Indestructibility, Measurability, and Degrees of Supercompactness\(^*\)

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Abstract

Suppose that \(\kappa\) is indestructibly supercompact and there is a measurable cardinal \(\lambda > \kappa\). It then follows that \(A_1 = \{\delta < \kappa \mid \delta\) is measurable, \(\delta\) is not a limit of measurable cardinals, and \(\delta\) is not \(\delta^+\) supercompact\}\) is unbounded in \(\kappa\). If in addition \(\lambda\) is \(2^\lambda\) supercompact, then \(A_2 = \{\delta < \kappa \mid \delta\) is measurable, \(\delta\) is not a limit of measurable cardinals, and \(\delta\) is \(\delta^+\) supercompact\}\) is unbounded in \(\kappa\) as well. The large cardinal hypotheses on \(\lambda\) are necessary, as we further demonstrate by constructing two distinct models in which either \(A_1 = \emptyset\) or \(A_2 = \emptyset\). In each of these models, there is an indestructibly supercompact cardinal \(\kappa\), and a restricted large cardinal structure above \(\kappa\). If we weaken the indestructibility requirement on \(\kappa\) to indestructibility under partial orderings which are both \(\kappa\)-directed closed and \((\kappa^+, \infty)\)-distributive, then it is possible to construct a model containing a supercompact cardinal \(\kappa\) witnessing this degree of indestructibility in which every measurable cardinal \(\delta < \kappa\) is (at least) \(\delta^+\) supercompact.

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1 Introduction and Preliminaries

It is a very interesting fact that the large cardinal structure of the universe above a supercompact cardinal $\kappa$ with suitable indestructibility properties can affect what happens at large cardinals below $\kappa$. On the other hand, it is possible to mitigate these effects if the universe contains relatively few large cardinals. These sorts of occurrences have been previously investigated in [1, 2, 3, 4, 5, 8].

The purpose of this paper is to continue studying this phenomenon, but in the context of investigating the degree of supercompactness certain measurable cardinals can manifest in universes containing a supercompact cardinal with various indestructibility properties. We begin with the following theorem, where as in [16], $\kappa$ is indestructibly supercompact if $\kappa$’s supercompactness is preserved by arbitrary $\kappa$-directed closed forcing.

**Theorem 1** Suppose that $\kappa$ is indestructibly supercompact and there is a measurable cardinal $\lambda > \kappa$. Then $A_1 = \{\delta < \kappa \mid \delta$ is measurable, $\delta$ is not a limit of measurable cardinals, and $\delta$ is not $\delta^+$ supercompact$\}$ is unbounded in $\kappa$.

In fact, if we assume additional hypotheses on $\lambda$, then it is possible to infer even more. Specifically, we have:

**Theorem 2** Suppose that $\kappa$ is indestructibly supercompact and there is a cardinal $\lambda > \kappa$ which is $2^\lambda$ supercompact. Then besides $A_1$ being unbounded in $\kappa$, $A_2 = \{\delta < \kappa \mid \delta$ is measurable, $\delta$ is not a limit of measurable cardinals, and $\delta$ is $\delta^+$ supercompact$\}$ is unbounded in $\kappa$ as well.

With a limited large cardinal structure above $\kappa$, Theorems 1 and 2 need not be true. Specifically, we have:

**Theorem 3** Suppose $V \models "ZFC + \kappa$ is supercompact + No cardinal $\delta > \kappa$ is $2^\delta$ supercompact”. There is then a partial ordering $\mathbb{P} \subseteq V$ such that $V^\mathbb{P} \models "ZFC + No cardinal $\delta > \kappa$ is $2^\delta$ supercompact + $\kappa$ is indestructibly supercompact + If $\delta < \kappa$ is a measurable cardinal which is not a limit of measurable cardinals, then $\delta$ is not $\delta^+$ supercompact”.


Theorem 4 Suppose $V \models \text{"ZFC + } \kappa \text{ is supercompact + No cardinal } \delta > \kappa \text{ is measurable". There is then a partial ordering } \mathbb{P} \subseteq V \text{ such that } V^\mathbb{P} \models \text{"ZFC + No cardinal } \delta > \kappa \text{ is measurable + } \kappa \text{ is indestructibly supercompact + If } \delta < \kappa \text{ is a measurable cardinal which is not a limit of measurable cardinals, then } \delta \text{ is } \delta^+ \text{ supercompact".}$

If we are willing to weaken the amount of indestructibility on our supercompact cardinal $\kappa$, then it is possible to obtain an improved version of Theorem 4. In particular:

Theorem 5 Suppose $V \models \text{"ZFC + } \kappa \text{ is supercompact + No cardinal } \delta > \kappa \text{ is measurable". There is then a partial ordering } \mathbb{P} \subseteq V \text{ such that } V^\mathbb{P} \models \text{"ZFC + No cardinal } \delta > \kappa \text{ is measurable + } \kappa \text{ is a supercompact cardinal whose supercompactness is indestructible under partial orderings which are both } \kappa\text{-directed closed and } (\kappa^+, \infty)\text{-distributive + If } \delta < \kappa \text{ is a measurable cardinal, then } \delta \text{ is (at least) } \delta^+ \text{ supercompact".}$

Of course, reflection easily yields that if $\kappa$ is supercompact, then $\{ \delta < \kappa \mid \delta \text{ is (at least) } \delta^+ \text{ supercompact} \}$ must be unbounded in $\kappa$. It is therefore not possible to obtain an analogue of Theorem 5 for Theorem 3.

We take this opportunity to make a few additional remarks concerning Theorems 1 – 5. The limited amount of indestructibility forced in Theorem 5 is due to the necessity in our proofs of preserving a nontrivial degree of supercompactness. However, if we weaken the requirement in Theorem 5 of all measurable cardinals $\delta < \kappa$ being $\delta^+$ supercompact to only measurable cardinals $\delta < \kappa$ which are not themselves limits of measurable cardinals being $\delta^+$ supercompact, then Theorem 4 shows that it is possible for $\kappa$ to be a fully indestructible supercompact cardinal. Also, as our proof will show, the degrees of indestructibility mentioned in the statement of Theorems 1 and 2 can be weakened. In particular, $\kappa$’s supercompactness can be indestructible under $\kappa$-directed closed, $(\kappa^+, \infty)$-distributive forcing. This provides a nice balance between Theorems 1 and 5, which complements the balance between Theorems 2 and 3 and Theorems 1 and 4.

We conclude Section 1 with a very brief discussion of some preliminary material. We presume a basic knowledge of large cardinals and forcing. A good reference in this regard is [15]. When
forcing, $q \geq p$ means that $q$ is stronger than $p$. When $G$ is $V$-generic over $\mathbb{P}$, we abuse notation slightly and take both $V[G]$ and $V^\mathbb{P}$ as being the generic extension of $V$ by $\mathbb{P}$. We also abuse notation slightly by occasionally confusing terms with the sets they denote, especially for ground model sets and variants of the generic object. For $\alpha < \beta$ ordinals, $[\alpha, \beta]$, $(\alpha, \beta]$, $[\alpha, \beta)$, and $(\alpha, \beta)$ are as in standard interval notation.

Suppose $\kappa$ is a regular cardinal. For $\alpha$ an arbitrary ordinal, the partial ordering $\text{Add}(\kappa, \alpha)$ is the standard Cohen partial ordering for adding $\alpha$ many Cohen subsets of $\kappa$. The partial ordering $\mathbb{P}$ is $\kappa$-directed closed if for every directed set $D \subseteq \mathbb{P}$ of size less than $\kappa$, there is a condition in $\mathbb{P}$ extending each member of $D$. $\mathbb{P}$ is $\kappa$-strategically closed if in the two person game in which the players construct an increasing sequence $\langle p_\alpha \mid \alpha \leq \kappa \rangle$, where player I plays odd stages and player II plays even stages, player II has a strategy ensuring the game can always be continued. $\mathbb{P}$ is $<\kappa$-strategically closed if $\mathbb{P}$ is $\delta$-strategically closed for every $\delta < \kappa$. $\mathbb{P}$ is $(\kappa, \infty)$-distributive if the intersection of $\kappa$ many dense open subsets of $\mathbb{P}$ is dense open. It follows that forcing with any partial ordering $\mathbb{P}$ which is $(\kappa^+, \infty)$-distributive preserves the $\kappa^+$ supercompactness of $\kappa$, since forcing with $\mathbb{P}$ adds no new subsets of $P_\kappa(\kappa^+)$.  

A corollary of Hamkins’ work on gap forcing found in [13, 14] will be employed in the proof of Theorems 2 – 5. We therefore state as a separate theorem what is relevant for this paper, along with some associated terminology, quoting from [13, 14] when appropriate. Suppose $\mathbb{P}$ is a partial ordering which can be written as $Q * R$, where $|Q| < \delta$, $Q$ is nontrivial, and $\mathbb{P} \models \check{R}$ is $\delta^+$-directed closed”. In Hamkins’ terminology of [13, 14], $\mathbb{P}$ admits a gap at $\delta$. Also, as in the terminology of [13, 14] and elsewhere, an embedding $j : V \rightarrow M$ is amenable to $V$ when $j \upharpoonright A \in V$ for any $A \in V$. The specific corollary of Hamkins’ work from [13, 14] we will be using is then the following.

**Theorem 6 (Hamkins)** Suppose that $V[G]$ is a generic extension obtained by forcing with $\mathbb{P}$ that admits a gap at some regular $\delta < \kappa$. Suppose further that $j : V[G] \rightarrow M[j(G)]$ is an embedding with critical point $\kappa$ for which $M[j(G)] \subseteq V[G]$ and $M[j(G)]^\delta \subseteq M[j(G)]$ in $V[G]$. Then $M \subseteq V$; indeed, $M = V \cap M[j(G)]$. If the full embedding $j$ is amenable to $V[G]$, then the restricted embedding $j \upharpoonright V : V \rightarrow M$ is amenable to $V$. If $j$ is definable from parameters (such as a measure or extender)
in $V[G]$, then the restricted embedding $j \upharpoonright V$ is definable from the names of those parameters in $V$.

## 2 A Key Forcing Notion

The proof of Theorem 1 will depend on the existence of a certain partial ordering $\mathbb{P}(\delta, \kappa)$. We isolate the existence of this key forcing notion in the following theorem.

**Theorem 7** Let $V \models "\delta < \kappa"$ be such that $\delta$ is a regular cardinal and $\kappa$ is $2^\kappa$ supercompact. There is then a $\delta$-directed closed partial ordering $\mathbb{P}(\delta, \kappa) \in V$ such that $V^{\mathbb{P}(\delta, \kappa)} \models "\kappa$ is the least measurable cardinal greater than $\delta + 2^\kappa = 2^{\kappa^+} = \kappa^{++}$ and $\kappa$ is $\kappa^+$ supercompact”.

**Proof:** Assume $\delta$ and $\kappa$ are as in the hypotheses of Theorem 7. We define $\mathbb{P}(\delta, \kappa)$ as $\mathbb{P}^1 \ast \mathbb{P}^2 \ast \mathbb{P}^3$. Let $\rho = (2^\kappa)^V$. $\mathbb{P}^1$ is defined as the reverse Easton iteration $\langle \langle P_n, \dot{Q}_n \rangle \mid n < \omega \rangle$, where $P_0 = \text{Add}(\rho^+, 1)$. For each $n < \omega$, if $\models_{P_n} "\text{There is a cardinal greater than } \rho \text{ violating GCH}"$, then $\models_{P_n} "\dot{Q}_n = \text{Add}(\gamma^+, 1)"$, where $\gamma$ is the least cardinal greater than $\rho$ violating GCH$. If this is not the case, i.e., if $\models_{P_n} "\text{All cardinals greater than } \rho \text{ satisfy GCH}"$, then $\models_{P_n} "\dot{Q}_n$ is trivial forcing”.

Standard arguments (see [15, Exercise 15.16] and [6, Lemma 4, Case 2]) show that if $\gamma$ is a cardinal, then after forcing with $\text{Add}(\gamma^+, 1)$, all cardinals less than or equal to $\gamma^+$ are preserved, $2^\gamma = \gamma^+$, $2^\gamma$ of the ground model is collapsed to $\gamma^+$, and all cardinals greater than or equal to $(2^\gamma)^+$ of the ground model are preserved. This means that $V^{\mathbb{P}^1} \models "\text{For every } n < \omega, 2^{\rho^+ + n} = \rho^{+n+1}"$. Since $\mathbb{P}^1$ is $\rho^+ = ((2^\kappa)^+)^V$-directed closed, $(2^\kappa)^V = (2^\kappa)^V^{\mathbb{P}^1} = \rho$. In addition, it follows that $V^{\mathbb{P}^1} \models "\kappa$ is $2^\kappa$ supercompact”.

Work now in $V_* = V^{\mathbb{P}^1}$. $\mathbb{P}^2$ is defined as $\mathbb{P}_\kappa \ast \text{Add}(\kappa^+, 1)$, where $\mathbb{P}_\kappa$ is the reverse Easton iteration $\langle \langle P_\alpha, \dot{Q}_\alpha \rangle \mid \alpha < \kappa \rangle$ of length $\kappa$ which does trivial forcing except for those cardinals $\lambda \in (\delta, \kappa)$ which are inaccessible in $V_*$. In this case, $P_{\lambda+1} = P_\lambda \ast \dot{Q}_\lambda$, where $\models_{P_\lambda \ast \dot{Q}_\lambda} "\text{For every } n < \omega, 2^{\lambda^+ + n} = \lambda^{+n+1}"$. Assuming $\dot{Q}_\lambda$ is a term for the analogue of the iteration given in the preceding paragraph, $\dot{Q}_\lambda$ may be written as $\text{Add}(\lambda^+, 1) \ast \dot{R}_\lambda$, where $\models_{P_\lambda \ast \text{Add}(\lambda^+, 1)} "\dot{R}_\lambda$ is $(2^\lambda)^+\text{-directed closed}"$, i.e., $\models_{P_\lambda \ast \text{Add}(\lambda^+, 1)} "$\dot{R}_\lambda$ is $\lambda^{++}\text{-directed closed}"$.

\[1\]We slightly abuse notation here when we write $\models_{P_n}$, since we always assume we are forcing above the relevant condition when necessary.
Standard arguments once again show that \( V_{\text{P}^2} \models "2^\gamma = \gamma^+ \" if \( \lambda \in (\delta, \kappa] \) is inaccessible and \( \gamma \in [\lambda, \lambda^+] \)". It is also the case that \( V_{\text{P}^2} \models "\kappa is 2^\kappa = \kappa^+ supercompact". To see this, let \( j : V_* \rightarrow M \) be an elementary embedding witnessing the \( \rho \) supercompactness of \( \kappa \) in \( V_* \) generated by a supercompact ultrafilter \( \mathcal{P}_\kappa(\rho) \). In particular, \( M^\rho \subseteq M \). We use a standard lifting argument (a form of which is given, e.g., in the proof of \([5, \text{Lemma 2.2}]\)) to show that \( j \) lifts in \( V_{\text{P}^\kappa * \text{Add}(\kappa^+, 1)} \) to \( j : V_{\text{P}^\kappa * \text{Add}(\kappa^+, 1)} \rightarrow M^j(\text{P}_\kappa * \text{Add}(\kappa^+, 1)) \). Specifically, let \( G_0 \) be \( V_* \)-generic over \( \mathbb{P}_\kappa \), and let \( G_1 \) be \( V_*[G_0] \)-generic over \( \text{Add}(\kappa^+, 1) \). Observe that \( j(\mathbb{P}_\kappa * \text{Add}(\kappa^+, 1)) = \mathbb{P}_\kappa * \text{Add}(\kappa^+, 1) * \hat{Q} * \text{Add}(j(\kappa^+), 1) \). Working in \( V_*[G_0][G_1] \), we first note that since \( \mathbb{P}_\kappa * \text{Add}(\kappa^+, 1) \) is \( \rho^+ \)-c.c., \( M[G_0][G_1] \) remains \( \rho \) closed with respect to \( V_*[G_0][G_1] \). This means that \( \hat{Q} \) is \( \rho^+ \)-directed closed in both \( M[G_0][G_1] \) and \( V_*[G_0][G_1] \).

Since \( M[G_0][G_1] \models "|\hat{Q}| = j(\kappa)" \), the number of dense open subsets of \( \hat{Q} \) present in \( M[G_0][G_1] \) is \( (2^{j(\kappa)})^M \). In \( V_* \), since \( M \) is given via an ultrapower by a supercompact ultrafilter over \( \text{P}_\kappa(2^\kappa) \), this is calculated as \(|\{ f : 2^\kappa < f < 2^\kappa \}| = |\{ f : 2^\kappa < f \}| = 2^{2^\kappa} = 2^{\rho^+} \). Since \( V_* \models "2^{2^\kappa} = (2^\kappa)^+ = \rho^+" \) and \( \rho^+ \) is preserved from \( V_* \) to \( V_*[G_0][G_1] \), we may let \( \langle D_\alpha \mid \alpha < \rho^+ \rangle \in V_*[G_0][G_1] \) enumerate the dense open subsets of \( \hat{Q} \) present in \( M[G_0][G_1] \). We may now use the fact that \( \hat{Q} \) is \( \rho^+ \)-directed closed in \( V_*[G_0][G_1] \) to meet each \( D_\alpha \) and thereby construct in \( V_*[G_0][G_1] \) an \( M[G_0][G_1] \)-generic object \( H_0 \) over \( \hat{Q} \). Our construction guarantees that \( j''G_0 \subseteq G_0 * G_1 * H_0 \), so \( j \) lifts in \( V_*[G_0][G_1] \) to \( j : V_*[G_0] \rightarrow M[G_0][G_1][H_0] \).

It remains to lift \( j \) in \( V_*[G_0][G_1] \) through \( \text{Add}(\kappa^+, 1) \). Because \( V_*[G_0] \models "|\text{Add}(\kappa^+, 1)| = 2^\kappa = (2^\kappa)^{V_*} = \rho^+", M[G_0][G_1][H_0] \models "|\text{Add}(j(\kappa^+), 1)| = 2^{j(\kappa^+)} = (2^{j(\kappa)})^M \). Therefore, since \( M[G_0][G_1][H_0] \) remains \( \rho \) closed with respect to \( V_*[G_0][G_1] \), \( M[G_0][G_1][H_0] \models "\text{Add}(j(\kappa^+), 1) \) is \( j(\kappa^+) \)-directed closed". And \( j(\kappa^+) > j(\kappa) > \rho \), there is a master condition \( q \in \text{Add}(j(\kappa^+), 1) \) for \( j''\{ p \mid p \in G_1 \} \).

Further, the number of dense open subsets of \( \text{Add}(j(\kappa^+), 1) \) present in \( M[G_0][G_1][H_0] \) is \( (2^{j(\kappa)})^M \). This is calculated in \( V_* \) as \(|\{ f : 2^\kappa < f < 2^{2^\kappa} \}| = |\{ f : 2^\kappa < f < (2^\kappa)^+ \}| = |\{ f : \rho < f < \rho^+ \}| = 2^\rho = (2^\kappa)^+ = \rho^+ \). Working in \( V_*[G_0][G_1] \), since \( \text{Add}(j(\kappa^+), 1) \) is \( \rho^+ \)-directed closed in both \( M[G_0][G_1][H_0] \) and \( V_*[G_0][G_1] \), we may consequently use the arguments of the preceding paragraph to construct an \( M[G_0][G_1][H_0] \)-generic object \( H_1 \) over \( \text{Add}(j(\kappa^+), 1) \) containing \( q \). Since by
the definition of $H_1$, $j''(G_0 \ast G_1) \subseteq G_0 \ast G_1 \ast H_0 \ast H_1$, $j$ lifts in $V[G_0][G_1]$ to $j : V[G_0][G_1] \rightarrow M[G_0][G_1][H_0][H_1]$. As $V[G_0][G_1] \models "|\rho| = \kappa^+"$, this means that $V[G_0][G_1] \models \kappa$ is $2\kappa = \kappa^+$ supercompact”. Note that by its definition, $P_1 \ast \check{P}_2$ is $\delta$-directed closed.

Work now in $V^{P_1 \ast \check{P}_2} = V$. Fix $\lambda \in (\delta, \kappa]$ an inaccessible cardinal. We define three notions of forcing. In particular, we describe now a specific form of the partial orderings of [11, Section 4]. Following the notation of [11, Section 4], we will denote these partial orderings by $P_{0, \lambda^+}^{++}[S]$, and $P_{0, \lambda^+}^{++}[S]$. So that readers are not overly burdened, we abbreviate our definitions and descriptions somewhat. Full details may be found by consulting [11], along with the relevant portions of [10]. We do mention explicitly, however, that (more than) the amount of GCH required for the definitions of $P_{0, \lambda^+}^{++}$, $P_{0, \lambda^+}^{++}[S]$, and $P_{0, \lambda^+}^{++}[S]$ to be given and for these partial orderings to have the properties described below has been forced by $P_1 \ast \check{P}_2$.

The first notion of forcing $P_{0, \lambda^+}^{++}$ is just the standard notion of forcing for adding a nonreflecting stationary set of ordinals $S$ of cofinality $\delta$ to $\lambda^+$. For further details on the definition of this partial ordering, we refer readers to [10] or [11]. We note only that $P_{0, \lambda^+}^{++}$ is $\delta$-directed closed. Next, work in $V_1 = V^{P_0^{\check{P}_2}}$, letting $\hat{S}$ be a term always forced to denote $S$. $P_{0, \lambda^+}^{++}[S]$ is the standard notion of forcing for introducing a club set $C$ which is disjoint to $S$ (and therefore makes $S$ nonstationary).

We fix now in $V_1$ a ♦($S$) sequence $X = \langle x_\alpha \mid \alpha \in S \rangle$, the existence of which is given by [10, Lemma 1] and [11, Lemma 1]. We are ready to define in $V_1$ the partial ordering $P_{1, \lambda^+}^{[S]}$. First, since each element of $S$ has cofinality $\delta$, the proof of Lemma 1 of [10] and [11] shows each $x \in X$ can be assumed to be such that order-type($x$) = $\delta$. Then, $P_{1, \lambda^+}^{[S]}$ is defined as the set of all 4-tuples $\langle w, \alpha, \bar{r}, Z \rangle$ satisfying the following properties.

1. $w \in [\lambda^+]^{<\lambda}$.

2. $\alpha < \lambda$.

3. $\bar{r} = \langle r_i \mid i \in w \rangle$ is a sequence of functions from $\alpha$ to $\{0, 1\}$, i.e., a sequence of subsets of $\alpha$.

4. $Z \subseteq \{x_\beta \mid \beta \in S \}$ is a set such that if $z \in Z$, then for some $y \in [w]^\delta$, $y \subseteq z$ and $z - y$ is bounded in the $\beta$ such that $z = x_\beta$. 

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The ordering on $\mathbb{P}_{\lambda,\lambda^+}^1[\dot{S}]$ is given by $\langle w^1, \alpha^1, \bar{r}^1, Z^1 \rangle \leq \langle w^2, \alpha^2, \bar{r}^2, Z^2 \rangle$ iff the following hold.

1. $w^1 \subseteq w^2$.
2. $\alpha^1 \leq \alpha^2$.
3. If $i \in w^1$, then $r^1_i \subseteq r^2_i$.
4. $Z^1 \subseteq Z^2$.
5. If $z \in Z^1 \cap [w^1]^{\delta}$ and $\alpha^1 \leq \alpha < \alpha^2$, then $|\{i \in z \mid r^2_i(\alpha) = 0\}| = |\{i \in z \mid r^2_i(\alpha) = 1\}| = \delta$.

The proof of [10, Lemma 4] shows that $\mathbb{P}_{\lambda,\lambda^+}^0[\mathbb{P}^1_{\lambda,\lambda^+}[\dot{S}] \times \mathbb{P}^2_{\lambda,\lambda^+}[\dot{S}]]$ is equivalent to $\text{Add}(\lambda^+, 1) * \text{Add}(\lambda, \lambda^+)$. The proofs of [10, Lemmas 3 and 5] and [11, Lemma 6] show that $\mathbb{P}_{\lambda,\lambda^+}^0 * \mathbb{P}_{\lambda,\lambda^+}^1[\dot{S}]$ preserves cardinals and cofinalities, is $\lambda^+-c.c.$, is $<\lambda$-strategically closed, and is such that $V^\mathbb{P}_{\lambda,\lambda^+}^0 * \mathbb{P}_{\lambda,\lambda^+}^1[\dot{S}] \models \text{"} 2^\lambda = \lambda^+, \ 2^{\lambda^+} = \lambda^{++}, \text{ and } \lambda \text{ is nonmeasurable".}$ By the remarks in [10, middle of page 108], $\mathbb{P}_{\lambda,\lambda^+}^0 * \mathbb{P}_{\lambda,\lambda^+}^1[\dot{S}]$ is $\delta$-directed closed.

Let $\mathbb{P}^3$ be the reverse Easton iteration of length $\kappa + 1$ which forces with $\mathbb{P}_{\lambda,\lambda^+}^0 * \mathbb{P}_{\lambda,\lambda^+}^1[\dot{S}]$ whenever $\lambda \in (\delta, \kappa)$ is inaccessible, forces with $\mathbb{P}_{\kappa,\kappa^+}^0 * (\mathbb{P}_{\kappa,\kappa^+}^1[\dot{S}] \times \mathbb{P}_{\kappa,\kappa^+}^2[\dot{S}])$ at stage $\kappa$, and does trivial forcing otherwise. By the facts mentioned in the preceding paragraph, $\mathbb{P}^3$ is $\delta$-directed closed. The proof of [11, Lemma 9] in conjunction with the facts mentioned in the preceding paragraph show that $V^{\mathbb{P}^3} \models \text{"} \kappa \text{ is the least measurable cardinal greater than } \delta + 2^\kappa = 2^{\kappa^+} = \kappa^{++} + \kappa \text{ is } \kappa^+ \text{ supercompact".}$ If we now define $\mathbb{P}(\delta, \kappa) = \mathbb{P}^1 * \mathbb{P}^2 * \mathbb{P}^3$, then $\mathbb{P}(\delta, \kappa)$, which is $\delta$-directed closed, is our desired partial ordering. This completes the proof of Theorem 7.

We conclude Section 2 by observing that the definitions of $\mathbb{P}^1$ and $\mathbb{P}^2$ given above may be changed. All that is necessary is that enough GCH is forced to allow the arguments of [10] and [11] to be used to establish that after forcing with $\mathbb{P}^3$, $\kappa$ has become the least measurable cardinal greater than $\delta$ and $\kappa$ remains $\kappa^+$ supercompact.
3 The Proofs of Theorems 1 – 5

We begin with the proof of Theorem 1, after which the proof of Theorem 2 follows immediately.

Proof: We follow the proofs of [1, Theorem 2] and [5, Theorem 1]. Suppose that \( \kappa \) is indestructibly supercompact and there is a measurable cardinal \( \lambda > \kappa \). We show that \( A_1 = \{ \delta < \kappa \mid \delta \) is measurable, \( \delta \) is not a limit of measurable cardinals, and \( \delta \) is not \( \delta^+ \) supercompact\} is unbounded in \( \kappa \). Let \( \eta > \kappa \) be the least measurable cardinal. Force with \( \text{Add}(\eta^+, 1) \). After this forcing, which is both \( \kappa \)-directed closed and \((\kappa^+, \infty)\)-distributive, \( 2^\eta = \eta^+ \) and \( \eta \) remains the least measurable cardinal above \( \kappa \). In particular, after the forcing, \( \eta \) is a measurable cardinal which is not a limit of measurable cardinals, so automatically, \( \eta \) is not \( 2^\eta = \eta^+ \) supercompact. Since \( \kappa \)'s supercompactness is suitably indestructible, by reflection, \( A_1 = \{ \delta < \kappa \mid \delta \) is measurable, \( \delta \) is not a limit of measurable cardinals, and \( \delta \) is not \( \delta^+ \) supercompact\} is unbounded in \( \kappa \) after the forcing has been performed. Once more, we infer by the fact \( \text{Add}(\eta^+, 1) \) is \( \kappa \)-directed closed that \( A_1 \) is unbounded in \( \kappa \) in the ground model. This completes the proof of Theorem 1.

\[\square\]

Proof: We argue in analogy to the proof of Theorem 1. Suppose that \( \kappa \) is indestructibly supercompact and there is a cardinal \( \lambda > \kappa \) which is \( 2^\lambda \) supercompact. To show that \( A_2 = \{ \delta < \kappa \mid \delta \) is measurable, \( \delta \) is not a limit of measurable cardinals, and \( \delta \) is \( \delta^+ \) supercompact\} is unbounded in \( \kappa \), force with \( \mathbb{P}(\kappa^{++}, \lambda) \). By Theorem 7, after this forcing, which is both \( \kappa \)-directed closed and \((\kappa^+, \infty)\)-distributive, \( \lambda \) has become the least measurable cardinal greater than both \( \kappa \) and \( \kappa^{++} \), and \( \lambda \) is \( \lambda^+ \) supercompact. In particular, after this forcing, \( \lambda \) is a measurable cardinal which is not a limit of measurable cardinals. We now argue as in the proof of Theorem 1. Since \( \kappa \)'s supercompactness is suitably indestructible, by reflection, \( A_2 = \{ \delta < \kappa \mid \delta \) is measurable, \( \delta \) is not a limit of measurable cardinals, and \( \delta \) is \( \delta^+ \) supercompact\} is unbounded in \( \kappa \) after the forcing has been performed. As before, we infer by the fact \( \mathbb{P}(\kappa^{++}, \lambda) \) is \( \kappa \)-directed closed that \( A_2 \) is unbounded in \( \kappa \) in the ground model. This completes the proof of Theorem 2.

\[\square\]
Having completed the proofs of Theorems 1 and 2, we turn now to the proof of Theorem 3.

**Proof:** Suppose $V \models \text{“ZFC + } \kappa \text{ is supercompact + No cardinal } \delta > \kappa \text{ is } 2^\delta \text{ supercompact”}$. Without loss of generality, by first doing a preliminary forcing if necessary, we assume in addition that $V \models \text{GCH}$. This is accomplished using a standard argument. In particular, it is possible to force GCH via the class reverse Easton iteration $\langle \langle P_\alpha, \dot{Q}_\alpha \rangle \mid \alpha \in \text{Ord} \rangle$, where $P_0 = \text{Add}(\omega, 1)$. For any ordinal $\alpha$, if $\models P_\alpha \text{”There is a cardinal violating GCH”}$, then $\models \dot{P}_\alpha \text{”} \dot{Q}_\alpha = \text{Add}(\gamma^+, 1) \text{”}$ where $\gamma$ is the least cardinal violating GCH. If this is not the case, i.e., if $\models \dot{P}_\alpha \text{”All cardinals satisfy GCH”}$, then $\models \dot{P}_\alpha \text{”} \dot{Q}_\alpha \text{ is trivial forcing”}$. Since this is a closure point forcing in Hamkins’ sense of [9], by Hamkins’ results of [12], no new $\delta$ which is $2^\delta$ supercompact (or indeed, no new cardinal which is measurable) is created. Hence, if $V \models \text{“}\delta \text{ is } 2^\delta \text{ supercompact”}$, then $V \models \text{“}\delta \text{ is a limit of measurable cardinals”}$.

Let $f$ be a Laver function [16] for $\kappa$, i.e., $f : \kappa \to V_\kappa$ is such that for every $x \in V$ and every $\lambda \geq |\text{TC}(x)|$, there is an elementary embedding $j : V \to M$ generated by a supercompact ultrafilter over $P_\kappa(\lambda)$ such that $j(f)(\kappa) = x$. Our partial ordering $P$ is the reverse Easton iteration of length $\kappa$ which begins by forcing with $\text{Add}(\omega, 1)$ and then (possibly) does nontrivial forcing only at cardinals $\delta < \kappa$ which are both limits of cardinals $\eta$ which are $2^\eta$ supercompact in $V$ and are at least $2^\delta$ supercompact in $V$. At such a stage $\delta$, if $f(\delta) = \hat{Q}$ and $\models \dot{P}_\delta \text{’} \hat{Q} \text{ is a } \delta\text{-directed closed partial ordering having rank below the least } \eta > \delta \text{ such that } \eta \text{ is } 2^\eta \text{ supercompact in } V”$, then $P_{\delta+1} = P_\delta * \hat{Q}$. If this is not the case, then $P_{\delta+1} = P_\delta * \hat{Q}$, where $\hat{Q}$ is a term for trivial forcing.

**Lemma 3.1** $V^P \models \text{“}\kappa \text{ is indestructibly supercompact”}$. 

**Proof:** We follow the proofs of [1, Lemma 2.1] and [5, Lemma 2.1]. Let $Q \in V^P$ be such that $V^P \models \text{“}Q \text{ is } \kappa\text{-directed closed”}$. Take $\hat{Q}$ as a term for $Q$ such that $\models \dot{P} \text{”} \hat{Q} \text{ is } \kappa\text{-directed closed”}$. Suppose $\lambda \geq \max(\kappa^+, |\text{TC}(\hat{Q})|)$ is an arbitrary cardinal, and let $\gamma = 2^{[\lambda]^{<\kappa}}$. Take $j : V \to M$ as an elementary embedding witnessing the $\gamma$ supercompactness of $\kappa$ generated by a supercompact ultrafilter over $P_\kappa(\gamma)$ such that $j(f)(\kappa) = \hat{Q}$. Since $V \models \text{“No cardinal } \delta \text{ above } \kappa \text{ is } 2^\delta \text{ supercompact”}$, $\gamma \geq 2^{[\kappa^+]^{<\kappa}}$, and $M^\gamma \subseteq M$, $M \models \text{“}\kappa \text{ is both } 2^\kappa = \kappa^+ \text{ supercompact and a limit of cardinals } \eta \text{ which} \ldots \”$
are $2^n$ supercompact, and no cardinal $\delta$ in the interval $(\kappa, \gamma]$ is $2^\delta$ supercompact”. Hence, the definition of $\mathbb{P}$ implies that $j(P \ast \dot{Q}) = P \ast \dot{Q} \ast \dot{R} \ast j(\dot{Q})$, where the first stage at which $\dot{R}$ is forced to do nontrivial forcing is well above $\gamma$. Laver’s original argument from [16] now applies and shows $V^P \ast \dot{Q} \Vdash \kappa is a \lambda supercompact”. (Simply let $G_0 \ast G_1 \ast G_2$ be $V$-generic over $P \ast \dot{Q} \ast \dot{R}$, lift $j$ in $V[G_0][G_1][G_2]$ to $j$ in $V[G_0][G_1][G_2][G_3]$, and show by the $\gamma^+$-directed closure of $\mathbb{R} \ast j(\dot{Q})$ that the supercompactness measure over $(P_\kappa(\lambda))^{V[G_0][G_1]}$ generated by $j$ is actually a member of $V[G_0][G_1]$.)

As $\lambda$ and $\mathbb{Q}$ were arbitrary, this completes the proof of Lemma 3.1.

\[\square\]

**Lemma 3.2** If $V \Vdash \delta < \kappa$ is a $2^\delta$ supercompact cardinal which is not a limit of cardinals $\eta$ which are $2^n$ supercompact”, then $V^P \Vdash \delta is a 2^\delta supercompact”.

**Proof:** Suppose that $V \Vdash \delta < \kappa$ is a $2^\delta$ supercompact cardinal which is not a limit of cardinals $\eta$ which are $2^n$ supercompact”. Write $P = P_\delta \ast \dot{P}_\delta$. By the definition of $P$, $|P_\delta| < \delta$ and $\|P_\delta\| \dot{\ast} \dot{P}_\delta$ is (at least) $\beth_\omega(\delta)$-directed closed”. Therefore, the Lévy-Solovay results [17] show that $V^{P_\delta} \Vdash \delta is 2^\delta supercompact”, so $V^{P_\delta \ast \dot{P}_\delta} = V^P \Vdash \delta is a 2^\delta supercompact”. This completes the proof of Lemma 3.2.

\[\square\]

**Lemma 3.3** $V^P \Vdash \text{“If } \delta < \kappa \text{ is a measurable cardinal which is not a limit of measurable cardinals, then } \delta is not } \delta^+ \text{ supercompact”}.$

**Proof:** We prove the contrapositive. Suppose that $V^P \Vdash \text{“If } \delta < \kappa \text{ is } \delta^+ \text{ supercompact”}$. Write $\mathbb{P} = \mathbb{P}' \ast \dot{\mathbb{P}}'$, where $|\mathbb{P}'| = \omega$, $\mathbb{P}'$ is nontrivial, and $\|P\| \dot{\ast} \dot{\mathbb{P}}'$ is $\aleph_2$-directed closed”. By Theorem 6, $V \Vdash \text{“If } \delta is (\delta^+)^{V^P} \text{ supercompact”}$. Consequently, since $(\delta^+)^{V^P} \geq (\delta^+)^V$, $V \Vdash \text{“If } \delta is \delta^+ = 2^\delta \text{ supercompact”}$. This allows us now to consider the following two cases.
Case 1: $V \vDash \text{“} \delta \text{ is not a limit of cardinals } \eta \text{ which are } 2^\eta \text{ supercompact} \text{”}. In this case, by Lemma 3.2, $V^P \vDash \text{“} \delta \text{ is } 2^\delta \text{ supercompact} \text{”}$. From this, we immediately infer that $V^P \vDash \text{“} \delta \text{ is a limit of measurable cardinals} \text{”}$.

Case 2: $V \vDash \text{“} \delta \text{ is a limit of cardinals } \eta \text{ which are } 2^\eta \text{ supercompact} \text{”}$. In particular, $V \vDash \text{“} \delta \text{ is a limit of cardinals } \eta \text{ which are } 2^\eta \text{ supercompact such that each } \eta \text{ is not a limit of cardinals } \gamma \text{ which are } 2^\gamma \text{ supercompact} \text{”}$. By Lemma 3.2, such $\eta$ are preserved to $V^P$, i.e., $V^P \vDash \text{“} \delta \text{ is a limit of cardinals } \eta \text{ which are } 2^\eta \text{ supercompact} \text{”}$. In other words, $V^P \vDash \text{“} \delta \text{ is a limit of measurable cardinals} \text{”}$.

Cases 1 and 2 complete the proof of Lemma 3.3.

□

Since trivial forcing is $\kappa$-directed closed, Lemma 3.1 implies that $V^P \vDash \text{“} \kappa \text{ is supercompact} \text{”}$. Also, because $\mathbb{P}$ may be defined so that $|\mathbb{P}| = \kappa$, the arguments of [17] show that $V^P \vDash \text{“} \text{No cardinal } \delta > \kappa \text{ is } 2^\delta \text{ supercompact} \text{”}$. These remarks, together with Lemmas 3.1 – 3.3, complete the proof of Theorem 3.

□

Having completed the proof of Theorem 3, we turn now to the proof of Theorem 4.

Proof: Suppose $V \vDash \text{“} \text{ZFC + } \kappa \text{ is supercompact + No cardinal } \delta > \kappa \text{ is measurable} \text{”}$. Without loss of generality, by first forcing GCH and then doing the forcing of [7, Theorem 1], we may assume in addition that $V \vDash \text{“} \text{If } \delta \leq \kappa \text{ is measurable, then } 2^\delta = 2^{\delta^+} = \delta^{++} \text{ and } \delta \text{ is } \delta^+ \text{ supercompact} \text{”}$. (The aforementioned property of $V$ may be assumed to hold because as we have already observed, forcing GCH will not create any new measurable cardinals. Since the forcing of [7, Theorem 1] may be defined so as to have size $\kappa$, by the results of [17], it will not create any new measurable cardinals greater than $\kappa$.) We then define $\mathbb{P}$ as in the proof of Theorem 3, except that at each nontrivial stage of forcing $\delta < \kappa$ (so in particular, $\mathbb{P}$ (possibly) does nontrivial forcing only at cardinals $\delta < \kappa$ which are both $2^\delta$ supercompact in $V$ and are limits of cardinals $\eta$ which are $2^\eta$ supercompact in $V$), we require that for our Laver function $f$, $f(\delta) = \dot{Q}$ and $\models_{\mathbb{P}_\delta} \text{“} \dot{Q} \text{ is a } \delta\text{-directed closed partial ordering having rank below the least } V\text{-measurable cardinal greater than } \delta \text{”}$. 

\[12\]
same arguments as used in the proof of Theorem 3, replacing both instances in the proof of Lemma 3.1 of δ not being $2^\delta$ supercompact with δ not being measurable, will now show that $V^P \models \kappa$ is indestructibly supercompact + No cardinal $\delta > \kappa$ is measurable”. The proof of Theorem 4 will therefore be complete once we have established the following lemma.

**Lemma 3.4** $V^P \models \text{“If } \delta < \kappa \text{ is a measurable cardinal which is not a limit of measurable cardinals, then } \delta \text{ is } \delta^+ \text{ supercompact”}.$

**Proof:** Suppose that $V^P \models \text{“} \delta < \kappa \text{ is a measurable cardinal which is not a limit of measurable cardinals”}. By the factorization of $\mathbb{P}$ given in the proof of Lemma 3.3 and Theorem 6, $V \models \text{“} \delta \text{ is a measurable cardinal”}. If $V \models \text{“} \delta \text{ is a measurable cardinal which is a limit of measurable cardinals”}, then in particular, $V \models \text{“} \delta \text{ is a measurable cardinal which is a limit of measurable cardinals which are not themselves limits of measurable cardinals”}. Observe now that essentially the same argument as given in the proof of Lemma 3.2 remains valid and shows that if $V \models \text{“} \eta < \kappa \text{ is a measurable cardinal which is not a limit of measurable cardinals”}, then $V^P \models \text{“} \eta \text{ is } \eta^+ \text{ supercompact”}. Thus, $V^P \models \text{“} \delta \text{ is a measurable cardinal which is a limit of measurable cardinals”}, a contradiction. Hence, if $V^P \models \text{“} \delta < \kappa \text{ is a measurable cardinal which is not a limit of measurable cardinals”}, then $V \models \text{“} \delta \text{ is a measurable cardinal which is not a limit of measurable cardinals”}. From this, we infer as earlier in the proof of this lemma that $V^P \models \text{“} \delta \text{ is } \delta^+ \text{ supercompact”}. This completes the proof of both Lemma 3.4 and Theorem 4.

□

□

We finish our proofs with the proof of Theorem 5.

**Proof:** Suppose $V \models \text{“} \text{ZFC + } \kappa \text{ is supercompact + No cardinal } \delta > \kappa \text{ is measurable”}. As in the proof of Theorem 4, we assume in addition that $V \models \text{“If } \delta \leq \kappa \text{ is measurable, then } 2^\delta = 2^{\delta^+} = \delta^{++} \text{ and } \delta \text{ is } \delta^+ \text{ supercompact”}$. We then define $\mathbb{P}$ as in the proof of Theorem 4, except that at each nontrivial stage of forcing $\delta < \kappa$, we require that for our Laver function $f$, $f(\delta) = \check{\delta}$ and $\models_{P_{\delta}} \text{“} \check{\delta} \text{ is } \delta^+ \text{ supercompact”}. We...
a $\delta$-directed closed, $(\delta^+, \infty)$-distributive partial ordering having rank below the least $V$-measurable cardinal greater than $\delta$". The same arguments as used in the proofs of Theorems 3 and 4 will now show that $V^P \models \kappa$ is a supercompact cardinal whose supercompactness is indestructible under partial orderings which are both $\kappa$-directed closed and $(\kappa^+, \infty)$-distributive + No cardinal $\delta > \kappa$ is measurable". The proof of Theorem 5 will therefore be complete once we have established the following lemma.

**Lemma 3.5** $V^P \models \text{"If } \delta < \kappa \text{ is a measurable cardinal, then } \delta \text{ is (at least) } \delta^+ \text{ supercompact"}.$

**Proof:** Suppose that $V^P \models \text{"} \delta < \kappa \text{ is a measurable cardinal"}$. As in the proof of Lemma 3.4, $V \models \text{"} \delta \text{ is a measurable cardinal"}$. This allows us to write $\mathbb{P} = \mathbb{P}_\delta \ast \mathbb{P}^\delta$ and consider the following two cases.

**Case 1:** $|\mathbb{P}_\delta| < \delta$. In this case, by the results of [17], because $V \models \text{"} \delta \text{ is (at least) } \delta^+ \text{ supercompact"}$, $V^\mathbb{P}_\delta \models \text{"} \delta \text{ is (at least) } \delta^+ \text{ supercompact"}$. Since by the definition of $\mathbb{P}$, $\models_{\mathbb{P}_\delta} \text{"} \mathbb{P}^\delta \text{ is both } \delta \text{-directed closed and } (\delta^+, \infty)\text{-distributive"}$, $V^{\mathbb{P}_\delta \ast \mathbb{P}^\delta} = V^P \models \text{"} \delta \text{ is (at least) } \delta^+ \text{ supercompact"}.

**Case 2:** $|\mathbb{P}_\delta| = \delta$. In this case, by our assumptions on $V$, let $j : V \rightarrow M$ be an elementary embedding witnessing the $\delta^+$ supercompactness of $\delta$ generated by a supercompact ultrafilter over $P_\delta(\delta^+)$ such that $M^{\delta^+} \subseteq M$ and $M \models \text{"} \delta \text{ is not } \delta^+ \text{ supercompact"}$. We then have that $j(\mathbb{P}_\delta) = \mathbb{P}_\delta \ast \check{\mathbb{Q}}$, where the first ordinal at which $\check{\mathbb{Q}}$ is forced to do nontrivial forcing is well beyond $\delta^+$. We may then use a simplified version of the standard lifting argument given in the proof of Theorem 7 to show that $j$ lifts in $V^{\mathbb{P}_\delta}$ to $j : V^{\mathbb{P}_\delta} \rightarrow M^{j(\mathbb{P}_\delta)}$. For completeness, we give the details. Let $G$ be $V$-generic over $\mathbb{P}_\delta$. Working in $V[G]$, we first note that since $\mathbb{P}_\delta$ is $\delta$-c.c., $M[G]$ remains $\delta^+$ closed with respect to $V[G]$. This means that $\mathbb{Q}$ is $\delta^{++}$-directed closed in both $M[G]$ and $V[G]$. As before, because $M[G] \models \text{"} |\mathbb{Q}| = j(\delta)\text{"}$, the number of dense open subsets of $\mathbb{Q}$ present in $M[G]$ is $(2^{j(\delta)})^M$. Since $V \models \text{"} 2^\delta = 2^{\delta^+} = 2^{\delta^{++}}\text{"}$ and $M$ is given via an ultrapower by a supercompact ultrafilter over $P_\delta(\delta^+)$, this is calculated as $|\{ f : f : [\delta^+]^{< \delta} \rightarrow 2^\delta \}| = |\{ f : f : \delta^+ \rightarrow \delta^{++} \}| = \delta^{++}$. We may therefore let $\langle D_\alpha \mid \alpha < \delta^{++} \rangle \in V[G]$ enumerate the dense open subsets of $\mathbb{Q}$ present in $M[G]$. We may now use the fact that $\mathbb{Q}$ is $\delta^{++}$-directed closed in $V[G]$ to meet each $D_\alpha$ and thereby construct in $V[G]$ an
$M[G]$-generic object $H$ over $\mathbb{Q}$. Our construction guarantees that $j''G \subseteq G \ast H$, so $j$ lifts in $V[G]$ to $j : V[G] \rightarrow M[G][H]$. Hence, $V^{P_δ} \models \text{"δ is (at least) $δ^+$ supercompact".}$ As in Case 1 above, since $V^{P_δ} \models \text{"$P_δ$ is both $δ$-directed closed and ($δ^+, \infty$)-distributive"}$, $V^{P_δ\ast\check{P}_δ} = V^P \models \text{"δ is (at least) $δ^+$ supercompact".}$ This completes the proof of both Lemma 3.5 and Theorem 5.

□

We conclude by remarking that other than the fact that the proof of Lemma 3.5 requires ($δ^+, \infty$)-distributivity at each nontrivial stage of forcing $δ$ in the definition of $P$, there is no reason prima facie to believe that this restriction must be present. We therefore end by asking if the proof of Theorem 5 can be reworked so that $V^P \models \text{"$κ$ is indestructibly supercompact".}$

References


