Level by Level Inequivalence, Strong Compactness, and GCH *†

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Abstract

We construct three models containing exactly one supercompact cardinal in which level by level inequivalence between strong compactness and supercompactness holds. In the first two models, below the supercompact cardinal κ, there is a non-supercompact strongly compact cardinal. In the last model, any suitably defined ground model Easton function is realized.

1 Introduction and Preliminaries

Say that a model containing at least one supercompact cardinal satisfies *level by level inequivalence between strong compactness and supercompactness* if for every non-supercompact measurable cardinal δ, there is some $\gamma > \delta$ such that δ is γ strongly compact yet δ is not γ supercompact.

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Models containing exactly one supercompact cardinal in which level by level inequivalence between strong compactness and supercompactness holds have been constructed in [3, Theorem 2], [5, Theorem 2], and [2, Theorem 1]. (See also [9]. Note that the dual notion of *level by level equivalence between strong compactness and supercompactness* was first studied by the author and Shelah in [10], to which we refer readers for additional details.) Key features of each of these models, however, are rather restricted large cardinal structures and fairly arbitrary GCH patterns. In particular, it is not possible to infer that there are any strongly compact cardinals below the supercompact cardinal in any of these models (although in the model of [2, Theorem 1], there are finitely many non-supercompact strongly compact cardinals above the supercompact cardinal). In addition, GCH holds in the models of both [3, Theorem 2] and [5, Theorem 2], and the GCH pattern of the model of [2, Theorem 1] is controlled by ground model indestructible supercompact cardinals. This prompts us to ask the following two questions.

Question 1: Is it possible to construct models containing at least one supercompact cardinal in which level by level inequivalence between strong compactness and supercompactness holds, and in which there is a non-supercompact strongly compact cardinal below some supercompact cardinal?

Question 2: Is it possible to construct a model containing at least one supercompact cardinal in which level by level inequivalence between strong compactness and supercompactness holds, and in which the GCH pattern on regular cardinals is precisely controlled?

The purpose of this paper is to answer the above questions in the affirmative. Specifically, we will prove the following three theorems, where we take as notation for this paper that if α is an ordinal, then σ_α is the least inaccessible cardinal above α.

Theorem 1 *Suppose $V \vDash$ "ZFC + GCH + $\lambda < \kappa_1 < \kappa_2$ are such that λ and κ_1 are both supercompact and κ_2 is inaccessible". There is then a partial ordering $\mathbb{P} \in V$, a submodel $\overline{V} \subseteq V^{\mathbb{P}}$, and $\kappa \in (\lambda, \kappa_1)$ such that $\overline{V} \vDash$ "ZFC + GCH + κ is supercompact + Level by level inequivalence between strong compactness and supercompactness holds + No cardinal $\delta > \kappa$ is inaccessible". In $\overline{V}$, λ is both the least strongly compact and least measurable cardinal.*

Theorem 2 *Suppose* $V \models$ *"ZFC + GCH +* $\lambda < \kappa_1 < \kappa_2$ *are such that* λ *and* κ_1 *are both supercompact and* κ_2 *is inaccessible". There is then a partial ordering* $\mathbb{P} \in V$*, a submodel* $\overline{V} \subseteq V^{\mathbb{P}}$*, and* $\kappa \in (\lambda, \kappa_1)$ *such that* $\overline{V} \models$ *"ZFC + GCH +* κ *is supercompact + Level by level inequivalence between strong compactness and supercompactness holds + No cardinal* $\delta > \kappa$ *is inaccessible". In* $\overline{V}$*,* λ *is the least strongly compact cardinal,* λ *is not supercompact (in fact,* λ *is not* 2^λ *supercompact), and* λ *is a limit of measurable cardinals.*

Theorem 3 *Suppose* $V \models$ *"ZFC + GCH +* $\kappa_1 < \kappa_2$ *are such that* κ_1 *is supercompact and* κ_2 *is inaccessible + No cardinal* $\delta > \kappa_1$ *is measurable". Let* F *be a class function defined on the regular cardinals with range a subset of the cardinals satisfying the following properties.*

1. *If* $\delta_1 < \delta_2$*, then* $F(\delta_1) \leq F(\delta_2)$*.*

2. $F(\delta) \in (\delta, \sigma_\delta)$ *(or if* σ_δ *does not exist, then* $F(\delta) > \delta$*).*

3. $\mathrm{cof}(F(\delta)) > \delta$*.*

4. F *is definable by a* Δ_2 *function.*

There is then a partial ordering $\mathbb{P} \subseteq V$*, a submodel* $\overline{V} \subseteq V^{\mathbb{P}}$*, and* $\kappa < \kappa_1$ *such that* $\overline{V} \models$ *"ZFC +* κ *is supercompact + Level by level inequivalence between strong compactness and supercompactness holds + No cardinal* $\delta > \kappa$ *is inaccessible". In* $\overline{V}$*,* κ *is the least strongly compact cardinal, and for every regular cardinal* δ*,* $2^\delta = F(\delta)$*.*

We take this opportunity to make several remarks concerning Theorem 3. We begin by observing that restriction (4) above on the Easton function F is as a result of Menas' proof of [17]. Restriction (2) is made so that inaccessible cardinals are preserved between V and $\overline{V}$, thereby simplifying the proof of Theorem 3. (Restrictions (1) and (3) are of course standard to any Easton function.) These constraints are fairly innocuous, however, and still allow us to contruct models $\overline{V}$ witnessing level by level inequivalence between strong compactness and supercompactness in which $2^\delta = \delta^{++}$ for every regular cardinal δ, $2^\delta = \delta^{+17}$ for every regular cardinal δ, etc. This is in sharp contrast to models in which level by level equivalence between strong compactness and supercompactness holds and GCH fails significantly (such as, e.g., the models constructed in [6]), where currently available techniques seem to allow far less flexibility in what can be forced to occur.

Before beginning the proof of Theorems 1 – 3, we elaborate briefly on our notation and terminology. For $\alpha < \beta$ ordinals, (α, β), $[\alpha, \beta)$, and $(\alpha, \beta]$ are as in standard interval notation. Suppose $\kappa < \lambda$ are cardinals. The partial ordering $\mathbb{P}$ is *κ-directed closed* if every directed subset of $\mathbb{P}$ of cardinality less than κ has a common extension. We will abuse notation slightly and use $V^{\mathbb{P}}$ to denote the generic extension of V by $\mathbb{P}$. κ is *$<\lambda$ supercompact ($<\lambda$ strongly compact)* if κ is δ supercompact (δ strongly compact) for every $\delta < \lambda$.

2 The Proofs of Theorems 1 – 3

We turn now to the proof of Theorem 1.

Proof: Let $V \vDash$ "ZFC + GCH + There exist cardinals $\lambda < \kappa_1 < \kappa_2$ such that λ and κ_1 are both supercompact and κ_2 is inaccessible". Without loss of generality, by [4, Theorem 2] and the remarks at the end of [4], we may assume in addition that $V \vDash$ "Every measurable cardinal δ is σ_δ strongly compact".

Let $\mathbb{P}$ be Magidor's partial ordering of [16, Theorem 3.5] which iteratively changes the cofinality of every measurable cardinal $\delta < \lambda$ to ω via Prikry forcing. By the work of [16], $V^{\mathbb{P}} \vDash$ "GCH + λ is both the least strongly compact and least measurable cardinal". Since $\mathbb{P}$ may be defined so that $|\mathbb{P}| = \lambda$, by the Lévy-Solovay results [15], $V^{\mathbb{P}} \vDash$ "Every measurable cardinal $\delta > \lambda$ is σ_δ strongly compact + κ_1 is κ_2 supercompact and κ_2 is inaccessible". By reflection, we may therefore let $\kappa \in (\lambda, \kappa_1)$ be the least cardinal such that $V^{\mathbb{P}} \vDash$ "κ is $<\sigma_\kappa$ supercompact". It is then the case that for $\overline{V} = (V_{\sigma_\kappa})^{V^{\mathbb{P}}}$, $\overline{V} \vDash$ "ZFC + GCH + κ is supercompact + No cardinal $\delta > \kappa$ is inaccessible". Further, in $\overline{V}$, λ is both the least strongly compact and least measurable cardinal. As any cardinal δ which is 2^δ supercompact must be a limit of measurable cardinals, this means that $\overline{V} \vDash$ "λ is not $2^\lambda = \lambda^+$ supercompact". Consequently, because $\overline{V} \vDash$ "Every measurable cardinal $\delta \in (\lambda, \kappa)$ is σ_δ strongly compact", $\overline{V} \vDash$ "Level by level inequivalence between strong compactness and supercompactness holds". This completes the proof of Theorem 1.

□

We observe that by replacing the partial ordering $\mathbb{P}$ used in the proof of Theorem 1 with the partial ordering of [8, Theorem 1], it is possible to assume that in addition, λ has its strong compactness indestructible under λ-directed closed forcing. This is the exact analogue of Laver's result of [14] for strongly compact, rather than supercompact, cardinals. If this has been done, GCH will no longer hold below λ in the final model. GCH will, however, continue to be true at and above λ, since the partial ordering of [8, Theorem 1] can be defined so as to have cardinality λ. In particular, it will also be the case (as it was in Theorem 1) that λ is not $2^\lambda = \lambda^+$ supercompact.

Turning now to the proof of Theorem 2, once again, let $V \vDash$ "ZFC + GCH + There exist cardinals $\lambda < \kappa_1 < \kappa_2$ such that λ and κ_1 are both supercompact and κ_2 is inaccessible". As in the proof of Theorem 1, we also assume that $V \vDash$ "Every measurable cardinal δ is σ_δ strongly compact".

Let $A = \{\delta < \lambda \mid \delta$ is a measurable cardinal which is a limit of measurable cardinals$\}$. Let $\mathbb{P}$ be Magidor's partial ordering of [16] which iteratively changes the cofinality of every $\delta \in A$ to ω via Prikry forcing. By the work of [16] (in particular, by [16, Theorem 3.4]), $V^{\mathbb{P}} \vDash$ "GCH + λ is a strongly compact cardinal".

Lemma 2.1 $V^{\mathbb{P}} \vDash$ "*λ is not $2^\lambda = \lambda^+$ supercompact*".

Proof: By [16, Theorem 3.1], any cardinal which is measurable in $V^{\mathbb{P}}$ had to have been measurable in V. Thus, if $V^{\mathbb{P}} \vDash$ "$\delta < \lambda$ is a measurable cardinal which is a limit of measurable cardinals", then $V \vDash$ "δ is a measurable cardinal which is a limit of measurable cardinals". However, by the definition of $\mathbb{P}$, $V^{\mathbb{P}} \vDash$ "$\mathrm{cof}(\delta) = \omega$". This means that $V^{\mathbb{P}} \vDash$ "Below λ, there are no measurable limits of measurable cardinals". Consequently, $V^{\mathbb{P}} \vDash$ "λ is not $2^\lambda = \lambda^+$ supercompact", since if it were, then $V^{\mathbb{P}} \vDash$ "λ is a limit of measurable limits of measurable cardinals". This completes the proof of Lemma 2.1.

□

Lemma 2.2 $V^{\mathbb{P}} \models$ "*No cardinal* $\delta < \lambda$ *is strongly compact*".

Proof: We follow the proof of [7, Lemma 3.1]. As was mentioned in the proof of Lemma 2.1, since $V \models$ "λ is 2^λ supercompact", $V \models$ "λ is a limit of measurable limits of measurable cardinals". Hence, by the definition of $\mathbb{P}$, in $V^{\mathbb{P}}$, unboundedly in λ many $\delta < \lambda$ contain Prikry sequences. However, by [11, Theorem 11.1], the presence of a Prikry sequence implies the presence of a non-reflecting stationary set of ordinals of cofinality ω. Therefore, since [18, Theorem 4.8] and the succeeding remarks imply such a set cannot exist above a strongly compact cardinal, we may now immediately infer that no cardinal $\delta < \lambda$ is strongly compact. This completes the proof of Lemma 2.2.

$\square$

Lemma 2.3 $V^{\mathbb{P}} \models$ "*If* $\delta < \lambda$, $\delta \notin A$ *is measurable in* V, *then* δ *is* σ_δ *strongly compact and is not* $2^\delta = \delta^+$ *supercompact*".

Proof: Suppose δ is as in the hypotheses of Lemma 2.3. Write $\mathbb{P} = \mathbb{P}_\delta * \dot{\mathbb{P}}^\delta$, where $\mathbb{P}_\delta$ is the portion of $\mathbb{P}$ defined on measurable cardinals below δ, and $\dot{\mathbb{P}}^\delta$ is a term for the rest of $\mathbb{P}$. Since $V \models$ "δ is a measurable cardinal which is not a limit of measurable cardinals", it follows that $|\mathbb{P}_\delta| < \delta$ and the first ordinal at which $\dot{\mathbb{P}}^\delta$ is forced to do nontrivial forcing is a V-measurable cardinal above δ. The results of [15] therefore imply that $V^{\mathbb{P}_\delta} \models$ "δ is σ_δ strongly compact and is not a limit of measurable cardinals". Then, because [16, Lemma 2.4] implies that $\Vdash_{\mathbb{P}_\delta}$ "Forcing with $\dot{\mathbb{P}}^\delta$ adds no new subsets of σ_δ", $V^{\mathbb{P}_\delta * \dot{\mathbb{P}}^\delta} = V^{\mathbb{P}} \models$ "δ is σ_δ strongly compact and is not a limit of measurable cardinals". Since as we have already observed in the proof of Theorem 1, any δ which is 2^δ supercompact must be a limit of measurable cardinals, it immediately follows that $V^{\mathbb{P}} \models$ "δ is not $2^\delta = \delta^+$ supercompact". This completes the proof of Lemma 2.3.

$\square$

We now argue as in the proof of Theorem 1 to complete the proof of Theorem 2. Since $\mathbb{P}$ may be defined so that $|\mathbb{P}| = \lambda$, we may define $\overline{V}$ as in the proof of Theorem 1 and infer that $\overline{V} \models$ "ZFC +

GCH + κ is supercompact + No cardinal $\delta > \kappa$ is inaccessible". By Lemmas 2.1 – 2.3, in $\overline{V}$, λ is the least strongly compact cardinal, λ is not 2^λ supercompact, λ is a limit of measurable cardinals, and every measurable cardinal $\delta < \lambda$ is σ_δ strongly compact yet is not δ^+ supercompact. Consequently, as was the case previously, $\overline{V} \models$ "Level by level inequivalence between strong compactness and supercompactness holds". This completes the proof of Theorem 2.

□

We remark that in each of the models $\overline{V}$ constructed above, there are no supercompact cardinals below κ. This is since if there were, then there would be some $\delta < \kappa$ which is $<\sigma_\delta$ supercompact in both $\overline{V}$ and $V^{\mathbb{P}}$. This, of course, contradicts the choice of κ. On the other hand, the question of whether there can be more than one non-supercompact strongly compact cardinal below κ is quite intriguing. If we begin our constructions by forcing over a model V such that $V \models$ "ZFC + GCH + $\lambda < \kappa_1 < \kappa_2$ are such that λ and κ_1 are both supercompact and κ_2 is inaccessible + λ and κ_1 are the only strongly compact cardinals" (such as a model in [10]), and then force as in [4] to obtain the additional property that each measurable cardinal δ is σ_δ strongly compact, then the answer is no. This follows from Hamkins' gap forcing results of [12, 13], since the forcing of [4] is both "mild" and can be formulated to "admit a low enough gap" (both in the sense of [12, 13]) so that it cannot create any new strongly compact cardinals. However, when the ground model V does not satisfy this additional property, we do not currently know an answer to this question.

Turning now to the proof of Theorem 3, let $V \models$ "ZFC + GCH + $\kappa_1 < \kappa_2$ are such that κ_1 is supercompact and κ_2 is inaccessible + No cardinal $\delta > \kappa_1$ is measurable". Suppose F is a function satisfying properties (1) – (4) of the hypotheses of Theorem 3. By [17, Theorem 18], we further assume that V has been extended to a model V_* via class forcing $\mathbb{Q}$ such that $V_* \models$ "ZFC + $\kappa_1 < \kappa_2$ are such that κ_1 is supercompact + No cardinal $\delta > \kappa_1$ is measurable + For every regular cardinal δ, $2^\delta = F(\delta)$". Property (2) of F implies that $V_* \models$ "κ_2 is inaccessible" as well.

Let $A = \{\delta < \kappa_1 \mid \delta$ is a measurable cardinal which is not $<\sigma_\delta$ supercompact$\}$. Let $\mathbb{R}$ be Magidor's partial ordering of [16] which iteratively changes the cofinality of every $\delta \in A$ to ω via Prikry forcing.

Lemma 2.4 $V_*^{\mathbb{R}} \models$ "κ_1 *is supercompact".*

Proof: Suppose $\lambda \ge \kappa_2$ is arbitrary. Let $j : V_* \to M$ be an elementary embedding witnessing the λ supercompactness of κ_1 generated by a supercompact ultrafilter over $P_{\kappa_1}(\lambda)$. Since $M^\lambda \subseteq M$ and $V_* \models$ "No cardinal $\delta > \kappa_1$ is measurable", $M \models$ "κ_1 is $<\kappa_2$ supercompact + κ_2 is inaccessible + No cardinal $\delta \in (\kappa_1, \lambda]$ is measurable". Thus, since j is generated by a supercompact ultrafilter over $P_{\kappa_1}(\lambda)$, by the definition of $\mathbb{R}$, $j(\mathbb{R}) = \mathbb{R} * \dot{\mathbb{R}}'$, where the first ordinal γ at which $\dot{\mathbb{R}}'$ is forced to do nontrivial forcing is well above λ.

We follow now the proof of [2, Lemma 1.2] (which itself follows the proof of the Lemma of [1]) to show that $V_*^{\mathbb{R}} \models$ "κ_1 is λ supercompact". Let $|\ \ |$ be the distance function of [16]. Define a term $\dot{\mathcal{U}}$ in V_* by $p \Vdash$ "$\dot{B} \in \dot{\mathcal{U}}$" iff $p \Vdash$ "$\dot{B} \subseteq (P_{\kappa_1}(\lambda))^{V_*^{\mathbb{R}}}$" and there is $q \in j(\mathbb{R})$ such that $q \ge j(p)$ (q extends $j(p)$), $|j(p) - q| = 0$, $j(p) \restriction \gamma = q \restriction \gamma = j(p) \restriction \kappa_1 = q \restriction \kappa_1 = p$, and $q \Vdash$ "$\langle j(\beta) \mid \beta < \lambda \rangle \in j(\dot{B})$". By [16, Theorem 3.4], $\dot{\mathcal{U}}$ is a well-defined term for a strongly compact measure over $(P_{\kappa_1}(\lambda))^{V_*^{\mathbb{R}}}$ in $V_*^{\mathbb{R}}$.

To see that $\Vdash_{\mathbb{R}}$ "$\dot{\mathcal{U}}$ is normal", let $p \Vdash$ "$\dot{f} : (P_{\kappa_1}(\lambda))^{V_*^{\mathbb{R}}} \to \lambda$ is a function such that $\dot{f}(s) \in s$ for all $s \in \dot{B}$ where $\dot{B} \in \dot{\mathcal{U}}$". Let φ_α for $\alpha < \lambda$ be the statement "$j(\dot{f})(\langle j(\beta) \mid \beta < \lambda \rangle) = j(\alpha)$" in the forcing language with respect to $j(\mathbb{R})$, and consider the sequence $\langle \varphi_\alpha \mid \alpha < \lambda \rangle$. Since $M^\lambda \subseteq M$, $\langle \varphi_\alpha \mid \alpha < \lambda \rangle \in M$. Thus, since γ is the least M cardinal in the half-open interval $[\kappa_1, j(\kappa_1))$ at which $\dot{\mathbb{R}}'$ is forced to do nontrivial forcing and $\gamma > \lambda$, we can apply [16, Lemma 2.4] in M to $\langle \varphi_\alpha \mid \alpha < \lambda \rangle$ and obtain a condition $q \ge j(p)$, $q \in j(\mathbb{R})$ such that $|j(p) - q| = 0$, $j(p) \restriction \gamma = q \restriction \gamma = j(p) \restriction \kappa_1 = q \restriction \kappa_1 = p$, and if $q' \ge q$, q' decides φ_α for some $\alpha < \lambda$, then $q' \restriction \kappa_1 \cup (q - p)$ decides φ_α in the same way. Hence, since $p \Vdash$ "$\dot{B} \in \dot{\mathcal{U}}$" implies we can assume (by extending q if necessary) that $q \Vdash$ "$\langle j(\beta) \mid \beta < \lambda \rangle \in j(\dot{B})$", there must be some $\alpha < \lambda$ such that for some $q' \ge q$, $q' \Vdash \varphi_\alpha$, i.e., such that $q' \Vdash$ "$j(\dot{f})(\langle j(\beta) \mid \beta < \lambda \rangle) = j(\alpha)$". By choice of q, $q' \restriction \kappa_1 \cup (q - p) \Vdash \varphi_\alpha$, i.e., $q' \restriction \kappa_1 \ge p$ is such that for some $r \in j(\mathbb{R})$ (r can be taken as $q' \restriction \kappa_1 \cup (q - p)$), $|j(q' \restriction \kappa_1) - r| = 0$, $j(q' \restriction \kappa_1) \restriction \kappa_1 = r \restriction \kappa_1 = q' \restriction \kappa_1$, and $r \Vdash \varphi_\alpha$. Since $r \Vdash \varphi_\alpha$, $r \Vdash$ "$\langle j(\beta) \mid \beta < \lambda \rangle \in j(\{s \in \dot{B} \mid \dot{f}(s) = \alpha\})$", so $q' \restriction \kappa_1 \ge p$ is such that $q' \restriction \kappa_1 \Vdash$ "$\{s \in \dot{B} \mid \dot{f}(s) = \alpha\} \in \dot{\mathcal{U}}$". Thus, $V_*^{\mathbb{R}} \models$ "κ_1 is λ supercompact". Since $\lambda \ge \kappa_2$ was arbitrary, this completes the proof of Lemma 2.4.

□

Lemma 2.5 $V_*^{\mathbb{R}} \models$ *"If* $\delta < \kappa_1$ *is measurable, then* δ *is* $<\sigma_\delta$ *strongly compact".*

Proof: Suppose $V_*^{\mathbb{R}} \models$ "$\delta < \kappa_1$ is measurable". By [16, Theorem 3.1], $V_* \models$ "δ is measurable". Further, it must be the case that $V_* \models$ "δ is $<\sigma_\delta$ supercompact". This is since otherwise, by the definition of $\mathbb{R}$, $V_*^{\mathbb{R}} \models$ "$\mathrm{cof}(\delta) = \omega$", contradicting the measurability of δ in $V_*^{\mathbb{R}}$. Therefore, again by the definition of $\mathbb{R}$, as in Lemma 2.3, we may write $\mathbb{R} = \mathbb{R}_\delta * \dot{\mathbb{R}}^\delta$. By the proof of [16, Theorem 3.4], $V_*^{\mathbb{R}_\delta} \models$ "δ is $<\sigma_\delta$ strongly compact". Then, again as in Lemma 2.3, because [16, Lemma 2.4] implies that $\Vdash_{\mathbb{R}_\delta}$ "Forcing with $\dot{\mathbb{R}}^\delta$ adds no new subsets of σ_δ", $V_*^{\mathbb{R}_\delta * \dot{\mathbb{R}}^\delta} = V_*^{\mathbb{R}} \models$ "δ is $<\sigma_\delta$ strongly compact". This completes the proof of Lemma 2.5.

□

Since $\mathbb{R}$ may be defined so that $|\mathbb{R}| = \kappa_1$, $V_*^{\mathbb{R}} \models$ "κ_2 is inaccessible". Therefore, since $V_*^{\mathbb{R}} \models$ "κ_1 is supercompact", by reflection, let $\kappa < \kappa_1$ be the least cardinal which is $<\sigma_\kappa$ supercompact. By [16, Theorem 3.1], $V_* \models$ "κ is measurable". Further, because $V_*^{\mathbb{R}} \models$ "$\delta < \kappa_1$ has cofinality ω if δ is measurable but not $<\sigma_\delta$ supercompact in V_*", $V_* \models$ "κ is $<\sigma_\kappa$ supercompact". Consequently, $A \cap \kappa$ is unbounded in κ, and as in the proof of Lemma 2.2, $V_*^{\mathbb{R}} \models$ "No cardinal $\delta < \kappa$ is strongly compact". As in the proofs of Theorems 1 and 2, if we now let $\overline{V} = (V_{\sigma_\kappa})^{V_*^{\mathbb{R}}}$, $\overline{V} \models$ "ZFC + κ is supercompact + κ is the least strongly compact cardinal + Level by level inequivalence between strong compactness and supercompactness holds + No cardinal $\delta > \kappa$ is inaccessible". In addition, since by the work of [16], the Magidor iteration of Prikry forcing preserves both cardinals and the sizes of their power sets, in $V_*^{\mathbb{R}}$ and $\overline{V}$, for every regular cardinal δ, $2^\delta = F(\delta)$. This completes the proof of Theorem 3 (with $\mathbb{P}$ defined as $\mathbb{Q} * \dot{\mathbb{R}}$).

□

3 Concluding Remarks

We observe that a key difference between the proof of Theorem 3 and the proofs of Theorems 1 and 2 is that in Theorems 1 and 2, we are able to assume initially, without loss of generality, that every measurable cardinal δ is σ_δ strongly compact. This is accomplished by forcing over a model satisfying GCH. Without GCH, the proofs from [4] do not go through and allow us to assume that every measurable cardinal satisfies this degree of strong compactness. Since GCH will be false in V_*, a different approach is used in order to prove Theorem 3.

We note that it is possible to prove Theorems 1 – 3 using slightly weaker hypotheses. Theorems 1 and 2 may be established using the existence of cardinals $\lambda < \kappa_1 < \kappa_2$ such that λ and κ_1 are both κ_2 supercompact and κ_2 is inaccessible. Theorem 3 may be established using the existence of cardinals $\kappa_1 < \kappa_2$ such that κ_1 is κ_2 supercompact and κ_2 is inaccessible. To avoid excessive technicalities and simplify our exposition, however, we have established these theorems using the hypotheses previously mentioned.

Finally, it is of course the case that each of the models $\overline{V}$ constructed above has a rather limited large cardinal structure. By slightly modifying the proofs of Theorems 1 – 3 and truncating the universe not at σ_κ but at the least weakly compact cardinal above κ, the least Ramsey cardinal above κ, or in general, at some suitable large cardinal which is provably below the least measurable cardinal above κ, it is possible to assume that $\overline{V}$ has a nontrival, although still rather restricted, large cardinal structure above κ. We consequently conclude by asking in general what the possible large cardinal structures are in a universe containing supercompact cardinals which satisfies level by level inequivalence between strong compactness and supercompactness.

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