\alpha < \kappa$ be such that $\delta = \delta_{\alpha+1}$. Write $\mathbb{P} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha * \dot{\mathbb{R}} = \mathbb{P}_{\alpha+1} * \dot{\mathbb{R}}$. As we have already observed, $V^{\mathbb{P}_{\alpha+1}} \models$ "$\delta_{\alpha+1}$ is inaccessible and $V_{\delta_{\alpha+1}} \models T_k$". Since $\Vdash_{\mathbb{P}_{\alpha+1}}$ "$\dot{\mathbb{R}}$ is $\delta_{\alpha+1}$-directed closed", $V^{\mathbb{P}_{\alpha+1} * \dot{\mathbb{R}}} = V^{\mathbb{P}} \models$ "$\delta_{\alpha+1}$ is inaccessible and $V_{\delta_{\alpha+1}} \models T_k$". In addition, because $V \models$ "$\delta_{\alpha+1}$ is an inaccessible cardinal which is not a limit of inaccessible cardinals", $V^{\mathbb{P}} \models$ "$\delta_{\alpha+1}$ is an inaccessible cardinal which is not a limit of inaccessible cardinals" as well. Consequently, the proof of Lemma 1.2 will be complete once we have shown that if $V^{\mathbb{P}} \models$ "δ is an inaccessible cardinal which is not a limit of inaccessible cardinals", then $V \models$ "δ is an inaccessible cardinal which is not a limit of inaccessible cardinals". If not, $V \models$ "δ is an inaccessible limit of inaccessible cardinals", so $V \models$ "δ is an inaccessible limit of inaccessible cardinals which are not themselves limits of inaccessible cardinals". As we have just shown, such cardinals are preserved to $V^{\mathbb{P}}$, so $V^{\mathbb{P}} \models$ "δ is an inaccessible limit of inaccessible cardinals". This contradiction completes the proof of Lemma 1.2.

$\square$

Since by Lemma 1.2, $A_i = \emptyset$, and since forcing cannot create a new inaccessible cardinal, Lemmas 1.1 and 1.2 complete the proof of Theorem 2.

□

We observe that in the proof we have just given for Theorem 2, $A_j = \emptyset$ for $j \neq k$. Our method of proof allows for other possible values for the A_j, which we leave for readers to work out for themselves. Further, our method of proof shows that if $k = 1$, then $A_1 = \{\delta < \kappa \mid \delta$ is an inaccessible cardinal which is not a limit of inaccessible cardinals and $V_\delta \vDash \text{CCA}\}$.

Note that by definition, the A_i are mutually disjoint, and regardless if κ is also indestructible, $\bigcup_{i=1,\ldots,4} A_i = \{\delta < \kappa \mid \delta$ is an inaccessible cardinal which is not a limit of inaccessible cardinals$\}$. Observe also that in spite of Theorem 2, by the last sentence, at least one of the A_i must be unbounded in κ if κ is supercompact.

One may wonder if Theorem 1 can be improved. To make this more precise, one may ask if it is possible to infer anything along the lines of Theorem 1 without the additional assumption of an inaccessible cardinal above the supercompact cardinal κ. By Theorem 2, one would then of necessity have to work with inaccessible limits of inaccessible cardinals. In fact, this question has a positive answer, as the following theorem shows.

Theorem 3 *Suppose κ is indestructibly supercompact. Then $V_\kappa \vDash T_1$, so by reflection, $B_1 =_{\text{df}} \{\delta < \kappa \mid \delta$ is an inaccessible limit of inaccessible cardinals and $V_\delta \vDash T_1\}$ is unbounded in κ.*

Proof: As has already been mentioned, ZFC $\vdash$ "CCA $\Longrightarrow$ GA + V = HOD". Thus, if κ is indestructibly supercompact, to prove Theorem 3, it suffices to show that $V_\kappa \vDash \text{CCA}$.

To do this, fix $\alpha < \kappa$ and $x \subseteq \alpha$. Let $\mathbb{P}$ be the κ-directed closed partial ordering which first forces GCH for all cardinals δ in the closed interval $[\kappa, \kappa^{+\alpha+\alpha}]$ and then forces failures of GCH in this interval so that $\beta \in x$ iff for every $\beta < \alpha$, $2^{\aleph_{\kappa+\beta+1}} = \aleph_{\kappa+\beta+2}$. By indestructibility, let $\lambda > \kappa$ be sufficiently large with $j : V^{\mathbb{P}} \to M$ an elementary embedding witnessing the λ supercompactness of κ generated by a supercompact ultrafilter $\mathcal{U} \in V^{\mathbb{P}}$ over $P_\kappa(\lambda)$ such that $M \vDash$ "$\beta \in x$ iff for every $\beta < \alpha$, $2^{\aleph_{\kappa+\beta+1}} = \aleph_{\kappa+\beta+2}$". Since $\alpha < \kappa$ and κ is the critical point of j, by reflection, there are unboundedly many $\theta \in (\alpha, \kappa)$ such that in both $V^{\mathbb{P}}$ and $(V_\kappa)^{V^{\mathbb{P}}}$, $\beta \in x$ iff for every $\beta < \alpha$, $2^{\aleph_{\theta+\beta+1}} = \aleph_{\theta+\beta+2}$. Since $\mathbb{P}$ is κ-directed closed, this last fact must be true in V and V_κ as well. Thus, $V_\kappa \vDash$ CCA. This completes the proof of Theorem 3.

$\square$

In light of Theorem 3, one may also wonder if Theorem 2 can be improved. To be more explicit, one may ask if it is possible, assuming κ is supercompact and no cardinal $\delta > \kappa$ is inaccessible, to obtain a model of ZFC in which κ is indestructibly supercompact and for *every* inaccessible cardinal $\delta < \kappa$, $V_\delta \vDash T_1$. Once again, this question has a positive answer, as the following theorem shows.

Theorem 4 *Suppose $V \vDash$ "ZFC + κ is supercompact + No cardinal $\delta > \kappa$ is inaccessible". Then there is a partial ordering $\mathbb{P} \in V$ such that $V^{\mathbb{P}} \vDash$ "ZFC + κ is indestructibly supercompact + No cardinal $\delta > \kappa$ is inaccessible + For every inaccessible cardinal $\delta < \kappa$, $V_\delta \vDash T_1$".*

Proof: Suppose $V \vDash$ "ZFC + κ is supercompact + No cardinal $\delta > \kappa$ is inaccessible". As in the proof of Theorem 2, we assume in addition that $V \vDash$ GCH. We then let $\mathbb{P}$ be the partial ordering used in the proof of Theorem 2 as defined when $k = 1$. From this, the proof of Theorem 2 and the succeeding remarks allow us to infer immediately that $V^{\mathbb{P}} \vDash$ "ZFC + κ is indestructibly supercompact + No cardinal $\delta > \kappa$ is inaccessible + $A_1 = \{\delta < \kappa \mid \delta$ is an inaccessible cardinal which is not a limit of inaccessible cardinals$\} = \{\delta < \kappa \mid \delta$ is an inaccessible cardinal which is not a limit of inaccessible cardinals and $V_\delta \vDash$ CCA$\}$". Therefore, as in the proof of Theorem 3, to prove Theorem 4, it suffices to show that if $V^{\mathbb{P}} \vDash$ "δ is an inaccessible cardinal which is a limit of inaccessible cardinals", then $(V_\delta)^{V^{\mathbb{P}}} \vDash$ CCA.

To see this, work for the rest of the proof in $V^{\mathbb{P}}$. Let $\alpha < \delta$ with $x \subseteq \alpha$, and let ρ be the least inaccessible cardinal greater than α. Clearly, $\rho < \delta$, $\rho \in A_1$, and $x \in V_\rho$. Since $V_\rho \vDash$ CCA, there is some $\theta \in (\alpha, \rho)$ such that $V_\rho \vDash$ "$\beta \in x$ iff for every $\beta < \alpha$, $2^{\aleph_{\theta+\beta+1}} = \aleph_{\theta+\beta+2}$". But then, $\theta \in (\alpha, \delta)$ and $V_\delta \vDash$ "$\beta \in x$ iff for every $\beta < \alpha$, $2^{\aleph_{\theta+\beta+1}} = \aleph_{\theta+\beta+2}$". Thus, $V_\delta \vDash$ CCA. This completes the proof of Theorem 4.

$\square$

Observe that by Theorem 4, it is impossible to improve Theorem 3. In other words, in ZFC alone, only B_1 need be unbounded in κ if κ is indestructibly supercompact, and not $B_i =_{\rm df} \{\delta < \kappa \mid \delta$ is an inaccessible limit of inaccessible cardinals and $V_\delta \models T_i\}$ for $i = 2, \ldots, 4$.

In conclusion to this paper, we note that results analogous to Theorems 1 – 4 hold if κ is either an indestructible strong cardinal in Gitik and Shelah's sense of [9] or an indestructible strongly unfoldable cardinal in Johnstone's sense of [11, 12]. (See [11, 12] for the definition of strongly unfoldable cardinal.) We leave it to readers to work out the details for themselves.

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