Mixed Levels of Indestructibility *†

Arthur W. Apter‡
Department of Mathematics
Baruch College of CUNY
New York, New York 10010 USA
and
The CUNY Graduate Center, Mathematics
365 Fifth Avenue
New York, New York 10016 USA
http://faculty.baruch.cuny.edu/aapter
awapter@alum.mit.edu

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Abstract

Starting from a supercompact cardinal κ, we force and construct a model in which κ is both the least strongly compact and least supercompact cardinal and κ exhibits mixed levels of indestructibility. Specifically, κ's strong compactness, but not its supercompactness, is indestructible under any κ-directed closed forcing which also adds a Cohen subset of κ. On the other hand, in this model, κ's supercompactness is indestructible under any κ-directed closed forcing which doesn't add a Cohen subset of κ.

1 Introduction and Preliminaries

In [3], the following theorem was proven.

Theorem 1 *Let $V \models$ "ZFC + κ is supercompact". There is then a partial ordering $\mathbb{P} \subseteq V$ such that $V^{\mathbb{P}} \models$ "κ is both supercompact and the least strongly compact cardinal". For any $\mathbb{Q} \in V^{\mathbb{P}}$ which is κ-directed closed, $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ "κ is strongly compact". Further, there is $\mathbb{R} \in V^{\mathbb{P}}$ which is κ-directed*

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closed and nontrivial such that $V^{\mathbb{P}*\dot{\mathbb{R}}} \vDash$ *"κ is not supercompact". Moreover, for this* $\mathbb{R}$, $V^{\mathbb{P}*\dot{\mathbb{R}}} \vDash$ *"κ has trivial Mitchell rank".*

The partial ordering $\mathbb{R}$ of Theorem 1 turns out to be $(\mathrm{Add}(\kappa, 1))^{V^{\mathbb{P}}}$ (where for any regular cardinal δ, $\mathrm{Add}(\delta, 1)$ is the standard partial ordering for adding a single Cohen subset of δ). We use this to motivate the terminology that for a model $\overline{V}$ of ZFC, partial ordering $\mathbb{Q} \in \overline{V}$, and regular cardinal δ of $\overline{V}$, $\mathbb{Q}$ *adds a Cohen subset of* δ means that in $\overline{V}^{\mathbb{Q}}$, there is a subset of δ which is $\overline{V}$-generic for $((\mathrm{Add}(\delta, 1))^{\overline{V}}$.

Theorem 1 may be thought of as being complementary to Laver's celebrated result of [11], where it is shown that any supercompact cardinal κ can have its supercompactness forced to be indestructible under arbitrary κ-directed closed forcing. Theorem 1 and the work of [11], however, together raise the following

Question: Is it possible to force a supercompact cardinal κ to have its strong compactness, but not its supercompactness, indestructible under κ-directed closed partial orderings in a certain class $\mathcal{C}$, and also have its supercompactness indestructible under κ-directed closed partial orderings lying in the complement of $\mathcal{C}$?

The purpose of this paper is to answer the above question in the affirmative. Specifically, we will prove the following theorem.

Theorem 2 *Let* $V \vDash$ *"ZFC + κ is supercompact". There is then a partial ordering* $\mathbb{P} \subseteq V$ *such that* $V^{\mathbb{P}} \vDash$ *"κ is both supercompact and the least strongly compact cardinal". For any* $\mathbb{Q} \in V^{\mathbb{P}}$ *which is κ-directed closed and adds a Cohen subset of κ,* $V^{\mathbb{P}*\dot{\mathbb{Q}}} \vDash$ *"κ is strongly compact but not supercompact". In fact,* $V^{\mathbb{P}*\dot{\mathbb{Q}}} \vDash$ *"κ has trivial Mitchell rank". On the other hand, for any* $\mathbb{Q} \in V^{\mathbb{P}}$ *which is κ-directed closed and doesn't add a Cohen subset of κ,* $V^{\mathbb{P}*\dot{\mathbb{Q}}} \vDash$ *"κ is supercompact".*

Forcing to obtain a model in which the least strongly compact cardinal is the same as the least supercompact cardinal was of course first done by Magidor in [12].

Before beginning the proof of our theorem, we briefly mention some preliminary information and terminology. Essentially, our notation and terminology are standard, and when this is not the case, this will be clearly noted. When forcing, $q \geq p$ will mean that q *is stronger than* p. If G is V-generic over $\mathbb{P}$, we will abuse notation slightly and use both $V[G]$ and $V^{\mathbb{P}}$ to indicate the universe obtained by forcing with $\mathbb{P}$. If $x \in V[G]$, then $\dot{x}$ will be a term in V for x. We may, from time to time, confuse terms with the sets they denote and write x when we actually mean $\dot{x}$ or $\check{x}$, especially when x is some variant of the generic set G, or x is in the ground model V. The abuse of notation mentioned above will be compounded by writing $x \in V^{\mathbb{P}}$ instead of $\dot{x} \in V^{\mathbb{P}}$. Any term for trivial forcing will always be taken as a term for the partial ordering $\{\emptyset\}$. If φ is a formula in the forcing language with respect to $\mathbb{P}$ and $p \in \mathbb{P}$, then $p \parallel \varphi$ means that p *decides* φ.

If $\mathbb{P}$ is an arbitrary partial ordering and κ is a regular cardinal, $\mathbb{P}$ is κ*-directed closed* if for every cardinal $\delta < \kappa$ and every directed set $\langle p_\alpha \mid \alpha < \delta \rangle$ of elements of $\mathbb{P}$ (where $\langle p_\alpha \mid \alpha < \delta \rangle$ is *directed* if every two elements p_ρ and p_ν have a common upper bound of the form p_σ) there is an upper bound $p \in \mathbb{P}$. $\mathbb{P}$ is κ*-strategically closed* if in the two person game of length $\kappa + 1$ in which the players construct an increasing sequence $\langle p_\alpha \mid \alpha \leq \kappa \rangle$, where player I plays odd stages and player II plays even stages (choosing the trivial condition at stage 0), player II has a strategy which ensures the game can always be continued. $\mathbb{P}$ is $<\kappa$*-strategically closed* if $\mathbb{P}$ is δ-strategically closed for every $\delta < \kappa$. Note that if $\mathbb{P}$ is κ-directed closed, then $\mathbb{P}$ is $<\kappa$-strategically closed. We adopt Hamkins' terminology of [8, 7, 6] and say that $x \subseteq \kappa$ *is a fresh subset of* κ *with respect to* $\mathbb{P}$ if $\mathbb{P}$ is nontrivial forcing, $x \in V^{\mathbb{P}}$, $x \notin V$, yet $x \cap \alpha \in V$ for every $\alpha < \kappa$.

From time to time within the course of our discussion, we will refer to partial orderings $\mathbb{P}$ as being *Gitik iterations*. By this we will mean an Easton support iteration as first given by Gitik in [5], to which we refer readers for a discussion of the basic properties of and terminology associated with such an iteration. For the purposes of this paper, at any stage δ at which a nontrivial forcing is done in a Gitik iteration, we assume the partial ordering $\mathbb{Q}_\delta$ with which we force has the form $\mathbb{R}_\delta * \dot{\mathbb{R}}'_\delta$, where $\mathbb{R}_\delta$ is δ-directed closed and $\dot{\mathbb{R}}_\delta$ is a term for either trivial forcing or Prikry forcing defined with respect to a normal measure over δ (although other types of partial orderings may be used in the general case — see [5] for additional details).

We recall for the benefit of readers the definition given by Hamkins in [9, Section 3] of the lottery sum of a collection of partial orderings. If $\mathfrak{A}$ is a collection of partial orderings, then the *lottery sum* is the partial ordering $\oplus\mathfrak{A} = \{\langle \mathbb{P}, p\rangle \mid \mathbb{P} \in \mathfrak{A} \text{ and } p \in \mathbb{P}\} \cup \{0\}$, ordered with 0 below everything and $\langle \mathbb{P}, p\rangle \le \langle \mathbb{P}', p'\rangle$ iff $\mathbb{P} = \mathbb{P}'$ and $p \le p'$. Intuitively, if G is V-generic over $\oplus\mathfrak{A}$, then G first selects an element of $\mathfrak{A}$ (or as Hamkins says in [9], "holds a lottery among the posets in $\mathfrak{A}$") and then forces with it.[1]

Key to the proof of Theorem 2 (specifically the fact that κ's supercompactness is not indestructible when forcing with any κ-directed closed partial ordering adding a Cohen subset of κ) is the following result due to Gitik [3, Proposition 1.1].

Proposition 1.1 *Suppose κ is a Mahlo cardinal and $\mathbb{P} = \langle\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha\rangle \mid \alpha \le \kappa\rangle$ is an Easton support iteration of length $\kappa + 1$ satisfying the following properties.*

1. $\mathbb{P}_0 = \{\emptyset\}$.

2. *For each $\alpha < \kappa$, $\Vdash_{\mathbb{P}_\alpha}$ "$|\dot{\mathbb{Q}}_\alpha| < \kappa$".*

3. *$\Vdash_{\mathbb{P}_\kappa}$ "$\dot{\mathbb{Q}}_\kappa$ is $<\kappa$-strategically closed".*

4. *For some $\alpha, \delta < \kappa$, $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ adds a new subset of δ".*

5. *κ is Mahlo in $V^{\mathbb{P}_{\kappa+1}} = V^{\mathbb{P}}$.*

Then in $V^{\mathbb{P}}$, there are no fresh subsets of κ.

We note that Proposition 1.1 is an analogue of results due to Hamkins (see [8, 7, 6]). Adopting the terminology of these papers, Hamkins shows that for a suitably large cardinal κ (measurable, supercompact, etc.) and an iteration $\mathbb{P}$ *admitting a gap below* κ (i.e., for some $\delta < \kappa$, $\mathbb{P}$ can be written as $\mathbb{Q} * \dot{\mathbb{R}}$, where $|\mathbb{Q}| < \delta$, $\mathbb{Q}$ is nontrivial, and $\Vdash_{\mathbb{Q}}$ "$\dot{\mathbb{R}}$ is δ-strategically closed"), after forcing

[1] The terminology "lottery sum" is due to Hamkins, although the concept of the lottery sum of partial orderings has been around for quite some time and has been referred to at different junctures via the names "disjoint sum of partial orderings," "side-by-side forcing," and "choosing which partial ordering to force with generically."

with $\mathbb{P}$, there are no fresh subsets of κ. The iterations we consider will not be gap forcings, yet they retain this crucial property vital to the proof of Theorem 2.

Finally, we mention that we are assuming familiarity with the large cardinal notions of measurability, strongness, strong compactness, and supercompactness. Interested readers may consult [10] or [13] for further details. We do note, however, that κ is said to be *supercompact up to the cardinal* λ if κ is δ supercompact for every $\delta < \lambda$. The measurable cardinal κ is said to have *trivial Mitchell rank* if there is no elementary embedding $j : V \to M$ generated by a normal measure $\mathcal{U}$ over κ such that $M \models$ "κ is a measurable cardinal". We explicitly observe that if κ has trivial Mitchell rank, then κ is not supercompact (and in fact, if κ has trivial Mitchell rank, then κ is not even 2^κ supercompact).

2 The Proof of Theorem 2

We turn now to the proof of Theorem 2.

Proof: Let $V \models$ "ZFC + κ is supercompact". Without loss of generality, we assume that $V \models$ GCH as well. For any ordinal δ, let δ' be the least V-strong cardinal above δ.

The partial ordering $\mathbb{P} = \langle \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle \mid \alpha < \kappa \rangle$ to be used in the proof of Theorem 2 is a modification of the one used in the proof of [3, Theorem 1]. Specifically, $\mathbb{P}$ is the Gitik iteration of length κ which has the following properties.

1. $\mathbb{P}$ begins by forcing with $\mathrm{Add}(\omega, 1)$, i.e., $\mathbb{P}_0 = \{\emptyset\}$ and $\Vdash_{\mathbb{P}_0}$ "$\dot{\mathbb{Q}}_0 = \dot{\mathrm{Add}}(\omega, 1)$".

2. The only stages at which $\mathbb{P}$ (possibly) does nontrivial forcing are those ordinals δ which are, in V, Mahlo limits of strong cardinals. At such a stage δ, $\mathbb{P}_{\delta+1} = \mathbb{P}_\delta * \dot{\mathbb{L}}_\delta * \dot{\mathbb{S}}_\delta$, where $\dot{\mathbb{L}}_\delta$ is a term for the lottery sum of all δ-directed closed partial orderings having rank below δ'.

3. If either $V^{\mathbb{P}_\delta * \dot{\mathbb{L}}_\delta} \models$ "There is no subset of δ which is $V^{\mathbb{P}_\delta}$-generic for $(\mathrm{Add}(\delta, 1))^{V^{\mathbb{P}_\delta}}$", or $V^{\mathbb{P}_\delta * \dot{\mathbb{L}}_\delta} \models$ "δ is not measurable", then $\dot{\mathbb{S}}_\delta$ is a term for trivial forcing.

4. If $V^{\mathbb{P}_\delta * \dot{\mathbb{L}}_\delta} \models$ "There is a subset of δ which is $V^{\mathbb{P}_\delta}$-generic for $(\mathrm{Add}(\delta, 1))^{V^{\mathbb{P}_\delta}}$" and $V^{\mathbb{P}_\delta * \dot{\mathbb{L}}_\delta} \models$ "δ is measurable", then $\dot{\mathbb{S}}_\delta$ is a term for Prikry forcing defined with respect to some normal

measure over δ.

The intuition behind the above definition of $\mathbb{P}$ is as follows. $\mathbb{P}$ begins by forcing with $\mathrm{Add}(\omega, 1)$ to ensure that Proposition 1.1 is applicable. The fact that no Prikry forcing is done when the forcing at stage δ doesn't add a Cohen subset of δ ensures that in $V^{\mathbb{P}}$, κ's supercompactness is indestructible under any κ-directed closed partial ordering not adding a Cohen subset of κ. Since Prikry forcing is performed when a nontrivial forcing at stage δ both adds a Cohen subset of δ and preserves the measurability of δ, we ensure that κ's strong compactness, but not its supercompactness, is indestructible in $V^{\mathbb{P}}$ under any κ-directed closed partial ordering adding a Cohen subset of κ. Because unboundedly many in κ Prikry sequences will have been added by $\mathbb{P}$, $V^{\mathbb{P}} \vDash$ "No cardinal below κ is strongly compact", i.e., $V^{\mathbb{P}} \vDash$ "κ is the least strongly compact cardinal".

The following lemmas show that $\mathbb{P}$ is as desired.

Lemma 2.1 *Suppose $\mathbb{Q} \in V^{\mathbb{P}}$ is a partial ordering which is κ-directed closed and adds a Cohen subset of κ. Then $V^{\mathbb{P} * \dot{\mathbb{Q}}} \vDash$ "κ is strongly compact".*

Proof: We follow the proof of [3, Lemma 2.2], quoting verbatim when appropriate. Suppose $\mathbb{Q} \in V^{\mathbb{P}}$ is κ-directed closed and adds a Cohen subset of κ. Let $\lambda > \max(2^\kappa, |\mathrm{TC}(\dot{\mathbb{Q}})|)$ be an arbitrary regular cardinal large enough so that $(2^{[\lambda]^{<\kappa}})^V = \rho = (2^{[\lambda]^{<\kappa}})^{V^{\mathbb{P} * \dot{\mathbb{Q}}}}$ and ρ is regular in both V and $V^{\mathbb{P} * \dot{\mathbb{Q}}}$, and let $\sigma = \rho^+ = 2^\rho$. Take $j : V \to M$ as an elementary embedding witnessing the σ supercompactness of κ such that $M \vDash$ "κ is not σ supercompact". By [1, Lemma 2.1], κ is a Mahlo limit of strong cardinals in M. Consequently, by the choice of σ, it is possible to opt for $\mathbb{Q}$ in the stage κ lottery held in M in the definition of $j(\mathbb{P})$. Further, $M \vDash$ "No cardinal δ in the half-open interval $(\kappa, \sigma]$ is strong". This is since otherwise, in M, κ is supercompact up to a strong cardinal, so by the proof of [1, Lemma 2.4], κ is supercompact in M. Therefore, the next nontrivial forcing in the definition of $j(\mathbb{P})$ takes place well above σ. Thus, in M, above the appropriate condition, because forcing with $\mathbb{Q}$ adds a Cohen subset of κ, $j(\mathbb{P} * \dot{\mathbb{Q}})$ is forcing equivalent to $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{S}}_\kappa * \dot{\mathbb{R}} * j(\dot{\mathbb{Q}})$, where $\Vdash_{\mathbb{P} * \dot{\mathbb{Q}}}$ "$\dot{\mathbb{S}}_\kappa$ is a term for Prikry forcing".

The remainder of the proof of Lemma 2.1 is as in the proof of [2, Lemma 2]. We outline the argument, and refer readers to the proof of [2, Lemma 2] for any missing details. By the last two sentences of the preceding paragraph, as in [2, Lemma 2], there is a term $\tau \in M$ in the language of forcing with respect to $j(\mathbb{P})$ such that if $G * H$ is either V-generic or M-generic over $\mathbb{P} * \dot{\mathbb{Q}}$, $\Vdash_{j(\mathbb{P})}$ "τ extends every $j(\dot{q})$ for $\dot{q} \in \dot{H}$". In other words, τ is a term for a "master condition" for $\dot{\mathbb{Q}}$. Thus, if $\langle \dot{A}_\alpha \mid \alpha < \rho < \sigma \rangle$ enumerates in V the canonical $\mathbb{P} * \dot{\mathbb{Q}}$ names of subsets of $(P_\kappa(\lambda))^{V[G*H]}$, we can define in M a sequence of $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{S}}_\kappa$ names of elements of $\dot{\mathbb{R}} * j(\dot{\mathbb{Q}})$, $\langle \dot{p}_\alpha \mid \alpha \leq \rho \rangle$, such that $\dot{p}_0$ is a term for $\langle 0, \tau \rangle$ (where 0 represents the trivial condition with respect to $\mathbb{R}$), $\Vdash_{\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{S}}_\kappa}$ "$\dot{p}_{\alpha+1}$ is a term for an Easton extension of $\dot{p}_\alpha$[2] deciding '$\langle j(\beta) \mid \beta < \lambda \rangle \in j(\dot{A}_\alpha)$' ", and for $\eta \leq \rho$ a limit ordinal, $\Vdash_{\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{S}}_\kappa}$ "$\dot{p}_\eta$ is a term for an Easton extension of each member of the sequence $\langle \dot{p}_\beta \mid \beta < \eta \rangle$". In $V[G * H]$, define a set $\mathcal{U} \subseteq 2^{[\lambda]^{<\kappa}}$ by $X \in \mathcal{U}$ iff $X \subseteq P_\kappa(\lambda)$ and for some $\langle r, q \rangle \in G * H$ and $q' \in \mathbb{S}_\kappa$ of the form $\langle \emptyset, B \rangle$, in M, $\langle r, \dot{q}, \dot{q}', \dot{p}_\rho \rangle \Vdash_{j(\mathbb{P} * \dot{\mathbb{Q}})}$ "$\langle j(\beta) \mid \beta < \lambda \rangle \in \dot{X}$" for a name $\dot{X}$ of X. As in [2, Lemma 2], $\mathcal{U}$ is a κ-additive, fine ultrafilter over $(P_\kappa(\lambda))^{V[G*H]}$, i.e., $V[G * H] \models$ "κ is λ strongly compact". Since λ was arbitrary, this completes the proof of Lemma 2.1.

$\square$

Lemma 2.2 *Suppose $\mathbb{Q} \in V^{\mathbb{P}}$ is a partial ordering which is κ-directed closed and doesn't add a Cohen subset of κ. Then $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ "κ is supercompact".*

Proof: Let $\mathbb{Q} \in V^{\mathbb{P}}$ be such that $\mathbb{Q}$ is κ-directed closed and in $V^{\mathbb{P} * \dot{\mathbb{Q}}}$, there is no subset of κ which is $V^{\mathbb{P}}$-generic for $(\mathrm{Add}(\kappa, 1))^{V^{\mathbb{P}}}$. As in Lemma 2.1, suppose $\lambda > \max(2^\kappa, |\mathrm{TC}(\dot{\mathbb{Q}})|)$ is an arbitrary regular cardinal large enough so that $(2^{[\lambda]^{<\kappa}})^V = \rho = (2^{[\lambda]^{<\kappa}})^{V^{\mathbb{P} * \dot{\mathbb{Q}}}}$ and ρ is regular in both V and $V^{\mathbb{P} * \dot{\mathbb{Q}}}$, and let $\sigma = \rho^+ = 2^\rho$. Take $j : V \to M$ as an elementary embedding witnessing the σ supercompactness of κ such that $M \models$ "κ is not σ supercompact". Again as in Lemma 2.1, by [1, Lemma 2.1], κ is a Mahlo limit of strong cardinals in M. Consequently, by the choice of σ, it is possible to opt for $\mathbb{Q}$ in the stage κ lottery held in M in the definition of $j(\mathbb{P})$. Further, once more

[2]Roughly speaking, p_β *is an Easton extension of* p_α means that p_β extends p_α as in a usual Easton support iteration, except that no stems of any components of p_α which are conditions in Prikry forcing are extended. For a more precise definition, readers are urged to consult either [5] or [2].

as in Lemma 2.1, since $M \vDash$ "No cardinal $\delta \in (\kappa, \sigma]$ is strong", the next nontrivial forcing in the definition of $j(\mathbb{P})$ takes place well above σ. Thus, in M, above the appropriate condition, because forcing with $\mathbb{Q}$ doesn't add a Cohen subset of κ, $j(\mathbb{P} * \dot{\mathbb{Q}})$ is forcing equivalent to $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{S}}_\kappa * \dot{\mathbb{R}} * j(\dot{\mathbb{Q}})$, where $\Vdash_{\mathbb{P} * \dot{\mathbb{Q}}}$ "$\dot{\mathbb{S}}_\kappa$ is a term for trivial forcing". With a slight abuse of notation, we will henceforth say that in M, above the appropriate condition, $j(\mathbb{P} * \dot{\mathbb{Q}})$ is forcing equivalent to $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}} * j(\dot{\mathbb{Q}})$.

As in the proof of Lemma 2.1, there is a term $\tau \in M$ in the language of forcing with respect to $j(\mathbb{P})$ such that if $G * H$ is either V-generic or M-generic over $\mathbb{P} * \dot{\mathbb{Q}}$, $\Vdash_{j(\mathbb{P})}$ "τ extends every $j(\dot{q})$ for $\dot{q} \in \dot{H}$". In other words, τ is once again a term for a "master condition" for $\dot{\mathbb{Q}}$. Thus, if as before, $\langle \dot{A}_\alpha \mid \alpha < \rho < \sigma \rangle$ enumerates in V the canonical $\mathbb{P} * \dot{\mathbb{Q}}$ names of subsets of $(P_\kappa(\lambda))^{V[G*H]}$, we can define in M a sequence of $\mathbb{P} * \dot{\mathbb{Q}}$ names of elements of $\dot{\mathbb{R}} * j(\dot{\mathbb{Q}})$, $\langle \dot{p}_\alpha \mid \alpha \le \rho \rangle$, such that $\dot{p}_0$ is a term for $\langle 0, \tau \rangle$ (where 0 once more represents the trivial condition with respect to $\mathbb{R}$), $\Vdash_{\mathbb{P} * \dot{\mathbb{Q}}}$ "$\dot{p}_{\alpha+1}$ is a term for an Easton extension of $\dot{p}_\alpha$ deciding '$\langle j(\beta) \mid \beta < \lambda \rangle \in j(\dot{A}_\alpha)$' ", and for $\eta \le \rho$ a limit ordinal, $\Vdash_{\mathbb{P} * \dot{\mathbb{Q}}}$ "$\dot{p}_\eta$ is a term for an Easton extension of each member of the sequence $\langle \dot{p}_\beta \mid \beta < \eta \rangle$". In $V[G * H]$, define a set $\mathcal{U} \subseteq 2^{[\lambda]^{<\kappa}}$ by $X \in \mathcal{U}$ iff $X \subseteq P_\kappa(\lambda)$ and for some $\langle r, q \rangle \in G * H$, in M, $\langle r, \dot{q}, \dot{p}_\rho \rangle \Vdash_{j(\mathbb{P} * \dot{\mathbb{Q}})}$ "$\langle j(\beta) \mid \beta < \lambda \rangle \in \dot{X}$" for some name $\dot{X}$ of X. As in [2, Lemma 2], $\mathcal{U}$ is a κ-additive, fine ultrafilter over $(P_\kappa(\lambda))^{V[G*H]}$. We show that $\mathcal{U}$ is normal as well.

To do this, suppose $\langle r, q \rangle \in G * H$ is such that $\langle r, \dot{q} \rangle \Vdash$ "$\dot{f} : (P_\kappa(\lambda))^{V[G*H]} \to \lambda$ is such that $\{s \mid \dot{f}(s) \in s\} \in \dot{\mathcal{U}}$". By the definition of $\mathcal{U}$ just given, we may assume that in M, $\langle r, \dot{q}, \dot{p}_\rho \rangle \Vdash$ "$\langle j(\alpha) \mid \alpha < \lambda \rangle \in \{s \mid j(\dot{f}(s)) \in s\}$". Let $\langle \varphi_\alpha \mid \alpha < \lambda \rangle$ be such that φ_α is the statement in the forcing language with respect to $j(\mathbb{P} * \dot{\mathbb{Q}})$ given by "$j(\dot{f})(\langle j(\beta) \mid \beta < \lambda \rangle) = j(\alpha)$". Since $\sigma > \lambda$ and $M^\sigma \subseteq M$, $\langle \varphi_\alpha \mid \alpha < \lambda \rangle \in M$. Therefore, by forcing above $\langle r, \dot{q} \rangle$ and arguing as in the definition of $\dot{p}_\rho$, we may assume that $\dot{p}'_\rho$ is a term for an Easton extension of p_ρ such that for every $\alpha < \lambda$, $\langle r, \dot{q}, \dot{p}'_\rho \rangle \parallel \varphi_\alpha$ (so in $M[G * H]$, $p'_\rho \parallel \varphi_\alpha$, where we assume that φ_α has been rewritten in the appropriate forcing language). Because $\langle r, \dot{q}, \dot{p}_\rho \rangle \Vdash$ "$\langle j(\alpha) \mid \alpha < \lambda \rangle \in \{s \mid j(\dot{f}(s)) \in s\}$", there must be some fixed $\alpha_0 < \lambda$ such that $p'_\rho \Vdash \varphi_{\alpha_0}$. In other words, by extending $\langle r, \dot{q} \rangle$ if necessary (and abusing notation by denoting the extended condition by $\langle r, \dot{q} \rangle$ as well), we may assume that $\langle r, \dot{q} \rangle \Vdash$ "$\{s \mid \dot{f}(s) = \alpha_0\} = \dot{A}_\gamma$" and $\langle r, \dot{q}, \dot{p}'_\rho \rangle \Vdash$ "$\langle j(\beta) \mid \beta < \lambda \rangle \in \{s \mid j(\dot{f}(s)) = j(\alpha_0)\}$" for some fixed $\alpha_0 < \lambda$ and fixed $\gamma < \rho$. It must consequently be the case that $\langle r, \dot{q}, \dot{p}_\rho \rangle \Vdash$ "$\langle j(\beta) \mid \beta < \lambda \rangle \in j(\dot{A}_\gamma)$". This is since otherwise, by the definition of $\dot{p}_\rho$, $\langle r, \dot{q}, \dot{p}_\rho \rangle \Vdash$ "$\langle j(\beta) \mid \beta < \lambda \rangle \notin j(\dot{A}_\gamma)$". However, $\langle r, \dot{q}, \dot{p}'_\rho \rangle \geq \langle r, \dot{q}, \dot{p}_\rho \rangle$ and $\langle r, \dot{q}, \dot{p}'_\rho \rangle \Vdash$ "$\langle j(\beta) \mid \beta < \lambda \rangle \in j(\dot{A}_\gamma)$". Thus, $\langle r, \dot{q}, \dot{p}_\rho \rangle \Vdash$ "$\{s \mid \dot{f}(s) = \alpha_0\} \in \dot{\mathcal{U}}$". This completes the proof of Lemma 2.2.

$\square$

Lemma 2.3 $V^{\mathbb{P}} \models$ *"No cardinal $\delta < \kappa$ is strongly compact".*

Proof: We follow the proof of [3, Lemma 2.3], again quoting verbatim when appropriate. Let $\lambda = \kappa^{+\omega}$. Take $j : V \to M$ as an elementary embedding witnessing the λ supercompactness of κ. Suppose $\mathbb{Q} \in V^{\mathbb{P}}$ is $\mathrm{Add}(\kappa, 1)$. By Lemma 2.1, $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ "κ is measurable" (since $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ "κ is strongly compact"). Because λ has been chosen large enough, it therefore follows that $M^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ "κ is measurable". In addition, as in Lemma 2.1, it is possible to opt for $\mathbb{Q}$ in the stage κ lottery held in M in the definition of $j(\mathbb{P})$. Therefore, by the definition of $\mathbb{P}$, since $\mathbb{Q} = \mathrm{Add}(\kappa, 1)$ and so of course adds a Cohen subset of κ, above the appropriate condition, $(j(\mathbb{P} * \dot{\mathbb{Q}}))_{\kappa+1} = \mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa = \mathbb{P}_{\kappa+1}$ is forcing equivalent in M to $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{S}}_\kappa$, where $\Vdash_{\mathbb{P} * \dot{\mathbb{Q}}}$ "$\dot{\mathbb{S}}_\kappa$ is Prikry forcing defined over κ". This means that in M, $\Vdash_{\mathbb{P}_\kappa}$ "By forcing above a condition $\dot{p}^*_\kappa$ ensuring that $\dot{\mathrm{Add}}(\kappa, 1)$ is chosen in the stage κ lottery held in the definition of $j(\mathbb{P})$, $\dot{\mathbb{Q}}_\kappa$ is forcing equivalent to a partial ordering adding a Prikry sequence to κ". Consequently, by reflection, for unboundedly many $\delta < \kappa$, $\Vdash_{\mathbb{P}_\delta}$ "By forcing above a condition $\dot{p}^*_\delta$ ensuring that $\dot{\mathrm{Add}}(\delta, 1)$ is chosen in the stage δ lottery held in the definition of $\mathbb{P}$, $\dot{\mathbb{Q}}_\delta$ is forcing equivalent to a partial ordering adding a Prikry sequence to δ".

It now follows that $\Vdash_{\mathbb{P}}$ "Unboundedly many $\delta < \kappa$ contain Prikry sequences". To see this, let $\gamma < \kappa$ be fixed but arbitrary. Choose $p = \langle \dot{p}_\alpha \mid \alpha < \kappa \rangle \in \mathbb{P}$. Since $\mathbb{P}$ is an Easton support iteration, let $\rho > \gamma$ be such that for every $\alpha \geq \rho$, $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{p}_\alpha$ is a term for the trivial condition". We may now find $\delta > \rho > \gamma$ such that $\Vdash_{\mathbb{P}_\delta}$ "By forcing above a condition $\dot{p}^*_\delta$ ensuring that $\dot{\mathrm{Add}}(\delta, 1)$ is chosen in the stage δ lottery held in the definition of $\mathbb{P}$, $\dot{\mathbb{Q}}_\delta$ is forcing equivalent to a partial ordering adding a Prikry sequence to δ". This means that we may find $q \geq p$ such that $q \Vdash$ "δ contains a Prikry sequence". Thus, $\Vdash_{\mathbb{P}}$ "Unboundedly many $\delta < \kappa$ contain Prikry sequences". Hence, by [4, Theorem 11.1], $V^{\mathbb{P}} \vDash$ "Unboundedly many $\delta < \kappa$ (i.e., the successors of those cardinals having Prikry sequences) contain non-reflecting stationary sets of ordinals of cofinality ω". By [13, Theorem 4.8] and the succeeding remarks, it thus follows that $V^{\mathbb{P}} \vDash$ "No cardinal $\delta < \kappa$ is strongly compact". This completes the proof of Lemma 2.3.

$\square$

Lemma 2.4 *Suppose $\mathbb{Q} \in V^{\mathbb{P}}$ is κ-directed closed and adds a Cohen subset of κ. Then $V^{\mathbb{P} * \dot{\mathbb{Q}}} \vDash$ "κ is not supercompact". In fact, in $V^{\mathbb{P} * \dot{\mathbb{Q}}}$, κ has trivial Mitchell rank.*

Proof: We slightly modify the proof of [3, Lemma 2.4], still quoting verbatim when appropriate. Let $G * H$ be V-generic over $\mathbb{P} * \dot{\mathbb{Q}}$. Let $H' \subseteq \kappa$, $H' \in V[G * H]$ be such that H' is $V[G]$-generic over $(\mathrm{Add}(\kappa, 1))^{V[G]}$. If $V[G * H] \vDash$ "κ does not have trivial Mitchell rank", then let $j : V[G * H] \to M[j(G * H)]$ be an elementary embedding generated by a normal measure $\mathcal{U} \in V[G * H]$ over κ such that $M[j(G * H)] \vDash$ "κ is measurable". Note that since $M = \bigcup_{\alpha \in \mathrm{Ord}} j(V_\alpha)$, $j \restriction V : V \to M$ is elementary. Therefore, because $j \restriction \kappa = \mathrm{id}$, we may infer that $(V_\kappa)^V = (V_\kappa)^M$. However, by Proposition 1.1, we may further infer that $(V_{\kappa+1})^M \subseteq (V_{\kappa+1})^V$. To see this, let $x \subseteq \kappa$, $x \in M$. Since $M \subseteq M[j(G * H)] \subseteq V[G * H]$, $x \in V[G * H]$. In addition, because $(V_\kappa)^V = (V_\kappa)^M$, we know that $x \cap \alpha \in V$ for every $\alpha < \kappa$. This means that if $x \notin V$, then x is a fresh subset of κ with respect to $\mathbb{P} * \dot{\mathbb{Q}}$. Since by Lemma 2.1, κ is strongly compact and hence both measurable and Mahlo in $V[G * H]$, this contradicts Proposition 1.1. Thus, $x \in V$, so $(\wp(\kappa))^M \subseteq (\wp(\kappa))^V$. From this, it of course immediately follows that $(V_{\kappa+1})^M \subseteq (V_{\kappa+1})^V$.

Let $I = j(G * H)$. Note that if $V \vDash$ "$\delta < \kappa$ is a strong cardinal", then $M \vDash$ "$j(\delta) = \delta$ is a strong cardinal". Also, $M \vDash$ "κ is a Mahlo limit of strong cardinals", since $M[j(G * H)] \vDash$ "κ is a Mahlo cardinal", and forcing can't create a new Mahlo cardinal. Hence, by the results of the preceding paragraph, it follows as well that $j(\mathbb{P}) \restriction \kappa = \mathbb{P}_\kappa = \mathbb{P}$ and $I_\kappa = G$. Further, as $V[G * H] \vDash$ "$M[I]^\kappa \subseteq M[I]$", $H' \in M[I]$. We know in addition that in M, $\Vdash_{\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa}$ "The forcing beyond stage κ adds no new subsets of 2^κ" and κ is a stage at which nontrivial forcing in $j(\mathbb{P})$ can take place. Consequently, $H' \in M[I_{\kappa+1}] = M[G][I(\kappa)]$, and $M[I_{\kappa+1}] \vDash$ "κ is measurable".

Note that since $\mathbb{P}$ is defined by taking Easton supports, $\mathbb{P}$ is κ-c.c. in both V and M. Because $\mathbb{P}$ is a Gitik iteration of suitably directed closed partial orderings together with Prikry forcing and $(V_\kappa)^V = (V_\kappa)^M$, $(V_\kappa)^{V[G]} = (V_\kappa)^{M[G]}$. It must therefore be the case that $(\mathrm{Add}(\kappa, 1))^{V[G]} = (\mathrm{Add}(\kappa, 1))^{M[G]}$. In addition, since $(V_{\kappa+1})^M \subseteq (V_{\kappa+1})^V$, the fact $\mathbb{P}$ is κ-c.c. in M yields that $(V_{\kappa+1})^{M[G]} \subseteq (V_{\kappa+1})^{V[G]}$. This means that H' is $M[G]$-generic over $(\mathrm{Add}(\kappa, 1))^{M[G]}$, since H' is $V[G]$-generic over $(\mathrm{Add}(\kappa, 1))^{V[G]} = (\mathrm{Add}(\kappa, 1))^{M[G]}$, and a dense open subset of $(\mathrm{Add}(\kappa, 1))^{M[G]}$ in $M[G]$ is a member of $(V_{\kappa+1})^{M[G]}$. Hence, H' must be added by the stage κ forcing done in $M[G] = M[I_\kappa]$, i.e., the stage κ lottery held in $M[I_\kappa]$ must opt for some forcing adding a Cohen subset of κ. By the definition of $\mathbb{P}$ and $j(\mathbb{P})$, we must then have that $M[I_{\kappa+1}] \vDash$ "κ contains a Prikry sequence". This contradiction to the fact that $M[I_{\kappa+1}] \vDash$ "κ is measurable" completes the proof of Lemma 2.4.

$\square$

Lemmas 2.1 – 2.4 complete the proof of Theorem 2.

$\square$

We conclude this paper with two questions. First, we ask what other classes of κ-directed closed partial orderings $\mathcal{C}$ will provide additional answers to our Question posed in Section 1. Finally, as in [3], we finish by asking if it is possible to get a model witnessing the conclusions of Theorem 2 in which κ is not the least strongly compact cardinal. Since Prikry forcing above a strongly compact cardinal destroys strong compactness, an answer to this question would require a different sort of iteration from the one used in the proof of Theorem 1.

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