Level by Level Inequivalence beyond Measurability [*†]

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Abstract

We construct models containing exactly one supercompact cardinal in which level by level inequivalence between strong compactness and supercompactness holds. In each model, above the supercompact cardinal, there are finitely many strongly compact cardinals, and the strongly compact and measurable cardinals precisely coincide.

Say that a model containing supercompact cardinals satisfies *level by level inequivalence between strong compactness and supercompactness* if for every non-supercompact measurable cardinal κ, there is some $\lambda > \kappa$ such that κ is λ strongly compact yet κ is not λ supercompact. Models containing exactly one supercompact cardinal in which level by level inequivalence between strong compactness and supercompactness holds have been constructed in [3, Theorem 2] and [5, Theorem 2].[1] (See also [7].) A key feature of each of these models, however, is a rather restricted large

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[1]Note that the dual notion of *level by level equivalence between strong compactness and supercompactness* was first studied by the author and Shelah in [8], to which we refer readers for additional details.

cardinal structure. In particular, there do not exist in any of these models cardinals $\kappa < \lambda$ such that κ is λ supercompact and λ is measurable, nor do there exist in any of these models non-supercompact strongly compact cardinals. This prompts us to ask

Question 1: Is it possible to construct models containing at least one supercompact cardinal in which level by level inequivalence between strong compactness and supercompactness holds, and in which there are measurable cardinals which are supercompact up to (and even beyond) a measurable cardinal?

Question 2: Is it possible to construct models containing at least one supercompact cardinal in which level by level inequivalence between strong compactness and supercompactness holds, and in which there are non-supercompact strongly compact cardinals?

The purpose of this paper is to answer the above questions in the affirmative. Specifically, we will prove the following theorem.

Theorem 1 *Suppose $V \models$ "ZFC + $\langle \kappa_i \mid i < \omega \rangle$ are the first ω supercompact cardinals". Let $n \in \omega$, $n \geq 1$ be fixed but arbitrary. There is then a partial ordering $\mathbb{P} \subseteq V$, a model $\overline{V} \subseteq V^{\mathbb{P}}$, and a sequence of cardinals $\lambda_0 < \cdots < \lambda_n$ such that $\overline{V} \models$ "λ_0 is supercompact + Level by level inequivalence between strong compactness and supercompactness holds". In $\overline{V}$, $\lambda_1, \ldots, \lambda_n$ are the first n measurable cardinals above λ_0, each λ_i for $i = 1, \ldots, n$ is strongly compact, and there are no measurable cardinals above λ_n.*

In $\overline{V}$, the supercompact cardinal λ_0 is clearly an example of a measurable cardinal which is supercompact beyond a finite number of measurable cardinals. However, by reflection, there are unboundedly many in λ_0 measurable cardinals which are also supercompact beyond a finite number of measurable cardinals. In fact, as our proof will show, *every* measurable cardinal $\kappa < \lambda_0$ will be strongly compact beyond a finite number of measurable cardinals (and much more). We will comment further on this towards the end of the paper.

Before beginning the proof of Theorem 1, we elaborate briefly on our terminology. Suppose $\kappa < \lambda$ are cardinals. The partial ordering $\mathbb{P}$ is *κ-directed closed* if every directed subset of $\mathbb{P}$ of cardinality less than κ has a common extension. κ is $<\lambda$ *strongly compact* if κ is δ strongly compact for every $\delta < \lambda$.

Turning to the proof of Theorem 1, let V be as in the hypotheses of this theorem. Without loss of generality, by first forcing GCH and then forcing with a partial ordering such as the ones described in [4] and [2], we may assume in addition that $V \models$ "For each $i < \omega$, κ_i has its supercompactness indestructible under κ_i-directed closed forcing [10] and $2^{\kappa_i} = \kappa_i^+$".

We now describe the first partial ordering $\mathbb{Q}$ used in the proof of Theorem 1. For any $i < \omega$, let $\mathbb{Q}_i$ be the reverse Easton iteration of length κ_{i+1} which adds a non-reflecting stationary set of ordinals of cofinality κ_i to each measurable cardinal in the open interval (κ_i, κ_{i+1}). (See, e.g., [2], [6], or [8] for a more complete description of this partial ordering.) Next, take $\mathbb{Q} = \prod_{i<\omega} \mathbb{Q}_i$ as the full support product.

By its definition, $\mathbb{Q}$ is κ_0-directed closed. Consequently, by our hypotheses on V, $V^{\mathbb{Q}} \models$ "κ_0 is supercompact". However, the following is also true.

Lemma 1.1 *For each $i < \omega$, $V^{\mathbb{Q}} \models$ "No cardinal $\delta \in (\kappa_i, \kappa_{i+1})$ is measurable, and κ_{i+1} is $<\kappa_{i+2}$ strongly compact".*

Proof: Let $i < \omega$ be fixed but arbitrary. Write $\mathbb{Q} = \mathbb{Q}^i \times \mathbb{Q}_i \times \mathbb{Q}_{<i}$, where $\mathbb{Q}^i = \prod_{j>i} \mathbb{Q}_j$ and $\mathbb{Q}_{<i} = \prod_{j<i} \mathbb{Q}_j$. Because by its definition, $\mathbb{Q}^i$ is in fact κ_{i+1}^+-directed closed, $V^{\mathbb{Q}^i} \models$ "κ_{i+1} is supercompact, and $2^{\kappa_{i+1}} = \kappa_{i+1}^+$". Therefore, the same argument as mentioned in [6, page 1908, last paragraph] literally unchanged now shows that $V^{\mathbb{Q}} \models$ "No cardinal $\delta \in (\kappa_i, \kappa_{i+1})$ is measurable, and κ_{i+1} is $<\kappa_{i+2}$ strongly compact". This completes the proof of Lemma 1.1.

$\square$

Let $V_* = V^{\mathbb{Q}}$. By Lemma 1.1 and the sentence immediately preceding its statement, $V_* \models$ "$\langle \kappa_i \mid i < \omega \rangle$ is a sequence of measurable cardinals such that for each $i < \omega$, κ_{i+1} is the least measurable cardinal greater than κ_i, and κ_i is $<\kappa_{i+1}$ strongly compact". Therefore, since $V_* \models$ "κ_0 is supercompact", let $j : V_* \to M$ be a λ supercompactness embedding for λ sufficiently large with the property that $M \models$ "$\langle \kappa_i \mid i < \omega \rangle$ is a sequence of measurable cardinals such that for each $i < \omega$,

κ_{i+1} is the least measurable cardinal greater than κ_i, and κ_i is $<\kappa_{i+1}$ strongly compact". Working in V_*, by reflection, for each $\gamma < \kappa_0$, there is a sequence $\langle \delta_i \mid i < \omega \rangle$ of measurable cardinals with $\delta_0 > \gamma$ such that for each $i < \omega$, δ_{i+1} is the least measurable cardinal greater than δ_i, and δ_i is $<\delta_{i+1}$ strongly compact. This now allows us to define $\mathbb{R} \in V_*$ as the Magidor iteration of Prikry forcing [12] of length κ_0 which adds a Prikry sequence to each measurable cardinal $\delta < \kappa_0$ for which for $\langle \delta_i \mid i < \omega \rangle$ the first ω many measurable cardinals greater than or equal to δ (with $\delta = \delta_0$), there is some $i < \omega$ such that δ_i is not $<\delta_{i+1}$ strongly compact. In other words, $\mathbb{R}$ iteratively destroys via a Magidor iteration of Prikry forcing of length κ_0 any measurable cardinal $\delta < \kappa_0$ which cannot be the first member of a sequence of measurable cardinals reflecting the aforementioned properties of $\langle \kappa_i \mid i < \omega \rangle$.

Lemma 1.2 *Let $\lambda = \sup_{i<\omega} \kappa_i$. Then $V_*^{\mathbb{R}} \models$ "κ_0 is λ supercompact".*

Proof: Let $j : V_* \to M$ be an elementary embedding witnessing the λ supercompactness of κ_0 generated by a supercompact ultrafilter over $P_{\kappa_0}(\lambda)$. Since $M^\lambda \subseteq M$, $M \models$ "$\langle \kappa_i \mid i < \omega \rangle$ is a sequence of measurable cardinals such that for each $i < \omega$, κ_{i+1} is the least measurable cardinal greater than κ_i, and κ_i is $<\kappa_{i+1}$ strongly compact". Thus, since j is generated by a supercompact ultrafilter over $P_{\kappa_0}(\lambda)$, by the definition of $\mathbb{R}$, $j(\mathbb{R}) = \mathbb{R} * \dot{\mathbb{R}}'$, where the first ordinal γ at which $\dot{\mathbb{R}}'$ is forced to do nontrivial forcing is well above λ.

We follow now the proof of the Lemma of [1]. Let $| \quad |$ be the distance function of [12]. Define a term $\dot{\mathcal{U}}$ in V_* by $p \Vdash$ "$\dot{B} \in \dot{\mathcal{U}}$" iff $p \Vdash$ "$\dot{B} \subseteq (P_{\kappa_0}(\lambda))^{V_*^{\mathbb{R}}}$" and there is $q \in j(\mathbb{R})$ such that $q \geq j(p)$ (q extends $j(p)$), $|j(p) - q| = 0$, $j(p) \restriction \gamma = q \restriction \gamma = j(p) \restriction \kappa_0 = q \restriction \kappa_0 = p$, and $q \Vdash$ "$\langle j(\beta) \mid \beta < \lambda \rangle \in j(\dot{B})$". By [12, Theorem 3.4], $\dot{\mathcal{U}}$ is a well-defined term for a strongly compact measure over $(P_{\kappa_0}(\lambda))^{V_*^{\mathbb{R}}}$ in $V_*^{\mathbb{R}}$.

To see that $\Vdash_{\mathbb{R}}$ "$\dot{\mathcal{U}}$ is normal", let $p \Vdash$ "$\dot{f} : (P_{\kappa_0}(\lambda))^{V_*^{\mathbb{R}}} \to \lambda$ is a function such that $\dot{f}(s) \in s$ for all $s \in \dot{B}$ where $\dot{B} \in \dot{\mathcal{U}}$". Let φ_α for $\alpha < \lambda$ be the statement "$j(\dot{f})(\langle j(\beta) \mid \beta < \lambda \rangle) = j(\alpha)$" in the forcing language with respect to $j(\mathbb{R})$, and consider the sequence $\langle \varphi_\alpha \mid \alpha < \lambda \rangle$. Since $M^\lambda \subseteq M$, $\langle \varphi_\alpha \mid \alpha < \lambda \rangle \in M$. Thus, since γ is the least M cardinal in the half-open interval $[\kappa_0, j(\kappa_0))$ at which $j(\mathbb{R})$ is forced to do nontrivial forcing and $\gamma > \lambda$, we can apply [12, Lemma

2.4] in M to $\langle \varphi_\alpha \mid \alpha < \lambda \rangle$ and obtain a condition $q \geq j(p)$, $q \in j(\mathbb{R})$ such that $|j(p) - q| = 0$, $j(p) \restriction \gamma = q \restriction \gamma = j(p) \restriction \kappa_0 = q \restriction \kappa_0 = p$, and if $q' \geq q$, q' decides φ_α for some $\alpha < \lambda$, then $q' \restriction \kappa_0 \cup (q - p)$ decides φ_α in the same way. Hence, since $p \Vdash$ "$\dot{B} \in \dot{\mathcal{U}}$" implies we can assume (by extending q if necessary) that $q \Vdash$ "$\langle j(\beta) \mid \beta < \lambda \rangle \in j(\dot{B})$", there must be some $\alpha < \lambda$ such that for some $q' \geq q$, $q' \Vdash \varphi_\alpha$, i.e., such that $q' \Vdash$ "$j(\dot{f})(\langle j(\beta) \mid \beta < \lambda \rangle) = j(\alpha)$". By choice of q, $q' \restriction \kappa_0 \cup (q - p) \Vdash \varphi_\alpha$, i.e., $q' \restriction \kappa_0 \geq p$ is such that for some $r \in j(\mathbb{R})$ (r can be taken as $q' \restriction \kappa_0 \cup (q - p)$), $|j(q' \restriction \kappa_0) - r| = 0$, $j(q' \restriction \kappa_0) \restriction \kappa_0 = r \restriction \kappa_0 = q' \restriction \kappa_0$, and $r \Vdash \varphi_\alpha$. Since $r \Vdash \varphi_\alpha$, $r \Vdash$ "$\langle j(\beta) \mid \beta < \lambda \rangle \in j(\{s \in \dot{B} \mid \dot{f}(s) = \alpha\})$", so $q' \restriction \kappa_0 \geq p$ is such that $q' \restriction \kappa_0 \Vdash$ "$\{s \in \dot{B} \mid \dot{f}(s) = \alpha\} \in \dot{\mathcal{U}}$". This completes the proof of Lemma 1.2.

□

Let $V_{**} = V_*^{\mathbb{R}}$. We observe that the proof of Lemma 1.2 shows that if $V_* \models$ "There are no measurable cardinals above λ", then $V_{**} \models$ "κ_0 is supercompact".

Lemma 1.3 $V_{**} \models$ *"If $\kappa < \kappa_0$ is measurable, then κ is $<\kappa'$ strongly compact for κ' the least measurable cardinal greater than κ".*

Proof: Suppose $V_{**} \models$ "$\kappa < \kappa_0$ is measurable". By [12, Theorem 3.1], $V_* \models$ "κ is measurable" as well. Therefore, by its definition, we may write $\mathbb{R} = \mathbb{R}_0 * \dot{\mathbb{R}}_1$, where $|\mathbb{R}_0| \leq \kappa$, and the first ordinal at which $\dot{\mathbb{R}}_1$ is forced to do nontrivial forcing is greater than ρ, the supremum of the first ω many V_*-measurable cardinals greater than κ. Because $|\mathbb{R}_0| \leq \kappa$, by the Lévy-Solovay results [11], the first ω many measurable cardinals greater than κ are the same in both V_* and $V_*^{\mathbb{R}_0}$. By [12, Lemma 2.1], $\Vdash_{\mathbb{R}_0}$ "Forcing with $\dot{\mathbb{R}}_1$ adds no subsets of ρ". Consequently, it is the case that $V_{**} = V_*^{\mathbb{R}} \models$ "κ is $<\kappa'$ strongly compact" iff $V_*^{\mathbb{R}_0} \models$ "κ is $<\kappa'$ strongly compact". For this last fact, we consider the following two cases.

Case 1: $|\mathbb{R}_0| < \kappa$. In this situation, by the results of [11], since $V_* \models$ "κ is $<\kappa'$ strongly compact", $V_*^{\mathbb{R}_0} \models$ "κ is $<\kappa'$ strongly compact".

Case 2: $|\mathbb{R}_0| = \kappa$. In this situation, by [12, Theorem 3.4], forcing with $\mathbb{R}_0$ preserves any degree of strong compactness κ exhibits in V_*. Hence, as in Case 1, $V_*^{\mathbb{R}_0} \models$ "κ is $<\kappa'$ strongly compact".

Cases 1 and 2 complete the proof of Lemma 1.3.

□

Fix now an arbitrary $n \in \omega$, $n \geq 1$. Because $|\mathbb{R}| = \kappa_0$, again by the results of [11], the first ω many measurable cardinals greater than κ_0 are the same in both V_* and $V_{**} = V_*^{\mathbb{R}}$. Consequently, by Lemma 1.2 and reflection, in V_{**}, we can find $\lambda_0 < \kappa_0$ such that λ_0 is the least cardinal which is λ_{n+1} supercompact for λ_{n+1} the $(n+1)^{\text{st}}$ measurable cardinal greater than λ_0. Denote by $\lambda_0 < \lambda_1 < \cdots < \lambda_n < \lambda_{n+1} < \kappa_0$ the sequence composed of the first $n+2$ many consecutive measurable cardinals starting with λ_0. Let $\overline{V} = (V_{\lambda_{n+1}})^{V_{**}}$. Clearly, $\overline{V} \vDash$ "λ_0 is supercompact + $\lambda_1, \ldots, \lambda_n$ are the first n measurable cardinals above λ_0 + There are no measurable cardinals above λ_n".

Lemma 1.4 $\overline{V} \vDash$ "*Each λ_i for $i = 1, \ldots, n$ is strongly compact*".

Proof: By Lemma 1.3, $V_{**} \vDash$ "λ_i is $<\lambda_{i+1}$ strongly compact for $i = 1, \ldots, n$". Therefore, $\overline{V} = (V_{\lambda_{n+1}})^{V_{**}} \vDash$ "λ_n is strongly compact". By Ketonen's characterization of strong compactness [9][2], if α is $<\beta$ strongly compact, β is γ strongly compact, and γ is regular, then α is γ strongly compact. Applying this theorem finitely often and doing a "downwards induction" going from λ_{n-1} to λ_1 then tells us that $\overline{V} \vDash$ "Each λ_i for $i = n-1, \ldots, 1$ is strongly compact". This completes the proof of Lemma 1.4.

□

Lemma 1.5 $\overline{V} \vDash$ "*Level by level inequivalence between strong compactness and supercompactness holds*".

Proof: Because $\overline{V} \vDash$ "The λ_i for $i = 1, \ldots n$ are both strongly compact and the first n measurable cardinals above λ_0", $\overline{V} \vDash$ "No λ_i for $i = 1, \ldots, n$ is supercompact". Thus, $\overline{V} \vDash$ "Level by level inequivalence between strong compactness and supercompactness holds above λ_0". Since $\overline{V} \vDash$ "λ_0

[2]Ketonen characterized strong compactness in [9] by showing that for $\kappa \leq \lambda$ regular cardinals, κ is λ strongly compact iff for every regular cardinal δ such that $\kappa \leq \delta \leq \lambda$, there is a κ-additive, uniform ultrafilter over δ.

is supercompact", the proof of Lemma 1.5 will be complete once we have shown that $\overline{V} \models$ "Level by level inequivalence between strong compactness and supercompactness holds below λ_0". To do this, let $\kappa < \lambda_0 < \kappa_0$ be measurable. By Lemma 1.3 and finitely many applications of Ketonen's characterization of strong compactness of [9], $\overline{V} \models$ "κ is γ strongly compact for γ the $(n+1)^{\text{st}}$ measurable cardinal above κ". By the choice of λ_0, this means that there is a cardinal $\gamma' > \kappa$, $\gamma' \leq \gamma < \lambda_0$ such that $\overline{V} \models$ "κ is γ' strongly compact yet κ is not γ' supercompact". This completes the proof of Lemma 1.5.

□

With $\mathbb{P} = \mathbb{Q} * \dot{\mathbb{R}}$, Lemmas 1.1 – 1.5 and the intervening remarks complete the proof of Theorem 1.

□

We observe that the proof of Lemma 1.5 actually shows that in $\overline{V}$, for $\kappa < \lambda_0$ measurable, κ is $<\rho$ strongly compact, where ρ is the supremum of the first ω many measurable cardinals greater than κ. Thus, κ exhibits a significant degree of level by level inequivalence between strong compactness and supercompactness. Also, suppose we build $\overline{V}$ by taking $n = 0$ in the above construction. We explicitly note that although there are no measurable cardinals above λ_0 (so λ_0 of course is not supercompact beyond a measurable cardinal), every measurable cardinal $\kappa < \lambda_0$ both exhibits level by level inequivalence between strong compactness and supercompactness and is $<\rho$ strongly compact.

We finish with two questions. First, we ask if it is possible to prove Theorem 1 using somewhat weaker hypotheses. Our current methods of proof seem to require something along the lines of the existence of an ω sequence of supercompact cardinals. Second, we note that the large cardinal structure of the model witnessing the conclusions of Theorem 1 remains somewhat limited. We conclude by asking if it is possible to remove the restrictions inherent to our proof, and obtain results analogous to those of this paper in which the large cardinal structure of the universe can be arbitrary.

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