Indestructible Strong Compactness and Level by Level Equivalence with No Large Cardinal Restrictions *†

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Abstract

We construct a model for the level by level equivalence between strong compactness and supercompactness with an arbitrary large cardinal structure in which the least supercompact cardinal κ has its strong compactness indestructible under κ-directed closed forcing. This is in analogy to and generalizes [3, Theorem 1], but without the restriction that no cardinal is supercompact up to an inaccessible cardinal.

1 Introduction and Preliminaries

In [3], the following theorem was proven.

Theorem 1 *Suppose $V \models$ "ZFC + There is a supercompact cardinal". There is then a model $\overline{V} \models$ "ZFC + There is a supercompact cardinal κ + Level by level equivalence between strong*

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compactness and supercompactness holds" in which the strong compactness of κ is indestructible under κ-directed closed forcing.

In $\overline{V}$, it is the case that no cardinal is supercompact up to an inaccessible cardinal. Consequently, κ of necessity must be the only supercompact cardinal in $\overline{V}$, and $\overline{V}$ does not contain a measurable cardinal above κ. Thus, $\overline{V}$ has a rather restricted large cardinal structure. This raises the following

Question: Is it possible to prove an analogue to Theorem 1, but in a universe with no restrictions on its large cardinal structure?

The purpose of this paper is to answer the above Question in the affirmative. Specifically, we will prove the following theorem.

Theorem 2 *Suppose $V \vDash$ "ZFC + $\mathcal{K} \neq \emptyset$ is the (possibly proper) class of supercompact cardinals + κ is the least supercompact cardinal". There is then a partial ordering $\mathbb{P} \subseteq V$ such that $V^{\mathbb{P}} \vDash$ "ZFC + $\mathcal{K}$ is the class of supercompact cardinals". In $V^{\mathbb{P}}$, level by level equivalence between strong compactness and supercompactness holds, and the strong compactness of κ is indestructible under κ-directed closed forcing.*

We observe that since $V \vDash$ "κ is the least supercompact cardinal + $\mathcal{K}$ is the class of supercompact cardinals" and $V^{\mathbb{P}} \vDash$ "$\mathcal{K}$ is the class of supercompact cardinals", it automatically follows in Theorem 2 that $V^{\mathbb{P}} \vDash$ "κ is the least supercompact cardinal".

Note that [8, Theorem 5] shows that if κ is indestructibly supercompact and level by level equivalence between strong compactness and supercompactness holds, then no cardinal $\lambda > \kappa$ is 2^λ supercompact. Thus, in any universe in which level by level equivalence between strong compactness and supercompactness holds and there is an indestructibly supercompact cardinal, there must of necessity be a restricted number of large cardinals. This is in sharp contrast to our Theorem 2, where we have level by level equivalence between strong compactness and supercompactness holding in a universe with an arbitrary large cardinal structure, together with the least supercompact cardinal κ having its *strong compactness* indestructible under any κ-directed closed forcing notion.

We now very briefly give some preliminary information concerning notation and terminology. For anything left unexplained, readers are urged to consult [3]. When forcing, $q \geq p$ means that q *is stronger than* p. For $\alpha \leq \beta$ ordinals, $[\alpha, \beta]$ and $(\alpha, \beta]$ are as in standard interval notation. For κ a cardinal, the partial ordering $\mathbb{P}$ is κ*-directed closed* if every directed set of conditions of size less than κ has an upper bound. If κ is a regular cardinal, $\mathrm{Add}(\kappa, 1)$ is the standard partial ordering for adding a single Cohen subset of κ. If G is V-generic over $\mathbb{P}$, we will abuse notation slightly and use both $V[G]$ and $V^{\mathbb{P}}$ to indicate the universe obtained by forcing with $\mathbb{P}$. We will, from time to time, confuse terms with the sets they denote and write x when we actually mean $\dot{x}$ or $\check{x}$. For $p \in \mathbb{P}$ and φ a formula in the forcing language with respect to $\mathbb{P}$, $p \parallel \varphi$ means that p *decides* φ.

We recall for the benefit of readers the definition given by Hamkins in [11, Section 3] of the lottery sum of a collection of partial orderings. If $\mathfrak{A}$ is a collection of partial orderings, then the *lottery sum* is the partial ordering $\oplus \mathfrak{A} = \{\langle \mathbb{P}, p \rangle \mid \mathbb{P} \in \mathfrak{A} \text{ and } p \in \mathbb{P}\} \cup \{0\}$, ordered with 0 below everything and $\langle \mathbb{P}, p \rangle \leq \langle \mathbb{P}', p' \rangle$ iff $\mathbb{P} = \mathbb{P}'$ and $p \leq p'$. Intuitively, if G is V-generic over $\oplus \mathfrak{A}$, then G first selects an element of $\mathfrak{A}$ (or as Hamkins says in [11], "holds a lottery among the posets in $\mathfrak{A}$") and then forces with it.[1]

Suppose V is a model of ZFC in which for all regular cardinals $\kappa < \lambda$, κ is λ strongly compact iff κ is λ supercompact, except possibly if κ is a measurable limit of cardinals δ which are λ supercompact. Such a model will be said to witness *level by level equivalence between strong compactness and supercompactness*. The exception is provided by a theorem of Menas [17], who showed that if κ is a measurable limit of cardinals δ which are λ supercompact, the κ is λ strongly compact but need not be λ supercompact. Models in which level by level equivalence between strong compactness and supercompactness holds nontrivially were first constructed in [9].

The partial ordering $\mathbb{P}$ which will be used in the proof of Theorem 2 is a *Gitik iteration*. By this we will mean an Easton support iteration as first given by Gitik in [10], to which we refer readers for a discussion of the basic properties of and terminology associated with such an iteration. For the purposes of this paper, each component $\dot{\mathbb{Q}}_\delta$ of the iteration used at a nontrivial stage δ has

[1]The terminology "lottery sum" is due to Hamkins, although the concept of the lottery sum of partial orderings has been around for quite some time and has been referred to at different junctures via the names "disjoint sum of partial orderings," "side-by-side forcing," and "choosing which partial ordering to force with generically."

the form $\dot{\mathbb{Q}}^0_\delta * \dot{\mathbb{Q}}^1_\delta$, where $\dot{\mathbb{Q}}^0_\delta$ is a term for a δ-directed closed partial ordering and $\dot{\mathbb{Q}}^1_\delta$ is a term for either trivial forcing or a Magidor iteration [16] of Prikry forcing (although other types of partial orderings may be used in the general case — see [10] for additional details).

We assume familiarity with the large cardinal notions of measurability, strongness, strong compactness, and supercompactness. Readers are urged to consult [12] for further details. We do wish to point out explicitly, however, that an *indestructibly supercompact cardinal* κ is one as in [14], i.e., a supercompact cardinal which remains supercompact after κ-directed closed forcing. Also, we say that κ *is supercompact (strong) up to an inaccessible cardinal* λ if κ is δ supercompact (δ strong) for every $\delta < \lambda$. For any cardinal δ, we adopt as our notation that δ' is the least strong cardinal greater than δ in our ground model V. A measurable cardinal κ is said to have *trivial Mitchell rank* if there is no normal measure $\mathcal{U}$ over κ with associated elementary embedding $j : V \to M$ such that $M \vDash$ "κ is measurable".

2 The Proof of Theorem 2

We turn now to the proof of Theorem 2.

Proof: Suppose $V \vDash$ "ZFC + $\mathcal{K} \neq \emptyset$ is the (possibly proper) class of supercompact cardinals + κ is the least supercompact cardinal". Without loss of generality, by first forcing GCH and then doing the forcing of [9], we assume in addition that $V \vDash$ "GCH + Level by level equivalence between strong compactness and supercompactness holds".

The partial ordering $\mathbb{P}$ used in the proof of Theorem 2 is defined as $\mathbb{P} = \langle\langle \mathbb{P}_\delta, \dot{\mathbb{Q}}_\delta\rangle \mid \delta < \kappa\rangle$, the Gitik iteration of length κ which possibly does nontrivial forcing only at those stages $\delta < \kappa$ which are in V measurable limits of strong cardinals. At such a δ, we let $\dot{\mathbb{Q}}_\delta = \dot{\mathbb{Q}}^0_\delta * \dot{\mathbb{Q}}^1_\delta$, where $\dot{\mathbb{Q}}^0_\delta$ is a term for the lottery sum of all partial orderings in $V^{\mathbb{P}_\delta}$ which are δ-directed closed and have rank below δ'. If trivial forcing is selected in the stage δ lottery, we do nothing, i.e., $\Vdash_{\mathbb{P}_\delta * \dot{\mathbb{Q}}^0_\delta}$ "If $V^{\mathbb{P}_\delta * \dot{\mathbb{Q}}^0_\delta} = V^{\mathbb{P}_\delta}$, then $\dot{\mathbb{Q}}^1_\delta$ is trivial forcing". If this is not the case, then let $\mathbb{R}_\delta$ be the partial ordering selected by the stage δ lottery. Let $\gamma = \max(\delta, |\mathbb{R}_\delta|)$. $\dot{\mathbb{Q}}^1_\delta$ is now a term for the Magidor iteration of Prikry forcing defined in $V^{\mathbb{P}_\delta * \dot{\mathbb{Q}}^0_\delta} = V^{\mathbb{P}_\delta * \dot{\mathbb{R}}_\delta}$ which adds a Prikry sequence to each measurable cardinal in the closed interval $[\delta, \gamma]$, i.e., $\Vdash_{\mathbb{P}_\delta * \dot{\mathbb{Q}}^0_\delta}$ "If $V^{\mathbb{P}_\delta * \dot{\mathbb{Q}}^0_\delta} \neq V^{\mathbb{P}_\delta}$, then $\dot{\mathbb{Q}}^1_\delta$ is the Magidor iteration of Prikry forcing adding a Prikry sequence to each measurable cardinal in the closed interval $[\check{\delta}, \dot{\gamma}]$, where $\dot{\mathbb{R}}_\delta$ is the partial ordering selected in the stage δ lottery and $\dot{\gamma} = \max(\check{\delta}, |\dot{\mathbb{R}}_\delta|)$".

Lemma 2.1 $V^{\mathbb{P}} \models$ *"κ is supercompact".*

Proof: We follow the proof of [2, Lemma 2.1] and [7, Lemma 2.1], quoting verbatim when appropriate. Let $\lambda \geq \kappa^+$ be an arbitrary regular cardinal, and let $j : V \to M$ be an elementary embedding witnessing the λ supercompactness of κ generated by a supercompact ultrafilter over $P_\kappa(\lambda)$ such that $M \models$ "κ is not λ supercompact". It is the case that $M \models$ "No cardinal $\delta \in (\kappa, \lambda]$ is strong". This is since otherwise, κ is supercompact up to a strong cardinal in M, and thus, by the proof of [5, Lemma 2.4], $M \models$ "κ is supercompact", a contradiction. Further, because $\lambda \geq \kappa^+ = 2^\kappa$, by [5, Lemma 2.1], $M \models$ "κ is a measurable limit of strong cardinals". Thus, κ is a stage in M at which either trivial or nontrivial forcing might possibly occur. This means that by forcing above a condition opting for trivial forcing in the stage κ lottery held in M in the definition of $j(\mathbb{P})$, we may assume that $j(\mathbb{P})$ is forcing equivalent to $\mathbb{P} * \dot{\mathbb{Q}}$, where the first nontrivial stage in $\dot{\mathbb{Q}}$ takes place well above λ.

We now show that $V^{\mathbb{P}} \models$ "κ is λ supercompact" as in the proof of [2, Lemma 2.1]. Specifically, we apply the argument of [10, Lemma 1.5]. In particular, let G be V-generic over $\mathbb{P}$. Since $2^\lambda = \lambda^+$ in both V and $V[G]$, we may let $\langle \dot{x}_\alpha \mid \alpha < \lambda^+ \rangle$ be an enumeration in V of all of the canonical $\mathbb{P}$-names of subsets of $P_\kappa(\lambda)$. Because $\mathbb{P}$ is a Gitik iteration of length κ, $\mathbb{P}$ is κ-c.c. Consequently, $M[G]$ remains λ closed with respect to $V[G]$. Therefore, by [10, Lemmas 1.4 and 1.2] and the fact $M[G]^\lambda \subseteq M[G]$, we may define in $V[G]$ an increasing sequence $\langle p_\alpha \mid \alpha < \lambda^+ \rangle$ of elements of $j(\mathbb{P})/G$ such that if $\alpha < \beta < \lambda^+$, p_β is an Easton extension of p_α,[2] every initial segment of the sequence is in $M[G]$, and for every $\alpha < \lambda^+$, $p_{\alpha+1} \parallel$ "$\langle j(\beta) \mid \beta < \lambda \rangle \in j(\dot{x}_\alpha)$". The remainder of the argument of [10, Lemma 1.5] remains valid and shows that a supercompact ultrafilter $\mathcal{U}$ over $(P_\kappa(\lambda))^{V[G]}$ may be defined in $V[G]$ by $x \in \mathcal{U}$ iff $x \subseteq (P_\kappa(\lambda))^{V[G]}$ and for some $\alpha < \lambda^+$ and some $\mathbb{P}$-name $\dot{x}$ of x,

[2] Roughly speaking, this means that p_β extends p_α as in a usual Easton support iteration, except that no stems of any components of p_α which are conditions in a Magidor iteration of Prikry forcing are extended. For a more precise definition, readers are urged to consult either [10] or [6].

in $M[G]$, $p_\alpha \Vdash_{j(\mathbb{P})/G}$ "$\langle j(\beta) \mid \beta < \lambda\rangle \in j(\dot{x})$". (The fact that $j''G = G$ tells us $\mathcal{U}$ is well-defined.) Thus, $V^{\mathbb{P}} \vDash$ "κ is λ supercompact". Since λ was arbitrary, this completes the proof of Lemma 2.1.

$\square$

Lemma 2.2 *Suppose $\mathbb{Q} \in V^{\mathbb{P}}$ is a partial ordering which is κ-directed closed. Then $V^{\mathbb{P} * \dot{\mathbb{Q}}} \vDash$ "κ is strongly compact".*

Proof: We follow the proof of [7, Lemma 2.2], again quoting verbatim when appropriate. Suppose $\mathbb{Q} \in V^{\mathbb{P}}$ is κ-directed closed. Let $\lambda > \max(\kappa, |\mathrm{TC}(\dot{\mathbb{Q}})|)$ be an arbitrary regular cardinal large enough so that $(2^{[\lambda]^{<\kappa}})^V = \rho = (2^{[\lambda]^{<\kappa}})^{V^{\mathbb{P} * \dot{\mathbb{Q}}}}$ and ρ is regular in both V and $V^{\mathbb{P} * \dot{\mathbb{Q}}}$, and let $\sigma = \rho^+ = 2^\rho$. Take $j : V \to M$ as an elementary embedding witnessing the σ supercompactness of κ such that $M \vDash$ "κ is not σ supercompact". As in Lemma 2.1, by [5, Lemma 2.1] and the fact $\sigma > 2^\kappa$, κ is a measurable limit of strong cardinals in M. Consequently, by the choice of σ, it is possible to opt for $\mathbb{Q}$ in the stage κ lottery held in M in the definition of $j(\mathbb{P})$. Further, as in Lemma 2.1, since $M \vDash$ "No cardinal $\delta \in (\kappa, \sigma]$ is strong", the next nontrivial forcing in the definition of $j(\mathbb{P})$ takes place well above σ. Thus, in M, above the appropriate condition, $j(\mathbb{P} * \dot{\mathbb{Q}})$ is forcing equivalent to $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{S}}_\kappa * \dot{\mathbb{R}} * j(\dot{\mathbb{Q}})$, where $\Vdash_{\mathbb{P} * \dot{\mathbb{Q}}}$ "$\dot{\mathbb{S}}_\kappa$ is a term for either trivial forcing or a Magidor iteration of Prikry forcing".

The remainder of the proof of Lemma 2.2 is as in the proof of [6, Lemma 2]. As in the proof of Lemma 2.1, we outline the argument, and refer readers to the proof of [6, Lemma 2] for any missing details. By the last two sentences of the preceding paragraph, as in [6, Lemma 2], there is a term $\tau \in M$ in the language of forcing with respect to $j(\mathbb{P})$ such that if $G * H$ is either V-generic or M-generic over $\mathbb{P} * \dot{\mathbb{Q}}$, $\Vdash_{j(\mathbb{P})}$ "τ extends every $j(\dot{q})$ for $\dot{q} \in \dot{H}$". In other words, τ is a term for a "master condition" for $\dot{\mathbb{Q}}$. Thus, if $\langle \dot{A}_\alpha \mid \alpha < \rho < \sigma\rangle$ enumerates in V the canonical $\mathbb{P} * \dot{\mathbb{Q}}$ names of subsets of $(P_\kappa(\lambda))^{V[G*H]}$, we can define in M a sequence of $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{S}}_\kappa$ names of elements of $\dot{\mathbb{R}} * j(\dot{\mathbb{Q}})$, $\langle \dot{p}_\alpha \mid \alpha \le \rho\rangle$, such that $\dot{p}_0$ is a term for $\langle 0, \tau\rangle$ (where 0 represents the trivial condition with respect to $\mathbb{R}$), $\Vdash_{\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{S}}_\kappa}$ "$\dot{p}_{\alpha+1}$ is a term for an Easton extension of $\dot{p}_\alpha$ deciding '$\langle j(\beta) \mid \beta < \lambda\rangle \in j(\dot{A}_\alpha)$' ", and for $\eta \leq \rho$ a limit ordinal, $\Vdash_{\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{S}}_\kappa}$ "$\dot{p}_\eta$ is a term for an Easton extension of each member of the sequence $\langle \dot{p}_\beta \mid \beta < \eta \rangle$". Define now in $V[G * H]$ a set $\mathcal{U} \subseteq 2^{[\lambda]^{<\kappa}}$ by $X \in \mathcal{U}$ iff $X \subseteq P_\kappa(\lambda)$ and for some $\langle r, q \rangle \in G * H$ and some $q' \in \mathbb{S}_\kappa$ either the trivial condition (if $\mathbb{S}_\kappa$ is trivial forcing) or of the form $\langle \langle \emptyset, \dot{B}_\delta \rangle \mid \delta \in [\kappa, \max(\kappa, |\mathbb{Q}|)]$ is measurable$\rangle$ (if $\mathbb{S}_\kappa$ is a Magidor iteration of Prikry forcing), in M, $\langle r, \dot{q}, \dot{q}', \dot{p}_\rho \rangle \Vdash$ "$\langle j(\beta) \mid \beta < \lambda \rangle \in \dot{X}$" for some name $\dot{X}$ of X. As in [6, Lemma 2], $\mathcal{U}$ is a κ-additive, fine ultrafilter over $(P_\kappa(\lambda))^{V[G*H]}$, i.e., $V[G * H] \models$ "κ is λ strongly compact". Since λ was arbitrary, this completes the proof of Lemma 2.2.

$\square$

Lemma 2.3 $V^{\mathbb{P}} \models$ "*Level by level equivalence between strong compactness and supercompactness holds*".

Proof: Because $V \models$ "Level by level equivalence between strong compactness and supercompactness holds" and $|\mathbb{P}| = \kappa$, by the Lévy-Solovay results [15], $V^{\mathbb{P}} \models$ "Level by level equivalence between strong compactness and supercompactness holds above κ". By Lemma 2.1, $V^{\mathbb{P}} \models$ "κ is supercompact", which means that $V^{\mathbb{P}} \models$ "Level by level equivalence between strong compactness and supercompactness holds at κ". Thus, to complete the proof of Lemma 2.3, it suffices to show that $V^{\mathbb{P}} \models$ "Level by level equivalence between strong compactness and supercompactness holds below κ".

To do this, let $\delta < \kappa$ and $\lambda > \delta$ be regular such that $V^{\mathbb{P}} \models$ "δ is λ strongly compact". Consider now the following two cases.

Case 1: δ is a stage in the definition of $\mathbb{P}$ at which only trivial forcing can take place, i.e., δ is not in V a measurable limit of strong cardinals. Let $\gamma = \sup(\{\sigma < \delta \mid \sigma$ is a stage of forcing in the definition of $\mathbb{P}$ at which nontrivial forcing might occur (so σ is in V a measurable limit of strong cardinals)$\})$. Write $\mathbb{P} = \mathbb{P}_{\gamma+1} * \dot{\mathbb{Q}}$. By the fact that only trivial forcing occurs at stage δ in the definition of $\mathbb{P}$, $\gamma < \delta$. If γ is non-measurable, then γ must be either singular or inaccessible. Therefore, by the definition of $\mathbb{P}$, $|\mathbb{P}_\gamma| < \delta$ and only trivial forcing is possible at stage γ. This means that $\mathbb{P}_{\gamma+1}$ is forcing equivalent to $\mathbb{P}_\gamma$ and $|\mathbb{P}_{\gamma+1}| < \delta$. However, if γ is measurable, i.e., if γ is a stage at which nontrivial forcing could occur, then note that inductively, it is the case that $|\mathbb{P}_\gamma| \le \gamma$. In particular, $|\mathbb{P}_\gamma| < \delta$. It consequently follows that $\mathbb{P}_{\gamma+1}$ is forcing equivalent to a partial ordering having cardinality less than δ. This is since otherwise, nontrivial forcing must be selected in the stage γ lottery held in the definition of $\mathbb{P}$ (because if not, i.e., if trivial forcing is selected in the stage γ lottery held in the definition of $\mathbb{P}$, then $\mathbb{P}_{\gamma+1}$ is forcing equivalent to $\mathbb{P}_\gamma$, a partial ordering having cardinality less than δ). Under these circumstances, we must have that $\Vdash_{\mathbb{P}_\gamma}$ "$\dot{\mathbb{Q}}_\gamma$ is forcing equivalent to a partial ordering of the form $\dot{\mathbb{R}}_\gamma * \dot{\mathbb{Q}}^1_\gamma$ where $\dot{\mathbb{R}}_\gamma$ is nontrivial" and $\Vdash_{\mathbb{P}_\gamma * \dot{\mathbb{R}}_\gamma}$ "$\dot{\mathbb{Q}}^1_\gamma$ is the Magidor iteration of Prikry forcing which adds a Prikry sequence to every measurable cardinal in the closed interval $[\gamma, \max(\gamma, \dot{\mathbb{R}}_\gamma)]$". Since by hypothesis, $\Vdash_{\mathbb{P}_\gamma}$ "$|\dot{\mathbb{R}}_\gamma * \dot{\mathbb{Q}}^1_\gamma| \ge \delta$", we must have that $\Vdash_{\mathbb{P}_\gamma}$ "$|\dot{\mathbb{R}}_\gamma| \ge \delta$" (because if $p \Vdash_{\mathbb{P}_\gamma}$ "$|\dot{\mathbb{R}}_\gamma| < \delta$", then by the definition of the Magidor iteration of Prikry forcing, $p \Vdash_{\mathbb{P}_\gamma}$ "$|\dot{\mathbb{R}}_\gamma * \dot{\mathbb{Q}}^1_\gamma| < \delta$" as well). It then follows that $V^{\mathbb{P}_\gamma * \dot{\mathbb{R}}_\gamma * \dot{\mathbb{Q}}^1_\gamma} = V^{\mathbb{P}_\gamma * \dot{\mathbb{Q}}_\gamma} \vDash$ "δ is non-measurable (because it either contains a Prikry sequence or is non-measurable in $V^{\mathbb{P}_\gamma * \dot{\mathbb{R}}_\gamma}$ and hence, by the work of [16], remains non-measurable in $V^{\mathbb{P}_\gamma * \dot{\mathbb{R}}_\gamma * \dot{\mathbb{Q}}^1_\gamma}$)". Since $\Vdash_{\mathbb{P}_{\gamma+1}}$ "Forcing with $\dot{\mathbb{Q}}$ adds no new subsets of 2^δ", $V^{\mathbb{P}_{\gamma+1} * \dot{\mathbb{Q}}} = V^{\mathbb{P}} \vDash$ "δ is non-measurable", a contradiction. Note that the argument given in the preceding two sentences actually shows that if nontrivial forcing is selected at stage γ in the definition of $\mathbb{P}$, then $V^{\mathbb{P}} \vDash$ "γ is non-measurable". This is since any nontrivial γ-directed closed forcing selected at stage γ must have cardinality at least γ, which has as a consequence that each occurrence of δ can be replaced by an occurrence of γ to obtain the same contradiction.

We now know that $\mathbb{P}_{\gamma+1}$ is forcing equivalent to a partial ordering having cardinality less than δ. We may also infer that $\lambda < \delta'$. This is since otherwise, if $\lambda \ge \delta'$, then $V^{\mathbb{P}} \vDash$ "δ is δ' strongly compact". Because $\Vdash_{\mathbb{P}_{\gamma+1}}$ "Forcing with $\dot{\mathbb{Q}}$ adds no new subsets of $2^{[\delta']^{<\delta}}$", it must be the case that $\Vdash_{\mathbb{P}_{\gamma+1}}$ "δ is δ' strongly compact". However, by the results of [15], $V \vDash$ "δ is δ' strongly compact". As $V \vDash$ "δ' is strong", again by the proof of [5, Lemma 2.4], $V \vDash$ "$\delta < \kappa$ is strongly compact". This contradicts the fact that $V \vDash$ "κ is the least supercompact cardinal and level by level equivalence between strong compactness and supercompactness holds (so in particular, there are no strongly compact cardinals less than κ)".

Note that the argument just given showing that if $V^{\mathbb{P}} \vDash$ "δ is δ' strongly compact", then $V \vDash$ "δ is δ' strongly compact" remains valid for any $\lambda < \delta'$. Therefore, we may infer that $V \vDash$ "δ is λ strongly compact". Hence, because level by level equivalence between strong compactness and supercompactness holds in V, $V \vDash$ "Either δ is λ supercompact, or δ is a measurable limit of cardinals which are λ supercompact". The preceding analysis tells us both that $\mathbb{P}_{\gamma+1}$ is forcing equivalent to a partial ordering having cardinality less than δ and $\Vdash_{\mathbb{P}_{\gamma+1}}$ "Forcing with $\dot{\mathbb{Q}}$ adds no new subsets of $2^{[\lambda]^{<\delta}}$". This, together with the results of [15], then allow us to infer that in each of $V^{\mathbb{P}_{\gamma+1}}$ and $V^{\mathbb{P}_{\gamma+1} * \dot{\mathbb{Q}}} = V^{\mathbb{P}}$, either δ is λ supercompact, or δ is a measurable limit of cardinals which are λ supercompact. Thus, in $V^{\mathbb{P}}$, δ cannot witness a failure of level by level equivalence between strong compactness and supercompactness.

Case 2: δ is a stage in the definition of $\mathbb{P}$ at which nontrivial forcing can take place, i.e., δ is in V a measurable limit of strong cardinals. As we observed in the proof given in Case 1 above, if nontrivial forcing is selected at stage δ, then $V^{\mathbb{P}} \vDash$ "δ is non-measurable". Since by hypothesis, $V^{\mathbb{P}} \vDash$ "δ is λ strongly compact", it must therefore be true that trivial forcing is selected in the stage δ lottery held in the definition of $\mathbb{P}$.

We show that as in Case 1, $\lambda < \delta'$. To see this, we note that if $\lambda \geq \delta'$, then the same reasoning as earlier tells us that $\Vdash_{\mathbb{P}_{\delta+1}}$ "δ is δ' strongly compact". However, because trivial forcing is selected at stage δ in the definition of $\mathbb{P}$, meaning that $\mathbb{P}_{\delta+1}$ and $\mathbb{P}_\delta$ are forcing equivalent, it is actually the case that $\Vdash_{\mathbb{P}_\delta}$ "δ is δ' strongly compact". Arguing now as in the proof of [3, Lemma 2.1] (and quoting verbatim when appropriate), note that because δ is measurable in V, $\mathbb{P}_\delta$ is the direct limit of $\langle \mathbb{P}_\alpha \mid \alpha < \delta \rangle$. Further, $\mathbb{P}_\delta$ satisfies δ-c.c. in $V^{\mathbb{P}_\delta}$, since δ is measurable and hence Mahlo in $V^{\mathbb{P}_\delta}$ and $\mathbb{P}_\delta$ is a subordering of the direct limit of $\langle \mathbb{P}_\alpha \mid \alpha < \delta \rangle$ as calculated in $V^{\mathbb{P}_\delta}$. Hence, by the proofs of [4, Lemma 3] or [1, Lemma 8], every δ-additive uniform ultrafilter over a regular cardinal $\gamma \geq \delta$ present in $V^{\mathbb{P}_\delta}$ must be an extension of a δ-additive uniform ultrafilter over γ in V. Therefore, since the δ' strong compactness of δ in $V^{\mathbb{P}_\delta}$ implies that every $V^{\mathbb{P}_\delta}$-regular cardinal $\gamma \in [\delta, \delta']$ carries a δ-additive uniform ultrafilter in $V^{\mathbb{P}_\delta}$, and since the fact $\mathbb{P}_\delta$ is the direct limit of $\langle \mathbb{P}_\alpha \mid \alpha < \delta \rangle$ tells us the regular cardinals at or above δ in $V^{\mathbb{P}_\delta}$ are the same as those in V, the preceding sentence implies that every V-regular cardinal $\gamma \in [\delta, \delta']$ carries a δ-additive uniform ultrafilter in V. Ketonen's theorem of [13] then implies that δ is δ' strongly compact in V, which gives the same contradiction as in Case 1.

We now know that $\lambda < \delta'$. Further, by the argument just given, $V \vDash$ "δ is λ strongly compact". Therefore, by the fact that level by level equivalence between strong compactness and supercompactness holds in V, $V \vDash$ "Either δ is λ supercompact, or δ is a measurable limit of cardinals which are λ supercompact". However, the latter cannot occur, since if it did, some cardinal $\gamma < \delta < \kappa$ would have to be γ' supercompact. Once more, the proof of [5, Lemma 2.4] yields that $V \vDash$ "γ is supercompact", contradicting the fact that κ is the least supercompact cardinal in V. This means that $V \vDash$ "δ is λ supercompact", so we may apply the argument found in the proof of Lemma 2.1 to show that in $V^{\mathbb{P}_\delta} = V^{\mathbb{P}_{\delta+1}}$, δ is λ supercompact. Once again, write $\mathbb{P} = \mathbb{P}_{\delta+1} * \dot{\mathbb{Q}}$. Because $\Vdash_{\mathbb{P}_{\delta+1}}$ "Forcing with $\dot{\mathbb{Q}}$ adds no new subsets of $2^{[\lambda]^{<\delta}}$", $V^{\mathbb{P}_{\delta+1} * \dot{\mathbb{Q}}} = V^{\mathbb{P}} \vDash$ "δ is λ supercompact". Consequently, level by level equivalence between strong compactness and supercompactness holds at δ in $V^{\mathbb{P}}$. This completes the proof of Lemma 2.3.

□

Lemma 2.4 $V^{\mathbb{P}} \vDash$ "*$\mathcal{K}$ is the class of supercompact cardinals*".

Proof: By Lemma 2.1, $V^{\mathbb{P}} \vDash$ "κ is supercompact". If $\delta < \kappa$ is such that $V^{\mathbb{P}} \vDash$ "δ is supercompact", then in particular, $V^{\mathbb{P}} \vDash$ "δ is δ' supercompact". Since $V^{\mathbb{P}} \vDash$ "δ is δ' strongly compact" as well, by the proof of Lemma 2.3, $V \vDash$ "δ is δ' strongly compact". This gives the same contradiction as in Lemma 2.3, so $V^{\mathbb{P}} \vDash$ "κ is the least supercompact cardinal". Because by its definition, $|\mathbb{P}| = \kappa$, by the results of [15], $V^{\mathbb{P}} \vDash$ "$\mathcal{K} - \{\kappa\}$ is the class of supercompact cardinals above κ". Thus, $V^{\mathbb{P}} \vDash$ "$\mathcal{K}$ is the class of supercompact cardinals". This completes the proof of Lemma 2.4.

□

Lemmas 2.1 – 2.4 complete the proof of Theorem 2.

□

As mentioned in Section 1, [8, Theorem 5] tells us that if there are sufficiently large cardinals present in the universe, then level by level equivalence between strong compactness and supercompactness is incompatible with an indestructibly supercompact cardinal. Thus, if the universe $V^{\mathbb{P}}$ witnessing the conclusions of Theorem 2 has a rich enough large cardinal structure, we automatically know that κ's supercompactness is not indestructible by some κ-directed closed forcing $\mathbb{R}$. By the proof of [8, Theorem 5], $\mathbb{R}$'s rank is fairly large. In fact, [7, Lemma 2.4] tells us that no matter the nature of the large cardinals present, after forcing with $\mathrm{Add}(\kappa, 1)$, not only is κ not supercompact, but it has trivial Mitchell rank. Thus, we conclude by reiterating a question first asked in [8] and still open, namely whether it is possible to have a universe where there is an indestructibly supercompact cardinal but containing relatively few large cardinals in which level by level equivalence between strong compactness and supercompactness holds.

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