Inaccessible Cardinals, Failures of GCH, and Level by Level Equivalence *†

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April 16, 2012
(revised February 16, 2013)

Abstract

We construct models for the level by level equivalence between strong compactness and supercompactness containing failures of GCH at inaccessible cardinals. In one of these models, no cardinal is supercompact up to an inaccessible cardinal, and for every inaccessible cardinal δ, $2^\delta > \delta^{++}$. In another of these models, no cardinal is supercompact up to an inaccessible cardinal, and the only inaccessible cardinals at which GCH holds are also measurable. These results extend and generalize [1, Theorem 3].

1 Introduction and Preliminaries

In [1], the following theorem was proven.

*2010 Mathematics Subject Classifications: 03E35, 03E55.

†Keywords: Supercompact cardinal, strongly compact cardinal, inaccessible cardinal, GCH, level by level equivalence between strong compactness and supercompactness.

‡The author's research was partially supported by PSC-CUNY grants.

§The author wishes to thank Brent Cody for helpful discussions on the subject matter of this paper.

¶The author wishes to thank the referee, for helpful comments and suggestions which have been incorporated into the current version of the paper.

Theorem 1 ([1, Theorem 3]) *Suppose $V \models$ "ZFC + $\mathcal{K} \neq \emptyset$ is the class of supercompact cardinals". There is then a partial ordering $\mathbb{P} \subseteq V$ such that $V^{\mathbb{P}} \models$ "ZFC + $\mathcal{K}$ is the class of supercompact cardinals". In $V^{\mathbb{P}}$, $2^\delta = \delta^{++}$ if δ is inaccessible, and $2^\delta = \delta^+$ if δ is not inaccessible. Further, in $V^{\mathbb{P}}$, for every pair of regular cardinals $\kappa < \lambda$, κ is λ strongly compact iff κ is λ supercompact, except possibly if κ is a measurable limit of cardinals δ which are λ supercompact, or λ is inaccessible.*

If our ground model V satsifies level by level equivalence between strong compactness and supercompactness and in addition is such that $V \models$ "ZFC + GCH + κ is supercompact + No cardinal is supercompact up to an inaccessible cardinal", then we have the next result as an immediate corollary.

Theorem 2 *Suppose $V \models$ "ZFC + GCH + κ is supercompact". Assume in addition that in V, no cardinal is supercompact up to an inaccessible cardinal, and level by level equivalence between strong compactness and supercompactness holds. There is then a partial ordering $\mathbb{P} \in V$ such that $V^{\mathbb{P}} \models$ "ZFC + κ is supercompact". In $V^{\mathbb{P}}$, no cardinal is supercompact up to an inaccessible cardinal, and level by level equivalence between strong compactness and supercompactness holds. Further, in $V^{\mathbb{P}}$, for every inaccessible cardinal δ, $2^\delta = \delta^{++}$, and for every cardinal δ which is not inaccessible, $2^\delta = \delta^+$.*

The techniques of [1], however, will not produce models analogous to the one for Theorem 2 in which $2^\delta > \delta^{++}$ for every inaccessible cardinal δ. In addition, [1, Theorem 3] says nothing about whether it is possible to obtain similar models in which only certain inaccessible cardinals violate GCH. This raises the following two questions.

Question 1: Is it possible to construct models such as those of Theorem 2 in which, for every inaccessible cardinal δ, $2^\delta > \delta^{++}$?

Question 2: Is it possible to construct models analogous to those of [1, Theorem 3] in which only certain inaccessible cardinals violate GCH?

The purpose of this paper is to answer the above questions in the affirmative. Specifically, we will prove the following theorems.

Theorem 3 *Suppose $V \models$ "ZFC + GCH + κ is supercompact". Assume in addition that in V, no cardinal is supercompact up to an inaccessible cardinal, and level by level equivalence between strong compactness and supercompactness holds. Let $h : \kappa + 1 \to \mathrm{Ord}$ satisfy the following four conditions.*

1. *$h(\delta) = 0$ if δ is not an inaccessible cardinal.*
2. *For δ an inaccessible cardinal, $h(\delta)$ has the properties that $h(\delta) > \delta^+$, $h(\delta)$ is the successor of a cardinal of cofinality greater than δ, and $h(\delta)$ is below the least inaccessible cardinal above δ.*
3. *Let ρ_δ be the cardinal predecessor of $h(\delta)$. If $\delta < \kappa$ is ρ_δ supercompact, there is $j_\delta : V \to M$ witnessing the ρ_δ supercompactness of δ which is generated by a supercompact ultrafilter over $P_\delta(\rho_\delta)$ with $j_\delta(h)(\delta) = h(\delta)$.*
4. *If $\delta \leq \kappa$ is γ supercompact and $\gamma \geq h(\delta)$, there is $j_{\delta,\gamma} : V \to M$ witnessing the γ supercompactness of δ which is generated by a supercompact ultrafilter over $P_\delta(\gamma)$ with $j_{\delta,\gamma}(h)(\delta) = h(\delta)$.*

There is then a partial ordering $\mathbb{P} \in V$ such that $V^{\mathbb{P}} \models$ "ZFC + κ is supercompact + No cardinal is supercompact up to an inaccessible cardinal". In $V^{\mathbb{P}}$, level by level equivalence between strong compactness and supercompactness holds, and for every $\delta \leq \kappa$ which is an inaccessible cardinal, $2^\gamma = h(\delta)$ for all cardinals $\gamma \in [\delta, h(\delta))$. Further, in $V^{\mathbb{P}}$, δ is ρ_δ supercompact if δ is a measurable cardinal.

Theorem 4 *Suppose $V \models$ "ZFC + $\mathcal{K} \neq \emptyset$ is the class of supercompact cardinals". There is then a partial ordering $\mathbb{P} \subseteq V$ such that $V^{\mathbb{P}} \models$ "ZFC + $\mathcal{K}$ is the class of supercompact cardinals". In $V^{\mathbb{P}}$, $2^\delta = \delta^{++}$ if δ is a nonmeasurable inaccessible cardinal, and $2^\delta = \delta^+$ if δ is a measurable cardinal. Further, in $V^{\mathbb{P}}$, for every pair of regular cardinals $\kappa < \lambda$, κ is λ strongly compact iff κ is λ supercompact, except possibly if κ is a measurable limit of cardinals δ which are λ supercompact, or λ is a nonmeasurable inaccessible cardinal.*

As a corollary to the proof of Theorem 4, we will also have the following theorem.

Theorem 5 *Suppose $V \models$ "ZFC + GCH + κ is supercompact". Assume in addition that in V, no cardinal is supercompact up to an inaccessible cardinal, and level by level equivalence between strong compactness and supercompactness holds. There is then a partial ordering $\mathbb{P} \in V$ such that $V^{\mathbb{P}} \models$ "ZFC + κ is supercompact". In $V^{\mathbb{P}}$, no cardinal is supercompact up to an inaccessible cardinal, and level by level equivalence between strong compactness and supercompactness holds. Further, in $V^{\mathbb{P}}$, for every nonmeasurable inaccessible cardinal δ, $2^\delta = \delta^{+19}$, and for every measurable cardinal δ, $2^\delta = \delta^+$.*

We take this opportunity to make some remarks concerning the above theorems. First, we note that although the conditions on h in Theorem 3 appear to be rather technical in nature, they are actually satisfied by many naturally occurring functions. For instance, if δ is an inaccessible cardinal and $h(\delta)$ is defined as δ^{+19}, the successor of the least V-strong limit cardinal greater than δ of cofinality δ^{++}, $\delta^{+\omega+5}$, etc., then h satisfies the conditions of Theorem 3. In addition, we observe that in Theorems 2, 3, and 5, it immediately follows that κ is the least supercompact cardinal. This is because in each case, in V and $V^{\mathbb{P}}$, no cardinal is supercompact up to an inaccessible cardinal. Finally, in Theorem 4, we explicitly note that our techniques require it must be the case that $2^\delta = \delta^{++}$ if δ is an inaccessible cardinal which is not also measurable. However, as our proof will show, there are many different possible values for 2^δ in Theorem 5 (e.g., δ^{++}, δ^{+5}, the successor of the first $\aleph$ fixed point above δ, etc.) if δ is an inaccessible cardinal which is not also measurable.

We now give some preliminary information concerning notation and terminology. For anything left unexplained, readers are urged to consult [4] or [5]. When forcing, $q \geq p$ means that *q is stronger than p*. For κ a regular cardinal and λ an ordinal, $\mathrm{Add}(\kappa, \lambda)$ is the standard partial ordering for adding λ many Cohen subsets of κ. For $\alpha < \beta$ ordinals, $[\alpha, \beta]$, $[\alpha, \beta)$, $(\alpha, \beta]$, and (α, β) are as in standard interval notation. If G is V-generic over $\mathbb{P}$, we will abuse notation slightly and use both $V[G]$ and $V^{\mathbb{P}}$ to indicate the universe obtained by forcing with $\mathbb{P}$. We will, from time to time, confuse terms with the sets they denote and write x when we actually mean $\dot{x}$ or $\check{x}$.

The partial ordering $\mathbb{P}$ is *κ-directed closed* if every directed set of conditions of size less than κ has an upper bound. $\mathbb{P}$ is *κ-strategically closed* if in the two person game in which the players construct an increasing sequence $\langle p_\alpha : \alpha \le \kappa \rangle$, where player I plays odd stages and player II plays even stages (choosing the trivial condition at stage 0), player II has a strategy which ensures the game can always be continued. Note that if $\mathbb{P}$ is κ-strategically closed and $f : \kappa \to V$ is a function in $V^{\mathbb{P}}$, then $f \in V$. $\mathbb{P}$ is *$<\kappa$-strategically closed* if $\mathbb{P}$ is δ-strategically closed for all cardinals $\delta < \kappa$. It is in addition the case that if $\mathbb{P}$ is κ-directed closed, then $\mathbb{P}$ is $<\kappa$-strategically closed.

Suppose V is a model of ZFC in which for all regular cardinals $\kappa < \lambda$, κ is λ strongly compact iff κ is λ supercompact, except possibly if κ is a measurable limit of cardinals δ which are λ supercompact. Such a universe will be said to witness *level by level equivalence between strong compactness and supercompactness*. The exception is provided by a theorem of Menas [13], who showed that if κ is a measurable limit of cardinals δ which are λ strongly compact, then κ is λ strongly compact but need not be λ supercompact. Any model of ZFC with this property also witnesses the Kimchi-Magidor property [9] that the classes of strongly compact and supercompact cardinals coincide precisely, except at measurable limit points. Models in which GCH and level by level equivalence between strong compactness and supercompactness hold nontrivially were first constructed in [4].

We assume familiarity with the large cardinal notions of measurability, strong compactness, and supercompactness. Readers are urged to consult [8] for further details. We do note, however, that we will say κ *is supercompact up to the inaccessible cardinal* λ if κ is δ supercompact for every $\delta < \lambda$.

A corollary of Hamkins' work on gap forcing found in [6] and [7] will be employed in the proofs of Theorems 3 – 5. We therefore state as a separate theorem what is relevant for this paper, along with some associated terminology, quoting from [6] and [7] when appropriate. Suppose $\mathbb{P}$ is a partial ordering which can be written as $\mathbb{Q} * \dot{\mathbb{R}}$, where $|\mathbb{Q}| < \delta$, $\mathbb{Q}$ is nontrivial, and $\Vdash_{\mathbb{Q}}$ "$\dot{\mathbb{R}}$ is δ-strategically closed". In Hamkins' terminology of [6] and [7], $\mathbb{P}$ *admits a gap at* δ. In Hamkins' terminology of [6] and [7], $\mathbb{P}$ is *mild* with respect to a cardinal κ iff every set of ordinals x in $V^{\mathbb{P}}$ of size less than κ has a "nice" name τ in V of size less than κ, i.e., there is a set y in V, $|y| < \kappa$, such that any ordinal forced by a condition in $\mathbb{P}$ to be in τ is an element of y. Also, as in the terminology of [6], [7], and elsewhere, an embedding $j : \overline{V} \to \overline{M}$ is *amenable to* $\overline{V}$ when $j \restriction A \in \overline{V}$ for any $A \in \overline{V}$. The specific corollary of Hamkins' work from [6] and [7] we will be using is then the following.

Theorem 6 (Hamkins) *Suppose that $V[G]$ is a generic extension obtained by forcing that admits a gap at some regular $\delta < \kappa$. Suppose further that $j : V[G] \to M[j(G)]$ is an embedding with critical point κ for which $M[j(G)] \subseteq V[G]$ and $M[j(G)]^\delta \subseteq M[j(G)]$ in $V[G]$. Then $M \subseteq V$; indeed, $M = V \cap M[j(G)]$. If the full embedding j is amenable to $V[G]$, then the restricted embedding $j \restriction V : V \to M$ is amenable to V. If j is definable from parameters (such as a measure or extender) in $V[G]$, then the restricted embedding $j \restriction V$ is definable from the names of those parameters in V. Finally, if $\mathbb{P}$ is mild with respect to κ and κ is λ strongly compact in $V[G]$ for any $\lambda \geq \kappa$, then κ is λ strongly compact in V.*

Finally, at several junctures throughout the course of this paper, we will mention the "standard lifting techniques" for lifting a λ supercompactness embedding $j : V \to M$ generated by a supercompactness measure over $P_\kappa(\lambda)$ to a generic extension given by a suitably defined Easton support iteration. Although there are numerous references to this in the literature, we will use the argument found in [2, Theorem 4] as the basis for the sketch we are about to present. Very briefly, this argument assumes the following.

1. $V \models$ GCH.

2. λ is a regular cardinal.

3. $\mathbb{P} * \dot{\mathbb{Q}} = \langle \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle : \alpha \leq \kappa \rangle$ is an Easton support iteration having length $\kappa + 1$.

4. For any inaccessible cardinal $\delta \leq \kappa$, $\Vdash_{\mathbb{P}_\delta}$ "$\dot{\mathbb{Q}}_\delta$ is $<\delta$-strategically closed".

5. $G_0 * G_1$ is V-generic over $\mathbb{P} * \dot{\mathbb{Q}}$.

6. $\Vdash_{\mathbb{P}}$ "$|\dot{\mathbb{Q}}| \leq \lambda$ and $\dot{\mathbb{Q}}$ is κ-directed closed".

7. $j(\mathbb{P} * \dot{\mathbb{Q}}) = \mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}} * j(\dot{\mathbb{Q}})$.

Since $V \vDash$ GCH, $M[G_0][G_1] \vDash$ "$|\mathbb{R}| = j(\kappa)$", and $V \vDash$ "$|j(\kappa^+)| = |j(2^\kappa)| = |\{f : f : P_\kappa(\lambda) \to \kappa^+\}| = |\{f : f : \lambda \to \kappa^+\}| = |\{f : f : \lambda \to \lambda\}|$", $V[G_0][G_1] \vDash$ "There are $\lambda^+ = 2^\lambda = |j(\kappa^+)| = |j(2^\kappa)|$ many dense open subsets of $\mathbb{R}$ present in $M[G_0][G_1]$". Because $M[G_0][G_1]$ remains λ-closed with respect to $V[G_0][G_1]$ and $\mathbb{R}$ is λ-strategically closed in both $M[G_0][G_1]$ and $V[G_0][G_1]$, working in $V[G_0][G_1]$, it is possible to build an $M[G_0][G_1]$-generic object G_2 over $\mathbb{R}$ such that $j''G_0 \subseteq G_0 * G_1 * G_2$. Still working in $V[G_0][G_1]$, one then lifts j to $j : V[G_0] \to M[G_0][G_1][G_2]$. Since $M[G_0][G_1][G_2]$ remains λ-closed with respect to $V[G_0][G_1]$ and $V[G_0] \vDash$ "$|\mathbb{Q}| \le \lambda$", there is a master condition $q \in V[G_0][G_1]$ for $\{j(p) : p \in G_1\}$. Because $V \vDash$ "$|j(\lambda^+)| = |j(2^\lambda)| = |\{f : f : P_\kappa(\lambda) \to \lambda^+\}| = |\{f : f : \lambda \to \lambda^+\}| = \lambda^+$" and $M[G_0][G_1][G_2] \vDash$ "$|j(\mathbb{Q})| \le j(\lambda)$", we may then build in $V[G_0][G_1]$ an $M[G_0][G_1][G_2]$-generic object G_3 for $j(\mathbb{Q})$ containing q. It is then the case that $j''(G_0 * G_1) \subseteq G_0 * G_1 * G_2 * G_3$, so we may fully lift j in $V[G_0][G_1]$ to a λ supercompactness embedding $j : V[G_0][G_1] \to M[G_0][G_1][G_2][G_3]$. This argument remains valid (and in fact becomes even simpler) if no forcing is done at stage κ in V, i.e., if $\dot{\mathbb{Q}}$ is a term for trivial forcing.

2 Forcing Notions from [4] and [5]

In order to present in a meaningful way the iteration to be used in the proof of Theorem 3, we first recall the definitions and properties of the fundamental building blocks of this partial ordering. In particular, we describe now a specific form of the partial orderings $\mathbb{P}^0_{\delta,\lambda}$, $\mathbb{P}^1_{\delta,\lambda}[S]$, and $\mathbb{P}^2_{\delta,\lambda}[S]$ of [5, Section 4], where the fixed but arbitrary regular cardinal $\gamma < \delta$ is replaced by the specific regular cardinal ω. So that readers are not overly burdened, we abbreviate our definitions and descriptions somewhat. Full details may be found by consulting [5], along with the relevant portions of [4]. We quote nearly verbatim from [3, Section 2].

Fix $\delta < \lambda$, $\lambda > \delta^+$ regular cardinals in our ground model V, with δ inaccessible and λ the successor of a cardinal of cofinality greater than δ. We assume GCH holds for all cardinals $\eta \ge \delta$. The first notion of forcing $\mathbb{P}^0_{\delta,\lambda}$ is just the standard notion of forcing for adding a nonreflecting stationary set of ordinals S of cofinality ω to λ. Next, work in $V_1 = V^{\mathbb{P}^0_{\delta,\lambda}}$, letting $\dot{S}$ be a term always forced to denote S. $\mathbb{P}^2_{\delta,\lambda}[S]$ is the standard notion of forcing for introducing a club set C

which is disjoint to S (and therefore makes S nonstationary).

We fix now in V_1 a $\clubsuit(S)$ sequence $X = \langle x_\alpha : \alpha \in S \rangle$, the existence of which is given by [4, Lemma 1] and [5, Lemma 1]. We are ready to define in V_1 the partial ordering $\mathbb{P}^1_{\delta,\lambda}[S]$. First, since each element of S has cofinality ω, the proofs of [4, Lemma 1] and [5, Lemma 1] show each $x \in X$ can be assumed to be such that order-type$(x) = \omega$. Then, $\mathbb{P}^1_{\delta,\lambda}[S]$ is defined as the set of all 4-tuples $\langle w, \alpha, \bar{r}, Z \rangle$ satisfying the following properties.

1. $w \in [\lambda]^{<\delta}$.

2. $\alpha < \delta$.

3. $\bar{r} = \langle r_i : i \in w \rangle$ is a sequence of functions from α to $\{0, 1\}$, i.e., a sequence of subsets of α.

4. $Z \subseteq \{x_\beta : \beta \in S\}$ is a set such that if $z \in Z$, then for some $y \in [w]^\omega$, $y \subseteq z$ and $z - y$ is bounded in the β such that $z = x_\beta$.

The ordering on $\mathbb{P}^1_{\delta,\lambda}[S]$ is given by $\langle w^1, \alpha^1, \bar{r}^1, Z^1 \rangle \le \langle w^2, \alpha^2, \bar{r}^2, Z^2 \rangle$ iff the following hold.

1. $w^1 \subseteq w^2$.

2. $\alpha^1 \le \alpha^2$.

3. If $i \in w^1$, then $r^1_i \subseteq r^2_i$.

4. $Z^1 \subseteq Z^2$.

5. If $z \in Z^1 \cap [w^1]^\omega$ and $\alpha^1 \le \alpha < \alpha^2$, then $|\{i \in z : r^2_i(\alpha) = 0\}| = |\{i \in z : r^2_i(\alpha) = 1\}| = \omega$.

The proof of [4, Lemma 4] shows that $\mathbb{P}^0_{\delta,\lambda} * (\mathbb{P}^1_{\delta,\lambda}[\dot{S}] \times \mathbb{P}^2_{\delta,\lambda}[\dot{S}])$ is forcing equivalent to $\mathrm{Add}(\lambda, 1) * \dot{\mathrm{Add}}(\delta, \lambda)$. The proofs of [4, Lemmas 3 and 5] and [5, Lemma 6] show that $\mathbb{P}^0_{\delta,\lambda} * \mathbb{P}^1_{\delta,\lambda}[\dot{S}]$ preserves cardinals and cofinalities, is λ^+-c.c., is $<\delta$-strategically closed, and is such that $V^{\mathbb{P}^0_{\delta,\lambda} * \mathbb{P}^1_{\delta,\lambda}[\dot{S}]} \models$ "$2^\eta = \lambda$ for every cardinal $\eta \in [\delta, \lambda)$ and δ is nonmeasurable". These proofs are valid regardless of the cofinality of the ordinals in S, and in particular, hold when the fixed but arbitrary regular cardinal $\gamma < \delta$ found in the definitions given in [5, Section 4] is replaced by the specific regular cardinal ω.

3 The Proofs of Theorems 3 and 4

We turn now to the proof of Theorem 3.

Proof: Suppose V, h, and κ are as in the hypotheses for Theorem 3. In particular, recall that by condition (3) on h, ρ_δ is the cardinal predecessor of $h(\delta)$, so $h(\delta) = \rho_\delta^+$. The partial ordering $\mathbb{P}$ used in the proof of Theorem 3 is the Easton support iteration having length $\kappa + 1$ which begins by forcing with $\mathrm{Add}(\omega, 1)$ and then does trivial forcing, except at stages $\delta \le \kappa$ which are inaccessible cardinals in V. If such a δ is not ρ_δ supercompact, then the forcing done at stage δ is $\mathbb{P}^0_{\delta,h(\delta)} * \mathbb{P}^1_{\delta,h(\delta)}[\dot{S}_{h(\delta)}]$, where $\dot{S}_{h(\delta)}$ is a term for the nonreflecting stationary set of ordinals of cofinality ω introduced by $\mathbb{P}^0_{\delta,h(\delta)}$. If such a δ is ρ_δ supercompact, then the forcing done at stage δ is $\mathbb{P}^0_{\delta,h(\delta)} * (\mathbb{P}^1_{\delta,h(\delta)}[\dot{S}_{h(\delta)}] \times \mathbb{P}^2_{\delta,h(\delta)}[\dot{S}_{h(\delta)}])$, where $\dot{S}_{h(\delta)}$ is as in the previous sentence.

Lemma 3.1 $V^{\mathbb{P}} \models$ *"κ is supercompact".*

Proof: We modify the proof of [3, Lemma 3.1]. Suppose $\lambda > h(\kappa)$ is any regular cardinal, and $j : V \to M$ is any elementary embedding witnessing the λ supercompactness of κ which is generated by a supercompact ultrafilter over $P_\kappa(\lambda)$. Since $V \models$ "No cardinal is supercompact up to an inaccessible cardinal", $M \models$ "No cardinal in the half-open interval $(\kappa, \lambda]$ is inaccessible". From this, it immediately follows that $j(\mathbb{P}) = \mathbb{P} * \dot{\mathbb{Q}}$, where the first ordinal at which $\dot{\mathbb{Q}}$ is forced to act nontrivially is well above λ. Since $V \models$ GCH, the standard lifting arguments mentioned in Section 1 now apply and show that $V^{\mathbb{P}} \models$ "κ is λ supercompact". Since λ was arbitrary, this completes the proof of Lemma 3.1.

□

Lemma 3.2 $V^{\mathbb{P}} \models$ *"No cardinal is supercompact up to an inaccessible cardinal".*

Proof: Suppose $V^{\mathbb{P}} \models$ "There exists a cardinal which is supercompact up to an inaccessible cardinal". Write $\mathbb{P} = \mathbb{P}_0 * \dot{\mathbb{Q}}$, where $|\mathbb{P}_0| = \omega$, $\mathbb{P}_0$ is nontrivial, and $\Vdash_{\mathbb{P}_0}$ "$\dot{\mathbb{Q}}$ is $\aleph_1$-strategically closed". By Theorem 6, this factorization of $\mathbb{P}$ and the fact that forcing cannot create an inaccessible cardinal indicate that any δ which is supercompact up to an inaccessible cardinal in $V^{\mathbb{P}}$ had to have been supercompact up to an inaccessible cardinal in V. Since $V \models$ "No cardinal is supercompact up to an inaccessible cardinal", this is impossible. This completes the proof of Lemma 3.2.

$\square$

Lemma 3.3 $V^{\mathbb{P}} \models$ "*Level by level equivalence between strong compactness and supercompactness holds*".

Proof: We modify the proof of [3, Lemma 3.3]. Suppose $V^{\mathbb{P}} \models$ "$\delta < \lambda$ are regular cardinals such that δ is λ strongly compact". We begin by noting that $V \models$ "δ is λ supercompact". To see this, by the definition of $\mathbb{P}$, it is easily established that any subset of δ in $V^{\mathbb{P}}$ of size below δ has a name of size below δ in V. Therefore, by the factorization of $\mathbb{P}$ given in the proof of Lemma 3.2 and Theorem 6, $V \models$ "δ is λ strongly compact". Since $V \models$ "No cardinal is supercompact up to an inaccessible cardinal", $V \models$ "δ is not a measurable limit of cardinals γ which are λ supercompact". Thus, by level by level equivalence between strong compactness and supercompactness, $V \models$ "δ is λ supercompact". Further, $V \models$ "λ is below the least inaccessible cardinal ζ above δ".

Continuing with the proof of Lemma 3.3, because $V \models$ "No cardinal is supercompact up to an inaccessible cardinal and κ is supercompact", $V \models$ "No cardinal $\rho > \kappa$ is inaccessible". Consequently, $V^{\mathbb{P}} \models$ "No cardinal $\rho > \kappa$ is inaccessible" as well. From this, it immediately follows that $\delta \leq \kappa$. By Lemma 3.1, Lemma 3.3 is true if $\delta = \kappa$. It therefore suffices to prove Lemma 3.3 when $\delta < \kappa$, which we assume for the duration of the proof of this lemma.

Let $A = \{\gamma \leq \delta : \gamma \text{ is an inaccessible cardinal}\}$. Write $\mathbb{P} = \mathbb{P}_A * \dot{\mathbb{Q}}$, where $\mathbb{P}_A$ is the portion of $\mathbb{P}$ acting on ordinals at most δ, and $\dot{\mathbb{Q}}$ is a term for the rest of $\mathbb{P}$, i.e., the portion of $\mathbb{P}$ acting on ordinals above δ. Since $\lambda < \zeta$ and $\Vdash_{\mathbb{P}_A}$ "$\dot{\mathbb{Q}}$ is ζ-strategically closed", to complete the proof of Lemma 3.3, it hence suffices to show that $V^{\mathbb{P}_A} \models$ "δ is λ supercompact".

Consider now the following two cases.

Case 1: $\sup(A) = \sigma < \delta$. If this is true, then by the definition of $\mathbb{P}$, it must be the case that $|\mathbb{P}_A| < \delta$. Thus, by the Lévy-Solovay results [11], $V^{\mathbb{P}_A} \models$ "δ is λ supercompact" as well.

Case 2: $\sup(A) = \delta$. It must be the case that $V \models$ "δ is ρ_δ supercompact", because otherwise, by the definition of $\mathbb{P}$, $V^{\mathbb{P}_A} \models$ "δ is not measurable". However, by the arguments found in [5, next to last paragraph on page 2033], $V^{\mathbb{P}_A} \models$ "δ is ρ_δ supercompact". Hence, we may assume without loss of generality that $\lambda \geq h(\delta) = \rho_\delta^+$. Consequently, let $j : V \to M$ be an elementary embedding witnessing the λ supercompactness of δ satisfying condition (4) of Theorem 3 (so j is generated by a supercompact ultrafilter over $P_\delta(\lambda)$ and is such that $j(h)(\delta) = h(\delta)$). Since $\lambda \geq h(\delta)$, $\mathbb{P}_A$ is forcing equivalent to $\mathbb{P}_\delta * \dot{\mathbb{Q}}^*$, where $\Vdash_{\mathbb{P}_\delta}$ "$|\dot{\mathbb{Q}}^*| = h(\delta) \leq \lambda$ and $\dot{\mathbb{Q}}^*$ is δ-directed closed". ($\mathbb{Q}^*$ is forcing equivalent to $\mathrm{Add}(h(\delta), 1) * \dot{\mathrm{A}}\mathrm{dd}(\delta, h(\delta))$.) In addition, the same reasoning as found in the proof of Lemma 3.1 shows that $M \models$ "No cardinal in the half-open interval $(\delta, \lambda]$ is inaccessible". Thus, $j(\mathbb{P}_\delta * \dot{\mathbb{Q}}^*)$ is forcing equivalent to $\mathbb{P}_\delta * \dot{\mathbb{Q}}^* * \dot{\mathbb{R}} * j(\dot{\mathbb{Q}}^*)$, where the first ordinal at which $\dot{\mathbb{R}}$ is forced to act nontrivially is well above λ. As in the proof of Lemma 3.1, the standard lifting arguments mentioned in Section 1 are then once again applicable and show that $V^{\mathbb{P}_A} \models$ "δ is λ supercompact". This completes the proof of Case 2 and Lemma 3.3.

□

By the remarks in the last paragraph of Section 2, the fact that by condition (2) of Theorem 3, for any inaccessible cardinal δ, $h(\delta)$ is below the least inaccessible cardinal above δ, and the definition of $\mathbb{P}$, $V^{\mathbb{P}} \models$ "For every $\delta \leq \kappa$ which is an inaccessible cardinal, $2^\gamma = h(\delta)$ for all cardinals $\gamma \in [\delta, h(\delta))$". By the proof of Lemma 3.3, $V^{\mathbb{P}} \models$ "δ is ρ_δ supercompact if δ is a measurable cardinal". These observations, together with Lemmas 3.1 – 3.3, complete the proof of Theorem 3.

□

Having completed the proof of Theorem 3, we turn now to the proof of Theorem 4.

Proof: Let $V \models$ "ZFC + $\mathcal{K}$ is the class of supercompact cardinals". Without loss of generality, by first doing a preliminary forcing as in [4] if necessary, we may also assume that GCH and level by level equivalence between strong compactness and supercompactness hold in V. This allows us to define in V our partial ordering $\mathbb{P}$ as the Easton support iteration which begins by forcing with $\mathrm{Add}(\omega, 1)$ and then does nontrivial forcing only at stages δ which are inaccessible cardinals in V. If $V \models$ "δ is inaccessible but nonmeasurable", then $\mathbb{P}_{\delta+1} = \mathbb{P}_\delta * \dot{\mathbb{Q}}_\delta$, where $\dot{\mathbb{Q}}_\delta$ is a term for $\mathrm{Add}(\delta, \delta^{++})$.

If $V \models$ "δ is measurable", then $\mathbb{P}_{\delta+1} = \mathbb{P}_\delta * \dot{\mathbb{Q}}_\delta$, where $\dot{\mathbb{Q}}_\delta$ is a term for $\mathrm{Add}(\delta, \delta^+)$. Exactly the same arguments as in the proof of [1, Theorem 3] (i.e., standard arguments in tandem with Theorem 6) show that cardinals and cofinalities are preserved when forcing with $\mathbb{P}$ and $V^{\mathbb{P}} \models$ "ZFC + $\mathcal{K}$ is the class of supercompact cardinals". By the definition of $\mathbb{P}$, it is further the case that the inaccessible cardinals of V and $V^{\mathbb{P}}$ are precisely the same and $V^{\mathbb{P}} \models$ "$2^\delta = \delta^{++}$ if δ is inaccessible but nonmeasurable in V + $2^\delta = \delta^+$ if δ is measurable in V". Thus, the proof of Theorem 4 will be complete once we have established the following three lemmas.

Lemma 3.4 *If $V \models$ "$\kappa < \lambda$ are such that κ is λ supercompact and λ is a successor cardinal", then $V^{\mathbb{P}} \models$ "κ is λ supercompact".*

Proof: We follow the proof of [1, Lemma 3.1], quoting almost verbatim when appropriate. If κ and λ are as in the hypotheses of Lemma 3.4, then we consider the following two cases.

Case 1: Either λ is not the successor of an inaccessible cardinal or λ is the successor of a measurable cardinal. Write $\mathbb{P} = \mathbb{P}_\lambda * \dot{\mathbb{P}}^\lambda$, where $\mathbb{P}_\lambda$ acts nontrivially on ordinals below λ, and $\mathbb{P}^\lambda$ consists of the rest of $\mathbb{P}$. By the choice of λ, $|\mathbb{P}_\lambda| \le \lambda$. Suppose $j : V \to M$ is an elementary embedding witnessing the λ supercompactness of κ which is generated by a supercompact ultrafilter over $P_\kappa(\lambda)$ and that $\lambda = \delta^+$. Note that since $2^\delta = \delta^+ = \lambda$ and $M^\lambda \subseteq M$, $V \models$ "λ is the successor of a measurable cardinal" iff $M \models$ "λ is the successor of a measurable cardinal". Hence, by the definition of $\mathbb{P}_\lambda$, no matter which of the two clauses in Case 1 holds, $\mathbb{P}_\lambda$ is an initial segment of $j(\mathbb{P}_\lambda)$. Therefore, the standard lifting arguments mentioned in Section 1 once again show that $V^{\mathbb{P}_\lambda} \models$ "κ is λ supercompact". Since $\Vdash_{\mathbb{P}_\lambda}$ "$\dot{\mathbb{P}}^\lambda$ is $(2^{[\lambda]^{<\kappa}})^+$-directed closed", $V^{\mathbb{P}_\lambda * \dot{\mathbb{P}}^\lambda} = V^{\mathbb{P}} \models$ "κ is λ supercompact".

Case 2: λ is the successor of a nonmeasurable inaccessible cardinal. Once again, write $\mathbb{P} = \mathbb{P}_\lambda * \dot{\mathbb{P}}^\lambda$, where $\mathbb{P}_\lambda$ acts nontrivially on ordinals below λ, and $\mathbb{P}^\lambda$ is the rest of $\mathbb{P}$. In this instance, it is not the case that $|\mathbb{P}_\lambda| \le \lambda$, since for the δ such that $\lambda = \delta^+$, $|\mathbb{P}_\lambda| = \delta^{++} = \lambda^+ > \lambda$. However, arguments originally due to Magidor [12], which are also given in both [4, pages 119–120] and [1, Case 2 of

Lemma 3.1] and are found other places in the literature as well, will yield that $V^{\mathbb{P}_\lambda} \vDash$ "κ is $\lambda = \delta^+$ supercompact". For the convenience of readers, we present these arguments below.

Write $\mathbb{P}_\lambda = \mathbb{Q}_0 * \dot{\mathbb{Q}}_1 * \dot{\mathrm{Add}}(\delta, \delta^{++})$, where $\mathbb{Q}_0$ acts nontrivially on ordinals below κ, and $\dot{\mathbb{Q}}_1$ is forced to act nontrivially on all remaining ordinals in the interval $[\kappa, \delta)$. Let G be V-generic over $\mathbb{P}_\lambda$, with $G_0 * G_1 * G_2$ the corresponding factorization of G. Fix $j : V \to M$ an elementary embedding witnessing the $\lambda = \delta^+ = 2^\delta$ supercompactness of κ which is generated by a supercompact ultrafilter $\mathcal{U}$ over $P_\kappa(\lambda)$. Since $M \vDash$ "λ is the successor of a nonmeasurable inaccessible cardinal", we then have $j(\mathbb{P}_\lambda) = \mathbb{Q}_0 * \dot{\mathbb{Q}}_1 * \dot{\mathrm{Add}}(\delta, \delta^{++}) * \dot{\mathbb{R}}_0 * \dot{\mathbb{R}}_1$, where $\dot{\mathbb{R}}_1$ is a term for $\mathrm{Add}(j(\delta), j(\delta^{++}))$ as computed in $M^{\mathbb{Q}_0 * \dot{\mathbb{Q}}_1 * \dot{\mathrm{Add}}(\delta, \delta^{++}) * \dot{\mathbb{R}}_0}$. Therefore, as in [1, Case 2 of Lemma 3.1], since $M[G_0][G_1][G_2]$ remains λ-closed with respect to $V[G_0][G_1][G_2]$ and $V \vDash \mathrm{GCH}$, it is possible working in $V[G_0][G_1][G_2]$ to construct an $M[G_0][G_1][G_2]$-generic object G_3 over $\mathbb{R}_0$ and lift j to $j : V[G_0][G_1] \to M[G_0][G_1][G_2][G_3]$. It is then the case that $M[G_0][G_1][G_2][G_3]$ remains λ-closed with respect to $V[G_0][G_1][G_2]$.

For $\alpha \in (\delta, \delta^{++})$ and $p \in \mathrm{Add}(\delta, \delta^{++})$, let $p \restriction \alpha = \{\langle\langle \rho, \sigma\rangle, \eta\rangle \in p : \sigma < \alpha\}$ and $G_2 \restriction \alpha = \{p \restriction \alpha : p \in G_2\}$. Clearly, $V[G_0][G_1][G_2] \vDash$ "$|G_2 \restriction \alpha| \le \delta^+$ for all $\alpha \in (\delta, \delta^{++})$". Thus, since $\mathrm{Add}(j(\delta), j(\delta^{++}))^{M[G_0][G_1][G_2][G_3]}$ is $j(\delta)$-directed closed and $j(\delta) > \delta^{++}$, $q_\alpha = \bigcup\{j(p) : p \in G_2 \restriction \alpha\}$ is well-defined and is an element of $\mathrm{Add}(j(\delta), j(\delta^{++}))^{M[G_0][G_1][G_2][G_3]}$. Further, if $\langle \rho, \sigma\rangle \in \mathrm{dom}(q_\alpha) - \mathrm{dom}(\bigcup_{\beta<\alpha} q_\beta)$ ($\bigcup_{\beta<\alpha} q_\beta$ is well-defined by closure), then $\sigma \in [\bigcup_{\beta<\alpha} j(\beta), j(\alpha))$. To see this, assume to the contrary that $\sigma < \bigcup_{\beta<\alpha} j(\beta)$. Let β be minimal such that $\sigma < j(\beta)$. It must thus be the case that for some $p \in G_2 \restriction \alpha$, $\langle \rho, \sigma\rangle \in \mathrm{dom}(j(p))$. Since by elementarity and the definitions of $G_2 \restriction \beta$ and $G_2 \restriction \alpha$, for $p \restriction \beta = q \in G_2 \restriction \beta$, $j(q) = j(p) \restriction j(\beta) = j(p \restriction \beta)$, it must be the case that $\langle \rho, \sigma\rangle \in \mathrm{dom}(j(q))$. This means $\langle \rho, \sigma\rangle \in \mathrm{dom}(q_\beta)$, a contradiction.

Since $M[G_0][G_1][G_2][G_3] \vDash$ "GCH holds for all cardinals greater than or equal to $j(\delta)$", $M[G_0][G_1][G_2][G_3] \vDash$ "$\mathrm{Add}(j(\delta), j(\delta^{++}))$ is $j(\delta^+)$-c.c. and has $j(\delta^{++})$ many maximal antichains". This means that if $\mathcal{A} \in M[G_0][G_1][G_2][G_3]$ is a maximal antichain of $\mathrm{Add}(j(\delta), j(\delta^{++}))$, $\mathcal{A} \subseteq \mathrm{Add}(j(\delta), \beta)$ for some $\beta \in (j(\delta), j(\delta^{++}))$. Thus, since GCH in V and the fact j is generated by a supercompact ultrafilter over $P_\kappa(\delta^+)$ imply that $V \vDash$ "$|j(\delta^{++})| = \delta^{++}$", we can let $\langle \mathcal{A}_\alpha : \alpha \in (\delta, \delta^{++})\rangle \in V[G_0][G_1][G_2]$ be an enumeration of all of the maximal antichains of $\mathrm{Add}(j(\delta), j(\delta^{++}))$

present in $M[G_0][G_1][G_2][G_3]$.

Working in $V[G_0][G_1][G_2]$, we define now an increasing sequence $\langle r_\alpha : \alpha \in (\delta, \delta^{++})\rangle$ of elements of $\mathrm{Add}(j(\delta), j(\delta^{++}))$ such that $\forall \alpha \in (\delta, \delta^{++})[r_\alpha \geq q_\alpha$ and $r_\alpha \in \mathrm{Add}(j(\delta), j(\alpha))]$ and such that $\forall \mathcal{A} \in \langle \mathcal{A}_\alpha : \alpha \in (\delta, \delta^{++})\rangle \exists \beta \in (\delta, \delta^{++}) \exists r \in \mathcal{A}[r_\beta \geq r]$. Assuming we have such a sequence, $G_4 = \{p \in \mathrm{Add}(j(\delta), j(\delta^{++})) : \exists r \in \langle r_\alpha : \alpha \in (\delta, \delta^{++})\rangle [r \geq p]$ is an $M[G_0][G_1][G_2][G_3]$-generic object over $\mathrm{Add}(j(\delta), j(\delta^{++}))$. To define $\langle r_\alpha : \alpha \in (\delta, \delta^{++})\rangle$, if α is a limit, we let $r_\alpha = \bigcup_{\beta \in (\delta, \alpha)} r_\beta$. By the facts $\langle r_\beta : \beta \in (\delta, \alpha)\rangle$ is (strictly) increasing and $M[G_0][G_1][G_2][G_3]$ is δ^+-closed with respect to $V[G_0][G_1][G_2]$, this definition is valid. Assuming now r_α has been defined and we wish to define $r_{\alpha+1}$, let $\langle \mathcal{B}_\beta : \beta < \eta \leq \delta^+\rangle$ be the subsequence of $\langle \mathcal{A}_\beta : \beta \leq \alpha + 1\rangle$ containing each antichain $\mathcal{A}$ such that $\mathcal{A} \subseteq \mathrm{Add}(j(\delta), j(\alpha+1))$. Since $q_\alpha, r_\alpha \in \mathrm{Add}(j(\delta), j(\alpha))$, $q_{\alpha+1} \in \mathrm{Add}(j(\delta), j(\alpha+1))$, and $j(\alpha) < j(\alpha+1)$, the condition $r'_{\alpha+1} = r_\alpha \cup q_{\alpha+1}$ is well-defined, since by our earlier observations, any new elements of $\mathrm{dom}(q_{\alpha+1})$ won't be present in either $\mathrm{dom}(q_\alpha)$ or $\mathrm{dom}(r_\alpha)$. We can thus, using the fact $M[G_0][G_1][G_2][G_3]$ is δ^+-closed with respect to $V[G_0][G_1][G_2]$, define by induction an increasing sequence $\langle s_\beta : \beta < \eta\rangle$ such that $s_0 \geq r'_{\alpha+1}$, $s_\rho = \bigcup_{\beta<\rho} s_\beta$ if ρ is a limit ordinal, and $s_{\beta+1} \geq s_\beta$ is such that $s_{\beta+1}$ extends some element of $\mathcal{B}_\beta$. The just mentioned closure fact implies $r_{\alpha+1} = \bigcup_{\beta<\eta} s_\beta$ is a well-defined condition.

In order to show that G_4 is $M[G_0][G_1][G_2][G_3]$-generic over $\mathrm{Add}(j(\delta), j(\delta^{++}))$, we must show that $\forall \mathcal{A} \in \langle \mathcal{A}_\alpha : \alpha \in (\delta, \delta^{++})\rangle \exists \beta \in (\delta, \delta^{++}) \exists r \in \mathcal{A}[r_\beta \geq r]$. To do this, we first note that $\langle j(\alpha) : \alpha < \delta^{++}\rangle$ is unbounded in $j(\delta^{++})$. To see this, if $\beta < j(\delta^{++})$ is an ordinal, then for some $f : P_\kappa(\delta^+) \to M$ representing β, we can assume that for $p \in P_\kappa(\delta^+)$, $f(p) < \delta^{++}$. Thus, by the regularity of δ^{++} in V, $\beta_0 = \bigcup_{p \in P_\kappa(\delta^+)} f(p) < \delta^{++}$, and $j(\beta_0) > \beta$. This means by our earlier remarks that if $\mathcal{A} \in \langle \mathcal{A}_\alpha : \alpha < \delta^{++}\rangle$, $\mathcal{A} = \mathcal{A}_\rho$, then we can let $\beta \in (\delta, \delta^{++})$ be such that $\mathcal{A} \subseteq \mathrm{Add}(j(\delta), j(\beta))$. By construction, for $\eta > \max(\beta, \rho)$, there is some $r \in \mathcal{A}$ such that $r_\eta \geq r$. And, as any $p \in \mathrm{Add}(\delta, \delta^{++})$ is such that for some $\alpha \in (\delta, \delta^{++})$, $p = p \restriction \alpha$, G_4 is such that if $p \in G_2$, $j(p) \in G_4$. Thus, working in $V[G_0][G_1][G_2]$, we have shown that j lifts to $j : V[G_0][G_1][G_2] \to M[G_0][G_1][G_2][G_3][G_4]$, i.e., $V[G_0][G_1][G_2] \models$ "κ is $\lambda = \delta^+$ supercompact". Since as in Case 1, $\Vdash_{\mathbb{P}_\lambda}$ "$\dot{\mathbb{P}}^\lambda$ is $(2^{[\lambda]^{<\kappa}})^+$-directed closed", $V^{\mathbb{P}_\lambda * \dot{\mathbb{P}}^\lambda} = V^{\mathbb{P}} \models$ "κ is λ supercompact".

This completes the proof of Case 2 and Lemma 3.4.

□

Lemma 3.5 $V^{\mathbb{P}} \models$ "$2^\delta = \delta^+$ *if* δ *is a measurable cardinal*".

Proof: Suppose $V^{\mathbb{P}} \models$ "δ is a measurable cardinal". As in the proof of Lemma 3.2, write $\mathbb{P} = \mathbb{P}_0 * \dot{\mathbb{Q}}$, where $|\mathbb{P}_0| = \omega$, $\mathbb{P}_0$ is nontrivial, and $\Vdash_{\mathbb{P}_0}$ "$\dot{\mathbb{Q}}$ is $\aleph_1$-strategically closed". Again by Theorem 6, this factorization of $\mathbb{P}$ indicates that δ is measurable in V. As we have already observed, the measurability of δ in V implies that $V^{\mathbb{P}} \models$ "$2^\delta = \delta^+$". Thus, the proof of Lemma 3.5 will be complete once we have shown that $V^{\mathbb{P}} \models$ "δ is a measurable cardinal". However, since $\mathbb{P} = \mathbb{P}_{\delta+1} * \dot{\mathbb{P}}^{\delta+1}$, where $\mathbb{P}_{\delta+1}$ acts nontrivially on ordinals less than or equal to δ and $\Vdash_{\mathbb{P}_{\delta+1}}$ "$\dot{\mathbb{P}}^{\delta+1}$ is $(2^\delta)^+$-directed closed", it will suffice to show that $V^{\mathbb{P}_{\delta+1}} \models$ "δ is a measurable cardinal".

To do this, we combine the standard lifting arguments mentioned in Section 1 with Magidor's argument found in the proof of Case 2 of Lemma 3.4 above and an idea of Levinski found in [10]. Suppose $G * H$ is V-generic over $\mathbb{P}_{\delta+1} = \mathbb{P}_\delta * \dot{\mathrm{Add}}(\delta, \delta^+)$. Let $j : V \to M$ be an elementary embedding witnessing δ's measurability generated by a normal measure over δ such that $M \models$ "δ is nonmeasurable". Write $j(\mathbb{P}_{\delta+1}) = j(\mathbb{P}_\delta * \dot{\mathrm{Add}}(\delta, \delta^+)) = \mathbb{P}_\delta * \dot{\mathbb{Q}}_\delta * \dot{\mathbb{R}} * \dot{\mathrm{Add}}(j(\delta), j(\delta^+))$, where $\dot{\mathbb{Q}}_\delta$ is a term for the stage δ forcing done in $M^{\mathbb{P}_\delta}$ and $\dot{\mathbb{R}}$ is a term for the forcing done in $M^{\mathbb{P}_\delta * \dot{\mathbb{Q}}_\delta} = M^{\mathbb{P}_{\delta+1}}$ (strictly) between stages δ and $j(\delta)$. Because $M \models$ "δ is nonmeasurable", $\dot{\mathbb{Q}}_\delta$ is a term for $(\mathrm{Add}(\delta, \delta^{++}))^{M^{\mathbb{P}_\delta}}$.

We use now Levinski's ideas of [10] to show that it is possible to rearrange H to form an $M[G]$-generic object H' over $\mathbb{Q}_\delta$ in $V[G][H]$. Since $V \models \mathrm{GCH}$ and j is generated by an ultrafilter over δ, $V \models$ "$|(\delta^{++})^M| = \delta^+$". In addition, since $\mathbb{P}$ is an Easton support iteration, $\mathbb{P}_\delta$ is δ-c.c., which means that cardinals at and above δ are preserved from V to $V[G]$ and M to $M[G]$. Hence, $(\delta^{++})^{M[G]} = (\delta^{++})^M$, $(\delta^+)^{V[G]} = (\delta^+)^V$, and $V[G] \models$ "$|(\delta^{++})^{M[G]}| = \delta^+$". Let $(\delta^{++})^{M[G]} = \rho$. Working in $V[G]$, we may therefore let $f : \delta^+ \to \rho$ be a bijection. For any $p \in \mathrm{Add}(\delta, \delta^+)$, $g(p) = \{\langle\langle \alpha, f(\beta)\rangle, \gamma\rangle : \langle\langle \alpha, \beta\rangle, \gamma\rangle \in p\} \in (\mathrm{Add}(\delta, \rho))^{M[G]}$. As can be easily checked (see [10]), $H' = \{g(p) : p \in H\}$ is an $M[G]$-generic object over $(\mathrm{Add}(\delta, \rho))^{M[G]}$.

We continue with the lifting argument. Since M is δ-closed with respect to V, $\mathbb{P}_\delta * \dot{\mathrm{Add}}(\delta, \delta^+)$ is δ^+-c.c. in V, and $\mathbb{P}_\delta * \dot{\mathbb{Q}}_\delta$ is δ^+-c.c. in M, $M[G][H']$ remains δ-closed with respect to $V[G][H]$. Therefore, since j is generated by an ultrafilter over δ and $V \models$ GCH, the standard arguments mentioned in Section 1 show that it is possible to construct in $V[G][H]$ an $M[G][H']$-generic object H'' over $\mathbb{R}$ and lift j to $j : V[G] \to M[G][H'][H'']$. Because the first ordinal at which $\mathbb{R}$ does nontrivial forcing is above $(\delta^{++})^{M[G]}$, $M[G][H'][H'']$ remains δ-closed with respect to $V[G][H]$.

It remains to lift j in $V[G][H]$ through the stage δ forcing $\mathrm{Add}(\delta, \delta^+)$. However, Magidor's argument as given in the proof of Case 2 of Lemma 3.4 above for the construction of the generic object G_4, replacing the use of a normal measure over $P_\kappa(\lambda)$ with a normal measure over δ, allows us working in $V[G][H]$ to construct an $M[G][H'][H'']$-generic object H''' for $\mathrm{Add}(j(\delta), j(\delta^+))$ such that if $p \in H$, $j(p) \in H'''$. Thus, working in $V[G][H]$, we have shown that j lifts to $j : V[G][H] \to M[G][H'][H''][H''']$, i.e., $V[G][H] \models$ "δ is a measurable cardinal". This completes the proof of Lemma 3.5.

$\square$

Lemma 3.6 $V^{\mathbb{P}} \models$ "*For every pair of regular cardinals $\kappa < \lambda$, κ is λ strongly compact iff κ is λ supercompact, except possibly if κ is a measurable limit of cardinals δ which are λ supercompact, or λ is a nonmeasurable inaccessible cardinal*".

Proof: We significantly modify the proof of [1, Lemma 3.2]. Suppose $V^{\mathbb{P}} \models$ "$\kappa < \lambda$ are regular, λ is either a successor or measurable cardinal, κ is λ strongly compact, and κ is not a measurable limit of cardinals δ which are λ supercompact". By its definition, forcing with $\mathbb{P}$ preserves all cardinals and cofinalities. In addition, by the proof of Lemma 3.5, $V \models$ "λ is a measurable cardinal" iff $V^{\mathbb{P}} \models$ "λ is a measurable cardinal". Consequently, $V \models$ "λ is either a successor or measurable cardinal".

Consider now the following two cases.

Case 1: λ is a successor cardinal in both $V^{\mathbb{P}}$ and V. By the definition of $\mathbb{P}$, any subset of κ in $V^{\mathbb{P}}$ of size below κ has a name of size below κ in V. Thus, by the factorization of $\mathbb{P}$ given in the second sentence of the proof of Lemma 3.5 and Theorem 6, $V \vDash$ "κ is λ strongly compact". By Lemma 3.4, any cardinal δ such that $V \vDash$ "δ is λ supercompact" remains λ supercompact in $V^{\mathbb{P}}$. This means $V \vDash$ "κ is not a measurable limit of cardinals δ which are λ supercompact". Hence, by level by level equivalence between strong compactness and supercompactness in V, $V \vDash$ "κ is λ supercompact", so another application of Lemma 3.4 implies that $V^{\mathbb{P}} \vDash$ "κ is λ supercompact".

Case 2: λ is a measurable cardinal in both $V^{\mathbb{P}}$ and V. As in Case 1, $V \vDash$ "κ is λ strongly compact". It is in addition true that $V \vDash$ "κ is not a measurable limit of cardinals δ which are λ supercompact". To see this, assume not, and let $\delta < \kappa$ be such that $V \vDash$ "δ is λ supercompact". It is then true that $V \vDash$ "δ is γ supercompact for every successor cardinal $\gamma < \lambda$", so by Lemma 3.4 and the fact that forcing with $\mathbb{P}$ preserves cardinals and cofinalities, $V^{\mathbb{P}} \vDash$ "δ is γ supercompact for every successor cardinal $\gamma < \lambda$". By an application of the alternate proof sketched in [8, Exercise 22.9], since $V^{\mathbb{P}} \vDash$ "λ is a measurable cardinal", $V^{\mathbb{P}} \vDash$ "δ is λ supercompact". Thus, if $V \vDash$ "κ is a measurable limit of cardinals δ which are λ supercompact", then $V^{\mathbb{P}} \vDash$ "κ is a measurable limit of cardinals δ which are λ supercompact", a contradiction. Therefore, by level by level equivalence between strong compactness and supercompactness in V, $V \vDash$ "κ is λ supercompact". The argument just given then shows that $V^{\mathbb{P}} \vDash$ "κ is λ supercompact" as well. This completes the proof of Case 2 and Lemma 3.6.

$\square$

Lemmas 3.4 – 3.6 complete the proof of Theorem 4.

$\square$

We note that the definition of the partial ordering $\mathbb{P}$ used in the proof of Theorem 4 shows that $V^{\mathbb{P}} \vDash$ "$2^\delta = \delta^+$ if δ is a successor or singular cardinal". In addition, any cardinal κ in $V^{\mathbb{P}}$ which is a measurable limit of cardinals δ which are λ strongly compact where $\lambda > \kappa$ is regular and is either a successor or measurable cardinal must be in $V^{\mathbb{P}}$ a measurable limit of cardinals δ which are λ supercompact. This is since Theorem 6, which tells us that there are no new instances of measurability, strong compactness, or supercompactness in $V^{\mathbb{P}}$, implies that κ must be in V

a measurable limit of cardinals δ which are λ strongly compact. κ can then be written in V as a measurable limit of cardinals δ which are λ strongly compact where each such δ is not itself a measurable limit of cardinals γ which are λ strongly compact. By level by level equivalence between strong compactness and supercompactness in V, each such cardinal δ must be λ supercompact in V. Lemmas 3.4 – 3.6 then imply that each of these cardinals remains λ supercompact in $V^{\mathbb{P}}$.

We briefly indicate how Theorem 5 follows as a corollary of (the proof of) Theorem 4. Suppose $V \vDash$ "ZFC + GCH + κ is supercompact + No cardinal is supercompact up to an inaccessible cardinal + Level by level equivalence between strong compactness and supercompactness holds". Let $\mathbb{P}$ be defined as in the proof of Theorem 4, except that each use of $\mathrm{Add}(\delta, \delta^{++})$ is replaced by a use of $\mathrm{Add}(\delta, \delta^{+19})$. An analogous argument to the one found in the proof of Lemma 3.2 shows that $V^{\mathbb{P}} \vDash$ "No cardinal is supercompact up to an inaccessible cardinal". In addition, Levinski's ideas of [10] used in the proof of Lemma 3.5 remain valid if $\mathrm{Add}(\delta, \delta^{++})$ is replaced by $\mathrm{Add}(\delta, \delta^{+19})$. These key observations then allow us to infer as in the proof of Theorem 4 that $V^{\mathbb{P}} \vDash$ "κ is supercompact + Level by level equivalence between strong compactness and supercompactness holds + For every inaccessible cardinal which is not also measurable, $2^\delta = \delta^{+19}$ + For every measurable cardinal δ, $2^\delta = \delta^+$". This completes our discussion of the proof of Theorem 5.

□

Suppose λ is a measurable cardinal and $j : V \to M$ is an elementary embedding having critical point λ which is generated by a normal measure over λ. We remark that our application of the ideas of [10] only requires the existence of a function $f : \lambda \to \lambda$ such that $V \vDash$ "$\mathrm{cof}(f(\gamma)) > \gamma$ if $\gamma < \lambda$ is inaccessible and $|j(f)(\lambda)| = \lambda^+$". This allows for great flexibility in the proof of Theorem 5 when determining the possible values for 2^δ if $\delta < \kappa$ is a nonmeasurable inaccessible cardinal.

It is reasonable to hope that the partial orderings described in Section 2 which are used in the proof of Theorem 3 can also be employed to prove versions of Theorems 1 and 4 containing different violations of GCH for the relevant inaccessible cardinals. However, because the definition of $\mathbb{P}^0_{\delta,\lambda} * \mathbb{P}^1_{\delta,\lambda}[\dot{S}]$ implies that in $V^{\mathbb{P}^0_{\delta,\lambda} * \mathbb{P}^1_{\delta,\lambda}[\dot{S}]}$, δ is nonmeasurable, it will not be possible to force with these partial orderings and end up with a universe containing more than one supercompact cardinal. We therefore conclude by asking whether it is possible to establish alternate forms of Theorems 1 and 4 in which for the relevant inaccessible cardinals λ, $2^\lambda > \lambda^{++}$.

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