Indestructibility, Measurability, and Degrees of Supercompactness *†

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Abstract

Suppose that κ is indestructibly supercompact and there is a measurable cardinal $\lambda > \kappa$. It then follows that $A_1 = \{\delta < \kappa \mid \delta$ is measurable, δ is not a limit of measurable cardinals, and δ is not δ^+ supercompact$\}$ is unbounded in κ. If in addition λ is 2^λ supercompact, then $A_2 = \{\delta < \kappa \mid \delta$ is measurable, δ is not a limit of measurable cardinals, and δ is δ^+ supercompact$\}$ is unbounded in κ as well. The large cardinal hypotheses on λ are necessary, as we further demonstrate by constructing via forcing two distinct models in which either $A_1 = \emptyset$ or $A_2 = \emptyset$. In each of these models, there is an indestructibly supercompact cardinal κ, and a restricted large cardinal structure above κ. If we weaken the indestructibility requirement on κ to indestructibility under partial orderings which are both κ-directed closed and (κ^+, ∞)-distributive, then it is possible to construct a model containing a supercompact cardinal κ witnessing this degree of indestructibility in which *every* measurable cardinal $\delta < \kappa$ is (at least) δ^+ supercompact.

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1 Introduction and Preliminaries

It is a very interesting fact that the large cardinal structure of the universe above a supercompact cardinal κ with suitable indestructibility properties can affect what happens at large cardinals below κ. On the other hand, it is possible to mitigate these effects if the universe contains relatively few large cardinals. These sorts of occurrences have been previously investigated in [1, 2, 3, 4, 5, 8].

The purpose of this paper is to continue studying this phenomenon, but in the context of investigating the degree of supercompactness certain measurable cardinals can manifest in universes containing a supercompact cardinal with various indestructibility properties. We begin with the following theorem, where as in [16], κ is *indestructibly supercompact* if κ's supercompactness is preserved by arbitrary κ-directed closed forcing.

Theorem 1 *Suppose that κ is indestructibly supercompact and there is a measurable cardinal $\lambda > \kappa$. Then $A_1 = \{\delta < \kappa \mid \delta$ is measurable, δ is not a limit of measurable cardinals, and δ is not δ^+ supercompact$\}$ is unbounded in κ.*

In fact, if we assume additional hypotheses on λ, then it is possible to infer even more. Specifically, we have:

Theorem 2 *Suppose that κ is indestructibly supercompact and there is a cardinal $\lambda > \kappa$ which is 2^λ supercompact. Then besides A_1 being unbounded in κ, $A_2 = \{\delta < \kappa \mid \delta$ is measurable, δ is not a limit of measurable cardinals, and δ is δ^+ supercompact$\}$ is unbounded in κ as well.*

With a limited large cardinal structure above κ, Theorems 1 and 2 need not be true. Specifically, we have:

Theorem 3 *Suppose $V \models$ "ZFC + κ is supercompact + No cardinal $\delta > \kappa$ is 2^δ supercompact". There is then a partial ordering $\mathbb{P} \subseteq V$ such that $V^{\mathbb{P}} \models$ "ZFC + No cardinal $\delta > \kappa$ is 2^δ supercompact + κ is indestructibly supercompact + If $\delta < \kappa$ is a measurable cardinal which is not a limit of measurable cardinals, then δ is not δ^+ supercompact".*

Theorem 4 *Suppose* $V \models$ *"ZFC +* κ *is supercompact + No cardinal* $\delta > \kappa$ *is measurable". There is then a partial ordering* $\mathbb{P} \subseteq V$ *such that* $V^{\mathbb{P}} \models$ *"ZFC + No cardinal* $\delta > \kappa$ *is measurable +* κ *is indestructibly supercompact + If* $\delta < \kappa$ *is a measurable cardinal which is not a limit of measurable cardinals, then* δ *is* δ^+ *supercompact".*

If we are willing to weaken the amount of indestructibility on our supercompact cardinal κ, then it is possible to obtain an improved version of Theorem 4. In particular:

Theorem 5 *Suppose* $V \models$ *"ZFC +* κ *is supercompact + No cardinal* $\delta > \kappa$ *is measurable". There is then a partial ordering* $\mathbb{P} \subseteq V$ *such that* $V^{\mathbb{P}} \models$ *"ZFC + No cardinal* $\delta > \kappa$ *is measurable +* κ *is a supercompact cardinal whose supercompactness is indestructible under partial orderings which are both* κ*-directed closed and* (κ^+, ∞)*-distributive + If* $\delta < \kappa$ *is a measurable cardinal, then* δ *is (at least)* δ^+ *supercompact".*

Of course, reflection easily yields that if κ is supercompact, then $\{\delta < \kappa \mid \delta$ is (at least) δ^+ supercompact$\}$ must be unbounded in κ. It is therefore not possible to obtain an analogue of Theorem 5 for Theorem 3.

We take this opportunity to make a few additional remarks concerning Theorems 1 – 5. The limited amount of indestructibility forced in Theorem 5 is due to the necessity in our proofs of preserving a nontrivial degree of supercompactness. However, if we weaken the requirement in Theorem 5 of all measurable cardinals $\delta < \kappa$ being δ^+ supercompact to only measurable cardinals $\delta < \kappa$ which are not themselves limits of measurable cardinals being δ^+ supercompact, then Theorem 4 shows that it is possible for κ to be a fully indestructible supercompact cardinal. Also, as our proof will show, the degrees of indestructibility mentioned in the statement of Theorems 1 and 2 can be weakened. In particular, κ's supercompactness can be indestructible under κ-directed closed, (κ^+, ∞)-distributive forcing. This provides a nice balance between Theorems 1 and 5, which complements the balance between Theorems 2 and 3 and Theorems 1 and 4.

We conclude Section 1 with a very brief discussion of some preliminary material. We presume a basic knowledge of large cardinals and forcing. A good reference in this regard is [15]. When forcing, $q \geq p$ means that q *is stronger than* p. When G is V-generic over $\mathbb{P}$, we abuse notation slightly and take both $V[G]$ and $V^{\mathbb{P}}$ as being the generic extension of V by $\mathbb{P}$. We also abuse notation slightly by occasionally confusing terms with the sets they denote, especially for ground model sets and variants of the generic object. For $\alpha < \beta$ ordinals, $[\alpha, \beta]$, $(\alpha, \beta]$, $[\alpha, \beta)$, and (α, β) are as in standard interval notation.

Suppose κ is a regular cardinal. For α an arbitrary ordinal, the partial ordering $\mathrm{Add}(\kappa, \alpha)$ is the standard Cohen partial ordering for adding α many Cohen subsets of κ. The partial ordering $\mathbb{P}$ is κ*-directed closed* if for every directed set $D \subseteq \mathbb{P}$ of size less than κ, there is a condition in $\mathbb{P}$ extending each member of D. $\mathbb{P}$ is κ*-strategically closed* if in the two person game in which the players construct an increasing sequence $\langle p_\alpha \mid \alpha \leq \kappa \rangle$, where player I plays odd stages and player II plays even stages, player II has a strategy ensuring the game can always be continued. $\mathbb{P}$ is $<\kappa$*-strategically closed* if $\mathbb{P}$ is δ-strategically closed for every $\delta < \kappa$. $\mathbb{P}$ is (κ, ∞)*-distributive* if the intersection of κ many dense open subsets of $\mathbb{P}$ is dense open. It follows that forcing with any partial ordering $\mathbb{P}$ which is (κ^+, ∞)-distributive preserves the κ^+ supercompactness of κ, since forcing with $\mathbb{P}$ adds no new subsets of $P_\kappa(\kappa^+)$.

A corollary of Hamkins' work on gap forcing found in [13, 14] will be employed in the proof of Theorems 2 – 5. We therefore state as a separate theorem what is relevant for this paper, along with some associated terminology, quoting from [13, 14] when appropriate. Suppose $\mathbb{P}$ is a partial ordering which can be written as $\mathbb{Q} * \dot{\mathbb{R}}$, where $|\mathbb{Q}| < \delta$, $\mathbb{Q}$ is nontrivial, and $\Vdash_{\mathbb{Q}}$ "$\dot{\mathbb{R}}$ is δ^+-directed closed". In Hamkins' terminology of [13, 14], $\mathbb{P}$ *admits a gap at* δ. Also, as in the terminology of [13, 14] and elsewhere, an embedding $j : \overline{V} \to \overline{M}$ is *amenable to* $\overline{V}$ when $j \restriction A \in \overline{V}$ for any $A \in \overline{V}$. The specific corollary of Hamkins' work from [13, 14] we will be using is then the following.

Theorem 6 (Hamkins) *Suppose that $V[G]$ is a generic extension obtained by forcing with $\mathbb{P}$ that admits a gap at some regular $\delta < \kappa$. Suppose further that $j : V[G] \to M[j(G)]$ is an embedding with critical point κ for which $M[j(G)] \subseteq V[G]$ and $M[j(G)]^\delta \subseteq M[j(G)]$ in $V[G]$. Then $M \subseteq V$; indeed, $M = V \cap M[j(G)]$. If the full embedding j is amenable to $V[G]$, then the restricted embedding $j \restriction V : V \to M$ is amenable to V. If j is definable from parameters (such as a measure or extender)* *in $V[G]$, then the restricted embedding $j \restriction V$ is definable from the names of those parameters in V.*

2 A Key Forcing Notion

The proof of Theorem 1 will depend on the existence of a certain partial ordering $\mathbb{P}(\delta, \kappa)$. We isolate the existence of this key forcing notion in the following theorem.

Theorem 7 *Let $V \models$ "$\delta < \kappa$ are such that δ is a regular cardinal and κ is 2^κ supercompact". There is then a δ-directed closed partial ordering $\mathbb{P}(\delta, \kappa) \in V$ such that $V^{\mathbb{P}(\delta,\kappa)} \models$ "κ is the least measurable cardinal greater than δ + $2^\kappa = 2^{\kappa^+} = \kappa^{++}$ + κ is κ^+ supercompact".*

Proof: Assume δ and κ are as in the hypotheses of Theorem 7. We define $\mathbb{P}(\delta, \kappa)$ as $\mathbb{P}^1 * \dot{\mathbb{P}}^2 * \dot{\mathbb{P}}^3$. Let $\rho = (2^\kappa)^V$. $\mathbb{P}^1$ is defined as the reverse Easton iteration $\langle \langle \mathbb{P}_n, \dot{\mathbb{Q}}_n \rangle \mid n < \omega \rangle$, where $\mathbb{P}_0 = \mathrm{Add}(\rho^+, 1)$. For each $n < \omega$, if $\Vdash_{\mathbb{P}_n}$ "There is a cardinal greater than ρ violating GCH", then $\Vdash_{\mathbb{P}_n}$ "$\dot{\mathbb{Q}}_n = \dot{\mathrm{Add}}(\gamma^+, 1)$ where γ is the least cardinal greater than ρ violating GCH". If this is not the case, i.e., if $\Vdash_{\mathbb{P}_n}$ "All cardinals greater than ρ satisfy GCH", then $\Vdash_{\mathbb{P}_n}$ "$\dot{\mathbb{Q}}_n$ is trivial forcing".[1] Standard arguments (see [15, Exercise 15.16] and [6, Lemma 4, Case 2]) show that if γ is a cardinal, then after forcing with $\mathrm{Add}(\gamma^+, 1)$, all cardinals less than or equal to γ^+ are preserved, $2^\gamma = \gamma^+$, 2^γ of the ground model is collapsed to γ^+, and all cardinals greater than or equal to $(2^\gamma)^+$ of the ground model are preserved. This means that $V^{\mathbb{P}^1} \models$ "For every $n < \omega$, $2^{\rho^{+n}} = \rho^{+n+1}$". Since $\mathbb{P}^1$ is $\rho^+ = ((2^\kappa)^+)^V$-directed closed, $(2^\kappa)^V = (2^\kappa)^{V^{\mathbb{P}^1}} = \rho$. In addition, it follows that $V^{\mathbb{P}^1} \models$ "κ is 2^κ supercompact".

Work now in $V_* = V^{\mathbb{P}^1}$. $\mathbb{P}^2$ is defined as $\mathbb{P}_\kappa * \dot{\mathrm{Add}}(\kappa^+, 1)$, where $\mathbb{P}_\kappa$ is the reverse Easton iteration $\langle \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle \mid \alpha < \kappa \rangle$ of length κ which does trivial forcing except for those cardinals $\lambda \in (\delta, \kappa)$ which are inaccessible in V_*. In this case, $\mathbb{P}_{\lambda+1} = \mathbb{P}_\lambda * \dot{\mathbb{Q}}_\lambda$, where $\Vdash_{\mathbb{P}_\lambda * \dot{\mathbb{Q}}_\lambda}$ "For every $n < \omega$, $2^{\lambda^{+n}} = \lambda^{+n+1}$". Assuming $\dot{\mathbb{Q}}_\lambda$ is a term for the analogue of the iteration given in the preceding paragraph, $\dot{\mathbb{Q}}_\lambda$ may be written as $\dot{\mathrm{Add}}(\lambda^+, 1) * \dot{\mathbb{R}}_\lambda$, where $\Vdash_{\mathbb{P}_\lambda * \dot{\mathrm{Add}}(\lambda^+,1)}$ "$\dot{\mathbb{R}}_\lambda$ is $(2^\lambda)^+$-directed closed", i.e., $\Vdash_{\mathbb{P}_\lambda * \dot{\mathrm{Add}}(\lambda^+,1)}$ "$\dot{\mathbb{R}}_\lambda$ is λ^{++}-directed closed".

[1] We slightly abuse notation here when we write $\Vdash_{\mathbb{P}_n}$, since we always assume we are forcing above the relevant condition when necessary.

Standard arguments once again show that $V_*^{\mathbb{P}^2} \models$ "$2^\gamma = \gamma^+$ if $\lambda \in (\delta, \kappa]$ is inaccessible and $\gamma \in [\lambda, \lambda^{+\omega})$". It is also the case that $V_*^{\mathbb{P}^2} \models$ "κ is $2^\kappa = \kappa^+$ supercompact". To see this, let $j : V_* \to M$ be an elementary embedding witnessing the ρ supercompactness of κ in V_* generated by a supercompact ultrafilter over $P_\kappa(\rho)$. In particular, $M^\rho \subseteq M$. We use a standard lifting argument (a form of which is given, e.g., in the proof of [5, Lemma 2.2]) to show that j lifts in $V_*^{\mathbb{P}_\kappa * \dot{\mathrm{Add}}(\kappa^+,1)}$ to $j : V_*^{\mathbb{P}_\kappa * \dot{\mathrm{Add}}(\kappa^+,1)} \to M^{j(\mathbb{P}_\kappa * \dot{\mathrm{Add}}(\kappa^+,1))}$. Specifically, let G_0 be V_*-generic over $\mathbb{P}_\kappa$, and let G_1 be $V_*[G_0]$-generic over $\mathrm{Add}(\kappa^+, 1)$. Observe that $j(\mathbb{P}_\kappa * \dot{\mathrm{Add}}(\kappa^+, 1)) = \mathbb{P}_\kappa * \dot{\mathrm{Add}}(\kappa^+, 1) * \dot{\mathbb{Q}} * \dot{\mathrm{Add}}(j(\kappa^+), 1)$. Working in $V_*[G_0][G_1]$, we first note that since $\mathbb{P}_\kappa * \dot{\mathrm{Add}}(\kappa^+, 1)$ is ρ^+-c.c., $M[G_0][G_1]$ remains ρ closed with respect to $V_*[G_0][G_1]$. This means that $\mathbb{Q}$ is ρ^+-directed closed in both $M[G_0][G_1]$ and $V_*[G_0][G_1]$.

Since $M[G_0][G_1] \models$ "$|\mathbb{Q}| = j(\kappa)$", the number of dense open subsets of $\mathbb{Q}$ present in $M[G_0][G_1]$ is $(2^{j(\kappa)})^M$. In V_*, since M is given via an ultrapower by a supercompact ultrafilter over $P_\kappa(2^\kappa)$, this is calculated as $|\{f \mid f : [2^\kappa]^{<\kappa} \to 2^\kappa\}| = |\{f \mid f : 2^\kappa \to 2^\kappa\}| = 2^{2^\kappa} = 2^\rho$. Since $V_* \models$ "$2^{2^\kappa} = (2^\kappa)^+ = \rho^+$" and ρ^+ is preserved from V_* to $V_*[G_0][G_1]$, we may let $\langle D_\alpha \mid \alpha < \rho^+\rangle \in V_*[G_0][G_1]$ enumerate the dense open subsets of $\mathbb{Q}$ present in $M[G_0][G_1]$. We may now use the fact that $\mathbb{Q}$ is ρ^+-directed closed in $V_*[G_0][G_1]$ to meet each D_α and thereby construct in $V_*[G_0][G_1]$ an $M[G_0][G_1]$-generic object H_0 over $\mathbb{Q}$. Our construction guarantees that $j''G_0 \subseteq G_0 * G_1 * H_0$, so j lifts in $V_*[G_0][G_1]$ to $j : V_*[G_0] \to M[G_0][G_1][H_0]$.

It remains to lift j in $V_*[G_0][G_1]$ through $\mathrm{Add}(\kappa^+, 1)$. Because $V_*[G_0] \models$ "$|\mathrm{Add}(\kappa^+, 1)| = 2^\kappa = (2^\kappa)^{V_*} = \rho$", $M[G_0][G_1][H_0] \models$ "$|\mathrm{Add}(j(\kappa^+), 1)| = 2^{j(\kappa)} = (2^{j(\kappa)})^M$". Therefore, since $M[G_0][G_1][H_0]$ remains ρ closed with respect to $V_*[G_0][G_1]$, $M[G_0][G_1][H_0] \models$ "$\mathrm{Add}(j(\kappa^+), 1)$ is $j(\kappa^+)$-directed closed", and $j(\kappa^+) > j(\kappa) > \rho$, there is a master condition $q \in \mathrm{Add}(j(\kappa^+), 1)$ for $j''\{p \mid p \in G_1\}$. Further, the number of dense open subsets of $\mathrm{Add}(j(\kappa^+), 1)$ present in $M[G_0][G_1][H_0]$ is $(2^{2^{j(\kappa)}})^M$. This is calculated in V_* as $|\{f \mid f : [2^\kappa]^{<\kappa} \to 2^{2^\kappa}\}| = |\{f \mid f : 2^\kappa \to (2^\kappa)^+\}| = |\{f \mid f : \rho \to \rho^+\}| = 2^\rho = (2^\kappa)^+ = \rho^+$. Working in $V_*[G_0][G_1]$, since $\mathrm{Add}(j(\kappa^+), 1)$ is ρ^+-directed closed in both $M[G_0][G_1][H_0]$ and $V_*[G_0][G_1]$, we may consequently use the arguments of the preceding paragraph to construct an $M[G_0][G_1][H_0]$-generic object H_1 over $\mathrm{Add}(j(\kappa^+), 1)$ containing q. Since by the definition of H_1, $j''(G_0 * G_1) \subseteq G_0 * G_1 * H_0 * H_1$, j lifts in $V_*[G_0][G_1]$ to $j : V_*[G_0][G_1] \to M[G_0][G_1][H_0][H_1]$. As $V_*[G_0][G_1] \models$ "$|\rho| = \kappa^+$", this means that $V_*^{\mathbb{P}^2} \models$ "κ is $2^\kappa = \kappa^+$ supercompact". Note that by its definition, $\mathbb{P}^1 * \dot{\mathbb{P}}^2$ is δ-directed closed.

Work now in $V^{\mathbb{P}^1 * \dot{\mathbb{P}}^2} = \overline{V}$. Fix $\lambda \in (\delta, \kappa]$ an inaccessible cardinal. We define three notions of forcing. In particular, we describe now a specific form of the partial orderings of [11, Section 4]. Following the notation of [11, Section 4], we will denote these partial orderings by $\mathbb{P}^0_{\lambda,\lambda^{++}}$, $\mathbb{P}^1_{\lambda,\lambda^{++}}[S]$, and $\mathbb{P}^2_{\lambda,\lambda^{++}}[S]$. So that readers are not overly burdened, we abbreviate our definitions and descriptions somewhat. Full details may be found by consulting [11], along with the relevant portions of [10]. We do mention explicitly, however, that (more than) the amount of GCH required for the definitions of $\mathbb{P}^0_{\lambda,\lambda^{++}}$, $\mathbb{P}^1_{\lambda,\lambda^{++}}[S]$, and $\mathbb{P}^2_{\lambda,\lambda^{++}}[S]$ to be given and for these partial orderings to have the properties described below has been forced by $\mathbb{P}^1 * \dot{\mathbb{P}}^2$.

The first notion of forcing $\mathbb{P}^0_{\lambda,\lambda^{++}}$ is just the standard notion of forcing for adding a nonreflecting stationary set of ordinals S of cofinality δ to λ^{++}. For further details on the definition of this partial ordering, we refer readers to [10] or [11]. We note only that $\mathbb{P}^0_{\lambda,\lambda^{++}}$ is δ-directed closed. Next, work in $V_1 = \overline{V}^{\mathbb{P}^0_{\lambda,\lambda^{++}}}$, letting $\dot{S}$ be a term always forced to denote S. $\mathbb{P}^2_{\lambda,\lambda^{++}}[S]$ is the standard notion of forcing for introducing a club set C which is disjoint to S (and therefore makes S nonstationary).

We fix now in V_1 a $\clubsuit(S)$ sequence $X = \langle x_\alpha \mid \alpha \in S \rangle$, the existence of which is given by [10, Lemma 1] and [11, Lemma 1]. We are ready to define in V_1 the partial ordering $\mathbb{P}^1_{\lambda,\lambda^{++}}[S]$. First, since each element of S has cofinality δ, the proof of Lemma 1 of [10] and [11] shows each $x \in X$ can be assumed to be such that order-type$(x) = \delta$. Then, $\mathbb{P}^1_{\lambda,\lambda^{++}}[S]$ is defined as the set of all 4-tuples $\langle w, \alpha, \bar{r}, Z \rangle$ satisfying the following properties.

1. $w \in [\lambda^{++}]^{<\lambda}$.
2. $\alpha < \lambda$.
3. $\bar{r} = \langle r_i \mid i \in w \rangle$ is a sequence of functions from α to $\{0, 1\}$, i.e., a sequence of subsets of α.
4. $Z \subseteq \{x_\beta \mid \beta \in S\}$ is a set such that if $z \in Z$, then for some $y \in [w]^\delta$, $y \subseteq z$ and $z - y$ is bounded in the β such that $z = x_\beta$.

The ordering on $\mathbb{P}^1_{\lambda,\lambda^{++}}[S]$ is given by $\langle w^1, \alpha^1, \bar{r}^1, Z^1\rangle \le \langle w^2, \alpha^2, \bar{r}^2, Z^2\rangle$ iff the following hold.

1. $w^1 \subseteq w^2$.

2. $\alpha^1 \le \alpha^2$.

3. If $i \in w^1$, then $r^1_i \subseteq r^2_i$.

4. $Z^1 \subseteq Z^2$.

5. If $z \in Z^1 \cap [w^1]^\delta$ and $\alpha^1 \le \alpha < \alpha^2$, then $|\{i \in z \mid r^2_i(\alpha) = 0\}| = |\{i \in z \mid r^2_i(\alpha) = 1\}| = \delta$.

The proof of [10, Lemma 4] shows that $\mathbb{P}^0_{\lambda,\lambda^{++}} * (\mathbb{P}^1_{\lambda,\lambda^{++}}[\dot{S}] \times \mathbb{P}^2_{\lambda,\lambda^{++}}[\dot{S}])$ is equivalent to $\mathrm{Add}(\lambda^{++}, 1) * \dot{\mathrm{Add}}(\lambda, \lambda^{++})$. The proofs of [10, Lemmas 3 and 5] and [11, Lemma 6] show that $\mathbb{P}^0_{\lambda,\lambda^{++}} * \mathbb{P}^1_{\lambda,\lambda^{++}}[\dot{S}]$ preserves cardinals and cofinalities, is λ^{+++}-c.c., is $<\lambda$-strategically closed, and is such that $V^{\mathbb{P}^0_{\lambda,\lambda^{++}} * \mathbb{P}^1_{\lambda,\lambda^{++}}[\dot{S}]} \models$ "$2^\lambda = \lambda^{++}$, $2^{\lambda^+} = \lambda^{++}$, and λ is nonmeasurable". By the remarks in [10, middle of page 108], $\mathbb{P}^0_{\lambda,\lambda^{++}} * \mathbb{P}^1_{\lambda,\lambda^{++}}[\dot{S}]$ is δ-directed closed.

Let $\mathbb{P}^3$ be the reverse Easton iteration of length $\kappa + 1$ which forces with $\mathbb{P}^0_{\lambda,\lambda^{++}} * \mathbb{P}^1_{\lambda,\lambda^{++}}[\dot{S}]$ whenever $\lambda \in (\delta, \kappa)$ is inaccessible, forces with $\mathbb{P}^0_{\kappa,\kappa^{++}} * (\mathbb{P}^1_{\kappa,\kappa^{++}}[\dot{S}] \times \mathbb{P}^2_{\kappa,\kappa^{++}}[\dot{S}])$ at stage κ, and does trivial forcing otherwise. By the facts mentioned in the preceding paragraph, $\mathbb{P}^3$ is δ-directed closed. The proof of [11, Lemma 9] in conjunction with the facts mentioned in the preceding paragraph show that $\overline{V}^{\mathbb{P}^3} \models$ "κ is the least measurable cardinal greater than δ + $2^\kappa = 2^{\kappa^+} = \kappa^{++}$ + κ is κ^+ supercompact". If we now define $\mathbb{P}(\delta, \kappa) = \mathbb{P}^1 * \dot{\mathbb{P}}^2 * \dot{\mathbb{P}}^3$, then $\mathbb{P}(\delta, \kappa)$, which is δ-directed closed, is our desired partial ordering. This completes the proof of Theorem 7.

$\square$

We conclude Section 2 by observing that the definitions of $\mathbb{P}^1$ and $\mathbb{P}^2$ given above may be changed. All that is necessary is that enough GCH is forced to allow the arguments of [10] and [11] to be used to establish that after forcing with $\mathbb{P}^3$, κ has become the least measurable cardinal greater than δ and κ remains κ^+ supercompact.

3 The Proofs of Theorems 1 – 5

We begin with the proof of Theorem 1, after which the proof of Theorem 2 follows immediately.

Proof: We follow the proofs of [1, Theorem 2] and [5, Theorem 1]. Suppose that κ is indestructibly supercompact and there is a measurable cardinal $\lambda > \kappa$. We show that $A_1 = \{\delta < \kappa \mid \delta$ is measurable, δ is not a limit of measurable cardinals, and δ is not δ^+ supercompact$\}$ is unbounded in κ. Let $\eta > \kappa$ be the least measurable cardinal. Force with $\mathrm{Add}(\eta^+, 1)$. After this forcing, which is both κ-directed closed and (κ^+, ∞)-distributive, $2^\eta = \eta^+$ and η remains the least measurable cardinal above κ. In particular, after the forcing, η is a measurable cardinal which is not a limit of measurable cardinals, so automatically, η is not $2^\eta = \eta^+$ supercompact. Since κ's supercompactness is suitably indestructible, by reflection, $A_1 = \{\delta < \kappa \mid \delta$ is measurable, δ is not a limit of measurable cardinals, and δ is not δ^+ supercompact$\}$ is unbounded in κ after the forcing has been performed. Once more, we infer by the fact $\mathrm{Add}(\eta^+, 1)$ is κ-directed closed that A_1 is unbounded in κ in the ground model. This completes the proof of Theorem 1.

□

Proof: We argue in analogy to the proof of Theorem 1. Suppose that κ is indestructibly supercompact and there is a cardinal $\lambda > \kappa$ which is 2^λ supercompact. To show that $A_2 = \{\delta < \kappa \mid \delta$ is measurable, δ is not a limit of measurable cardinals, and δ is δ^+ supercompact$\}$ is unbounded in κ, force with $\mathbb{P}(\kappa^{++}, \lambda)$. By Theorem 7, after this forcing, which is both κ-directed closed and (κ^+, ∞)-distributive, λ has become the least measurable cardinal greater than both κ and κ^{++}, and λ is λ^+ supercompact. In particular, after this forcing, λ is a measurable cardinal which is not a limit of measurable cardinals. We now argue as in the proof of Theorem 1. Since κ's supercompactness is suitably indestructible, by reflection, $A_2 = \{\delta < \kappa \mid \delta$ is measurable, δ is not a limit of measurable cardinals, and δ is δ^+ supercompact$\}$ is unbounded in κ after the forcing has been performed. As before, we infer by the fact $\mathbb{P}(\kappa^{++}, \lambda)$ is κ-directed closed that A_2 is unbounded in κ in the ground model. This completes the proof of Theorem 2.

□

Having completed the proofs of Theorems 1 and 2, we turn now to the proof of Theorem 3.

Proof: Suppose $V \models$ "ZFC + κ is supercompact + No cardinal $\delta > \kappa$ is 2^δ supercompact". Without loss of generality, by first doing a preliminary forcing if necessary, we assume in addition that $V \models$ GCH. This is accomplished using a standard argument. In particular, it is possible to force GCH via the class reverse Easton iteration $\langle \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle \mid \alpha \in \mathrm{Ord} \rangle$, where $\mathbb{P}_0 = \mathrm{Add}(\omega, 1)$. For any ordinal α, if $\Vdash_{\mathbb{P}_\alpha}$ "There is a cardinal violating GCH", then $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha = \dot{\mathrm{Add}}(\gamma^+, 1)$ where γ is the least cardinal violating GCH". If this is not the case, i.e., if $\Vdash_{\mathbb{P}_\alpha}$ "All cardinals satisfy GCH", then $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ is trivial forcing". Since this is a closure point forcing in Hamkins' sense of [9], by Hamkins' results of [12], no new δ which is 2^δ supercompact (or indeed, no new cardinal which is measurable) is created. Hence, if $V \models$ "δ is δ^+ supercompact", then $V \models$ "δ is 2^δ supercompact". This has as a consequence that if $V \models$ "δ is δ^+ supercompact", then $V \models$ "δ is a limit of measurable cardinals".

Let f be a Laver function [16] for κ, i.e., $f : \kappa \to V_\kappa$ is such that for every $x \in V$ and every $\lambda \geq |\mathrm{TC}(x)|$, there is an elementary embedding $j : V \to M$ generated by a supercompact ultrafilter over $P_\kappa(\lambda)$ such that $j(f)(\kappa) = x$. Our partial ordering $\mathbb{P}$ is the reverse Easton iteration of length κ which begins by forcing with $\mathrm{Add}(\omega, 1)$ and then (possibly) does nontrivial forcing only at cardinals $\delta < \kappa$ which are both limits of cardinals η which are 2^η supercompact in V and are at least 2^δ supercompact in V. At such a stage δ, if $f(\delta) = \dot{\mathbb{Q}}$ and $\Vdash_{\mathbb{P}_\delta}$ "$\dot{\mathbb{Q}}$ is a δ-directed closed partial ordering having rank below the least $\eta > \delta$ such that η is 2^η supercompact in V", then $\mathbb{P}_{\delta+1} = \mathbb{P}_\delta * \dot{\mathbb{Q}}$. If this is not the case, then $\mathbb{P}_{\delta+1} = \mathbb{P}_\delta * \dot{\mathbb{Q}}$, where $\dot{\mathbb{Q}}$ is a term for trivial forcing.

Lemma 3.1 $V^{\mathbb{P}} \models$ "*κ is indestructibly supercompact*".

Proof: We follow the proofs of [1, Lemma 2.1] and [5, Lemma 2.1]. Let $\mathbb{Q} \in V^{\mathbb{P}}$ be such that $V^{\mathbb{P}} \models$ "$\mathbb{Q}$ is κ-directed closed". Take $\dot{\mathbb{Q}}$ as a term for $\mathbb{Q}$ such that $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is κ-directed closed". Suppose $\lambda \geq \max(\kappa^+, |\mathrm{TC}(\dot{\mathbb{Q}})|)$ is an arbitrary cardinal, and let $\gamma = 2^{|[\lambda]^{<\kappa}|}$. Take $j : V \to M$ as an elementary embedding witnessing the γ supercompactness of κ generated by a supercompact ultrafilter over $P_\kappa(\gamma)$ such that $j(f)(\kappa) = \dot{\mathbb{Q}}$. Since $V \models$ "No cardinal δ above κ is 2^δ supercompact", $\gamma \geq 2^{[\kappa^+]^{<\kappa}}$, and $M^\gamma \subseteq M$, $M \models$ "κ is both $2^\kappa = \kappa^+$ supercompact and a limit of cardinals η which are 2^η supercompact, and no cardinal δ in the interval $(\kappa, \gamma]$ is 2^δ supercompact". Hence, the definition of $\mathbb{P}$ implies that $j(\mathbb{P} * \dot{\mathbb{Q}}) = \mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}} * j(\dot{\mathbb{Q}})$, where the first stage at which $\dot{\mathbb{R}}$ is forced to do nontrivial forcing is well above γ. Laver's original argument from [16] now applies and shows $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ "κ is λ supercompact". (Simply let $G_0 * G_1 * G_2$ be V-generic over $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$, lift j in $V[G_0][G_1][G_2]$ to $j : V[G_0] \to M[G_0][G_1][G_2]$, take a master condition p for $j''G_1$ and a $V[G_0][G_1][G_2]$-generic object G_3 over $j(\mathbb{Q})$ containing p, lift j again in $V[G_0][G_1][G_2][G_3]$ to $j : V[G_0][G_1] \to M[G_0][G_1][G_2][G_3]$, and show by the γ^+-directed closure of $\mathbb{R} * j(\dot{\mathbb{Q}})$ that the supercompactness measure over $(P_\kappa(\lambda))^{V[G_0][G_1]}$ generated by j is actually a member of $V[G_0][G_1]$.) As λ and $\mathbb{Q}$ were arbitrary, this completes the proof of Lemma 3.1.

$\square$

Lemma 3.2 *If $V \models$ "$\delta < \kappa$ is a 2^δ supercompact cardinal which is not a limit of cardinals η which are 2^η supercompact", then $V^{\mathbb{P}} \models$ "δ is 2^δ supercompact".*

Proof: Suppose that $V \models$ "$\delta < \kappa$ is a 2^δ supercompact cardinal which is not a limit of cardinals η which are 2^η supercompact". Write $\mathbb{P} = \mathbb{P}_\delta * \dot{\mathbb{P}}^\delta$. By the definition of $\mathbb{P}$, $|\mathbb{P}_\delta| < \delta$ and $\Vdash_{\mathbb{P}_\delta}$ "$\dot{\mathbb{P}}^\delta$ is (at least) $\beth_\omega(\delta)$-directed closed". Therefore, the Lévy-Solovay results [17] show that $V^{\mathbb{P}_\delta} \models$ "δ is 2^δ supercompact", so $V^{\mathbb{P}_\delta * \dot{\mathbb{P}}^\delta} = V^{\mathbb{P}} \models$ "δ is 2^δ supercompact". This completes the proof of Lemma 3.2.

$\square$

Lemma 3.3 $V^{\mathbb{P}} \models$ "*If $\delta < \kappa$ is a measurable cardinal which is not a limit of measurable cardinals, then δ is not δ^+ supercompact*".

Proof: We prove the contrapositive. Suppose that $V^{\mathbb{P}} \models$ "$\delta < \kappa$ is δ^+ supercompact". Write $\mathbb{P} = \mathbb{P}' * \dot{\mathbb{P}}''$, where $|\mathbb{P}'| = \omega$, $\mathbb{P}'$ is nontrivial, and $\Vdash_{\mathbb{P}'}$ "$\dot{\mathbb{P}}''$ is $\aleph_2$-directed closed". By Theorem 6, $V \models$ "δ is $(\delta^+)^{V^{\mathbb{P}}}$ supercompact". Consequently, since $(\delta^+)^{V^{\mathbb{P}}} \geq (\delta^+)^V$, $V \models$ "δ is $\delta^+ = 2^\delta$ supercompact". This allows us now to consider the following two cases.

Case 1: $V \models$ "δ is not a limit of cardinals η which are 2^η supercompact". In this case, by Lemma 3.2, $V^{\mathbb{P}} \models$ "δ is 2^δ supercompact". From this, we immediately infer that $V^{\mathbb{P}} \models$ "δ is a limit of measurable cardinals".

Case 2: $V \models$ "δ is a limit of cardinals η which are 2^η supercompact". In particular, $V \models$ "δ is a limit of cardinals η which are 2^η supercompact such that each η is not a limit of cardinals γ which are 2^γ supercompact". By Lemma 3.2, such η are preserved to $V^{\mathbb{P}}$, i.e., $V^{\mathbb{P}} \models$ "δ is a limit of cardinals η which are 2^η supercompact". In other words, $V^{\mathbb{P}} \models$ "δ is a limit of measurable cardinals".

Cases 1 and 2 complete the proof of Lemma 3.3.

$\square$

Since trivial forcing is κ-directed closed, Lemma 3.1 implies that $V^{\mathbb{P}} \models$ "κ is supercompact". Also, because $\mathbb{P}$ may be defined so that $|\mathbb{P}| = \kappa$, the arguments of [17] show that $V^{\mathbb{P}} \models$ "No cardinal $\delta > \kappa$ is 2^δ supercompact". These remarks, together with Lemmas 3.1 – 3.3, complete the proof of Theorem 3.

$\square$

Having completed the proof of Theorem 3, we turn now to the proof of Theorem 4.

Proof: Suppose $V \models$ "ZFC + κ is supercompact + No cardinal $\delta > \kappa$ is measurable". Without loss of generality, by first forcing GCH and then doing the forcing of [7, Theorem 1], we may assume in addition that $V \models$ "If $\delta \le \kappa$ is measurable, then $2^\delta = 2^{\delta^+} = \delta^{++}$ and δ is δ^+ supercompact". (The aforementioned property of V may be assumed to hold because as we have already observed, forcing GCH will not create any new measurable cardinals. Since the forcing of [7, Theorem 1] may be defined so as to have size κ, by the results of [17], it will not create any new measurable cardinals greater than κ.) We then define $\mathbb{P}$ as in the proof of Theorem 3, except that at each nontrivial stage of forcing $\delta < \kappa$ (so in particular, $\mathbb{P}$ (possibly) does nontrivial forcing only at cardinals $\delta < \kappa$ which are both 2^δ supercompact in V and are limits of cardinals η which are 2^η supercompact in V), we require that for our Laver function f, $f(\delta) = \dot{\mathbb{Q}}$ and $\Vdash_{\mathbb{P}_\delta}$ "$\dot{\mathbb{Q}}$ is a δ-directed closed partial ordering having rank below the least V-measurable cardinal greater than δ". The same arguments as used in the proof of Theorem 3, replacing both instances in the proof of Lemma 3.1 of δ not being 2^δ supercompact with δ not being measurable, will now show that $V^{\mathbb{P}} \vDash$ "κ is indestructibly supercompact + No cardinal $\delta > \kappa$ is measurable". The proof of Theorem 4 will therefore be complete once we have established the following lemma.

Lemma 3.4 $V^{\mathbb{P}} \vDash$ "*If $\delta < \kappa$ is a measurable cardinal which is not a limit of measurable cardinals, then δ is δ^+ supercompact*".

Proof: Suppose that $V^{\mathbb{P}} \vDash$ "$\delta < \kappa$ is a measurable cardinal which is not a limit of measurable cardinals". By the factorization of $\mathbb{P}$ given in the proof of Lemma 3.3 and Theorem 6, $V \vDash$ "δ is a measurable cardinal". If $V \vDash$ "δ is a measurable cardinal which is a limit of measurable cardinals", then in particular, $V \vDash$ "δ is a measurable cardinal which is a limit of measurable cardinals which are not themselves limits of measurable cardinals". Observe now that essentially the same argument as given in the proof of Lemma 3.2 remains valid and shows that if $V \vDash$ "$\eta < \kappa$ is a measurable cardinal which is not a limit of measurable cardinals", then $V^{\mathbb{P}} \vDash$ "η is η^+ supercompact". Thus, $V^{\mathbb{P}} \vDash$ "δ is a measurable cardinal which is a limit of measurable cardinals", a contradiction. Hence, if $V^{\mathbb{P}} \vDash$ "$\delta < \kappa$ is a measurable cardinal which is not a limit of measurable cardinals", then $V \vDash$ "δ is a measurable cardinal which is not a limit of measurable cardinals". From this, we infer as earlier in the proof of this lemma that $V^{\mathbb{P}} \vDash$ "δ is δ^+ supercompact". This completes the proof of both Lemma 3.4 and Theorem 4.

□

□

We finish our proofs with the proof of Theorem 5.

Proof: Suppose $V \vDash$ "ZFC + κ is supercompact + No cardinal $\delta > \kappa$ is measurable". As in the proof of Theorem 4, we assume in addition that $V \vDash$ "If $\delta \le \kappa$ is measurable, then $2^\delta = 2^{\delta^+} = \delta^{++}$ and δ is δ^+ supercompact" We then define $\mathbb{P}$ as in the proof of Theorem 4, except that at each nontrivial stage of forcing $\delta < \kappa$, we require that for our Laver function f, $f(\delta) = \dot{\mathbb{Q}}$ and $\Vdash_{\mathbb{P}_\delta}$ "$\dot{\mathbb{Q}}$ is a δ-directed closed, (δ^+, ∞)-distributive partial ordering having rank below the least V-measurable cardinal greater than δ". The same arguments as used in the proofs of Theorems 3 and 4 will now show that $V^{\mathbb{P}} \models$ "κ is a supercompact cardinal whose supercompactness is indestructible under partial orderings which are both κ-directed closed and (κ^+, ∞)-distributive + No cardinal $\delta > \kappa$ is measurable". The proof of Theorem 5 will therefore be complete once we have established the following lemma.

Lemma 3.5 $V^{\mathbb{P}} \models$ "*If $\delta < \kappa$ is a measurable cardinal, then δ is (at least) δ^+ supercompact*".

Proof: Suppose that $V^{\mathbb{P}} \models$ "$\delta < \kappa$ is a measurable cardinal". As in the proof of Lemma 3.4, $V \models$ "δ is a measurable cardinal". This allows us to write $\mathbb{P} = \mathbb{P}_\delta * \dot{\mathbb{P}}^\delta$ and consider the following two cases.

Case 1: $|\mathbb{P}_\delta| < \delta$. In this case, by the results of [17], because $V \models$ "δ is (at least) δ^+ supercompact", $V^{\mathbb{P}_\delta} \models$ "δ is (at least) δ^+ supercompact". Since by the definition of $\mathbb{P}$, $\Vdash_{\mathbb{P}_\delta}$ "$\dot{\mathbb{P}}^\delta$ is both δ-directed closed and (δ^+, ∞)-distributive", $V^{\mathbb{P}_\delta * \dot{\mathbb{P}}^\delta} = V^{\mathbb{P}} \models$ "δ is (at least) δ^+ supercompact".

Case 2: $|\mathbb{P}_\delta| = \delta$. In this case, by our assumptions on V, let $j : V \to M$ be an elementary embedding witnessing the δ^+ supercompactness of δ generated by a supercompact ultrafilter over $P_\delta(\delta^+)$ such that $M^{\delta^+} \subseteq M$ and $M \models$ "δ is not δ^+ supercompact". We then have that $j(\mathbb{P}_\delta) = \mathbb{P}_\delta * \dot{\mathbb{Q}}$, where the first ordinal at which $\dot{\mathbb{Q}}$ is forced to do nontrivial forcing is well beyond δ^+. We may then use a simplified version of the standard lifting argument given in the proof of Theorem 7 to show that j lifts in $V^{\mathbb{P}_\delta}$ to $j : V^{\mathbb{P}_\delta} \to M^{j(\mathbb{P}_\delta)}$. For completeness, we give the details. Let G be V-generic over $\mathbb{P}_\delta$. Working in $V[G]$, we first note that since $\mathbb{P}_\delta$ is δ-c.c., $M[G]$ remains δ^+ closed with respect to $V[G]$. This means that $\mathbb{Q}$ is δ^{++}-directed closed in both $M[G]$ and $V[G]$. As before, because $M[G] \models$ "$|\mathbb{Q}| = j(\delta)$", the number of dense open subsets of $\mathbb{Q}$ present in $M[G]$ is $(2^{j(\delta)})^M$. Since $V \models$ "$2^\delta = 2^{\delta^+} = \delta^{++}$" and M is given via an ultrapower by a supercompact ultrafilter over $P_\delta(\delta^+)$, this is calculated as $|\{f \mid f : [\delta^+]^{<\delta} \to 2^\delta\}| = |\{f \mid f : \delta^+ \to \delta^{++}\}| = \delta^{++}$. We may therefore let $\langle D_\alpha \mid \alpha < \delta^{++} \rangle \in V[G]$ enumerate the dense open subsets of $\mathbb{Q}$ present in $M[G]$. We may now use the fact that $\mathbb{Q}$ is δ^{++}-directed closed in $V[G]$ to meet each D_α and thereby construct in $V[G]$ an

$M[G]$-generic object H over $\mathbb{Q}$. Our construction guarantees that $j''G \subseteq G * H$, so j lifts in $V[G]$ to $j : V[G] \to M[G][H]$. Hence, $V^{\mathbb{P}_\delta} \models$ "δ is (at least) δ^+ supercompact". As in Case 1 above, since $\Vdash_{\mathbb{P}_\delta}$ "$\dot{\mathbb{P}}^\delta$ is both δ-directed closed and (δ^+, ∞)-distributive", $V^{\mathbb{P}_\delta * \dot{\mathbb{P}}^\delta} = V^{\mathbb{P}} \models$ "δ is (at least) δ^+ supercompact". This completes the proof of both Lemma 3.5 and Theorem 5.

□

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We conclude by remarking that other than the fact that the proof of Lemma 3.5 requires (δ^+, ∞)-distributivity at each nontrivial stage of forcing δ in the definition of $\mathbb{P}$, there is no reason prima facie to believe that this restriction must be present. We therefore end by asking if the proof of Theorem 5 can be reworked so that $V^{\mathbb{P}} \models$ "κ is indestructibly supercompact".

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