A Note on Powers of Singular Strong Limit Cardinals *†

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Abstract

We show via a simple forcing argument that if $\kappa \geq \aleph_0$ is any cardinal such that $\kappa^{+\omega}$ is a strong limit cardinal, then $2^{\kappa^{+\omega}} < \kappa^{+\omega_4}$. Our proof makes use of pcf theory applied only at $\aleph_\omega$ and is generalizable to other contexts.

One of the most important results in all of set theory is Shelah's celebrated theorem [2] that if $\aleph_\omega$ is a strong limit cardinal, then $2^{\aleph_\omega} < \aleph_{\omega_4}$. This theorem is proven via pcf theory and generalizes earlier work by many people (for a more detailed discussion of the relevant history, see the survey article by Abraham and Magidor [1]).

The purpose of this note is to show via a simple forcing argument that the analogue of this ZFC provable fact is true about certain larger singular strong limit cardinals as well. In particular, we have the following theorem.

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§This paper is dedicated to the 60th birthdays of Peter Koepke and Philip Welch. It is truly a privilege to be able to contribute a paper to this Festschrift in their honor. Its main result was proven while the author was speaking with Koepke on March 21, 2013 late in the afternoon at the Cosi restaurant located at 31st Street and Park Avenue South in Manhattan.

Theorem 1 *If $\kappa \geq \aleph_0$ is any cardinal such that $\kappa^{+\omega}$ is a strong limit cardinal, then $2^{\kappa^{+\omega}} < \kappa^{+\omega_4}$.*

Note that as Shelah says in [2, page 359], his result about $\aleph_\omega$ holds for all cardinal non-fixed points. Consequently, Theorem 1 is not a new result. What is new, however, is that a simple forcing argument immediately implies Theorem 1 without any use of pcf theory beyond $\aleph_\omega$. We will comment on this further towards the end of the note.

Theorem 1 is proven by contradiction. Specifically, suppose that $\kappa \geq \aleph_0$ is any cardinal in some $V \vDash$ ZFC such that $\kappa^{+\omega}$ is a strong limit cardinal, and that in addition, $2^{\kappa^{+\omega}} \geq \kappa^{+\omega_4}$. Since we are giving a proof using pcf theory only at $\aleph_\omega$, we assume without loss of generality that $\kappa \geq \aleph_\omega$.

Force over V with $\mathbb{P} = \mathrm{Coll}(\omega_5, \kappa) = \{f \mid f : \omega_5 \to \kappa$ is a function such that $|\mathrm{dom}(f)| \leq \omega_4\}$, ordered as usual by inclusion. Since $\mathbb{P}$ is ω_4-closed (i.e., every increasing chain of length ω_4 has an upper bound), forcing with $\mathbb{P}$ preserves all cardinals less than or equal to ω_5. A standard density argument shows that forcing with $\mathbb{P}$ collapses κ to an ordinal having cardinality ω_5. In addition, $|\mathbb{P}| \leq |[\kappa]^{\omega_5}| \leq |\kappa^\kappa| = 2^\kappa$ (because by our earlier assumption, $\kappa \geq \aleph_\omega$), so since $\kappa^{+\omega}$ is a strong limit cardinal, $|\mathbb{P}| < \kappa^{+\omega}$. Consequently, it must be the case that for some fixed $k_0 < \omega$, $|\mathbb{P}| = \kappa^{+k_0}$, which of course immediately implies that $\mathbb{P}$ is κ^{+k_0+1}-c.c. From this, we may infer that there is some $n_0 \geq 1$, $n_0 < \omega$ such that for all natural numbers $0 \leq m < \omega$, $\kappa^{+n_0+m} = \omega_{6+m}$, i.e., that $\kappa^{+n_0} = \omega_6$, $\kappa^{+n_0+1} = \omega_7$, etc., and that in addition, $\kappa^{+\omega} = \aleph_\omega$. It further follows that $\kappa^{+\omega}$ remains a strong limit cardinal in $V^{\mathbb{P}}$, since for all $m \geq \max(k_0, n_0)$, $m < \omega$, $(2^{\kappa^{+m}})^{V^{\mathbb{P}}} \leq (2^{\kappa^{+m} \times |\mathbb{P}|})^V = (2^{\kappa^{+m}})^V < \kappa^{+\omega}$. But now, in $V^{\mathbb{P}}$, $\aleph_\omega$ is a strong limit cardinal such that $2^{\aleph_\omega} \geq (\aleph_\omega)^{+\omega_4}$, i.e., $V^{\mathbb{P}} \vDash$ "$2^{\aleph_\omega} \geq \aleph_{\omega_4}$". This contradiction to Shelah's theorem proves Theorem 1.

$\square$

Since as we observed above, Shelah's theorem generalizes to all cardinal non-fixed points, the proof of Theorem 1 also yields results such as the following.

Theorem 2 *If $\kappa \geq \aleph_0$ is any cardinal such that $\kappa^{+\omega_1}$ is a strong limit cardinal, then $2^{\kappa^{+\omega_1}} < \kappa^{+\omega_4}$.*

In analogy to Theorem 1, the proof of Theorem 2 requires no use of pcf theory beyound $\aleph_{\omega_1}$. By changing each occurrence of the ordinal ω to the ordinal ω_1, literally the same proof as given for Theorem 1 remains valid. Also, for the same reasons as with Theorem 1, Theorem 2 is not new. In addition, both Theorems 1 and 2 use Shelah's work as a significant black box, and in a real sense, use pcf theory as much as a direct proof not going through either $\aleph_\omega$ or $\aleph_{\omega_1}$ would. However, what is true is that the proof of Theorem 1 shows that any new ZFC bound on the size of the power set of $\aleph_\omega$, $\aleph_{\omega_1}$, etc., assuming these are strong limit cardinals, also automatically applies to additional, arbitrarily large singular strong limit cardinals as well. As a specific example, it is widely believed that eventually, some ZFC proof will show that if $\aleph_\omega$ is a strong limit cardinal, then $2^{\aleph_\omega} < \aleph_{\omega_1}$. If this were indeed the case, then the proof of Theorem 1 would show that this bound automatically transfers to all infinite cardinals κ such that $\kappa^{+\omega}$ is a strong limit cardinal. We consequently have, speaking in slogan form but still in a very concrete sense, that "the size of power sets of singular strong limit cardinals down low controls the size of power sets of certain singular strong limit cardinals up high." Further, our methods show in addition that any ZFC proof which *prima facie* only produces a bound on the size of the power set of $\aleph_\omega$, $\aleph_{\omega_1}$, etc. and doesn't appear to generalize to larger cardinals actually does produce a bound on the size of the power sets of certain larger singular strong limit cardinals. Hence, we have the quite interesting and significant fact that ZFC shows there is actually very little freedom on the size of the power sets of certain arbitrarily high singular strong limit cardinals.

References

[1] U. Abraham, M. Magidor, "Cardinal Arithmetic", in: **Handbook of Set Theory**, Springer-Verlag, Berlin and New York, 2010, 1149–1228.

[2] S. Shelah, **Cardinal Arithmetic**, *Oxford Logic Guides 29*, The Clarendon Press, Oxford University Press, New York, 1994.