On the Consistency Strength of Level by Level Inequivalence *†

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Abstract

We show that the theories "ZFC + There is a supercompact cardinal" and "ZFC + There is a supercompact cardinal + Level by level inequivalence between strong compactness and supercompactness holds" are equiconsistent.

1 Introduction and Preliminaries

Say that a model containing at least one supercompact cardinal satisfies *level by level inequivalence between strong compactness and supercompactness* if for every non-supercompact measurable cardinal δ, there is some $\gamma > \delta$ such that δ is γ strongly compact yet δ is not γ supercompact.

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This can alternatively be stated by saying that *level by level inequivalence between strong compactness and supercompactness holds at every measurable cardinal* δ. Models containing exactly one supercompact cardinal in which level by level inequivalence between strong compactness and supercompactness holds may be found in [4, Theorem 2], [5, Theorem 2], [2, Theorem 1], [3, Theorems 1–3], [6, Theorem 32(2)], and [1, Theorem 1.1]. (Note that the dual notion of *level by level equivalence between strong compactness and supercompactness* was first studied by the author and Shelah in [8], to which we refer readers for additional details.) A key feature of all of these constructions, however, is the use of hypotheses stronger in consistency strength than "ZFC + There is a supercompact cardinal". This prompts us to ask the following

Question: Are the theories "ZFC + There is a supercompact cardinal" and "ZFC + There is a supercompact cardinal + Level by level inequivalence between strong compactness and supercompactness holds" equiconsistent?

The purpose of this paper is to answer the above Question in the affirmative. Specifically, we prove the following.

Theorem 1 *The theories "ZFC + There is a supercompact cardinal" and "ZFC + There is a supercompact cardinal + Level by level inequivalence between strong compactness and supercompactness holds" are equiconsistent.*

The proof of Theorem 1 raises the question of whether there is something different about the models for level by level inequivalence between strong compactness and supercompactness constructed using hypotheses stronger in consistency strength than "ZFC + There is a supercompact cardinal". We will come back to this issue at the end of the paper.

Before beginning the proof of Theorem 1, we very briefly mention some preliminary information concerning notation and terminology. When forcing, $q \geq p$ means that *q is stronger than p*. For $\alpha < \beta$ ordinals, $[\alpha, \beta]$, $[\alpha, \beta)$, $(\alpha, \beta]$, and (α, β) are as in standard interval notation. If G is V-generic over $\mathbb{P}$, we will abuse notation slightly and use both $V[G]$ and $V^{\mathbb{P}}$ to indicate the universe obtained by forcing with $\mathbb{P}$. If $\mathbb{P}$ is a reverse Easton iteration such that at stage α, a nontrivial forcing is done adding a subset of δ, then we will say that δ *is in the field of* $\mathbb{P}$. We will, from time to time, confuse terms with the sets they denote and write x when we actually mean $\dot{x}$ or $\check{x}$.

Suppose $\kappa > \omega$ is a regular cardinal. The partial ordering $\mathbb{P}$ is *κ-strategically closed* if in the two person game in which the players construct an increasing sequence $\langle p_\alpha \mid \alpha \leq \kappa \rangle$, where player I plays odd stages and player II plays even (which of course includes limit) stages (choosing the trivial condition at stage 0), player II has a strategy which ensures the game can always be continued. $\mathbb{P}$ is *$\prec\kappa$-strategically closed* if in the two person game in which the players construct an increasing sequence $\langle p_\alpha \mid \alpha < \kappa \rangle$, where player I plays odd stages and player II plays even stages (again choosing the trivial condition at stage 0), player II has a strategy which ensures the game can always be continued. $\mathbb{P}$ is *$<\kappa$-strategically closed* if $\mathbb{P}$ is δ-strategically closed for every $\delta < \kappa$. Note that if $\mathbb{P}$ is $\prec\kappa$-strategically closed, then $\mathbb{P}$ is $<\kappa$-strategically closed as well.

An example of a partial ordering which is $\prec\kappa$-strategically closed and which will be used in the proof of Theorem 1 is the partial ordering $\mathbb{P}$ for adding a nonreflecting stationary set of ordinals of cofinality ω to κ. Specifically, $\mathbb{P} = \{p \mid$ For some $\alpha < \kappa$, $p : \alpha \to \{0, 1\}$ is a characteristic function of S_p, a subset of α not stationary at its supremum nor having any initial segment which is stationary at its supremum, such that $\beta \in S_p$ implies $\beta > \omega$ and $\mathrm{cof}(\beta) = \omega\}$, ordered by $q \geq p$ iff $q \supseteq p$ and $S_p = S_q \cap \sup(S_p)$, i.e., S_q is an end extension of S_p. For additional details, readers are urged to consult [8, second paragraph of Section 1, page 106].

We mention that we are assuming complete familiarity with the notions of measurability, strong compactness, and supercompactness. Interested readers may consult [14] for further details. We note only that all elementary embeddings witnessing the λ supercompactness of κ are presumed to come from some fine, κ-complete, normal ultrafilter $\mathcal{U}$ over $P_\kappa(\lambda) = \{x \subseteq \lambda \mid |x| < \kappa\}$, and all elementary embeddings witnessing the λ strong compactness of κ are presumed to come from some fine, κ-complete ultrafilter $\mathcal{U}$ over $P_\kappa(\lambda)$. An equivalent definition for κ being λ strongly compact is that there is an elementary embedding $j : V \to M$ having critical point κ such that for any $x \subseteq M$ with $|x| \leq \lambda$, there is some $y \in M$ such that $x \subseteq y$ and $M \vDash$ "$|y| < j(\kappa)$".

A corollary of Hamkins' work on gap forcing found in [12, 13] will be employed in the proof of Theorem 1. We therefore state as a separate theorem what is relevant for this paper, along with some associated terminology, quoting from [12, 13] when appropriate. Suppose $\mathbb{P}$ is a partial ordering which can be written as $\mathbb{Q} * \dot{\mathbb{R}}$, where $|\mathbb{Q}| < \delta$, $\mathbb{Q}$ is nontrivial, and $\Vdash_{\mathbb{Q}}$ "$\dot{\mathbb{R}}$ is δ-strategically closed". In Hamkins' terminology of [12, 13], $\mathbb{P}$ *admits a gap at* δ. In Hamkins' terminology of [12, 13], $\mathbb{P}$ is *mild with respect to a cardinal* κ iff every set of ordinals x in $V^{\mathbb{P}}$ of size less than κ has a "nice" name τ in V of size less than κ, i.e., there is a set y in V, $|y| < \kappa$, such that any ordinal forced by a condition in $\mathbb{P}$ to be in τ is an element of y. Also, as in the terminology of [12, 13] and elsewhere, an embedding $j : \overline{V} \to \overline{M}$ is *amenable to* $\overline{V}$ when $j \restriction A \in \overline{V}$ for any $A \in \overline{V}$. The specific corollary of Hamkins' work from [12, 13] we will be using is then the following.

Theorem 2 (Hamkins) *Suppose that $V[G]$ is a generic extension obtained by forcing that admits a gap at some regular $\delta < \kappa$. Suppose further that $j : V[G] \to M[j(G)]$ is an embedding with critical point κ for which $M[j(G)] \subseteq V[G]$ and $M[j(G)]^{\delta} \subseteq M[j(G)]$ in $V[G]$. Then $M \subseteq V$; indeed, $M = V \cap M[j(G)]$. If the full embedding j is amenable to $V[G]$, then the restricted embedding $j \restriction V : V \to M$ is amenable to V. If j is definable from parameters (such as a measure or extender) in $V[G]$, then the restricted embedding $j \restriction V$ is definable from the names of those parameters in V. Finally, if $\mathbb{P}$ is mild with respect to κ and κ is λ strongly compact in $V[G]$ for any $\lambda \geq \kappa$, then κ is λ strongly compact in V.*

2 The Proof of Theorem 1

We turn now to the proof of Theorem 1.

Proof: Let $V \vDash$ "ZFC + κ is supercompact". Without loss of generality, by doing a preliminary forcing and truncating the universe if necessary, we assume in addition that $V \vDash$ "GCH + There are no cardinals $\delta < \lambda$ such that δ is γ supercompact for every $\gamma < \lambda$ and λ is measurable". Note that this implies $V \vDash$ "No cardinal $\eta > \kappa$ is measurable".

Let $B = \{\delta < \kappa \mid \delta$ is measurable and level by level inequivalence between strong compactness and supercompactness holds at $\delta\}$. It follows by a theorem of Magidor (unpublished by him, but given as [8, Lemma 7]) that B is unbounded in κ. In fact, the proof of [8, Lemma 7] actually shows we may assume that $B \supseteq A$, where $A = \{\delta < \kappa \mid$ There exists $\lambda > \delta$, $\lambda < \kappa$ such that λ has cofinality δ, δ is γ supercompact for every $\gamma < \lambda$, δ is not λ supercompact, yet δ is λ strongly compact$\}$.[1]

We are now in a position to define the partial ordering $\mathbb{P}$ with which we will force to construct our model witnessing the conclusions of Theorem 1. $\mathbb{P}$ is the reverse Easton iteration having length κ which begins by adding a Cohen subset of ω and then does nontrivial forcing only when $\delta < \kappa$ is measurable and $\delta \notin A$. At such a stage, we force with the partial ordering adding a nonreflecting stationary set of ordinals of cofinality ω to δ. Since by its definition, forcing with $\mathbb{P}$ preserves all cofinalities (and hence all cardinals as well), the expression "$\text{cof}(\lambda) = \delta$" may be written without fear of ambiguity.

Lemma 2.1 $V^{\mathbb{P}} \vDash$ *"κ is supercompact".*

Proof: Let $\lambda > \kappa$ be an arbitrary singular strong limit cardinal having cofinality κ. Let $j : V \to M$ be an elementary embedding witnessing the λ supercompactness of κ such that $M \vDash$ "κ is not λ supercompact". Because $M^\lambda \subseteq M$ and λ is a strong limit cardinal in M, it follows that $M \vDash$ "κ is γ supercompact for every $\gamma < \lambda$". By the proof found in the first footnote, as $M \vDash$ "$\text{cof}(\lambda) = \kappa$", $M \vDash$ "κ is λ strongly compact". Thus, $M \vDash$ "Level by level inequivalence between strong compactness and supercompactness holds at κ and is witnessed by the singular strong limit cardinal λ of cofinality κ", i.e., $M \vDash$ "$\kappa \in j(A)$". Further, since $V \vDash$ "No cardinal $\eta > \kappa$ is measurable" and $M^\lambda \subseteq M$, $M \vDash$ "No cardinal in the half-open interval $(\kappa, \lambda]$ is measurable". This means that $j(\mathbb{P}) = \mathbb{P} * \dot{\mathbb{Q}}$, where κ is a trivial stage of forcing, and the first nontrivial stage of forcing $\delta > \kappa$ is such that $\delta > \lambda$.

If we now let G be V-generic over $\mathbb{P}$ and H be $V[G]$-generic over $\mathbb{Q}$, since $j''G \subseteq G * H$, standard arguments show that j lifts in $V[G][H]$ to $j : V[G] \to M[G][H]$. Further, because $\mathbb{P}$ is a reverse

[1] The proof that A is unbounded in κ is as follows. Let $\lambda > \kappa$ be a singular strong limit cardinal of cofinality κ. Let $j : V \to M$ be an elementary embedding witnessing the λ supercompactness of κ such that $M \vDash$ "κ is not λ supercompact". Because $M^\lambda \subseteq M$, $M \vDash$ "κ is γ supercompact for every $\gamma < \lambda$". Let now $\langle \gamma_\alpha \mid \alpha < \kappa \rangle$ be a sequence of regular cardinals cofinal in λ, $\langle \mu_\alpha \mid \alpha < \kappa \rangle$ be a sequence of κ-additive, fine ultrafilters over $P_\kappa(\gamma_\alpha)$, and μ be a κ-additive measure over κ. If for any $A \subseteq P_\kappa(\lambda)$ and any $\alpha < \kappa$ we define $A \restriction \gamma_\alpha = A \cap P_\kappa(\gamma_\alpha)$, then in analogy to [8, Lemma 7]) (see also the argument given in [10, (1) and (2) of Lemma 3]), the collection μ^* of subsets of $P_\kappa(\lambda)$ given by $A \in \mu^*$ iff $\{\alpha < \delta \mid A \restriction \gamma_\alpha \in \mu_\alpha\} \in \mu$ defines a κ-additive, fine ultrafilter over $P_\kappa(\lambda)$. Thus, $M \vDash$ "κ is λ strongly compact". By reflection, because $M \vDash$ "κ is not λ supercompact", the set A is unbounded in κ.

Easton iteration having length κ, $\mathbb{P}$ is κ-c.c. Thus, since $M^\lambda \subseteq M$ in V, $M[G]^\lambda \subseteq M[G]$ in $V[G]$ as well. As $\delta > \lambda$, the definition of $\mathbb{P}$ implies that $\mathbb{Q}$ is $<\lambda^+$-strategically closed in $M[G]$. The fact $M[G]^\lambda \subseteq M[G]$ in $V[G]$ implies that $\mathbb{Q}$ is also $<\lambda^+$-strategically closed in $V[G]$. Let $\gamma > \kappa$, $\gamma < \lambda$ be an arbitrary cardinal. Since λ is a strong limit cardinal in both $V[G]$ and $M[G]$ and $\mathbb{Q}$ is $<\lambda^+$-strategically closed in each of these models, the supercompact ultrafilter $\mathcal{U}_\gamma \in V[G][H]$ over $(P_\kappa(\gamma))^{V[G]}$ defined by $x \in \mathcal{U}_\gamma$ iff $\langle j(\alpha) \mid \alpha < \gamma \rangle \in j(x)$ is such that $\mathcal{U}_\gamma \in V[G]$. Hence, $V[G] \models$ "κ is γ supercompact for every $\gamma < \lambda$". As $\lambda > \kappa$ was arbitrary, $V[G] \models$ "κ is supercompact". This completes the proof of Lemma 2.1.

$\square$

Lemma 2.2 *Suppose $V^{\mathbb{P}} \models$ "$\delta < \kappa$ is measurable". Then for some $\lambda > \delta$, $\lambda < \kappa$ such that $\mathrm{cof}(\lambda) = \delta$, $V^{\mathbb{P}} \models$ "δ is λ strongly compact".*

Proof: Assume $\delta < \kappa$ is such that $V^{\mathbb{P}} \models$ "δ is measurable". Write $\mathbb{P} = \mathbb{P}^0 * \dot{\mathbb{P}}^1$, where $|\mathbb{P}^0| = \omega$, $\mathbb{P}^0$ is nontrivial, and $\Vdash_{\mathbb{P}^0}$ "$\dot{\mathbb{P}}^1$ is $\aleph_1$-strategically closed". By this factorization and its definition, $\mathbb{P}$ admits a gap at $\aleph_1$. Hence, by Theorem 2, it must also be the case that $V \models$ "δ is measurable". Further, it must be true that $\delta \in A$. This is since otherwise, by the definition of $\mathbb{P}$, $V^{\mathbb{P}} \models$ "δ contains a nonreflecting stationary set of ordinals of cofinality ω and hence is nonmeasurable". Consequently, there must be some $\lambda > \delta$, $\lambda < \kappa$ such that $\mathrm{cof}(\lambda) = \delta$ and $V \models$ "δ is ξ supercompact for every $\xi < \lambda$, δ is not λ supercompact, yet δ is λ strongly compact".

Let $\gamma < \lambda$ be such that $\gamma = \zeta^{++}$ for some cardinal $\zeta \geq \delta$. Since $\gamma < \lambda$, $V \models$ "δ is γ supercompact". Write $\mathbb{P} = \mathbb{P}_\delta * \dot{\mathbb{P}}^\delta$. By the definition of $\mathbb{P}$, it is the case that $\Vdash_{\mathbb{P}_\delta}$ "$\dot{\mathbb{P}}^\delta$ is $<\eta$-strategically closed for η the least measurable cardinal above δ". Since again by the definition of $\mathbb{P}$, $|\mathbb{P}_\delta| \leq \delta$, by the Lévy-Solovay results [15], η is the least measurable cardinal above δ in both V and $V^{\mathbb{P}_\delta}$. Consequently, by the fact that $V \models$ "There are no cardinals $\sigma < \rho$ such that σ is ξ supercompact for every $\xi < \rho$ and ρ is measurable", $\gamma, \lambda < \eta$. Thus, to show that either $V^{\mathbb{P}} \models$ "δ is γ strongly compact" or $V^{\mathbb{P}} \models$ "δ is λ strongly compact", it suffices to prove that either $V^{\mathbb{P}_\delta} \models$ "δ is γ strongly compact" or $V^{\mathbb{P}_\delta} \models$ "δ is λ strongly compact".

We first demonstrate that $V^{\mathbb{P}_\delta} \vDash$ "δ is γ strongly compact" by considering the following two cases.

Case 1: $|\mathbb{P}_\delta| < \delta$. By the results of [15], $V^{\mathbb{P}_\delta} \vDash$ "δ is γ supercompact" (and so of course, $V^{\mathbb{P}_\delta} \vDash$ "δ is γ strongly compact").

Case 2: $|\mathbb{P}_\delta| = \delta$. We use an argument analogous to the one found in the proof of [7, Lemma 2.3], from which we feel free to quote liberally when necessary. Specifically, let $k_1 : V \to M$ be an elementary embedding witnessing the γ supercompactness of δ such that $M \vDash$ "δ is not γ supercompact". Since $\gamma > \delta^+ = 2^\delta$, we know that δ is measurable in M. Therefore, there is a normal measure over δ in M yielding an embedding $k_2 : M \to N$ with critical point δ such that $N \vDash$ "δ is not measurable". It is easy to verify using the embedding definition of γ strong compactness given in Section 1 that the composed embedding $j = k_2 \circ k_1 : V \to N$ witnesses the γ strong compactness of δ. We will show that j lifts to $j : V^{\mathbb{P}_\delta} \to N^{j(\mathbb{P}_\delta)}$. This lifted embedding will witness the γ strong compactness of δ in $V^{\mathbb{P}_\delta}$.

To do this, factor $j(\mathbb{P}_\delta)$ as $\mathbb{P}_\delta * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$, where $\dot{\mathbb{Q}}$ is a term for the portion of $j(\mathbb{P}_\delta)$ from stage δ up to and including stage $k_2(\delta)$, and $\dot{\mathbb{R}}$ is a term for the rest of $j(\mathbb{P}_\delta)$, from stage $k_2(\delta) + 1$ up to $j(\delta)$. Since $N \vDash$ "δ is not measurable", we know that $\delta \notin \text{field}(\dot{\mathbb{Q}})$. Further, by GCH, $M \vDash$ "δ is ζ^+ supercompact". Consequently, because $M \vDash$ "δ is not $\gamma = \zeta^{++}$ supercompact", there cannot be some M-singular cardinal $\rho > \delta$ such that $M \vDash$ "δ is ξ supercompact for every $\xi < \rho$, δ is not ρ supercompact, yet δ is ρ strongly compact". This means that $\delta \notin k_1(A)$ and $k_2(\delta) \notin k_2(k_1(A))$. Thus, the field of $\dot{\mathbb{Q}}$ is composed of a subset of the N-measurable cardinals in the interval $(\delta, k_2(\delta)]$ (and in particular, $k_2(\delta)$ is in the field of $\dot{\mathbb{Q}}$), and the field of $\dot{\mathbb{R}}$ is composed of a subset of the N-measurable cardinals in the interval $(k_2(\delta), k_2(k_1(\delta)))$.

Let G_0 be V-generic over $\mathbb{P}_\delta$. We will construct in $V[G_0]$ an $N[G_0]$-generic object G_1 over $\mathbb{Q}$ and an $N[G_0][G_1]$-generic object G_2 over $\mathbb{R}$. Since $\mathbb{P}_\delta$ is an Easton support iteration of small forcing, with a direct limit at stage δ and no forcing right at stage δ, the construction of G_1 and G_2 ensures that $j''G_0 \subseteq G_0 * G_1 * G_2$. It follows that $j : V \to N$ lifts to $j : V[G_0] \to N[G_0][G_1][G_2]$ in $V[G_0]$.

To build G_1, note that since k_2 is generated by an ultrafilter $\mathcal{U}$ over δ and GCH holds in both V and M, we know $|k_2(2^\delta)| = |k_2(\delta^+)| = |\{f \mid f : \delta \to \delta^+ \text{ is a function}\}| = |[\delta^+]^\delta| = \delta^+$. Thus, as $N[G_0] \vDash$ "$|\wp(\mathbb{Q})| = k_2(2^\delta)$", we can let $\langle D_\alpha \mid \alpha < \delta^+ \rangle$ be an enumeration in $V[G_0]$ of the dense open subsets of $\mathbb{Q}$ present in $N[G_0]$. Since the δ closure of N with respect to either M or V implies that the least element of the field of $\mathbb{Q}$ is above δ^+, the definition of $\mathbb{Q}$ as the Easton support iteration which adds a nonreflecting stationary set of ordinals of cofinality ω to the appropriate subset of the N-measurable cardinals in the interval $(\delta, k_2(\delta)]$ implies that $N[G_0] \vDash$ "$\mathbb{Q}$ is $\prec\delta^+$-strategically closed". Since the standard arguments show that forcing with the δ-c.c. partial ordering $\mathbb{P}_\delta$ preserves that $N[G_0]$ remains δ-closed with respect to either $M[G_0]$ or $V[G_0]$, we know that $\mathbb{Q}$ is $\prec\delta^+$-strategically closed in both $M[G_0]$ and $V[G_0]$. We now construct G_1 in either $M[G_0]$ or $V[G_0]$ as follows. Fix a winning strategy $\mathcal{S}$ for player II in the game of length δ^+ for the partial ordering $\mathbb{Q}$ and use it to construct a play $\langle\, q_\alpha \mid \alpha < \delta^+ \,\rangle$ of the game. Since player II's moves are determined by $\mathcal{S}$, we need only specify the moves of the first player. Specifically, if player II has just played the condition $q_{2\alpha}$ at the (even) stage 2α, player I selects and then plays a condition $q_{2\alpha+1}$ above $q_{2\alpha}$ from the dense set D_α using $\mathcal{S}$. Since $\mathcal{S}$ is used at limit stages, this completes the construction of the play $\langle\, q_\alpha \mid \alpha < \delta^+ \,\rangle$. Let $G_1 = \{p \in \mathbb{Q} \mid \exists \alpha < \delta^+ \, (q_\alpha \geq p)\}$ be the filter generated by this increasing sequence of conditions. By construction, this filter meets all the dense sets D_α, and so it is $N[G_0]$-generic over $\mathbb{Q}$.

It remains to construct in $V[G_0]$ the desired $N[G_0][G_1]$-generic object G_2 over $\mathbb{R}$. To do this, we first observe that as $M \vDash$ "$\delta \notin k_1(A)$", we can factor $k_1(\mathbb{P}_\delta)$ as $\mathbb{P}_\delta * \dot{\mathbb{S}} * \dot{\mathbb{T}}$, where $\dot{\mathbb{S}}$ is a term for the partial ordering adding a nonreflecting stationary set of ordinals of cofinality ω to δ, and $\dot{\mathbb{T}}$ is a term for the rest of $k_1(\mathbb{P}_\delta)$.

Since $M \vDash$ "There are no measurable cardinals in the half-open interval $(\delta, \gamma]$" (which follows because $\gamma < \eta$), the field of $\dot{\mathbb{T}}$ is composed of a subset of the M-measurable cardinals in the interval $(\gamma, k_1(\delta))$. This implies that in M, $\Vdash_{\mathbb{P}_\delta * \dot{\mathbb{S}}}$ "$\dot{\mathbb{T}}$ is $\prec\gamma^+$-strategically closed". Further, since γ is regular, GCH implies that $|[\gamma]^{<\delta}| = \gamma$. By GCH, we know that $2^\gamma = \gamma^+$. Therefore, as k_1 is generated by an ultrafilter over $P_\delta(\gamma)$, we may calculate $|2^{k_1(\gamma)}|^M = |k_1(2^\gamma)| = |k_1(\gamma^+)| = |\{f \mid f : P_\delta(\gamma) \to \gamma^+ \text{ is a function}\}| = |[\gamma^+]^\gamma| = \gamma^+$.

Work until otherwise specified in M. Consider the "term forcing" partial ordering $\mathbb{T}^*$ (see [11] for the first published account of term forcing or [9, Section 1.2.5, page 8]; the notion is originally due to Laver) associated with $\dot{\mathbb{T}}$, i.e., $\tau \in \mathbb{T}^*$ essentially iff τ is a term in the forcing language with respect to $\mathbb{P}_\delta * \dot{\mathbb{S}}$ and $\Vdash_{\mathbb{P}_\delta * \dot{\mathbb{S}}}$ "$\tau \in \dot{\mathbb{T}}$", ordered by $\tau \geq \sigma$ iff $\Vdash_{\mathbb{P}_\delta * \dot{\mathbb{S}}}$ "$\tau \geq \sigma$". Since this definition, taken literally, would produce a proper class, we restrict the terms appearing in it to a sufficiently large set-sized collection (such that any term τ forced by the trivial condition to be in $\dot{\mathbb{T}}$ will be forced by the trivial condition to be equal to an element of $\mathbb{T}^*$) of size $k_1(\gamma)$ in M.[2] Since $\Vdash_{\mathbb{P}_\delta * \dot{\mathbb{S}}}$ "$\dot{\mathbb{T}}$ is $\prec\gamma^+$-strategically closed", it can easily be verified that $\mathbb{T}^*$ is also $\prec\gamma^+$-strategically closed in M and, as $M^\gamma \subseteq M$, in V as well. Because $M \vDash$ "$2^{k_1(\gamma)} = (k_1(\gamma))^+ = k_1(\gamma^+)$", we can let $\langle D_\alpha \mid \alpha < \gamma^+ \rangle$ be an enumeration in V of the dense open subsets of $\mathbb{T}^*$ found in M and argue as we did when constructing G_1 to build in V an M-generic object H_2 over $\mathbb{T}^*$.

Note that since N is an ultrapower of M via a normal ultrafilter $\mathcal{U} \in M$ over δ, [9, Fact 2, Section 1.2.2] tells us that $k_2'' H_2$ generates an N-generic object G_2^* over $k_2(\mathbb{T}^*)$. By elementarity, $k_2(\mathbb{T}^*)$ is the term forcing in N defined with respect to $k_2(k_1(\mathbb{P}_\delta)_{\delta+1}) = \mathbb{P}_\delta * \dot{\mathbb{Q}}$. Therefore, since $j(\mathbb{P}_\delta) = k_2(k_1(\mathbb{P}_\delta)) = \mathbb{P}_\delta * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$, G_2^* is N-generic over $k_2(\mathbb{T}^*)$, and $G_0 * G_1$ is N-generic over $k_2(\mathbb{P}_\delta * \dot{\mathbb{S}})$, we know by [9, Fact 1, Section 1.2.5] (see also [11]) that $G_2 = \{i_{G_0 * G_1}(\tau) \mid \tau \in G_2^*\}$ is $N[G_0][G_1]$-generic over $\mathbb{R}$. Thus, in $V[G_0]$, the embedding $j : V \to N$ lifts to $j : V[G_0] \to N[G_0][G_1][G_2]$. This means that $V[G_0] \vDash$ "δ is γ strongly compact".

Let now $\langle \lambda_\alpha^{++} \mid \alpha < \delta \rangle \in V^{\mathbb{P}_\delta}$ be a strictly increasing sequence cofinal in λ such that $\lambda_0^{++} > \delta$. By Cases 1 and 2 above, for each λ_α^{++}, we may choose a δ-additive, fine ultrafilter $\mu_\alpha \in V^{\mathbb{P}_\delta}$ over $P_\delta(\lambda_\alpha^{++})$. Because $V^{\mathbb{P}_\delta} \vDash$ "$\mathrm{cof}(\lambda) = \delta$", by the argument given in the first footnote, $V^{\mathbb{P}_\delta} \vDash$ "δ is λ strongly compact". This completes the proof of Lemma 2.2.

[2] In the official definition of $\mathbb{T}^*$, the basic idea is to include only the canonical terms. Since $\dot{\mathbb{T}}$ is forced to have cardinality $k_1(\gamma)$, there is a set $\{\tau_\alpha \mid \alpha < k_1(\gamma)\}$ of terms such that for any other term τ, if $\Vdash_{\mathbb{P}_\delta * \dot{\mathbb{S}}}$ "$\tau \in \dot{\mathbb{T}}$", then there is a dense set of conditions in $\mathbb{P}_\delta * \dot{\mathbb{S}}$ forcing "$\tau = \tau_\alpha$" for various α. While this collection of terms may not itself be adequate, we enlarge it as follows: For each maximal antichain $A \subseteq \mathbb{P}_\delta * \dot{\mathbb{S}}$ and each function $s : A \to \{\tau_\alpha \mid \alpha < k_1(\gamma)\}$, there is (by arguments from elementary forcing) a term τ_s such that $p \Vdash$ "$\tau_\alpha = \tau_{s(p)}$" for each $p \in A$; let $\mathbb{T}^*$ be the collection of all such terms τ_s, ranging over all maximal antichains of $\mathbb{P}_\delta * \dot{\mathbb{S}}$. Since $\mathbb{P}_\delta * \dot{\mathbb{S}}$ has size less than $k_1(\gamma)$ in M, the number of such terms is $k_1(\gamma)$. And finally, if a term τ is forced to be in $\dot{\mathbb{T}}$, then elementary forcing arguments establish that τ is forced to be equal to τ_s for some s.

$\square$

Lemma 2.3 $V^{\mathbb{P}} \vDash$ "*Level by level inequivalence between strong compactness and supercompactness holds*".

Proof: Suppose δ is such that $V^{\mathbb{P}} \vDash$ "δ is measurable". Since $\mathbb{P}$ may be defined such that $|\mathbb{P}| = \kappa$ and $V \vDash$ "No cardinal $\eta > \kappa$ is measurable", by the results of [15], $\delta \le \kappa$. Since by Lemma 2.1, $V^{\mathbb{P}} \vDash$ "κ is supercompact", we assume without loss of generality that $\delta < \kappa$. Further, by the proof of Lemma 2.2, we know that there must be some $\lambda > \delta$, $\lambda < \kappa$ such that $\mathrm{cof}(\lambda) = \delta$ and $V \vDash$ "δ is γ supercompact for every $\gamma < \lambda$, δ is not λ supercompact, yet δ is λ strongly compact". By Lemma 2.2, $V^{\mathbb{P}} \vDash$ "δ is λ strongly compact". In addition, by the factorization of $\mathbb{P}$ given in Lemma 2.2 and Theorem 2, if γ is any cardinal such that $V^{\mathbb{P}} \vDash$ "δ is γ supercompact", then $V \vDash$ "δ is γ supercompact" as well. Hence, since $V \vDash$ "δ is not λ supercompact", $\gamma < \lambda$, so every degree of supercompactness witnessed by δ in $V^{\mathbb{P}}$ must be below λ. This means that $V^{\mathbb{P}} \vDash$ "Level by level inequivalence between strong compactness and supercompactness holds at δ", so since δ was arbitrary, $V^{\mathbb{P}} \vDash$ "Level by level inequivalence between strong compactness and supercompactness holds". This completes the proof of Lemma 2.3.

$\square$

Since clearly Con(ZFC + There is a supercompact cardinal + Level by level inequivalence between strong compactness and supercompactness holds) $\Longrightarrow$ Con(ZFC + There is a supercompact cardinal), Lemmas 2.1 – 2.3 complete the proof of Theorem 1.

$\square$

In conclusion to this paper, we return to the question raised before of whether there is something different about the model witnessing the conclusions of Theorem 1 from the models constructed earlier witnessing level by level inequivalence between strong compactness and supercompactness. In fact, there is a key difference. All of the previous models have the property that when a non-supercompact measurable cardinal δ witnesses level by level inequivalence in $V^{\mathbb{P}}$ (or in some intermediate generic extension used to construct the final model $V^{\mathbb{P}}$),

(*) there is always some inaccessible cardinal $\lambda > \delta$ and some $\rho \in (\delta, \lambda)$ such that for every $\gamma \in [\rho, \lambda)$, δ is γ strongly compact yet δ is not γ supercompact.

The construction given here does not necessarily do this, and relies on singular instances of level by level inequivalence between strong compactness and supercompactness. To see that there may be measurable cardinals for which property (*) does not hold, we may suppose by the results of [8] and by truncating the universe if necessary that our ground model V is such that $V \vDash$ "ZFC + GCH + κ is supercompact + There are no cardinals $\delta < \lambda$ such that δ is γ supercompact for every $\gamma < \lambda$ and λ is inaccessible + For every pair of regular cardinals $\delta < \lambda$, δ is λ strongly compact iff δ is λ supercompact". Let $\mathbb{P}$ be as before. It will still be true that $V^{\mathbb{P}}$ is a model for level by level inequivalence between strong compactness and supercompactness. However, in $V^{\mathbb{P}}$, no measurable cardinal will satisfy property (*). This follows since if δ is measurable and there is some inaccessible cardinal $\lambda > \delta$ and some $\rho \in (\delta, \lambda)$ such that for every $\gamma \in [\rho, \lambda)$, δ is γ strongly compact yet δ is not γ supercompact, then by the factorization of $\mathbb{P}$ given in the proof of Lemma 2.2 and the fact $\mathbb{P}$ is mild with respect to δ, it must also be the case that for every $\gamma \in [\rho, \lambda)$, $V \vDash$ "δ is γ strongly compact". By our assumptions on V, we have in addition that $V \vDash$ "δ is γ supercompact". This means that in V, $\lambda > \delta$ is inaccessible and such that δ is γ supercompact for every $\gamma \in [\rho, \lambda)$, which contradicts our assumptions on V that there are no cardinals $\delta < \lambda$ such that δ is γ supercompact for every $\gamma < \lambda$ and λ is inaccessible. We therefore end by asking whether the theory "ZFC + There exists a supercompact cardinal + Level by level inequivalence between strong compactness and supercompactness holds + For every non-supercompact measurable cardinal δ, there is some inaccessible cardinal $\lambda > \delta$ and some $\rho \in (\delta, \lambda)$ such that for every $\gamma \in [\rho, \lambda)$, δ is γ strongly compact yet δ is not γ supercompact" is stronger in consistency strength than the theory "ZFC + There exists a supercompact cardinal + Level by level inequivalence between strong compactness and supercompactness holds". We conjecture that this is indeed the case.

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