Indestructibility and Destructible Measurable Cardinals [*†]

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Abstract

Say that κ's measurability is *destructible* if there exists a $<\kappa$-closed forcing adding a new subset of κ which destroys κ's measurability. For any δ, let $\lambda_\delta =_{\rm df}$ The least beth fixed point above δ. Suppose that κ is indestructibly supercompact and there is a measurable cardinal $\lambda > \kappa$. It then follows that $A_1 = \{\delta < \kappa \mid \delta$ is measurable, δ is not a limit of measurable cardinals, δ is not δ^+ strongly compact, and δ's measurability is destructible when forcing with partial orderings having rank below $\lambda_\delta\}$ is unbounded in κ. On the other hand, under the same hypotheses, $A_2 = \{\delta < \kappa \mid \delta$ is measurable, δ is not a limit of measurable cardinals, δ is not δ^+ strongly compact, and δ's measurability is indestructible when forcing with either $\mathrm{Add}(\delta, 1)$ or $\mathrm{Add}(\delta, \delta^+)\}$ is unbounded in κ as well. The large cardinal hypothesis on λ is necessary, as we further demonstrate by constructing via forcing two distinct models in which either $A_1 = \emptyset$ or $A_2 = \emptyset$. In each of these models, both of which have restricted large cardinal structures above κ, every measurable cardinal δ which is not a limit of measurable cardinals is δ^+ strongly compact, and there is an indestructibly supercompact cardinal κ. In the model in which $A_1 = \emptyset$, every measurable cardinal δ which is not a limit of measurable cardinals is

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$<\lambda_\delta$ strongly compact and has its $<\lambda_\delta$ strong compactness (and hence also its measurability) indestructible when forcing with δ-directed closed partial orderings having rank below λ_δ. The choice of the least beth fixed point above δ is arbitrary, and other values of λ_δ are also possible.

1 Introduction and Preliminaries

It is a very interesting fact that the large cardinal structure of the universe above a supercompact cardinal κ with suitable indestructibility properties can affect the large cardinal structure below κ in ways which are not immediately apparent. On the other hand, these effects may not be present if the universe contains relatively few large cardinals. These sorts of occurrences have previously been investigated in [1, 2, 3, 4, 5, 6, 12].

The purpose of this paper is to continue studying this phenomenon, but in the context of investigating destructibility and indestructibility properties certain measurable cardinals can manifest in universes containing an indestructibly supercompact cardinal. We begin with the following theorem, where as in [22], κ is *indestructibly supercompact* if κ's supercompactness is preserved by arbitrary κ-directed closed forcing. In analogy to [20], κ's measurability is *destructible* if there exists a $<\kappa$-closed forcing adding a new subset of κ which destroys κ's measurability. As in [20], κ's measurability is *superdestructible* if *every* $<\kappa$-closed forcing adding a new subset of κ destroys κ's measurability. κ's measurability is *indestructible when forcing with a partial ordering* $\mathbb{P}$ if after forcing with $\mathbb{P}$, κ remains measurable. For any δ, we take $\lambda_\delta =_{\mathrm{df}}$ The least beth fixed point above δ. We also say that κ *is* $<\lambda$ *supercompact* ($<\lambda$ *strongly compact) for* $\lambda > \kappa$ if κ is δ supercompact (δ strongly compact) for every $\delta < \lambda$.

Theorem 1 *Suppose that κ is indestructibly supercompact and there is a measurable cardinal $\lambda > \kappa$. Then $A_1 = \{\delta < \kappa \mid \delta$ is measurable, δ is not a limit of measurable cardinals, δ is not δ^+ strongly compact, and δ's measurability is destructible when forcing with partial orderings having rank below $\lambda_\delta\}$ and $A_2 = \{\delta < \kappa \mid \delta$ is measurable, δ is not a limit of measurable cardinals, δ is not δ^+ strongly compact, and δ's measurability is indestructible when forcing with either* $\mathrm{Add}(\delta, 1)$ *or* $\mathrm{Add}(\delta, \delta^+)\}$ *are both unbounded in κ.*

With a limited large cardinal structure above κ, Theorem 1 need not be true. Specifically, we have:

Theorem 2 *Suppose $V \models$ "ZFC + κ is supercompact + No cardinal $\zeta > \kappa$ is measurable". There is then a partial ordering $\mathbb{P} \subseteq V$ such that $V^{\mathbb{P}} \models$ "ZFC + No cardinal $\zeta > \kappa$ is measurable + κ is indestructibly supercompact + If $\delta < \kappa$ is a measurable cardinal which is not a limit of measurable cardinals, then δ is δ^+ strongly compact and δ's measurability is destructible when forcing with partial orderings having rank below λ_δ".*

Theorem 3 *Suppose $V \models$ "ZFC + κ is supercompact + No cardinal $\zeta > \kappa$ is inaccessible". There is then a partial ordering $\mathbb{P} \subseteq V$ such that $V^{\mathbb{P}} \models$ "ZFC + No cardinal $\zeta > \kappa$ is inaccessible + κ is indestructibly supercompact and is also the least strongly compact cardinal + Any measurable cardinal $\delta < \kappa$ which is not a limit of measurable cardinals is $<\lambda_\delta$ strongly compact and has its $<\lambda_\delta$ strong compactness (and hence also its measurability) indestructible when forcing with δ-directed closed partial orderings having rank below λ_δ".*

We take this opportunity to make a few remarks concerning Theorems 1 – 3. As our proofs will show, each relevant measurable cardinal δ in both Theorems 1 and 2 can have its measurability destructible when forcing with many more partial orderings than just those having rank below λ_δ. Also, in Theorem 2, it is possible for each measurable cardinal which is not a limit of measurable cardinals to be λ strongly compact for many different regular cardinals $\lambda > \delta^+$. Further, in Theorem 3, λ_δ can take on values different from the least beth fixed point above δ. We will comment on these issues later in the paper. In addition, with just the hypothesis of the existence of a measurable cardinal λ above an indestructibly supercompact cardinal κ, it does not seem possible to be able to infer, e.g., that $A_3 = \{\delta < \kappa \mid \delta$ is measurable, δ is not a limit of measurable cardinals, and δ's measurability is indestructible when forcing with either $\mathrm{Add}(\delta, 1)$, $\mathrm{Add}(\delta, \delta^+)$, or $\mathrm{Add}(\delta, \delta^{++})\}$ is unbounded in κ. Gitik's work of [15, 14] seems to suggest that stronger hypotheses on λ are required. Finally, Theorem 3 should be contrasted with [9, Theorem 1.1]. For its statement, we take as our notation that $\rho_\delta =_{\mathrm{df}}$ The least inaccessible cardinal above δ. In this theorem, relative to the existence of a model for GCH and a cardinal κ which is ρ_κ supercompact, a model V^* is constructed in which κ is indestructibly supercompact and is also the least strongly compact cardinal, no cardinal $\delta > \kappa$ is inaccessible, and for every measurable cardinal $\delta < \kappa$, δ is $<\rho_\delta$ strongly compact and has its $<\rho_\delta$ strong compactness indestructible when forcing with δ-directed closed partial orderings having rank below ρ_δ. Although this is *prima facie* a stronger result than Theorem 3 of this paper, V^* is constructed by truncating a forcing extension of V at ρ_κ. It is therefore only a set model, and not a proper class model as is the model witnessing the conclusions of the current Theorem 3. In addition, ZFC + GCH + There exists κ which is ρ_κ supercompact $\vdash$ Con(ZFC + GCH + There exists a supercompact cardinal with no inaccessible cardinals above it), which of course easily follows by considering V_{ρ_κ}. Thus, Theorem 3 has two advantages over [9, Theorem 1.1], in that Theorem 3's witnessing model is a proper class, and Theorem 3 is established using provably weaker hypotheses.

We conclude Section 1 with a very brief discussion of some preliminary material. We presume a basic knowledge of large cardinals and forcing. A good reference in this regard is [21]. When forcing, $q \geq p$ means that *q is stronger than p*. When G is V-generic over $\mathbb{P}$, we abuse notation slightly and take both $V[G]$ and $V^{\mathbb{P}}$ as being the generic extension of V by $\mathbb{P}$. We also abuse notation slightly by occasionally confusing terms with the sets they denote, especially for ground model sets and variants of the generic object. For $\alpha < \beta$ ordinals, $[\alpha, \beta]$, $(\alpha, \beta]$, $[\alpha, \beta)$, and (α, β) are as in standard interval notation.

Suppose $\kappa < \lambda$ are regular cardinals. For α an arbitrary ordinal, the partial ordering $\mathrm{Add}(\kappa, \alpha)$ is the standard Cohen partial ordering for adding α many Cohen subsets of κ. The partial ordering $\mathbb{P}$ is *κ-directed closed* if for every directed set $D \subseteq \mathbb{P}$ of size less than κ, there is a condition in $\mathbb{P}$ extending each member of D. $\mathbb{P}$ is *κ-closed* if every increasing chain of members of $\mathbb{P}$ of length κ has an upper bound. $\mathbb{P}$ is *$<\kappa$-closed* if $\mathbb{P}$ is δ-closed for every $\delta < \kappa$. $\mathbb{P}$ is *κ-strategically closed* if in the two person game in which the players construct an increasing sequence $\langle p_\alpha \mid \alpha \leq \kappa \rangle$, where player I plays odd stages and player II plays even stages, player II has a strategy ensuring the game can always be continued. $\mathbb{P}$ is *$\prec\kappa$-strategically closed* if in the two person game in which the players construct an increasing sequence $\langle p_\alpha \mid \alpha < \kappa \rangle$, where player I plays odd stages and player II plays even stages, player II has a strategy ensuring the game can always be continued. It therefore follows that any partial ordering $\mathbb{P}$ which is κ-directed closed is also $\prec\kappa$-strategically closed and consequently adds no new subsets of any cardinal $\delta < \kappa$.

In the proof of Theorem 1, we will use the partial ordering $\mathbb{P}(\kappa, \lambda)$, the standard notion of forcing for adding a nonreflecting stationary set of ordinals of cofinality κ to λ. For further details on the definition of this partial ordering, we refer readers to either [10] or [8]. We note only that $\mathbb{P}(\kappa, \lambda)$ is κ-directed closed and $\prec\lambda$-strategically closed.

In the proof of Theorem 3, we will refer to our partial ordering $\mathbb{P}$ as being a *Gitik style iteration of Prikry-like forcings.* By this we will mean an Easton support iteration as first given by Gitik in [13]. The ordering, roughly speaking, is the usual one associated with reverse Easton iterations, except that when extending Prikry conditions, we take larger stems only finitely often. For a more precise definition, we urge readers to consult either [13] or [11].

We recall for the benefit of readers the definition given by Hamkins in [19, Section 3] of the lottery sum of a collection of partial orderings. If $\mathfrak{A}$ is a collection of partial orderings, then the *lottery sum* is the partial ordering $\oplus\mathfrak{A} = \{\langle \mathbb{P}, p \rangle \mid \mathbb{P} \in \mathfrak{A} \text{ and } p \in \mathbb{P}\} \cup \{0\}$, ordered with 0 below everything and $\langle \mathbb{P}, p \rangle \le \langle \mathbb{P}', p' \rangle$ iff $\mathbb{P} = \mathbb{P}'$ and $p \le p'$. Intuitively, if G is V-generic over $\oplus\mathfrak{A}$, then G first selects an element of $\mathfrak{A}$ (or as Hamkins says in [19], "holds a lottery among the posets in $\mathfrak{A}$") and then forces with it.[1]

A corollary of Hamkins' work on gap forcing found in [17, 18] will be employed in the proof of Theorems 1 – 3. We therefore state as a separate theorem what is relevant for this paper, along with some associated terminology, quoting from [17, 18] when appropriate. Suppose $\mathbb{P}$ is a partial ordering which can be written as $\mathbb{Q} * \dot{\mathbb{R}}$, where $|\mathbb{Q}| < \delta$, $\mathbb{Q}$ is nontrivial, and $\Vdash_{\mathbb{Q}}$ "$\dot{\mathbb{R}}$ is δ^+-directed closed". In Hamkins' terminology of [17, 18], $\mathbb{P}$ *admits a gap at* δ. In Hamkins' terminology of [17, 18], $\mathbb{P}$ is *mild with respect to a cardinal* κ iff every set of ordinals x in $V^{\mathbb{P}}$ of size below κ has a "nice" name τ in V of size below κ, i.e., there is a set y in V, $|y| < \kappa$, such that any ordinal

[1]The terminology "lottery sum" is due to Hamkins, although the concept of the lottery sum of partial orderings has been around for quite some time and has been referred to at different junctures via the names "disjoint sum of partial orderings," "side-by-side forcing," and "choosing which partial ordering to force with generically."

forced by a condition in $\mathbb{P}$ to be in τ is an element of y. Also, as in the terminology of [17, 18] and elsewhere, an embedding $j : \overline{V} \to \overline{M}$ is *amenable to* $\overline{V}$ when $j \restriction A \in \overline{V}$ for any $A \in \overline{V}$. The specific corollary of Hamkins' work from [17, 18] we will be using is then the following.

Theorem 4 (Hamkins) *Suppose that $V[G]$ is a generic extension obtained by forcing with $\mathbb{P}$ that admits a gap at some regular $\delta < \kappa$. Suppose further that $j : V[G] \to M[j(G)]$ is an elementary embedding with critical point κ for which $M[j(G)] \subseteq V[G]$ and $M[j(G)]^\delta \subseteq M[j(G)]$ in $V[G]$. Then $M \subseteq V$; indeed, $M = V \cap M[j(G)]$. If the full embedding j is amenable to $V[G]$, then the restricted embedding $j \restriction V : V \to M$ is amenable to V. If j is definable from parameters (such as a measure or extender) in $V[G]$, then the restricted embedding $j \restriction V$ is definable from the names of those parameters in V. Finally, if $\mathbb{P}$ is mild with respect to κ and κ is λ strongly compact in $V[G]$ for any $\lambda \geq \kappa$, then κ is λ strongly compact in V.*

2 The Proofs of Theorems 1 – 3

We turn now to the proof of Theorem 1, which will be established via a sequence of lemmas.

Lemma 2.1 *If $\kappa < \lambda$ are such that κ is indestructibly supercompact and λ is measurable, then $A_1 = \{\delta < \kappa \mid \delta$ is measurable, δ is not a limit of measurable cardinals, δ is not δ^+ strongly compact, and δ's measurability is destructible when forcing with partial orderings having rank below $\lambda_\delta\}$ is unbounded in κ.*

Proof: We follow the proofs of [1, Theorem 2] and [5, Theorem 1]. Suppose that κ is indestructibly supercompact and that λ is the least measurable cardinal greater than κ. We show that A_1 is unbounded in κ. First force with $\mathbb{P}(\kappa, \lambda^+)$, the partial ordering which adds a nonreflecting stationary set of ordinals of cofinality κ to λ^+. By [25, Theorem 4.8] and the fact that $\mathbb{P}(\kappa, \lambda^+)$ is $\prec\lambda^+$-strategically closed, after this forcing, λ is not λ^+ strongly compact and is the least measurable cardinal above κ. Next, force with $\mathrm{Add}(\kappa, 1)$. After this forcing, which has cardinality $\kappa < \lambda$, by the results of [23], λ is not λ^+ strongly compact and λ remains the least measurable cardinal above κ. In particular, after the forcing, λ is a measurable cardinal which is not a limit of measurable cardinals. In addition, by [20, Theorem I], after the forcing, λ has become a superdestructible measurable cardinal. In particular, κ's measurability is destructible when forcing with partial orderings having rank below λ_κ. Since κ's supercompactness is indestructible when forcing with κ-directed closed partial orderings, $\mathbb{P}(\kappa, \lambda^+) * \dot{\mathrm{Add}}(\kappa, 1)$ is κ-directed closed, and the destructibility of λ when forcing with partial orderings having rank below λ_δ is detectible in a large enough V_η, by reflection, $A_1 = \{\delta < \kappa \mid \delta$ is measurable, δ is not a limit of measurable cardinals, δ is not δ^+ strongly compact, and δ's measurability is destructible when forcing with partial orderings having rank below $\lambda_\delta\}$ is unbounded in κ after the forcing has been performed. Once more, we infer by the fact $\mathbb{P}(\kappa, \lambda^+) * \dot{\mathrm{Add}}(\kappa, 1)$ is κ-directed closed that A_1 is unbounded in κ in the ground model. This completes the proof of Lemma 2.1.

□

We remark that in general, it is not possible to infer that if $\delta \in A_1$, then δ is a superdestructible measurable cardinal. To see this, suppose $\overline{V} \vDash$ "κ is indestructibly supercompact", $\overline{V} = V^{\mathbb{P}}$ for $\mathbb{P}$ the reverse Easton iteration of length κ of [22] which forces indestructibility for κ, and $j : \overline{V}^{\mathbb{P}(\kappa, \lambda^+) * \dot{\mathrm{Add}}(\kappa, 1)} \to \overline{M}$ is an elementary embedding witnessing a fixed but arbitrary degree of supercompactness of κ in $\overline{V}^{\mathbb{P}(\kappa, \lambda^+) * \dot{\mathrm{Add}}(\kappa, 1)}$. By elementarity, $\overline{M}$ will be a generic extension of a ground model M via the partial ordering $j(\mathbb{P}) * \dot{\mathbb{P}}(j(\kappa), j(\lambda^+)) * \dot{\mathrm{Add}}(j(\kappa), 1) = \mathbb{P} * \dot{\mathbb{P}}(\kappa, \lambda^+) * \dot{\mathrm{Add}}(\kappa, 1) * \dot{\mathbb{Q}} * \dot{\mathbb{P}}(j(\kappa), j(\lambda^+)) * \dot{\mathrm{Add}}(j(\kappa), 1)$, where $\dot{\mathbb{Q}}$ is a term for the forcing done between stages $\kappa + 1$ and $j(\kappa)$ in $M^{\mathbb{P} * \dot{\mathbb{P}}(\kappa, \lambda^+) * \dot{\mathrm{Add}}(\kappa, 1)}$. The nontrivial forcing $\mathbb{Q} * \dot{\mathbb{P}}(j(\kappa), j(\lambda^+)) * \dot{\mathrm{Add}}(j(\kappa), 1)$ could have caused λ not to be a superdestructible measurable cardinal in $\overline{M}$, the model over which reflection is done.[2]

Returning now to the proof of Theorem 1, in order to complete it, we first need to define a certain partial ordering $\mathbb{P}$. As in the proof of Lemma 2.1, assume that $\kappa < \lambda$ are such that κ is indestructibly supercompact and λ is the least measurable cardinal above κ in our ground model V. We then let $\mathbb{P} = \mathbb{P}^0 * \dot{\mathbb{P}}(\kappa, \lambda^+) * \dot{\mathbb{P}}^1$, where $\mathbb{P}^0 = \mathrm{Add}(\lambda^+, 1)$. $\dot{\mathbb{P}}^1$ will be a term for the reverse Easton iteration of length λ which begins by forcing with $\mathrm{Add}(\kappa, 1)$ and then performs nontrivial

[2]It should be noted that at the moment, it is unknown if the nontrivial forcing $\mathbb{Q} * \dot{\mathbb{P}}(j(\kappa), j(\lambda^+)) * \dot{\mathrm{Add}}(j(\kappa), 1)$ actually destroys λ's superdestructibility.

forcing only at those stages $\delta \in (\kappa, \lambda)$ which are inaccessible cardinals. At such a δ, we force with the lottery sum of trivial forcing $\{\emptyset\}$, $\mathrm{Add}(\delta, 1)$, and $\mathrm{Add}(\delta, \delta^+)$.

For the proofs of Lemmas 2.3 – 2.5, we use well known ideas. Our reference is [6, Theorem 2.1] for Lemmas 2.3 and 2.4 and [7, Lemma 2.2] for Lemma 2.5, from which we feel free to quote verbatim if necessary.

Lemma 2.2 $V^{\mathbb{P}} \vDash$ "*λ is not λ^+ strongly compact*".

Proof: Consider $V^{\mathbb{P}^0}$. Standard arguments (see, e.g., [21, Exercise 15.16]) show that in $V^{\mathbb{P}^0}$, $2^\lambda = \lambda^+$, all cardinals less than or equal to λ^+ are preserved, $(2^\lambda)^V$ is collapsed to λ^+, and all cardinals greater than or equal to $((2^\lambda)^+)^V$ are preserved. Since $\mathbb{P}(\kappa, \lambda^+)$ is $\prec\lambda^+$-strategically closed and $V^{\mathbb{P}^0} \vDash$ "$|\mathbb{P}(\kappa, \lambda^+)| = \lambda^+$", these facts are preserved to $\overline{V} = V^{\mathbb{P}^0 * \dot{\mathbb{P}}(\kappa, \lambda^+)}$ as well. Therefore, as in the proof of Lemma 2.1, by [25, Theorem 4.8], $\overline{V} \vDash$ "λ is not λ^+ strongly compact".

To complete the proof of Lemma 2.2, we must now show that $\overline{V}^{\mathbb{P}^1} = V^{\mathbb{P}} \vDash$ "λ is not λ^+ strongly compact". To do this, write $\mathbb{P}^1 = \mathrm{Add}(\kappa, 1) * \dot{\mathbb{R}}$. Since $|\mathrm{Add}(\kappa, 1)| = \kappa$, $\mathrm{Add}(\kappa, 1)$ is nontrivial, and $\Vdash_{\mathrm{Add}(\kappa,1)}$ "$\dot{\mathbb{R}}$ is κ^{++}-directed closed", it follows that $\mathbb{P}^1$ admits a gap at κ^+. Further, by its definition, $\mathbb{P}^1$ is mild with respect to λ. Therefore, by Theorem 4, if λ were λ^+ strongly compact in $\overline{V}^{\mathbb{P}^1} = V^{\mathbb{P}}$, it would have had to have been λ^+ strongly compact in $\overline{V}$. As we have just proven in the preceding paragraph, this is false. This completes the proof of Lemma 2.2.

$\square$

For the remainder of the proof of Theorem 1, we adopt the notation used in the proof of Lemma 2.2.

Lemma 2.3 $V^{\mathbb{P}} \vDash$ "*λ is the least measurable cardinal above κ*".

Proof: By its definition, $\mathrm{Add}(\lambda^+, 1) * \dot{\mathbb{P}}(\kappa, \lambda^+)$ is $\prec\lambda^+$-strategically closed and therefore adds no new subsets of λ. Since $V \vDash$ "λ is the least measurable cardinal above κ" and $\overline{V} = V^{\mathrm{Add}(\lambda^+,1) * \dot{\mathbb{P}}(\kappa,\lambda^+)}$ contains no new subsets of λ, $\overline{V} \vDash$ "λ is the least measurable cardinal above κ" as well. We can therefore let $j : \overline{V} \to M$ be an elementary embedding generated by a normal measure over λ. We can also let G be $\overline{V}$-generic over $\mathbb{P}^1$. Because $\mathbb{P}^1$ is a reverse Easton iteration having length λ, $\mathbb{P}^1$ is λ-c.c. Consequently, as $M^\lambda \subseteq M$, $M[G]^\lambda \subseteq M[G]$, and the cardinal structure in $\overline{V}[G]$ at and above λ is the same as in $\overline{V}$.

By opting for a condition in M which ensures that trivial forcing is chosen at stage λ in the definition of $j(\mathbb{P}^1)$, $j(\mathbb{P}^1)$ can be taken to be forcing equivalent to $\mathbb{P}^1 * \dot{\mathbb{Q}}$, where the first nontrivial stage in $\dot{\mathbb{Q}}$ occurs well after λ. Further, since j is generated by a normal measure over λ, $2^\lambda = \lambda^+$ in $\overline{V}$, and $M[G] \models$ "$|\mathbb{Q}| = j(\lambda)$", the number of dense open subsets of $\mathbb{Q}$ present in $M[G]$ is $(2^{j(\lambda)})^M = (2^{j(\lambda)})^{M[G]} = j(\lambda^+)$. This is calculated in either $\overline{V}$ or $\overline{V}[G]$ as $|\{f \mid f : \lambda \to 2^\lambda\}| = |\{f \mid f : \lambda \to \lambda^+\}| = [\lambda^+]^\lambda = \lambda^+$. We may consequently let $\langle D_\alpha \mid \alpha < \lambda^+ \rangle \in \overline{V}[G]$ enumerate the dense open subsets of $\mathbb{Q}$ present in $M[G]$. Because $M[G]^\lambda \subseteq M[G]$, by the definition of $\mathbb{P}^1$, $\mathbb{Q}$ is λ^+-directed closed in both $M[G]$ and $\overline{V}[G]$. We may hence in $\overline{V}[G]$ meet each D_α and thereby construct in $\overline{V}[G]$ an $M[G]$-generic object H over $\mathbb{Q}$. Since by construction, $j''G \subseteq G * H$, j lifts in $\overline{V}[G]$ to $j : \overline{V}[G] \to M[G][H]$. This means that $\overline{V}[G] \models$ "λ is a measurable cardinal", i.e., $V^{\mathbb{P}} \models$ "λ is a measurable cardinal". Finally, since $\mathbb{P}^1$ admits a gap at κ^+, by Theorem 4, any cardinal greater than κ^+ which is measurable in $\overline{V}[G] = V^{\mathbb{P}}$ had to have been measurable in $\overline{V}$. Since $\overline{V} \models$ "λ is the least measurable cardinal above κ", this means that $V^{\mathbb{P}} \models$ "λ is the least measurable cardinal above κ" as well. This completes the proof of Lemma 2.3.

□

Lemma 2.4 $V^{\mathbb{P} * \dot{\text{Add}}(\lambda, 1)} \models$ "*λ is a measurable cardinal*".

Proof: Consider as before $j : \overline{V} \to M$. By opting for a condition in M which ensures that $\text{Add}(\lambda, 1)$ is chosen at stage λ in the definition of $j(\mathbb{P}^1)$, $j(\mathbb{P}^1)$ can be taken to be forcing equivalent to $\mathbb{P}^1 * \dot{\text{Add}}(\lambda, 1) * \dot{\mathbb{Q}} * \dot{\text{Add}}(j(\lambda), 1)$, where once again, the first nontrivial stage in $\dot{\mathbb{Q}}$ takes place well after λ. Let $G_0 * G_1$ be $\overline{V}$-generic over $\mathbb{P}^1 * \dot{\text{Add}}(\lambda, 1)$. Since $\mathbb{P}^1 * \dot{\text{Add}}(\lambda, 1)$ is λ^+-c.c. and $M^\lambda \subseteq M$, $M[G_0][G_1]^\lambda \subseteq M[G_0][G_1]$ as well. This means that the same analysis as given in the proof of Lemma 2.3 allows us to build in $\overline{V}[G_0][G_1]$ an $M[G_0][G_1]$-generic object G_2 over $\mathbb{Q}$ and lift j in $\overline{V}[G_0][G_1]$ to $j : \overline{V}[G_0] \to M[G_0][G_1][G_2]$. Therefore, since $M[G_0][G_1][G_2]^\lambda \subseteq M[G_0][G_1][G_2]$

in $\overline{V}[G_0][G_1]$, $\overline{V}[G_0] \vDash$ "$|\mathrm{Add}(\lambda, 1)| = \lambda$", $M[G_0][G_1][G_2] \vDash$ "$\mathrm{Add}(j(\lambda), 1)$ is $j(\lambda)$-directed closed", and $j(\lambda) > \lambda^+$, there is a master condition $q \in \mathrm{Add}(j(\lambda), 1)$ for $j''G_1$. Further, the number of dense open subsets of $\mathrm{Add}(j(\lambda), 1)$ present in $M[G_0][G_1][G_2]$ is $(2^{j(\lambda)})^M$. As in the proof of Lemma 2.3, since $(2^{j(\lambda)})^M = (2^{j(\lambda)})^{M[G_0][G_1][G_2]}$, this is calculated in either $\overline{V}$ or $\overline{V}[G_0][G_1]$ as λ^+. Consequently, we can once again use the same argument as given in the proof of Lemma 2.3 and build in $\overline{V}[G_0][G_1]$ an $M[G_0][G_1][G_2]$-generic object G_3 over $\mathrm{Add}(j(\lambda), 1)$ containing q. Since by construction, $j''(G_0 * G_1) \subseteq G_0 * G_1 * G_2 * G_3$, j now fully lifts to $j : \overline{V}[G_0][G_1] \to M[G_0][G_1][G_2][G_3]$. Hence, $\overline{V}[G_0][G_1] \vDash$ "λ is a measurable cardinal", i.e., $V^{\mathbb{P} * \dot{\mathrm{A}}\mathrm{dd}(\lambda^+, 1)} \vDash$ "λ is a measurable cardinal". This completes the proof of Lemma 2.4.

$\square$

Lemma 2.5 $V^{\mathbb{P} * \dot{\mathrm{A}}\mathrm{dd}(\lambda, \lambda^+)} \vDash$ "*λ is a measurable cardinal*".

Proof: Consider again $j : \overline{V} \to M$. By opting for a condition in M which ensures that $\mathrm{Add}(\lambda, \lambda^+)$ is chosen at stage λ in the definition of $j(\mathbb{P}^1)$, $j(\mathbb{P}^1)$ can be taken to be forcing equivalent to $\mathbb{P}^1 * \dot{\mathrm{A}}\mathrm{dd}(\lambda, \lambda^+) * \dot{\mathbb{Q}} * \dot{\mathrm{A}}\mathrm{dd}(j(\lambda), j(\lambda^+))$, where as before, the first nontrivial stage in $\dot{\mathbb{Q}}$ takes place well after λ. Let $G_0 * G_1$ be $\overline{V}$-generic over $\mathbb{P}^1 * \dot{\mathrm{A}}\mathrm{dd}(\lambda, \lambda^+)$. Since $\mathbb{P}^1 * \dot{\mathrm{A}}\mathrm{dd}(\lambda, \lambda^+)$ is λ^+-c.c. and $M^\lambda \subseteq M$, as in Lemma 2.4, $M[G_0][G_1]^\lambda \subseteq M[G_0][G_1]$ as well. This means that the same analysis as given previously allows us to build in $\overline{V}[G_0][G_1]$ an $M[G_0][G_1]$-generic object G_2 over $\mathbb{Q}$ and lift j in $\overline{V}[G_0][G_1]$ to $j : \overline{V}[G_0] \to M[G_0][G_1][G_2]$. As in Lemma 2.4, $M[G_0][G_1][G_2]^\lambda \subseteq M[G_0][G_1][G_2]$ in $\overline{V}[G_0][G_1]$.

To lift j fully to $\overline{V}[G_0][G_1]$, we now use the idea found in [7, Lemma 2.2] (which has appeared elsewhere in the literature as well — readers may consult [7, Lemma 2.2] for additional references). We again feel free to quote verbatim as needed. We will construct in $\overline{V}[G_0][G_1]$ an $M[G_0][G_1][G_2]$-generic object over $\mathrm{Add}(j(\lambda), j(\lambda^+))$. For $\alpha \in (\lambda, \lambda^+)$ and $p \in \mathrm{Add}(\lambda, \lambda^+)$, let $p \restriction \alpha = \{\langle\langle\rho, \sigma\rangle, \eta\rangle \in p \mid \sigma < \alpha\}$ and $G_1 \restriction \alpha = \{p \restriction \alpha \mid p \in G_1\}$. Clearly, $\overline{V}[G_0][G_1] \vDash$ "$|G_1 \restriction \alpha| \leq \lambda$ for all $\alpha \in (\lambda, \lambda^+)$". Thus, since $(\mathrm{Add}(j(\lambda), j(\lambda^+))^{M[G_0][G_1][G_2]}$ is $j(\lambda)$-directed closed and $j(\lambda) > \lambda^+$, $q_\alpha = \bigcup\{j(p) \mid p \in G_1 \restriction \alpha\}$ is well-defined and is an element of $\mathrm{Add}(j(\lambda), j(\lambda^+))^{M[G_0][G_1][G_2]}$. Further, if

$\langle \rho, \sigma \rangle \in \mathrm{dom}(q_\alpha) - \mathrm{dom}(\bigcup_{\beta<\alpha} q_\beta)$ ($\bigcup_{\beta<\alpha} q_\beta$ is well-defined by closure), then $\sigma \in [\bigcup_{\beta<\alpha} j(\beta), j(\alpha))$. To see this, assume to the contrary that $\sigma < \bigcup_{\beta<\alpha} j(\beta)$. Let β be minimal such that $\sigma < j(\beta)$. It must thus be the case that for some $p \in G_1 \restriction \alpha$, $\langle \rho, \sigma \rangle \in \mathrm{dom}(j(p))$. Since by elementarity and the definitions of $G_1 \restriction \beta$ and $G_1 \restriction \alpha$, for $p \restriction \beta = q \in G_1 \restriction \beta$, $j(q) = j(p) \restriction j(\beta) = j(p \restriction \beta)$, it must be the case that $\langle \rho, \sigma \rangle \in \mathrm{dom}(j(q))$. This means $\langle \rho, \sigma \rangle \in \mathrm{dom}(q_\beta)$, a contradiction.

Since $M[G_0][G_1][G_2] \vDash$ "$2^{j(\lambda)} = j(\lambda^+)$", $M[G_0][G_1][G_2] \vDash$ "$\mathrm{Add}(j(\lambda), j(\lambda^+))$ is $j(\lambda^+)$-c.c. and has $j(\lambda^+)$ many maximal antichains". This means that if $\mathcal{A} \in M[G_0][G_1][G_2]$ is a maximal antichain of $\mathrm{Add}(j(\lambda), j(\lambda^+))$, $\mathcal{A} \subseteq \mathrm{Add}(j(\lambda), \beta)$ for some $\beta \in (j(\lambda), j(\lambda^+))$. Thus, since $\overline{V} \vDash$ "$2^\lambda = \lambda^+$" and the fact j is generated by a normal measure over λ imply that $\overline{V} \vDash$ "$|j(\lambda^+)| = \lambda^+$", we can let $\langle \mathcal{A}_\alpha \mid \alpha \in (\lambda, \lambda^+) \rangle \in \overline{V}[G_0][G_1]$ be an enumeration of all of the maximal antichains of $\mathrm{Add}(j(\lambda), j(\lambda^+))$ present in $M[G_0][G_1][G_2]$.

Working in $\overline{V}[G_0][G_1]$, we define now an increasing sequence $\langle r_\alpha \mid \alpha \in (\lambda, \lambda^+) \rangle$ of elements of $\mathrm{Add}(j(\lambda), j(\lambda^+))$ such that $\forall \alpha \in (\lambda, \lambda^+)[r_\alpha \geq q_\alpha$ and $r_\alpha \in \mathrm{Add}(j(\lambda), j(\alpha))]$ and such that $\forall \mathcal{A} \in \langle \mathcal{A}_\alpha \mid \alpha \in (\lambda, \lambda^+) \rangle \exists \beta \in (\lambda, \lambda^+) \exists r \in \mathcal{A}[r_\beta \geq r]$. Assuming we have such a sequence, $G_3 = \{p \in \mathrm{Add}(j(\lambda), j(\lambda^+)) \mid \exists r \in \langle r_\alpha \mid \alpha \in (\lambda, \lambda^+) \rangle [r \geq p]$ is an $M[G_0][G_1][G_2]$-generic object over $\mathrm{Add}(j(\lambda), j(\lambda^+))$. To define $\langle r_\alpha \mid \alpha \in (\lambda, \lambda^+) \rangle$, if α is a limit, we let $r_\alpha = \bigcup_{\beta \in (\lambda, \alpha)} r_\beta$. By the facts $\langle r_\beta \mid \beta \in (\lambda, \alpha) \rangle$ is (strictly) increasing and $M[G_0][G_1][G_2]^\lambda \subseteq M[G_0][G_1][G_2]$ in $\overline{V}[G_0][G_1]$, this definition is valid. Assuming now r_α has been defined and we wish to define $r_{\alpha+1}$, let $\langle \mathcal{B}_\beta \mid \beta < \eta < \lambda^+ \rangle$ be the subsequence of $\langle \mathcal{A}_\beta \mid \beta \leq \alpha + 1 \rangle$ containing each antichain $\mathcal{A}$ such that $\mathcal{A} \subseteq \mathrm{Add}(j(\lambda), j(\alpha + 1))$. Since $q_\alpha, r_\alpha \in \mathrm{Add}(j(\lambda), j(\alpha))$, $q_{\alpha+1} \in \mathrm{Add}(j(\lambda), j(\alpha + 1))$, and $j(\alpha) < j(\alpha + 1)$, the condition $r'_{\alpha+1} = r_\alpha \cup q_{\alpha+1}$ is well-defined, since by our earlier observations, any new elements of $\mathrm{dom}(q_{\alpha+1})$ won't be present in either $\mathrm{dom}(q_\alpha)$ or $\mathrm{dom}(r_\alpha)$. We can thus, using the fact $M[G_0][G_1][G_2]^\lambda \subseteq M[G_0][G_1][G_2]$ in $\overline{V}[G_0][G_1]$, define by induction an increasing sequence $\langle s_\beta \mid \beta < \eta \rangle$ such that $s_0 \geq r'_{\alpha+1}$, $s_\rho = \bigcup_{\beta<\rho} s_\beta$ if ρ is a limit ordinal, and $s_{\beta+1} \geq s_\beta$ is such that $s_{\beta+1}$ extends some element of $\mathcal{B}_\beta$. The just mentioned closure fact implies $r_{\alpha+1} = \bigcup_{\beta<\eta} s_\beta$ is a well-defined condition.

In order to show that G_3 is $M[G_0][G_1][G_2]$-generic over $\mathrm{Add}(j(\lambda), j(\lambda^+))$, we must show that $\forall \mathcal{A} \in \langle \mathcal{A}_\alpha \mid \alpha \in (\lambda, \lambda^+) \rangle \exists \beta \in (\lambda, \lambda^+) \exists r \in \mathcal{A}[r_\beta \geq r]$. To do this, we first note that $\langle j(\alpha) \mid \alpha < \lambda^+ \rangle$ is unbounded in $j(\lambda^+)$. To see this, if $\beta < j(\lambda^+)$ is an ordinal, then for some $f : \lambda \to M$ representing β, we can assume that for $\alpha < \lambda$, $f(\alpha) < \lambda^+$. Thus, by the regularity of λ^+ in $\overline{V}$, $\beta_0 = \bigcup_{\alpha < \lambda} f(\alpha) < \lambda^+$, and $j(\beta_0) \geq \beta$. This means by our earlier remarks that if $\mathcal{A} \in \langle \mathcal{A}_\alpha \mid \alpha < \lambda^+ \rangle$, $\mathcal{A} = \mathcal{A}_\rho$, then we can let $\beta \in (\lambda, \lambda^+)$ be such that $\mathcal{A} \subseteq \mathrm{Add}(j(\lambda), j(\beta))$. By construction, for $\eta > \max(\beta, \rho)$, there is some $r \in \mathcal{A}$ such that $r_\eta \geq r$. And, as any $p \in \mathrm{Add}(\lambda, \lambda^+)$ is such that for some $\alpha \in (\lambda, \lambda^+)$, $p = p \restriction \alpha$, G_3 is such that if $p \in G_1$, $j(p) \in G_3$. Thus, working in $\overline{V}[G_0][G_1]$, we have shown that j lifts to $j : \overline{V}[G_0][G_1] \to M[G_0][G_1][G_2][G_3]$. This means that $\overline{V}[G_0][G_1] \models$ "λ is a measurable cardinal", i.e., $V^{\mathbb{P} * \dot{\mathrm{A}}\mathrm{dd}(\lambda, \lambda^+)} \models$ "λ is a measurable cardinal". This completes the proof of Lemma 2.5.

□

We may now complete the proof of Theorem 1 by mimicking the proof of Lemma 2.1. We show that $A_2 = \{\delta < \kappa \mid \delta$ is measurable, δ is not a limit of measurable cardinals, δ is not δ^+ strongly compact, and δ's measurability is indestructible when forcing with either $\mathrm{Add}(\delta, 1)$ or $\mathrm{Add}(\delta, \delta^+)\}$ is unbounded in κ. Force with $\mathbb{P}$. By Lemma 2.2, after this forcing, λ is not λ^+ strongly compact. In addition, after this forcing, which is κ-directed closed, by Lemma 2.3, λ remains the least measurable cardinal above κ. In particular, after the forcing, λ is a measurable cardinal which is not a limit of measurable cardinals. In addition, by Lemmas 2.4 and 2.5, after the forcing, λ has become a measurable cardinal whose measurability is indestructible when forcing with either $\mathrm{Add}(\lambda, 1)$ or $\mathrm{Add}(\lambda, \lambda^+)$. Since κ's supercompactness is indestructible when forcing with κ-directed closed partial orderings and the indestructibility of λ when forcing with either $\mathrm{Add}(\lambda, 1)$ or $\mathrm{Add}(\lambda, \lambda^+)$ is detectible in a large enough V_η, by reflection, A_2 is unbounded in κ after the forcing has been performed. Once more, we infer by the fact $\mathbb{P}$ is κ-directed closed that A_2 is unbounded in κ in the ground model. This completes the proof of Theorem 1.

□

Having finished with the proof of Theorem 1, we turn now to the proof of Theorem 2. Recall that this theorem states that if $V \models$ "ZFC + κ is supercompact + No cardinal $\zeta > \kappa$ is measurable", then there is a partial ordering $\mathbb{P} \subseteq V$ such that $V^{\mathbb{P}} \models$ "ZFC + No cardinal $\zeta > \kappa$ is measurable + κ is indestructibly supercompact + If $\delta < \kappa$ is a measurable cardinal which is not a limit of measurable cardinals, then δ is δ^+ strongly compact and δ's measurability is destructible when forcing with partial orderings having rank below λ_δ".

Proof: Suppose V is as in the hypotheses for Theorem 2. Without loss of generality, by [8, Theorem 2] and the remarks at the end of [8], we assume in addition that $V \models$ "GCH + Every measurable cardinal δ is δ^+ strongly compact".[3]

Let f be a Laver function [22] for κ, i.e., $f : \kappa \to V_\kappa$ is such that for every $x \in V$ and every $\lambda \geq |\mathrm{TC}(x)|$, there is an elementary embedding $j : V \to M$ generated by a supercompact ultrafilter over $P_\kappa(\lambda)$ such that $j(f)(\kappa) = x$. Our partial ordering $\mathbb{P}$ is the reverse Easton iteration of length κ which begins by forcing with $\mathrm{Add}(\omega, 1)$ and then (possibly) does nontrivial forcing only at cardinals $\delta < \kappa$ which are measurable limits of measurable cardinals in V. At such a stage δ, if $f(\delta) = \dot{\mathbb{Q}}$ and $\Vdash_{\mathbb{P}_\delta}$ "$\dot{\mathbb{Q}}$ is a δ-directed closed partial ordering having rank below the least measurable cardinal above δ in V", then $\mathbb{P}_{\delta+1} = \mathbb{P}_\delta * \dot{\mathbb{Q}}$. If this is not the case, then $\mathbb{P}_{\delta+1} = \mathbb{P}_\delta * \dot{\mathbb{Q}}$, where $\dot{\mathbb{Q}}$ is a term for trivial forcing.

Lemma 2.6 $V^{\mathbb{P}} \models$ "*κ is indestructibly supercompact*".

Proof: We follow the proofs of [1, Lemma 2.1], [5, Lemma 2.1], and [6, Lemma 3.1], quoting verbatim when appropriate. Let $\mathbb{Q} \in V^{\mathbb{P}}$ be such that $V^{\mathbb{P}} \models$ "$\mathbb{Q}$ is κ-directed closed". Take $\dot{\mathbb{Q}}$ as a term for $\mathbb{Q}$ such that $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is κ-directed closed". Suppose $\lambda \geq \max(2^\kappa, |\mathrm{TC}(\dot{\mathbb{Q}})|)$ is an arbitrary cardinal, and let $\gamma = 2^{|[\lambda]^{<\kappa}|}$. Take $j : V \to M$ as an elementary embedding witnessing the γ supercompactness of κ generated by a supercompact ultrafilter over $P_\kappa(\gamma)$ such that $j(f)(\kappa) = \dot{\mathbb{Q}}$. Since $V \models$ "No cardinal $\delta > \kappa$ is measurable", $\gamma \geq 2^\kappa$, and $M^\gamma \subseteq M$, $M \models$ "κ is a measurable limit of measurable cardinals, and no cardinal δ in the interval $(\kappa, \gamma]$ is measurable". Hence, the definition of $\mathbb{P}$ implies that $j(\mathbb{P} * \dot{\mathbb{Q}}) = \mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}} * j(\dot{\mathbb{Q}})$, where the first stage at which $\dot{\mathbb{R}}$ is

[3] Forcing GCH may require the use of a proper class partial ordering. This in turn implies that the forcing conditions which are defined and used to prove Theorem 1 are in actuality a proper class as well.

forced to do nontrivial forcing is well above γ. Laver's original argument from [22] now applies and shows $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ "κ is λ supercompact". (Simply let $G_0 * G_1 * G_2$ be V-generic over $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$, lift j in $V[G_0][G_1][G_2]$ to $j : V[G_0] \to M[G_0][G_1][G_2]$, take a master condition p for $j''G_1$ and a $V[G_0][G_1][G_2]$-generic object G_3 over $j(\mathbb{Q})$ containing p, lift j again in $V[G_0][G_1][G_2][G_3]$ to $j : V[G_0][G_1] \to M[G_0][G_1][G_2][G_3]$, and show by the γ^+-directed closure of $\mathbb{R} * j(\dot{\mathbb{Q}})$ that the supercompactness measure over $(P_\kappa(\lambda))^{V[G_0][G_1]}$ generated by j is actually a member of $V[G_0][G_1]$.) As λ and $\mathbb{Q}$ were arbitrary, this completes the proof of Lemma 2.6.

$\square$

Lemma 2.7 $V^{\mathbb{P}} \models$ *"If $\delta < \kappa$ is a measurable cardinal which is not a limit of measurable cardinals, then δ is δ^+ strongly compact and δ's measurability is destructible when forcing with partial orderings having rank below λ_δ".*

Proof: Suppose $\delta < \kappa$ is such that $V \models$ "δ is a measurable cardinal which is not a limit of measurable cardinals". Write $\mathbb{P} = \mathbb{P}_\delta * \dot{\mathbb{P}}^\delta$. Since δ is not a limit of measurable cardinals in V, by the definition of $\mathbb{P}$, $\mathbb{P}_\delta$ is forcing equivalent to a partial ordering having cardinality less than δ. In addition, $\mathbb{P}_\delta$ is nontrivial. Consequently, by the results of [23] and [20, Theorem I], $V^{\mathbb{P}_\delta} \models$ "δ is a measurable cardinal which is not a limit of measurable cardinals, δ is δ^+ strongly compact, and δ's measurability is destructible when forcing with partial orderings having rank below λ_δ". Thus, since $\Vdash_{\mathbb{P}_\delta}$ "$\dot{\mathbb{P}}^\delta$ is η-directed closed for η the least measurable limit of measurable cardinals above δ" (which inductively is the same in both V and $V^{\mathbb{P}_\delta}$), $V^{\mathbb{P}_\delta * \dot{\mathbb{P}}^\delta} = V^{\mathbb{P}} \models$ "δ is a measurable cardinal which is not a limit of measurable cardinals, δ is δ^+ strongly compact, and δ's measurability is destructible when forcing with partial orderings having rank below λ_δ". To complete the proof of Lemma 2.7, it therefore suffices to show that any $V^{\mathbb{P}}$-measurable cardinal which is not a limit of measurable cardinals is a V-measurable cardinal which is not a limit of measurable cardinals as well.

To do this, suppose now that $\delta < \kappa$ is such that $V^{\mathbb{P}} \models$ "δ is a measurable cardinal which is not a limit of measurable cardinals". Write $\mathbb{P} = \mathrm{Add}(\omega, 1) * \dot{\mathbb{Q}}$. Since $|\mathrm{Add}(\omega, 1)| = \omega$, $\mathrm{Add}(\omega, 1)$

is nontrivial, and $\Vdash_{\mathrm{Add}(\omega,1)}$ "$\dot{\mathbb{Q}}$ is $\aleph_2$-directed closed", $\mathbb{P}$ admits a gap at $\aleph_1$. By Theorem 4, this means that $V \models$ "δ is a measurable cardinal". If $V \models$ "δ is a measurable cardinal which is a limit of measurable cardinals", then $V \models$ "δ is a measurable cardinal which is a limit of measurable cardinals which themselves are not limits of measurable cardinals". By the preceding paragraph, since such cardinals are preserved to $V^{\mathbb{P}}$, this means that $V^{\mathbb{P}} \models$ "δ is a measurable cardinal which is a limit of measurable cardinals", a contradiction. This completes the proof of Lemma 2.7.

$\square$

Since $\mathbb{P}$ may be defined so that $|\mathbb{P}| = \kappa$, the results of [23] show that $V^{\mathbb{P}} \models$ "No cardinal $\delta > \kappa$ is measurable". Lemmas 2.6 and 2.7 therefore complete the proof of Theorem 2.

$\square$

As we mentioned earlier, our methods of proof for both Theorems 1 and 2 show that the relevant measurable cardinals δ can have their measurability destructible when forcing with partial orderings other than just those having rank below the least beth fixed point above δ. For instance, the definition of λ_δ could be changed to the second beth fixed point above δ, etc. Further, the proof of Lemma 2.7 actually shows that any measurable cardinal which is not a limit of measurable cardinals has its measurability destructible when forcing with partial orderings having rank below the least V-measurable limit of measurable cardinals above it. In addition, the methods of [8] allow us to assume that in the ground model V for Theorem 2, each measurable cardinal δ is λ strongly compact, where λ is any fixed regular cardinal below the least measurable cardinal above δ (such as, e.g., δ^{++}, δ^{+17}, the least inaccessible or Ramsey cardinal above δ, etc.). Thus, in the model $V^{\mathbb{P}}$ witnessing the conclusions of Theorem 2, each measurable cardinal δ which is not a limit of measurable cardinals may also be λ strongly compact for the aforementioned values of λ. Finally, it is not possible to extend Theorem 2 and obtain a model in which *every* measurable cardinal $\delta < \kappa$ has its measurability destructible when forcing with partial orderings having rank below λ_δ. This follows since if κ is indestructibly supercompact and $j : V \to M$ is an elementary embedding witnessing the λ_κ supercompactness of κ, then $M \models$ "κ is a measurable cardinal which is a limit of measurable cardinals and κ's measurability is indestructible when forcing with partial orderings having rank below λ_κ". Hence, by reflection, $\{\delta < \kappa \mid \delta$ is a measurable cardinal which is a limit of measurable cardinals and δ's measurability is indestructible when forcing with partial orderings having rank below $\lambda_\delta\}$ must be unbounded in κ in V.

We turn now to the proof of Theorem 3. Recall that this theorem states that if $V \vDash$ "ZFC + κ is supercompact + No cardinal $\zeta > \kappa$ is inaccessible", then there is a partial ordering $\mathbb{P} \subseteq V$ such that $V^{\mathbb{P}} \vDash$ "ZFC + No cardinal $\zeta > \kappa$ is inaccessible + κ is indestructibly supercompact and is also the least strongly compact cardinal + Any measurable cardinal $\delta < \kappa$ which is not a limit of measurable cardinals is $<\lambda_\delta$ strongly compact and has its $<\lambda_\delta$ strong compactness (and hence also its measurability) indestructible when forcing with δ-directed closed partial orderings having rank below λ_δ".

Proof: Suppose V is as in the hypotheses for Theorem 3. Without loss of generality, by doing a preliminary forcing if necessary, we assume that $V \vDash$ GCH as well.

The proof of Theorem 3 now proceeds as a modification of the proof of [9, Theorem 1.1]. We will quote verbatim from the relevant portions of [9] in our presentation when appropriate. In particular, whereas the partial ordering found in the proof of [9, Theorem 1.1] has a uniform definition for every measurable cardinal δ (due to the strong hypotheses used, which as we have already indicated prove the consistency of the hypotheses found in the proof of Theorem 3), the definition of the partial ordering $\mathbb{P}$ employed in the proof of Theorem 3 splits into two cases. These will depend on whether a V-measurable cardinal δ is such that $V \vDash$ "δ is $<\lambda_\delta$ supercompact but is a limit of cardinals γ which are $<\lambda_\gamma$ supercompact" or the negation of this, i.e., if either $V \vDash$ "δ is $<\lambda_\delta$ supercompact and is not a limit of cardinals γ which are $<\lambda_\gamma$ supercompact" or $V \vDash$ "δ is measurable but is not $<\lambda_\delta$ supercompact". The difference between these two cases will be in how the lottery sum is formed. Specifically, the partial ordering $\mathbb{P}$ used in the proof of Theorem 3 is a Gitik style iteration of Prikry-like forcings of length κ, $\langle\langle\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha\rangle \mid \alpha < \kappa\rangle$, which possibly does a nontrivial forcing only at those ordinals $\delta < \kappa$ which are V-measurable cardinals. If $V \vDash$ "δ is $<\lambda_\delta$ supercompact and is a limit of cardinals γ which are $<\lambda_\gamma$ supercompact", we first force with $\mathbb{Q}^*_\delta$, the lottery sum of all partial orderings having rank less than the least inaccessible cardinal above δ

which are δ-directed closed in $V^{\mathbb{P}_\delta}$. If this is not the case, i.e., if either $V \models$ "δ is $<\lambda_\delta$ supercompact and is not a limit of cardinals γ which are $<\lambda_\gamma$ supercompact" or $V \models$ "δ is measurable but is not $<\lambda_\delta$ supercompact", we first force with $\mathbb{Q}^*_\delta$, the lottery sum of all partial orderings having rank less than λ_δ which are δ-directed closed in $V^{\mathbb{P}_\delta}$. If $V \models$ "δ is not $<\lambda_\delta$ supercompact" and $V^{\mathbb{P}_\delta * \dot{\mathbb{Q}}^*_\delta} \models$ "δ is measurable", then $\mathbb{Q}_\delta = \mathbb{Q}^*_\delta * \dot{\mathbb{S}}_\delta$, where $\dot{\mathbb{S}}_\delta$ is a term for Prikry forcing over δ defined using the appropriate normal measure. If the preceding conjunction of conditions does not hold, i.e., if either $V \models$ "δ is $<\lambda_\delta$ supercompact" or $V^{\mathbb{P}_\delta * \dot{\mathbb{Q}}^*_\delta} \models$ "δ is not measurable", then $\mathbb{Q}_\delta = \mathbb{Q}^*_\delta * \dot{\mathbb{S}}_\delta$, where $\dot{\mathbb{S}}_\delta$ is a term for trivial forcing.

Lemma 2.8 $V^{\mathbb{P}} \models$ "*κ is an indestructibly supercompact cardinal*".

Proof: Suppose $V^{\mathbb{P}} \models$ "$\mathbb{Q}$ is a κ-directed closed partial ordering". We show that $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ "κ is supercompact". To do this, we argue as in [9, Lemma 2.1] (which combines arguments found in the proofs of [11, Lemmas 2 and 3]). For any missing details, readers are urged to consult [11].

Let $\delta > \max(2^{[\lambda_\kappa]^{<\kappa}}, |\mathrm{TC}(\dot{\mathbb{Q}})|)$ be an arbitrary V-cardinal large enough such that $(2^{[\delta]^{<\kappa}})^V = \rho = (2^{[\delta]^{<\kappa}})^{V^{\mathbb{P} * \dot{\mathbb{Q}}}}$. Take $j : V \to M$ as an elementary embedding witnessing the ρ supercompactness of κ. Since $V \models$ "No cardinal $\zeta > \kappa$ is inaccessible", $M \models$ "There are no inaccessible cardinals in the interval $(\kappa, \rho]$". Further, because $\rho > 2^{[\lambda_\kappa]^{<\kappa}}$ and $V \models$ "κ is supercompact", $M \models$ "κ is $<\lambda_\kappa$ supercompact and is a limit of cardinals γ which are $<\lambda_\gamma$ supercompact". Hence, by the definition of $\mathbb{P}$, $\mathbb{Q}$ is an allowable choice in the stage κ lottery held in $M^{\mathbb{P}_\kappa} = M^{\mathbb{P}}$ in the definition of $j(\mathbb{P})$. In addition, as $M \models$ "κ is $<\lambda_\kappa$ supercompact", $j(\mathbb{P} * \dot{\mathbb{Q}})$ is forcing equivalent in M to $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{S}} * \dot{\mathbb{R}} * j(\dot{\mathbb{Q}})$, where $\dot{\mathbb{S}}$ is a term for trivial forcing. Because $M \models$ "There are no inaccessible cardinals in the interval $(\kappa, \rho]$", the next nontrivial stage in the definition of $j(\mathbb{P})$ after κ takes place well above ρ. Hence, as in [11, Lemma 2], there is a term $\tau \in M$ in the language of forcing with respect to $j(\mathbb{P})$ such that if $G * H$ is either V-generic or M-generic over $\mathbb{P} * \dot{\mathbb{Q}}$, $\Vdash_{j(\mathbb{P})}$ "τ extends every $j(\dot{q})$ for $\dot{q} \in \dot{H}$". In other words, τ is a term for a "master condition" for $j'' \dot{H}$.

Let $K = K_0 * K_1 * K_2$ be $V[G][H]$-generic over $\mathbb{S} * \dot{\mathbb{R}} * j(\dot{\mathbb{Q}})$. Define an embedding $j^* : V[G][H] \to M[G][H][K]$ by $j^*(i_{G*H}(\overline{\tau})) = i_{G*H*K}(j(\overline{\tau}))$ for any term $\overline{\tau}$ denoting a set in $V[G][H]$. Since the closure properties of M imply any term for a condition in K_2 can be assumed to extend the "master condition" τ above, as in [11, Lemma 3], j^* is a well-defined elementary embedding lifting j which can be used to define a supercompact ultrafilter $\mathcal{U} \in V[G][H][K]$ over $(P_\kappa(\delta))^{V[G][H]}$ by $X \in \mathcal{U}$ iff $\langle j(\alpha) \mid \alpha < \delta \rangle \in j^*(X)$. Since $\mathbb{P} * \dot{\mathbb{Q}}$ is ρ-c.c., the usual arguments show that $M[G][H]$ remains ρ closed with respect to $V[G][H]$. Hence, since forcing with $\mathbb{S} * \dot{\mathbb{R}} * j(\dot{\mathbb{Q}})$ over either $M[G][H]$ or $V[G][H]$ adds no subsets of $\rho, \mathcal{U} \in V[G][H]$, i.e., $V[G][H] \models$ "κ is δ supercompact". Since $\delta > \kappa$ was an arbitrarily large enough V-cardinal, and since trivial forcing is κ-directed closed, this completes the proof of Lemma 2.8.

$\square$

Lemma 2.9 $V^{\mathbb{P}} \models$ "*Any measurable cardinal* $\delta < \kappa$ *was* $<\lambda_\delta$ *supercompact in* V".

Proof: Suppose $V^{\mathbb{P}} \models$ "$\delta < \kappa$ is measurable". Write $\mathbb{P} = \mathbb{P}_{\delta+1} * \dot{\mathbb{P}}^{\delta+1}$. Since by the definition of $\mathbb{P}$, $\Vdash_{\mathbb{P}_{\delta+1}}$ "$\dot{\mathbb{P}}^{\delta+1}$ adds no bounded subsets of the least inaccessible cardinal above δ", it must be the case that $\Vdash_{\mathbb{P}_{\delta+1}}$ "δ is measurable".

Note now that $V \models$ "δ is measurable". For, if this were not the case, then again by the definition of $\mathbb{P}$, since $\dot{\mathbb{Q}}_\delta$ is a term for trivial forcing, it must be true that $\Vdash_{\mathbb{P}_\delta}$ "δ is measurable". In addition, observe that as any measurable cardinal is also Mahlo, $V^{\mathbb{P}} \models$ "δ is a Mahlo cardinal". Because forcing cannot create a new Mahlo cardinal, it must also be true that $V \models$ "δ is a Mahlo cardinal" as well. Therefore, since when a lottery sum is performed at a nontrivial stage of forcing γ, it is of partial orderings having rank below the least V-inaccessible cardinal above γ, the definition of $\mathbb{P}$ allows us to infer that $\mathbb{P}_\delta \subseteq V_\delta$. Hence, as $\mathbb{P}$ is a Gitik style iteration of Prikry-like forcings, $\mathbb{P}_\delta$ is the direct limit of $\langle \mathbb{P}_\alpha \mid \alpha < \delta \rangle$. This means that since $\mathbb{P}_\delta$ satisfies δ-c.c. in $V^{\mathbb{P}_\delta}$ (this follows because δ is measurable and hence Mahlo in $V^{\mathbb{P}_\delta}$ and $\mathbb{P}_\delta$ is a subordering of the Easton support product of $\langle \mathbb{Q}_\alpha \mid \alpha < \delta \rangle$ as calculated in $V^{\mathbb{P}_\delta}$), the proof of [10, Lemma 3] tells us that the restriction of every δ-additive ultrafilter over δ present in $V^{\mathbb{P}_\delta}$ to V is a δ-additive ultrafilter over δ which is a member of V. This contradiction to our supposition that $V \models$ "δ is not measurable" consequently yields that $V \models$ "δ is measurable". However, if it is not the case that δ is $<\lambda_\delta$ supercompact in V, then by the definition of $\mathbb{P}$, $V^{\mathbb{P}_{\delta+1}} \models$ "δ is not measurable". This completes the proof of Lemma 2.9.

$\square$

Lemma 2.10 $V^{\mathbb{P}} \vDash$ "*Any measurable cardinal* $\delta < \kappa$ *not a limit of cardinals* γ *which are* $<\lambda_\gamma$ *supercompact in* V *is* $<\lambda_\delta$ *strongly compact and has its* $<\lambda_\delta$ *strong compactness indestructible under* δ*-directed closed forcing having rank less than* λ_δ*".*

Proof: By Lemma 2.9, we know that any measurable cardinal $\delta < \lambda_\delta$ had to have been $<\lambda_\delta$ supercompact in V. As in Lemma 2.9, by the definition of $\mathbb{P}$, write $\mathbb{P} = \mathbb{P}_{\delta+1} * \dot{\mathbb{P}}^{\delta+1}$, where $\Vdash_{\mathbb{P}_{\delta+1}}$ "Forcing with $\dot{\mathbb{P}}^{\delta+1}$ does not add any subsets of λ_δ". Thus, to prove Lemma 2.10, it suffices to show that its conclusions hold in $V^{\mathbb{P}_{\delta+1}}$.

Towards this end, let $\mathbb{Q} \in V^{\mathbb{P}_{\delta+1}}$ be such that $V^{\mathbb{P}_{\delta+1}} \vDash$ "$\mathbb{Q}$ is δ-directed closed and has rank less than λ_δ". By the definition of $\mathbb{P}$, since $V \vDash$ "δ is $<\lambda_\delta$ supercompact and is not a limit of cardinals γ which are $<\lambda_\gamma$ supercompact", $\mathbb{P}_{\delta+1} * \dot{\mathbb{Q}}$ is forcing equivalent to $\mathbb{P}_\delta * \dot{\mathbb{Q}}' * \dot{\mathbb{S}}_\delta * \dot{\mathbb{Q}}$, where $\dot{\mathbb{Q}}'$ is a term for the partial ordering of rank less than λ_δ selected in the stage δ lottery held in the definition of $\mathbb{P}$, and $\dot{\mathbb{S}}_\delta$ is a term for trivial forcing. Since $\dot{\mathbb{Q}}' * \dot{\mathbb{S}}_\delta * \dot{\mathbb{Q}}$ can be taken to be a term in the forcing language with respect to $\mathbb{P}_\delta$ for a δ-directed closed partial ordering having rank less than λ_δ, we abuse notation in what follows and assume without loss of generality that what we will write as $\dot{\mathbb{Q}}$ is actually $\dot{\mathbb{Q}}' * \dot{\mathbb{S}}_\delta * \dot{\mathbb{Q}}$.

We proceed now in analogy to the argument given in the second and third paragraphs of the proof of Lemma 2.8. Specifically, the fact that $V \vDash$ GCH and the definition of $\mathbb{P}$ allow us to choose $\gamma > \max(\delta, |\mathrm{TC}(\dot{\mathbb{Q}})|)$, $\gamma < \lambda_\delta$ as an arbitrary V-regular cardinal large enough such that $(2^{[\gamma]^{<\delta}})^V = \rho = (2^{[\gamma]^{<\delta}})^{V^{\mathbb{P}_\delta * \dot{\mathbb{Q}}}}$ and ρ is regular in both V and $V^{\mathbb{P}_\delta * \dot{\mathbb{Q}}}$. Let $\sigma = \rho^+$. Take $j : V \to M$ as an elementary embedding witnessing the σ supercompactness of δ such that $M \vDash$ "δ is not σ supercompact". By the definition of $\mathbb{P}_\delta$, because in both V and M, $\sigma < \lambda_\delta$, $\mathbb{Q}$ is an allowable choice in the stage δ lottery held in $M^{\mathbb{P}_\delta}$ in the definition of $j(\mathbb{P}_\delta)$. Consequently, $j(\mathbb{P}_\delta * \dot{\mathbb{Q}})$ is forcing equivalent in M to $\mathbb{P}_\delta * \dot{\mathbb{Q}} * \dot{\mathbb{S}}'_\delta * \dot{\mathbb{R}} * j(\dot{\mathbb{Q}})$, where $\dot{\mathbb{S}}'_\delta$ is either a term for trivial forcing or for Prikry forcing over δ defined using the appropriate normal measure. Further, since $M \vDash$ "There are no inaccessible cardinals in the interval $(\delta, \sigma]$", as before, the next nontrivial stage in the definition of $j(\mathbb{P}_\delta)$ after δ takes place well above σ. Hence, as in Lemma 2.8 and [11, Lemma 2], there is a term $\tau \in M$ in the language of forcing with respect to $j(\mathbb{P}_\delta)$ such that if $G * H$ is either V-generic or

M-generic over $\mathbb{P}_\delta * \dot{\mathbb{Q}}$, $\Vdash_{j(\mathbb{P}_\delta)}$ "τ extends every $j(\dot{q})$ for $\dot{q} \in \dot{H}$". In other words, τ is once again a term for a "master condition" for $j''\dot{H}$. Thus, if $\langle \dot{A}_\alpha \mid \alpha < \rho < \sigma \rangle$ enumerates in V the canonical $\mathbb{P}_\delta * \dot{\mathbb{Q}}$ names of subsets of $(P_\delta(\gamma))^{V[G][H]}$, we can as is done in the proof of [11, Lemma 2] define in M a sequence of $\mathbb{P}_\delta * \dot{\mathbb{Q}} * \dot{\mathbb{S}}'_\delta$ names of elements of $\dot{\mathbb{R}} * j(\dot{\mathbb{Q}})$, $\langle \dot{p}_\alpha \mid \alpha \leq \rho \rangle$, such that $\dot{p}_0$ is a term for $\langle 0, \tau \rangle$ (where 0 represents the trivial condition with respect to $\mathbb{R}$), $\Vdash_{\mathbb{P}_\delta * \dot{\mathbb{Q}} * \dot{\mathbb{S}}'_\delta}$ "$\dot{p}_{\alpha+1}$ is a term for an Easton extension of $\dot{p}_\alpha$ deciding '$\langle j(\beta) \mid \beta < \gamma \rangle \in j(\dot{A}_\alpha)$' ",[4] and for $\eta \leq \rho$ a limit ordinal, $\Vdash_{\mathbb{P}_\delta * \dot{\mathbb{Q}} * \dot{\mathbb{S}}'_\delta}$ "$\dot{p}_\eta$ is a term for an Easton extension of each member of the sequence $\langle \dot{p}_\beta \mid \beta < \eta \rangle$". If we then in $V[G][H]$ define a set $\mathcal{U} \subseteq 2^{[\gamma]^{<\delta}}$ by $X \in \mathcal{U}$ iff $X \subseteq P_\delta(\gamma)$ and for some $\langle r, q \rangle \in G * H$ and some $q' \in \mathbb{S}'_\delta$ either the trivial condition (if $\mathbb{S}'_\delta$ is trivial forcing) or of the form $\langle \emptyset, B \rangle$ (if $\mathbb{S}'_\delta$ is Prikry forcing), in M, $\langle r, \dot{q}, \dot{q}', \dot{p}_\rho \rangle \Vdash$ "$\langle j(\beta) \mid \beta < \gamma \rangle \in j(\dot{X})$" for some name $\dot{X}$ of X, then as in [11, Lemma 2], $\mathcal{U}$ is a δ-additive, fine ultrafilter over $(P_\delta(\gamma))^{V[G][H]}$, i.e., $V[G][H] \models$ "δ is γ strongly compact". Since γ was arbitrary, and since trivial forcing is $<\lambda_\delta$-directed closed and can be defined so as to have rank less than λ_δ, this completes the proof of Lemma 2.10.

$\square$

Lemma 2.11 $V^{\mathbb{P}} \models$ "*Any measurable cardinal $\delta < \kappa$ which is not a limit of measurable cardinals was in V a $<\lambda_\delta$ supercompact cardinal which is not a limit of cardinals γ which are $<\lambda_\gamma$ supercompact*".

Proof: By Lemma 2.9, we know that $V \models$ "δ is $<\lambda_\delta$ supercompact". If $V \models$ "δ is a limit of cardinals γ which are $<\lambda_\gamma$ supercompact", then $V \models$ "δ is a limit of cardinals ζ which are $<\lambda_\zeta$ supercompact and each such ζ is not a limit of cardinals η which are $<\lambda_\eta$ supercompact". By the proof of Lemma 2.10, each such ζ is $<\lambda_\zeta$ strongly compact and hence measurable in $V^{\mathbb{P}}$. This means that $V^{\mathbb{P}} \models$ "δ is a limit of measurable cardinals", a contradiction. This completes the proof of Lemma 2.11.

$\square$

[4]Roughly speaking, Easton extension means that $p_\beta \geq p_\alpha$ as in a usual reverse Easton iteration, except that at coordinates at which Prikry forcing occurs in p_α, measure 1 sets are shrunk and stems are not extended. For a more precise definition, readers are urged to consult [13].

Lemma 2.12 $V^{\mathbb{P}} \models$ *"No cardinal $\gamma < \kappa$ is strongly compact".*

Proof: We argue in analogy to the proof of [11, Lemma 4] and [9, Lemma 2.4]. Let $\delta > \kappa^{++}$ be any sufficiently large cardinal below λ_κ, e.g., the least strong limit cardinal above κ. Take $j : V \to M$ as an elementary embedding witnessing the δ supercompactness of κ such that $M \models$ "κ is not δ supercompact". Since $\delta < \lambda_\kappa$, this means that $M \models$ "κ is not $<\lambda_\kappa$ supercompact". By the choice of δ, it is possible to opt for $\mathrm{Add}(\kappa, \kappa^{++})$ at stage κ in $M^{\mathbb{P}} = M^{\mathbb{P}_\kappa}$ in the definition of $j(\mathbb{P})$. Further, by Lemma 2.8 and the fact $M^\delta \subseteq M$, $M^{\mathbb{P}_\kappa * \dot{\mathrm{Add}}(\kappa, \kappa^{++})} \models$ "κ is measurable". By the definition of $\mathbb{P}$, this therefore means that above the appropriate condition, $j(\mathbb{P})$ is forcing equivalent in M to $\mathbb{P}_\kappa * \dot{\mathrm{Add}}(\kappa, \kappa^{++}) * \dot{\mathbb{S}}_\kappa * \dot{\mathbb{R}}$, where $\dot{\mathbb{S}}_\kappa$ is a term for Prikry forcing over κ defined with respect to the appropriate normal measure, and $\Vdash_{\mathbb{P}_\kappa * \dot{\mathrm{Add}}(\kappa, \kappa^{++}) * \dot{\mathbb{S}}_\kappa}$ "Forcing with $\dot{\mathbb{R}}$ does not add any subsets to κ". Consequently, $M^{\mathbb{P}_\kappa * \dot{\mathrm{Add}}(\kappa, \kappa^{++}) * \dot{\mathbb{S}}_\kappa * \dot{\mathbb{R}}} = M^{j(\mathbb{P})} \models$ "κ is a singular strong limit cardinal violating GCH". By reflection, this just means that $V^{\mathbb{P}} \models$ "There are unboundedly in κ many singular strong limit cardinals below κ violating GCH". By Solovay's theorem [24] that GCH must hold at any singular strong limit cardinal above a strongly compact cardinal, $V^{\mathbb{P}} \models$ "No cardinal $\gamma < \kappa$ is strongly compact". This completes the proof of Lemma 2.12.

□

Lemmas 2.8 – 2.12 complete the proof of Theorem 3.

□

As we mentioned earlier, our proof methods for Theorem 3 remain valid for different values of λ_δ, e.g., $\lambda_\delta =_{\mathrm{df}}$ The second beth fixed point above δ, etc. The arguments for Lemmas 2.8 – 2.12 will literally go through unchanged. In addition, the proof methods for Theorem 3 also remain valid if we assume that $V \models$ "ZFC + κ is supercompact + No cardinal $\zeta > \kappa$ is Mahlo". Here, we assume that in the definition of $\mathbb{P}$, the word "inaccessible" is replaced with the word "Mahlo", and λ_δ is, e.g., as it was originally (or in fact is replaced by any suitable value). The key point is that the proof of Lemma 2.9 is still sound. In particular, if $V^{\mathbb{P}} \models$ "$\delta < \kappa$ is measurable" and $V \models$ "δ is not measurable", it will still be the case that $\mathbb{P}_\delta \subseteq V_\delta$. This allows us to infer that in fact, $V \models$ "δ is measurable". However, if $V \models$ "ZFC + κ is supercompact + No cardinal $\zeta > \kappa$ is Ramsey", the definition of $\mathbb{P}$ is modified changing the word "inaccessible" to "Ramsey", λ_δ is as before or is anything suitable, $V^{\mathbb{P}} \models$ "$\delta < \kappa$ is measurable", and $V \models$ "δ is not measurable", we do not necessarily know that $\mathbb{P}_\delta \subseteq V_\delta$. It will of course still be true that $V \models$ "δ is a Mahlo cardinal". However, there could be some V-measurable cardinal $\gamma < \delta$ at which in the stage γ lottery, a partial ordering having size above δ were chosen. If this were indeed the case, then $\mathbb{P}_\delta \not\subseteq V_\delta$, and we cannot use our original methods to infer that $V \models$ "δ is measurable". Because $\mathbb{P}$ is a Gitik style iteration of Prikry-like forcings, we cannot use the results of [17, 18, 16] either to infer that $V \models$ "δ is measurable". Thus, the proof of Lemma 2.9, and hence also the proof of Theorem 3, break down. We therefore conclude by asking if there is some alternate method which will allow the appropriate analogue of Theorem 3 to be proven if our ground model V is such that $V \models$ "ZFC + κ is supercompact + No cardinal $\zeta > \kappa$ is Ramsey" (or even if V is such that $V \models$ "ZFC + κ is supercompact + No cardinal $\zeta > \kappa$ is measurable").

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