Indestructible Strong Compactness and Level by Level Inequivalence *†

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Abstract

If $\delta < \gamma$ are such that δ is indestructibly supercompact and γ is measurable, then it must be the case that level by level inequivalence between strong compactness and supercompactness fails. We prove a theorem which points to this result being best possible. Specifically, we show that relative to the existence of cardinals $\kappa_1 < \lambda$ such that κ_1 is λ supercompact and λ is inaccessible, there is a model for level by level inequivalence between strong compactness and supercompactness containing a supercompact cardinal $\kappa < \kappa_1$ in which κ's strong compactness, but not supercompactness, is indestructible under κ-directed closed forcing. In this model, κ is the least strongly compact cardinal, and no cardinal is supercompact up to an inaccessible cardinal.

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1 Introduction and Preliminaries

Say that a model of ZFC containing at least one supercompact cardinal κ satisfies *level by level inequivalence between strong compactness and supercompactness* if for every non-supercompact measurable cardinal δ, there is some $\gamma > \delta$ such that δ is γ strongly compact yet δ is not γ supercompact. By [2, Theorem 2], this is incompatible with κ being indestructibly supercompact, assuming there is a measurable cardinal $\lambda > \kappa$. Specifically, [2, Theorem 2] shows that if κ is indestructibly supercompact and $\lambda > \kappa$ is measurable, then $\{\delta < \kappa \mid \delta$ is measurable and δ is not δ^+ strongly compact$\} \subseteq \{\delta < \kappa \mid \delta$ is measurable and level by level inequivalence between strong compactness and supercompactness fails at $\delta\}$ is unbounded in κ. This raises the following

Question: Is it possible to obtain a model with a restricted large cardinal structure containing an indestructibly supercompact cardinal in which level by level inequivalence between strong compactness and supercompactness holds?

Unfortunately, an answer to this Question remains unknown. The purpose of this paper is to establish a result giving evidence that a positive answer to this Question is indeed plausible. Specifically, we prove the following theorem.

Theorem 1 *Suppose $V \vDash$ "ZFC + GCH + $\kappa_1 < \lambda$ are such that κ_1 is λ supercompact and λ is inaccessible". There is then a partial ordering $\mathbb{P} \in V$, a submodel $\overline{V} \subseteq V^{\mathbb{P}}$, and $\kappa < \kappa_1$ such that $\overline{V} \vDash$ "ZFC + κ is supercompact + No cardinal $\delta > \kappa$ is inaccessible". In $\overline{V}$, no cardinal is supercompact up to an inaccessible cardinal, and level by level inequivalence between strong compactness and supercompactness holds. Further, in $\overline{V}$, κ is the least strongly compact cardinal, and κ's strong compactness, but not supercompactness, is indestructible under κ-directed closed forcing.*

Theorem 1 should be contrasted with [3, Theorem 1]. In particular, a model witnessing the dual notion of *level by level equivalence between strong compactness and supercompactness*[1] which

[1]This notion was first introduced by the author and Shelah in [7]. For the purposes of this paper and [3], *level by level equivalence between strong compactness and supercompactness* means that for any two regular cardinals $\kappa < \lambda$, κ is λ strongly compact iff κ is λ supercompact. The general case is treated in [7].

otherwise satisfies the same conclusions as Theorem 1 of this paper is constructed in [3, Theorem 1], starting from a model of "ZFC + There exists a supercompact cardinal". Also, note that the only known restrictions on the large cardinal structure of any possible model witnessing a positive answer to our Question are provided by [2, Theorem 2]. We will touch upon this again at the end of the paper.

Before beginning the proof of Theorem 1, we briefly mention some preliminary information and terminology. Essentially, our notation and terminology are standard, and when this is not the case, this will be clearly noted. When forcing, $q \geq p$ will mean that q *is stronger than* p. If G is V-generic over $\mathbb{P}$, we will abuse notation slightly and use both $V[G]$ and $V^{\mathbb{P}}$ to indicate the universe obtained by forcing with $\mathbb{P}$. If $x \in V[G]$, then $\dot{x}$ will be a term in V for x. We may, from time to time, confuse terms with the sets they denote and write x when we actually mean $\dot{x}$ or $\check{x}$, especially when x is some variant of the generic set G, or x is in the ground model V. The abuse of notation mentioned above will be compounded by writing $x \in V^{\mathbb{P}}$ instead of $\dot{x} \in V^{\mathbb{P}}$. Any term for trivial forcing will always be taken as a term for the partial ordering $\{\emptyset\}$. If φ is a formula in the forcing language with respect to $\mathbb{P}$ and $p \in \mathbb{P}$, then $p \parallel \varphi$ means that p *decides* φ.

If $\kappa \geq \omega$ is a regular cardinal, then $\mathrm{Add}(\kappa, 1)$ is the standard partial ordering for adding a single Cohen subset of κ. If $\mathbb{P}$ is an arbitrary partial ordering, $\mathbb{P}$ is κ*-directed closed* if for every cardinal $\delta < \kappa$ and every directed set $\langle p_\alpha \mid \alpha < \delta \rangle$ of elements of $\mathbb{P}$ (where $\langle p_\alpha \mid \alpha < \delta \rangle$ is *directed* if every two elements p_ρ and p_ν have a common upper bound of the form p_σ) there is an upper bound $p \in \mathbb{P}$. $\mathbb{P}$ is κ*-strategically closed* if in the two person game of length $\kappa + 1$ in which the players construct an increasing sequence $\langle p_\alpha \mid \alpha \leq \kappa \rangle$, where player I plays odd stages and player II plays even stages (choosing the trivial condition at stage 0), player II has a strategy which ensures the game can always be continued. $\mathbb{P}$ is $<\kappa$*-strategically closed* if $\mathbb{P}$ is δ-strategically closed for every $\delta < \kappa$. Note that if $\mathbb{P}$ is κ-directed closed, then $\mathbb{P}$ is $<\kappa$-strategically closed (so since $\mathrm{Add}(\kappa, 1)$ is κ-directed closed, $\mathrm{Add}(\kappa, 1)$ is $<\kappa$-strategically closed as well). We adopt Hamkins' terminology of [12, 11, 10] and say that $x \subseteq \kappa$ *is a fresh subset of* κ *with respect to* $\mathbb{P}$ if $\mathbb{P}$ is nontrivial forcing, $x \in V^{\mathbb{P}}$, $x \notin V$, yet $x \cap \alpha \in V$ for every $\alpha < \kappa$.

The partial ordering $\mathbb{P}$ used in the proof of Theorem 1 will be a *Gitik iteration*. By this we will mean an Easton support iteration $\mathbb{P}$ as first given by Gitik in [9], to which we refer readers for a discussion of the basic properties of and terminology associated with such an iteration. For the purposes of this paper, at any stage δ at which a nontrivial forcing is done in a Gitik iteration, we assume the partial ordering $\mathbb{Q}_\delta$ with which we force is either δ-directed closed or is Prikry forcing defined with respect to a normal measure over δ (although other types of partial orderings may be used in the general case — see [9] for additional details). We do explicitly mention that if $p, q \in \mathbb{P}$, then $q \geq p$ roughly speaking means that $q \geq p$ as in a usual reverse Easton iteration, except that stems of Prikry conditions in p can only be extended nontrivially finitely often. If $q \geq p$ but no stems of Prikry conditions in p are extended, then q is called an *Easton extension of* p. For a more precise definition, readers are urged to consult [9].

Key to the proof of Lemma 2.5, which shows that the supercompactness of the cardinal κ of Theorem 1 is not indestructible under κ-directed closed forcing, is the following theorem due to Gitik.

Theorem 2 ([6, Proposition 1.1]) *Suppose κ is a Mahlo cardinal and $\mathbb{P} = \langle\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha\rangle \mid \alpha \leq \kappa\rangle$ is an Easton support iteration of length $\kappa + 1$ satisfying the following properties.*

1. $\mathbb{P}_0 = \{\emptyset\}$.
2. *For each $\alpha < \kappa$, $\Vdash_{\mathbb{P}_\alpha}$ "$|\dot{\mathbb{Q}}_\alpha| < \kappa$".*
3. *$\Vdash_{\mathbb{P}_\kappa}$ "$\dot{\mathbb{Q}}_\kappa$ is $<\kappa$-strategically closed".*
4. *For some $\alpha, \delta < \kappa$, $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ adds a new subset of δ".*
5. *κ is Mahlo in $V^{\mathbb{P}_{\kappa+1}} = V^{\mathbb{P}}$.*

Then in $V^{\mathbb{P}}$, there are no fresh subsets of κ.

We recall for the benefit of readers the definition given by Hamkins in [13, Section 3] of the lottery sum of a collection of partial orderings. If $\mathfrak{A}$ is a collection of partial orderings, then the *lottery sum* is the partial ordering $\oplus\mathfrak{A} = \{\langle \mathbb{P}, p\rangle \mid \mathbb{P} \in \mathfrak{A} \text{ and } p \in \mathbb{P}\} \cup \{0\}$, ordered with 0 below everything and $\langle \mathbb{P}, p\rangle \le \langle \mathbb{P}', p'\rangle$ iff $\mathbb{P} = \mathbb{P}'$ and $p \le p'$. Intuitively, if G is V-generic over $\oplus\mathfrak{A}$, then G first selects an element of $\mathfrak{A}$ (or as Hamkins says in [13], "holds a lottery among the posets in $\mathfrak{A}$") and then forces with it.[2]

Finally, we mention that we are assuming familiarity with the large cardinal notions of measurability, strong compactness, and supercompactness. Interested readers may consult [14] or [16] for further details. We do note, however, that the cardinal κ is said to be *supercompact (strongly compact) up to the cardinal* λ if κ is δ supercompact (δ strongly compact) for every $\delta < \lambda$. κ is said to be *indestructibly supercompact* if, as in [15], κ's supercompactness is indestructible under arbitrary κ-directed closed forcing. The measurable cardinal κ is said to have *trivial Mitchell rank* if there is no elementary embedding $j : V \to M$ generated by a normal measure $\mathcal{U}$ over κ such that $M \vDash$ "κ is a measurable cardinal". We explicitly note that if κ has trivial Mitchell rank, then κ is not supercompact (and in fact, if κ has trivial Mitchell rank, then κ is not even 2^κ supercompact).

2 The Proof of Theorem 1

We turn now to the proof of Theorem 1.

Proof: Let $V \vDash$ "ZFC + GCH + $\kappa_1 < \lambda$ are such that κ_1 is λ supercompact and λ is inaccessible". Without loss of generality, assume that $\lambda = \kappa_1'$, where for the duration of this paper, for any ordinal δ, δ' is the least inaccessible cardinal above δ in V.

The partial ordering $\mathbb{P} = \langle\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha\rangle \mid \alpha < \kappa_1\rangle$ to be used in the proof of Theorem 1 is the Gitik iteration of length κ_1 which has the following properties.

1. $\mathbb{P}$ begins by forcing with $\mathrm{Add}(\omega, 1)$, i.e., $\mathbb{P}_0 = \{\emptyset\}$ and $\Vdash_{\mathbb{P}_0}$ "$\dot{\mathbb{Q}}_0 = \dot{\mathrm{Add}}(\omega, 1)$".

2. The only other stages $\alpha > 0$ at which $\mathbb{P}$ (possibly) does nontrivial forcing are those ordinals δ which are, in V, Mahlo cardinals which are not supercompact up to δ'. (In particular, we emphasize that if $\delta < \kappa_1$ is such that $V \vDash$ "δ is supercompact up to δ'", then δ is a trivial

[2] The terminology "lottery sum" is due to Hamkins, although the concept of the lottery sum of partial orderings has been around for quite some time and has been referred to at different junctures via the names "disjoint sum of partial orderings," "side-by-side forcing," and "choosing which partial ordering to force with generically."

stage of forcing.) At such a stage δ, $\mathbb{P}_{\delta+1} = \mathbb{P}_\delta * \dot{\mathbb{L}}_\delta * \dot{\mathbb{R}}_\delta$, where $\dot{\mathbb{L}}_\delta$ is a term for the lottery sum of all δ-directed closed partial orderings having rank below δ'.

3. If $V^{\mathbb{P}_\delta * \dot{\mathbb{L}}_\delta} \models$ "δ is not measurable", then $\dot{\mathbb{R}}_\delta$ is a term for trivial forcing.

4. If $V^{\mathbb{P}_\delta * \dot{\mathbb{L}}_\delta} \models$ "δ is measurable", then $\dot{\mathbb{R}}_\delta$ is a term for Prikry forcing defined with respect to some normal measure over δ.

The intuition behind the above definition of $\mathbb{P}$ is as follows. The fact that nothing is done at stage δ unless δ is a Mahlo cardinal in V which is not supercompact up to δ', i.e., that no Prikry sequence is added, ensures that κ_1 is λ supercompact in $V^{\mathbb{P}}$. By reflection, this means that there is $\kappa < \kappa_1$ such that in $V^{\mathbb{P}}$, κ is supercompact up to κ'. Let $\overline{V} = (V_{\kappa'})^{V^{\mathbb{P}}}$. In $\overline{V}$, κ is supercompact. Since a Prikry sequence is added when a nontrivial forcing at stage δ preserves the measurability of δ, there will be a partial ordering $\mathbb{R} \in \overline{V}$ such that $\overline{V}^{\mathbb{R}} \models$ "κ is not supercompact". The lottery sum at stage δ, in conjunction with the Prikry forcing, will allow us to show that in $\overline{V}$, κ's strong compactness is preserved by nontrivial forcing. Because unboundedly many in κ Prikry sequences will have been added by $\mathbb{P}$, $\overline{V} \models$ "No cardinal below κ is strongly compact", i.e., $\overline{V} \models$ "κ is the least strongly compact cardinal". The definition of $\overline{V}$ will ensure that in $\overline{V}$, level by level inequivalence between strong compactness and supercompactness holds.

The following lemmas show that $\mathbb{P}$ is as desired.

Lemma 2.1 $V^{\mathbb{P}} \models$ "*κ_1 is λ supercompact*".

Proof: Take $j : V \to M$ as an elementary embedding witnessing the λ supercompactness of κ_1. Since $M^\lambda \subseteq M$, $M \models$ "κ_1 is supercompact up to $\kappa_1' = \lambda$". This means by the definition of $\mathbb{P}$ that only trivial forcing occurs at stage κ_1 in M in the definition of $j(\mathbb{P})$. Consequently, in M, above the appropriate condition, $j(\mathbb{P})$ is forcing equivalent to $\mathbb{P} * \dot{\mathbb{Q}}$, where the first nontrivial stage in $\dot{\mathbb{Q}}$ takes place well after λ.

We now follow the proofs of [1, Lemma 2.1] and [6, Lemma 2.1] and apply the argument of [9, Lemma 1.5]. Specifically, let G be V-generic over $\mathbb{P}$. Since GCH in V implies that $V \models$ "$2^\lambda = \lambda^+$", we may let $\langle \dot{x}_\alpha \mid \alpha < \lambda^+ \rangle$ be an enumeration in V of all of the canonical $\mathbb{P}$-names of subsets of $P_{\kappa_1}(\lambda)$. Because $\mathbb{P}$ is κ_1-c.c. and $M^\lambda \subseteq M$, $M[G]^\lambda \subseteq M[G]$. By [9, Lemmas 1.4 and 1.2], we may therefore define in $V[G]$ an increasing sequence $\langle p_\alpha \mid \alpha < \lambda^+ \rangle$ of elements of $j(\mathbb{P})/G$ such that if $\alpha < \beta < \lambda^+$, p_β is an Easton extension of p_α, every initial segment of the sequence is in $M[G]$, and for every $\alpha < \lambda^+$, $p_{\alpha+1} \parallel$ "$\langle j(\beta) \mid \beta < \lambda \rangle \in j(\dot{x}_\alpha)$". (For $\beta < \lambda^+$ a limit ordinal, p_β exists since $M[G]^\lambda \subseteq M[G]$ and the first nontrivial stage of forcing in p_α for $\alpha < \beta$ takes place well after λ. This means that it is possible to let p_β be an Easton extension of p_α for $\alpha < \beta$.) The remainder of the argument of [9, Lemma 1.5] remains valid and shows that a supercompact ultrafilter $\mathcal{U}$ over $(P_{\kappa_1}(\lambda))^{V[G]}$ may be defined in $V[G]$ by $x \in \mathcal{U}$ iff $x \subseteq (P_{\kappa_1}(\lambda))^{V[G]}$ and for some $\alpha < \lambda^+$ and some $\mathbb{P}$-name $\dot{x}$ of x, in $M[G]$, $p_\alpha \Vdash_{j(\mathbb{P})/G}$ "$\langle j(\beta) \mid \beta < \lambda \rangle \in j(\dot{x})$". (The fact that $j''G = G$ tells us $\mathcal{U}$ is well-defined.) Thus, $\Vdash_{\mathbb{P}}$ "κ_1 is λ supercompact". This completes the proof of Lemma 2.1.

$\square$

Lemma 2.2 *Suppose $V \vDash$ "$\kappa \le \kappa_1$ is supercompact up to κ'". Let $\mathbb{Q} \in V^{\mathbb{P}}$ be a partial ordering which is κ-directed closed and has rank below κ'. Then $V^{\mathbb{P} * \dot{\mathbb{Q}}} \vDash$ "κ is strongly compact up to κ'".*

Proof: We follow the proofs of [4, Lemma 2.2] and [6, Lemma 2.2], quoting verbatim when appropriate. Let $\kappa \le \kappa_1$ be such that $V \vDash$ "κ is supercompact up to κ'". Write $\mathbb{P} = \mathbb{P}_\kappa * \dot{\mathbb{P}}^\kappa$. Since by the definition of $\mathbb{P}$, $|\mathbb{P}_\kappa| = \kappa$ and $\Vdash_{\mathbb{P}_\kappa}$ "Forcing with $\dot{\mathbb{P}}^\kappa$ adds no new subsets of $(\kappa')^V$", κ' has the same meaning in V, $V^{\mathbb{P}_\kappa}$, and $V^{\mathbb{P}_\kappa * \dot{\mathbb{P}}^\kappa} = V^{\mathbb{P}}$. Writing now κ' without fear of ambiguity, it suffices to show that if $\mathbb{Q} \in V^{\mathbb{P}_\kappa}$ is κ-directed closed and has rank below κ', then $V^{\mathbb{P}_\kappa * \dot{\mathbb{Q}}} \vDash$ "κ is strongly compact up to κ'".

To do this, let $\zeta > \max(\kappa, |\mathrm{TC}(\dot{\mathbb{Q}})|)$, $\zeta < \kappa'$ be an arbitrary regular cardinal. By GCH in V and the choice of ζ, it is the case that $(2^{[\zeta]^{<\kappa}})^V = (2^{[\zeta]^{<\kappa}})^{V^{\mathbb{P}_\kappa * \dot{\mathbb{Q}}}} = (2^\zeta)^V = (2^\zeta)^{V^{\mathbb{P}_\kappa * \dot{\mathbb{Q}}}} = (\zeta^+)^V = (\zeta^+)^{V^{\mathbb{P}_\kappa * \dot{\mathbb{Q}}}}$. Without ambiguity, we may write $\rho = \zeta^+$ and $\sigma = \rho^+ = 2^\rho = 2^{\zeta^+} = 2^{2^\zeta}$. By assuming $j(\kappa)$ is minimal, we may take $j : V \to M$ as an elementary embedding witnessing the σ supercompactness of κ such that $M \vDash$ "κ is not σ supercompact". Since $\sigma < \kappa'$, by the choice of σ and the definition of $\mathbb{P}$, it is possible to opt for $\mathbb{Q}$ in the stage κ lottery held in M in the definition of $j(\mathbb{P}_\kappa)$. In addition, the next nontrivial forcing in the definition of $j(\mathbb{P}_\kappa)$ takes place well above σ. Thus, in M, above the appropriate condition, $j(\mathbb{P}_\kappa * \dot{\mathbb{Q}})$ is forcing equivalent to $\mathbb{P}_\kappa * \dot{\mathbb{Q}} * \dot{\mathbb{S}}_\kappa * \dot{\mathbb{R}} * j(\dot{\mathbb{Q}})$, where $\Vdash_{\mathbb{P}_\kappa * \dot{\mathbb{Q}}}$ "$\dot{\mathbb{S}}_\kappa$ is a term for either Prikry forcing or trivial forcing".

The remainder of the proof of Lemma 2.2 is as in the proof of [5, Lemma 2]. As in the proof of Lemma 2.1, we outline the argument, and refer readers to the proof of [5, Lemma 2] for any missing details. By the last two sentences of the preceding paragraph, as in [5, Lemma 2], there is a term $\tau \in M$ in the language of forcing with respect to $j(\mathbb{P}_\kappa)$ such that if $G * H$ is either V-generic or M-generic over $\mathbb{P}_\kappa * \dot{\mathbb{Q}}$, $\Vdash_{j(\mathbb{P}_\kappa)}$ "τ extends every $j(\dot{q})$ for $\dot{q} \in \dot{H}$". In other words, τ is a term for a "master condition" for $j(\dot{\mathbb{Q}})$. Thus, if $\langle \dot{A}_\alpha \mid \alpha < \rho < \sigma \rangle$ enumerates in V the canonical $\mathbb{P}_\kappa * \dot{\mathbb{Q}}$ names of subsets of $(P_\kappa(\zeta))^{V[G*H]}$, we can define in M a sequence of $\mathbb{P}_\kappa * \dot{\mathbb{Q}} * \dot{\mathbb{S}}_\kappa$ names of elements of $\dot{\mathbb{R}} * j(\dot{\mathbb{Q}})$, $\langle \dot{p}_\alpha \mid \alpha \le \rho \rangle$, such that $\dot{p}_0$ is a term for $\langle 0, \tau \rangle$ (where 0 represents the trivial condition with respect to $\mathbb{R}$), $\Vdash_{\mathbb{P}_\kappa * \dot{\mathbb{Q}} * \dot{\mathbb{S}}_\kappa}$ "$\dot{p}_{\alpha+1}$ is a term for an Easton extension of $\dot{p}_\alpha$ deciding '$\langle j(\beta) \mid \beta < \zeta \rangle \in j(\dot{A}_\alpha)$' ", and for $\eta \le \rho$ a limit ordinal, $\Vdash_{\mathbb{P}_\kappa * \dot{\mathbb{Q}} * \dot{\mathbb{S}}_\kappa}$ "$\dot{p}_\eta$ is a term for an Easton extension of each member of the sequence $\langle \dot{p}_\beta \mid \beta < \eta \rangle$". We can then in $V[G * H]$ define a set $\mathcal{U} \subseteq 2^{[\zeta]^{<\kappa}}$ by $X \in \mathcal{U}$ iff $X \subseteq P_\kappa(\zeta)$ and for some $\langle r, q \rangle \in G * H$ and some $q' \in \mathbb{S}_\kappa$ either the trivial condition (if $\mathbb{S}_\kappa$ is trivial forcing) or of the form $\langle \emptyset, B \rangle$ (if $\mathbb{S}_\kappa$ is Prikry forcing), in M, $\langle r, \dot{q}, \dot{q}', \dot{p}_\rho \rangle \Vdash$ "$\langle j(\beta) \mid \beta < \zeta \rangle \in \dot{X}$" for some name $\dot{X}$ of X. As in [5, Lemma 2], $\mathcal{U}$ is a κ-additive, fine ultrafilter over $(P_\kappa(\zeta))^{V[G*H]}$, i.e., $V[G * H] \vDash$ "κ is ζ strongly compact". Since $\zeta < \kappa'$ was arbitrary, this completes the proof of Lemma 2.2.

□

Lemma 2.3 $V^{\mathbb{P}} \vDash$ "*Any measurable cardinal* $\kappa \le \kappa_1$ *is strongly compact up to* κ'".

Proof: Suppose $V^{\mathbb{P}} \vDash$ "$\kappa \le \kappa_1$ is measurable". By Lemma 2.1, we assume without loss of generality that $\kappa < \kappa_1$. Since $V^{\mathbb{P}} \vDash$ "κ is a Mahlo cardinal" and forcing can't create a new Mahlo cardinal, $V \vDash$ "κ is a Mahlo cardinal" as well. Write $\mathbb{P} = \mathbb{P}_{\kappa+1} * \dot{\mathbb{P}}^{\kappa+1}$. We will show that $V \vDash$ "κ is supercompact up to κ'". Otherwise, by the definition of $\mathbb{P}$, if $V \vDash$ "κ is not supercompact up to κ'", $V^{\mathbb{P}_{\kappa+1}} \vDash$ "κ is not measurable". Hence, since $\Vdash_{\mathbb{P}_{\kappa+1}}$ "Forcing with $\dot{\mathbb{P}}^{\kappa+1}$ adds no new subsets of κ'", $V^{\mathbb{P}_{\kappa+1} * \dot{\mathbb{P}}^{\kappa+1}} = V^{\mathbb{P}} \vDash$ "κ is not measurable" as well. This is contrary to our assumptions, so $V \vDash$ "κ is supercompact up to κ'". Therefore, since trivial forcing is κ-directed closed and is taken to have rank below κ', by Lemma 2.2, $V^{\mathbb{P}} \vDash$ "κ is strongly compact up to κ'". This completes the proof of Lemma 2.3.

□

As one of the referees has pointed out, as will be seen in the final construction of the model witnessing the conclusions of Theorem 1, Lemmas 2.1 – 2.3 provide the existence of a model for level by level inequivalence between strong compactness and supercompactness containing a supercompact, indestructibly strongly compact cardinal. Lemma 2.4 will show that κ is the least strongly compact cardinal. Lemma 2.5 will show that there is a serious technical obstacle to using the techniques of this paper in answering our Question, i.e., obtaining a model for level by level inequivalence between strong compactness and supercompactness containing an indestructibly supercompact cardinal. We will return to this issue at the end of the paper.

Lemma 2.4 *Suppose $V^{\mathbb{P}} \vDash$ "$\kappa \leq \kappa_1$ is a measurable cardinal". Then $V^{\mathbb{P}} \vDash$ "No cardinal $\delta < \kappa$ is strongly compact up to κ".*

Proof: We modify the proof of [6, Lemma 2.3], once more quoting verbatim when appropriate. Let $\gamma = \kappa^{+\omega}$. By the proof of Lemma 2.3, $V \vDash$ "κ is supercompact up to κ'". Consequently, by assuming $j(\kappa)$ is minimal, we may once again take $j : V \to M$ as an elementary embedding witnessing the γ supercompactness of κ such that $M \vDash$ "κ is not γ supercompact". Suppose $\mathbb{Q} \in V^{\mathbb{P}_\kappa}$ is trivial forcing $\{\emptyset\}$. By the proof of Lemma 2.2, $V^{\mathbb{P}_\kappa * \dot{\mathbb{Q}}} \vDash$ "κ is measurable" (since $V^{\mathbb{P}_\kappa * \dot{\mathbb{Q}}} \vDash$ "κ is strongly compact up to κ'"). Because γ has been chosen large enough, it therefore follows that $M^{\mathbb{P}_\kappa * \dot{\mathbb{Q}}} \vDash$ "κ is measurable". In addition, as in Lemma 2.2, it is possible to opt for $\mathbb{Q}$ in the stage κ lottery held in M in the definition of $j(\mathbb{P}_\kappa)$. Therefore, by the definition of $\mathbb{P}$, above the appropriate condition, $(j(\mathbb{P}_\kappa * \dot{\mathbb{Q}}))_{\kappa+1} = (\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa)^M = (\mathbb{P}_{\kappa+1})^M$ is forcing equivalent in M to $\mathbb{P}_\kappa * \dot{\mathbb{S}}_\kappa$, where $\Vdash_{\mathbb{P}_\kappa}$ "$\dot{\mathbb{S}}_\kappa$ is Prikry forcing defined over κ". This means that in M, $\Vdash_{\mathbb{P}_\kappa}$ "By forcing above a condition $\dot{p}^*_\kappa$ ensuring that trivial forcing $\{\emptyset\}$ is chosen in the stage κ lottery held in the definition of $j(\mathbb{P}_\kappa)$, $\dot{\mathbb{Q}}_\kappa$ is forcing equivalent to Prikry forcing defined using some normal measure over κ". Consequently, by reflection, for unboundedly many $\delta < \kappa$, $\Vdash_{\mathbb{P}_\delta}$ "By forcing above a condition $\dot{p}^*_\delta$ ensuring that trivial forcing $\{\emptyset\}$ is chosen in the stage δ lottery held in the definition of $\mathbb{P}_\kappa$, $\dot{\mathbb{Q}}_\delta$ is forcing equivalent to Prikry forcing defined using some normal measure over δ".

It now follows that $\Vdash_{\mathbb{P}_\kappa}$ "Unboundedly many $\delta < \kappa$ contain Prikry sequences". To see this, let $\gamma < \kappa$ be fixed but arbitrary. Choose $p = \langle \dot{p}_\alpha \mid \alpha < \kappa \rangle \in \mathbb{P}_\kappa$. Since $\mathbb{P}_\kappa$ is an Easton support iteration, let $\rho > \gamma$ be such that for every $\alpha \geq \rho$, $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{p}_\alpha$ is a term for the trivial condition". We may now find $\delta > \rho > \gamma$, $\delta < \kappa$ such that $\Vdash_{\mathbb{P}_\delta}$ "By forcing above a condition $\dot{p}^*_\delta$ ensuring that trivial forcing $\{\emptyset\}$ is chosen in the stage δ lottery held in the definition of $\mathbb{P}_\kappa$, $\dot{\mathbb{Q}}_\delta$ is forcing equivalent to Prikry forcing defined using some normal measure over δ". This means that we may find $q \geq p$ such that $q \Vdash$ "δ contains a Prikry sequence". Thus, $\Vdash_{\mathbb{P}_\kappa}$ "Unboundedly many $\delta < \kappa$ contain Prikry sequences". Hence, by [8, Theorem 11.1], $V^{\mathbb{P}_\kappa} \models$ "Unboundedly many $\delta < \kappa$ (i.e., the successors of those cardinals having Prikry sequences) contain non-reflecting stationary sets of ordinals of cofinality ω". By [16, Theorem 4.8] and the succeeding remarks, it thus follows that $V^{\mathbb{P}_\kappa} \models$ "No cardinal $\delta < \kappa$ is strongly compact up to κ". Because $V \models$ "κ is supercompact up to κ'", only trivial forcing occurs at stage κ of the definition of $\mathbb{P}$. If we now write $\mathbb{P} = \mathbb{P}_\kappa * \dot{\mathbb{P}}^\kappa$, we may consequently infer that $\Vdash_{\mathbb{P}_\kappa}$ "Forcing with $\dot{\mathbb{P}}^\kappa$ adds no new subsets of κ'". From this, it immediately follows that $V^{\mathbb{P}} \models$ "No cardinal $\delta < \kappa$ is strongly compact up to κ". This completes the proof of Lemma 2.4.

$\square$

Lemma 2.5 *Suppose* $V^{\mathbb{P}} \models$ "*$\kappa \leq \kappa_1$ is a measurable cardinal*". *Then for* $\mathbb{R} = (\mathrm{Add}(\kappa, 1))^{V^{\mathbb{P}}}$, $V^{\mathbb{P} * \dot{\mathbb{R}}} \models$ "*κ is not supercompact*". *In fact, in* $V^{\mathbb{P} * \dot{\mathbb{R}}}$, *$\kappa$ has trivial Mitchell rank.*

Proof: As in the proof of Lemma 2.4, write $\mathbb{P} = \mathbb{P}_\kappa * \dot{\mathbb{P}}^\kappa$. Since $\Vdash_{\mathbb{P}_\kappa}$ "Forcing with $\dot{\mathbb{P}}^\kappa$ adds no new subsets of κ'", $\mathbb{R} \in V^{\mathbb{P}_\kappa}$, and $V^{\mathbb{P} * \dot{\mathbb{R}}} \models$ "κ has trivial Mitchell rank" iff $V^{\mathbb{P}_\kappa * \dot{\mathbb{R}}} \models$ "κ has trivial Mitchell rank". We therefore proceed by showing that $V^{\mathbb{P}_\kappa * \dot{\mathbb{R}}} \models$ "κ has trivial Mitchell rank".

The proof of this fact is a modified version of the proof of [6, Lemma 2.4]. In our argument, we again quote verbatim from [6] when appropriate. Let $G * H$ be V-generic over $\mathbb{P}_\kappa * \dot{\mathbb{R}}$. If $V[G * H] \models$ "κ

does not have trivial Mitchell rank", then let $j : V[G * H] \to M[j(G * H)]$ be an elementary embedding generated by a normal measure $\mathcal{U} \in V[G * H]$ over κ such that $M[j(G * H)] \vDash$ "κ is measurable". Note that since $M = \bigcup_{\alpha \in \mathrm{Ord}} j(V_\alpha)$, $j \restriction V : V \to M$ is elementary. Therefore, because $j \restriction \kappa = \mathrm{id}$, we may infer that $(V_\kappa)^V = (V_\kappa)^M$. However, by Theorem 2, we may further infer that $(V_{\kappa+1})^M \subseteq (V_{\kappa+1})^V$. To see this, let $x \subseteq \kappa$, $x \in M$. Since $M \subseteq M[j(G * H)] \subseteq V[G * H]$, $x \in V[G * H]$. In addition, because $(V_\kappa)^V = (V_\kappa)^M$, we know that $x \cap \alpha \in V$ for every $\alpha < \kappa$. This means that if $x \notin V$, then x is a fresh subset of κ with respect to $\mathbb{P}_\kappa * \dot{\mathbb{R}}$. By the proof of Lemma 2.3, $V \vDash$ "κ is supercompact up to κ'". Since by the proof of Lemma 2.2, κ must therefore be strongly compact up to κ' and hence both measurable and Mahlo in $V[G * H]$, this contradicts Theorem 2. Thus, $x \in V$, so $(\wp(\kappa))^M \subseteq (\wp(\kappa))^V$. From this, it of course immediately follows that $(V_{\kappa+1})^M \subseteq (V_{\kappa+1})^V$.

Let $I = j(G * H)$. Note that if $V \vDash$ "$\delta < \kappa$ is a Mahlo cardinal", then $M \vDash$ "$j(\delta) = \delta$ is a Mahlo cardinal". Also, $M \vDash$ "κ is a Mahlo cardinal", since $M[j(G * H)] \vDash$ "κ is a Mahlo cardinal", and forcing can't create a new Mahlo cardinal. Hence, by the results of the preceding paragraph, it follows as well that $j(\mathbb{P}_\kappa) \restriction \kappa = \mathbb{P}_\kappa$ and $I_\kappa = G$. Further, as $V[G * H] \vDash$ "$M[I]^\kappa \subseteq M[I]$", $H \in M[I]$. We know in addition that in M, $\Vdash_{\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa}$ "The forcing beyond stage κ adds no new subsets of 2^κ". Consequently, $H \in M[I_{\kappa+1}] = M[G * I(\kappa)]$, and $M[I_{\kappa+1}] \vDash$ "κ is measurable".

Note that since $\mathbb{P}_\kappa$ is defined by taking Easton supports, $\mathbb{P}_\kappa$ is κ-c.c. in both V and M. Because $\mathbb{P}_\kappa$ is a Gitik iteration of suitably directed closed partial orderings together with Prikry forcing and $(V_\kappa)^V = (V_\kappa)^M$, $(V_\kappa)^{V[G]} = (V_\kappa)^{M[G]}$. It must therefore be the case that $(\mathrm{Add}(\kappa, 1))^{V[G]} = (\mathrm{Add}(\kappa, 1))^{M[G]}$. In addition, since $(V_{\kappa+1})^M \subseteq (V_{\kappa+1})^V$, the fact $\mathbb{P}_\kappa$ is κ-c.c. in M yields that $(V_{\kappa+1})^{M[G]} \subseteq (V_{\kappa+1})^{V[G]}$. This means that H is $M[G]$-generic over $(\mathrm{Add}(\kappa, 1))^{M[G]}$, since H is $V[G]$-generic over $(\mathrm{Add}(\kappa, 1))^{V[G]} = (\mathrm{Add}(\kappa, 1))^{M[G]}$, and a dense open subset of $(\mathrm{Add}(\kappa, 1))^{M[G]}$ in $M[G]$ is a member of $(V_{\kappa+1})^{M[G]}$. Hence, H must be added by the stage κ forcing done in $M[G] = M[I_\kappa]$, i.e., there must be a stage κ lottery held in $M[I_\kappa]$ opting for some nontrivial forcing. As we have already observed, because by hypothesis $V[G * H] \vDash$ "κ does not have trivial Mitchell rank", in both $M[I] = M[j(G * H)]$ and $M[G * I(\kappa)] = M[I_{\kappa+1}]$, κ is measurable. Consequently, by the definition of $\mathbb{P}_\kappa$ and $j(\mathbb{P}_\kappa)$, we must then have that $M[I_{\kappa+1}] \models$ "κ contains a Prikry sequence". This contradiction to the fact that $M[I_{\kappa+1}] \models$ "κ is measurable" shows that $V^{\mathbb{P}_\kappa * \dot{\mathbb{R}}} \models$ "κ has trivial Mitchell rank". This completes the proof of Lemma 2.5.

$\square$

To complete the proof of Theorem 1, by Lemma 2.1 and reflection, let $\kappa < \kappa_1$ be the least cardinal such that $V^{\mathbb{P}} \models$ "κ is supercompact up to κ'". Let $\overline{V} = (V_{\kappa'})^{V^{\mathbb{P}}}$. It is then the case that $\overline{V} \models$ "ZFC + κ is supercompact + No cardinal $\delta > \kappa$ is inaccessible". By the leastness of κ and Lemma 2.3, in $\overline{V}$, no cardinal is supercompact up to an inaccessible cardinal, and level by level inequivalence between strong compactness and supercompactness holds. By Lemmas 2.2, 2.4, and 2.5, in $\overline{V}$, κ is the least strongly compact cardinal, and κ's strong compactness, but not supercompactness, is indestructible under κ-directed closed forcing. This completes the proof of Theorem 1.

$\square$

In conclusion to this paper, we pose two questions. First, we ask if it is possible to produce a model witnessing the conclusions of Theorem 1 in which κ is not the least strongly compact cardinal. Since Prikry forcing above a strongly compact cardinal destroys strong compactness, an answer to this question would require a different sort of iteration from the one used in the proof of Theorem 1. Finally, we note that if we assume stronger hypotheses on our ground model V, then it is possible to obtain models analogous to the one for Theorem 1 in which there are large cardinals above κ. As an example, if we start with a model in which $\kappa_1 < \lambda$ are such that κ_1 is λ supercompact and λ is Mahlo, then it is possible to force and construct a model in which κ is supercompact, no cardinal is supercompact up to a Mahlo cardinal, and the additional conclusions of Theorem 1 hold. (We simply change the definition of δ' to be the least V-Mahlo cardinal above δ and proceed as we did originally.) Such a model, of course, can contain inaccessible cardinals above κ. However, because of the appropriate analogue of Lemma 2.5, κ will not be indestructibly supercompact. We therefore conclude by reiterating our original Question, and ask whether it is possible to construct a model with a restricted large cardinal structure containing an indestructibly supercompact cardinal in which level by level inequivalence between strong compactness and supercompactness holds.

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