A Universal Indestructibility Theorem Compatible with Level by Level Equivalence [*][†]

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Abstract

We prove an indestructibility theorem for degrees of supercompactness that is compatible with level by level equivalence between strong compactness and supercompactness.

1 Introduction and Preliminaries

Say that a model V of ZFC satisfies *level by level equivalence between strong compactness and supercompactness* if for every pair of regular cardinals $\kappa < \lambda$, κ is λ strongly compact iff κ is λ supercompact, except possibly if κ is a measurable limit of cardinals which are λ supercompact. (A theorem of Menas [18] shows that if κ is a measurable limit of cardinals which are λ supercompact, then κ is λ strongly compact but need not be λ supercompact.) Models in which level by

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level equivalence between strong compactness and supercompactness holds nontrivially were first constructed by the author and Shelah in [10].[1]

It is known that in general, indestructibility for supercompactness in Laver's sense of [16] is incompatible with level by level equivalence between strong compactness and supercompactness, assuming the universe contains enough large cardinals. Indeed, [8, Theorem 5] shows that if $\kappa < \lambda$ are such that κ is indestructibly supercompact and λ is λ^+ supercompact, then level by level equivalence between strong compactness and supercompactness fails below κ. In spite of this, however, as has been done in [2, 3, 4, 6, 7, 8], it is possible to establish theorems which witness indestructibility in a restricted sense (either by placing limits on the number of large cardinals in the universe or constraining the amount of indestructibility witnessed) but which are compatible with level by level equivalence. A key feature of all of these theorems, however, is that there are always measurable cardinals κ in the witnessing universe which have their measurability destroyed by any $<\kappa$-closed forcing which adds a subset of κ (see [14] for further details on this phenomenon). This raises the following general

Question: Are there any indestructibility theorems which are compatible with level by level equivalence between strong compactness and supercompactness in which every measurable cardinal κ witnesses some indestructibility properties?

The purpose of this paper is to provide a positive answer to the aformentioned Question. Specifically, we prove the following theorem, where we say for regular cardinals $\kappa \le \lambda$ that the λ supercompactness of κ is *indestructible under forcing with a partial ordering* $\mathbb{P}$ if κ remains λ supercompact after forcing with $\mathbb{P}$. For $\alpha \ge 1$ any ordinal, $\mathrm{Add}(\kappa, \alpha)$ is the standard partial ordering for adding α many Cohen subsets of κ, i.e., $\mathrm{Add}(\kappa, \alpha) = \{f \mid f : \kappa \times \alpha \to \{0, 1\}$ is a function such that $|\mathrm{dom}(f)| < \kappa\}$, ordered by inclusion.

[1]In particular, it is shown in [10] that starting from a model V of ZFC + GCH in which the class of supercompact cardinals is nonempty but otherwise arbitrary, it is possible to force and obtain a model $\overline{V}$ of ZFC + GCH *containing exactly the same measurable and supercompact cardinals* in which level by level equivalence between strong compactness and supercompactness holds. Note that it is Hamkins' results of [11, 12], a corollary of which is stated here as Theorem 2, which imply that the classes of measurable and supercompact cardinals in V and $\overline{V}$ are the same.

Theorem 1 *Suppose $V \models$ "ZFC + GCH + The class of supercompact cardinals is nonempty + Level by level equivalence between strong compactness and supercompactness holds". There is then a cofinality preserving partial ordering $\mathbb{P} \subseteq V$ such that $V^{\mathbb{P}} \models$ "ZFC + GCH + Level by level equivalence between strong compactness and supercompactness holds", and in addition:*

1. *If $\kappa \le \lambda$ are regular cardinals, then $V \models$ "κ is λ supercompact" iff $V^{\mathbb{P}} \models$ "κ is λ supercompact" (so in particular, V and $V^{\mathbb{P}}$ contain the same measurable and supercompact cardinals).*

2. *If $V^{\mathbb{P}} \models$ "κ is λ supercompact and $\lambda \ge \kappa$ is inaccessible", then the λ supercompactness of κ is indestructible under forcing with either $\mathrm{Add}(\lambda, 1)$ or $\mathrm{Add}(\lambda, \lambda^+)$.*

We note that in Theorem 1, there are no restrictions whatsoever on the classes of measurable and supercompact cardinals. In particular, in both V and $V^{\mathbb{P}}$, there can be a proper class of measurable cardinals, a proper class of supercompact cardinals, etc. In addition, if $\lambda = \kappa$ in clause (2), then Theorem 1 states that the κ supercompactness of κ is indestructible under forcing with either $\mathrm{Add}(\kappa, 1)$ or $\mathrm{Add}(\kappa, \kappa^+)$. Since κ is measurable iff κ is κ supercompact iff κ is κ strongly compact, this means that in $V^{\mathbb{P}}$, every measurable cardinal κ has its measurability indestructible under forcing with either $\mathrm{Add}(\kappa, 1)$ or $\mathrm{Add}(\kappa, \kappa^+)$. Therefore, Theorem 1 may be regarded as providing a model for level by level equivalence between strong compactness and supercompactness with a very weak form of *universal indestructibility for measurability* (where as in [9], this means that for any measurable cardinal κ, κ's measurability is indestructible under arbitrary κ-directed closed forcing — see [9] for a discussion of the concept of universal indestructibility in general).

We conclude Section 1 with a very brief discussion of some preliminary material. We presume a basic knowledge of large cardinals and forcing. A good reference in this regard is [15]. When forcing, $q \ge p$ means that *q is stronger than p*. When G is V-generic over $\mathbb{P}$, we abuse notation slightly and take both $V[G]$ and $V^{\mathbb{P}}$ as being the generic extension of V by G. We also abuse notation slightly by occasionally confusing terms with the sets they denote, especially for ground model sets and variants of the generic object.

Suppose $\kappa < \lambda$ are regular cardinals. The partial ordering $\mathbb{P}$ is *κ-directed closed* if for every directed set $D \subseteq \mathbb{P}$ of size less than κ, there is a condition in $\mathbb{P}$ extending each member of D. $\mathbb{P}$ is

κ-*closed* if every increasing chain of members of $\mathbb{P}$ of length κ has an upper bound. $\mathbb{P}$ is $<\kappa$-*closed* if $\mathbb{P}$ is δ-closed for every $\delta < \kappa$.

We recall for the benefit of readers the definition given by Hamkins in [13, Section 3] of the lottery sum of a collection of partial orderings. If $\mathfrak{A}$ is a collection of partial orderings, then the *lottery sum* is the partial ordering $\oplus\mathfrak{A} = \{\langle \mathbb{P}, p\rangle \mid \mathbb{P} \in \mathfrak{A} \text{ and } p \in \mathbb{P}\} \cup \{0\}$, ordered with 0 below everything and $\langle \mathbb{P}, p\rangle \le \langle \mathbb{P}', p'\rangle$ iff $\mathbb{P} = \mathbb{P}'$ and $p \le p'$. Intuitively, if G is V-generic over $\oplus\mathfrak{A}$, then G first selects an element of $\mathfrak{A}$ (or as Hamkins says in [13], "holds a lottery among the posets in $\mathfrak{A}$") and then forces with it.[2]

A corollary of Hamkins' work on gap forcing found in [11, 12] will be employed in the proof of Theorem 1. We therefore state as a separate theorem what is relevant for this paper, along with some associated terminology, quoting from [11, 12] when appropriate. Suppose $\mathbb{P}$ is a partial ordering which can be written as $\mathbb{Q} * \dot{\mathbb{R}}$, where $|\mathbb{Q}| < \delta$, $\mathbb{Q}$ is nontrivial, and $\Vdash_{\mathbb{Q}}$ "$\dot{\mathbb{R}}$ is δ^+-directed closed". In Hamkins' terminology of [11, 12], $\mathbb{P}$ *admits a gap at* δ. In Hamkins' terminology of [11, 12], $\mathbb{P}$ is *mild with respect to a cardinal* κ iff every set of ordinals x in $V^{\mathbb{P}}$ of size below κ has a "nice" name $\dot{\tau}$ in V of size below κ, i.e., there is a set y in V, $|y| < \kappa$, such that $\Vdash_{\mathbb{P}}$ "$\dot{\tau} \subseteq \check{y}$". Also, as in the terminology of [11, 12] and elsewhere, an embedding $j : \overline{V} \to \overline{M}$ is *amenable to* $\overline{V}$ when $j \restriction A \in \overline{V}$ for any $A \in \overline{V}$. The specific corollary of Hamkins' work from [11, 12] we will be using is then the following.

Theorem 2 (Hamkins) *Suppose that $V[G]$ is a generic extension obtained by forcing with $\mathbb{P}$ that admits a gap at some regular $\delta < \kappa$. Suppose further that $j : V[G] \to M[j(G)]$ is an elementary embedding with critical point κ for which $M[j(G)] \subseteq V[G]$ and $M[j(G)]^{\delta} \subseteq M[j(G)]$ in $V[G]$. Then $M \subseteq V$; indeed, $M = V \cap M[j(G)]$. If the full embedding j is amenable to $V[G]$, then the restricted embedding $j \restriction V : V \to M$ is amenable to V. If j is definable from parameters (such as a measure or extender) in $V[G]$, then the restricted embedding $j \restriction V$ is definable from the names of those parameters in V. Finally, if $\mathbb{P}$ is mild with respect to κ and κ is λ strongly compact in*

[2]The terminology "lottery sum" is due to Hamkins, although the concept of the lottery sum of partial orderings has been around for quite some time and has been referred to at different junctures via the names "disjoint sum of partial orderings," "side-by-side forcing," and "choosing which partial ordering to force with generically."

$V[G]$ for any $\lambda \geq \kappa$, then κ is λ strongly compact in V.

2 The Proof of Theorem 1

We turn now to the proof of Theorem 1.

Proof: Suppose $V \models$ "ZFC + GCH + The class of supercompact cardinals in nonempty + Level by level equivalence between strong compactness and supercompactness holds". Let Ω be the class of all ordinals if there is a proper class of measurable cardinals, or the supremum of the set of all measurable cardinals otherwise. The partial ordering $\mathbb{P} = \langle\langle \mathbb{P}_\delta, \dot{\mathbb{Q}}_\delta\rangle \mid \delta \in \Omega\rangle$ used in the proof of Theorem 1 will be the (possibly proper class) reverse Easton iteration which begins by forcing with $\mathrm{Add}(\omega, 1)$ and then (potentially) performs nontrivial forcing only at those stages δ which are inaccessible cardinals. At such a δ, $\mathbb{P}_{\delta+1} = \mathbb{P}_\delta * \dot{\mathbb{Q}}_\delta$, where $\dot{\mathbb{Q}}_\delta$ is a term for the lottery sum of $\{\emptyset\}$, $\mathrm{Add}(\delta, 1)$, and $\mathrm{Add}(\delta, \delta^+)$. Standard arguments show that $V^{\mathbb{P}} \models$ ZFC and forcing with $\mathbb{P}$ preserves all cofinalities and GCH.

Lemma 2.1 *Suppose $\kappa \leq \lambda$ are regular cardinals such that $V \models$ "κ is λ supercompact". Then $V^{\mathbb{P}} \models$ "κ is λ supercompact".*

Proof: Write $\mathbb{P} = \mathbb{P}_{\lambda+1} * \dot{\mathbb{S}} = \mathbb{P}_\kappa * \dot{\mathbb{R}} * \dot{\mathbb{S}}$, where $\dot{\mathbb{R}}$ is a term for the portion of $\mathbb{P}$ acting on inaccessible cardinals in the closed interval $[\kappa, \lambda]$, and $\dot{\mathbb{S}}$ is a term for the portion of $\mathbb{P}$ acting on inaccessible cardinals above λ^+. Since $\Vdash_{\mathbb{P}_\kappa * \dot{\mathbb{R}}}$ "$\dot{\mathbb{S}}$ is η-directed closed for η the least strong limit cardinal above λ", to show that $V^{\mathbb{P}} \models$ "κ is λ supercompact", it suffices to show that $V^{\mathbb{P}_{\lambda+1}} = V^{\mathbb{P}_\kappa * \dot{\mathbb{R}}} \models$ "κ is λ supercompact".

To do this, let $j : V \to M$ be an elementary embedding witnessing the λ supercompactness of κ generated by a supercompact ultrafilter over $P_\kappa(\lambda)$. Let G_0 be V-generic over $\mathbb{P}_\kappa$ and G_1 be $V[G_0]$-generic over $\mathbb{R}$. We consider now the following two cases.

Case 1: $|G_1| \leq \lambda$, i.e., the part of $\mathbb{P}_\kappa * \dot{\mathbb{R}}$ above a suitable condition p_0 is forcing equivalent to a partial ordering having cardinality at most λ. Without loss of generality, but with an abuse of terminology and notation, we will take $\mathbb{P}_\kappa * \dot{\mathbb{R}}$ as $\mathbb{P}_\kappa * \dot{\mathbb{R}}$ above p_0. Write $j(\mathbb{P}_\kappa * \dot{\mathbb{R}}) = \mathbb{P}_\kappa * \dot{\mathbb{R}} * \dot{\mathbb{R}}' * j(\dot{\mathbb{R}})$, where $\dot{\mathbb{R}}'$ is a term for the portion of the forcing acting on M-inaccessible cardinals in the open interval $(\lambda, j(\kappa))$. Because $\mathbb{P}_\kappa * \dot{\mathbb{R}}$ is forcing equivalent to a partial ordering having cardinality at most λ, we may assume that $\mathbb{P}_\kappa * \dot{\mathbb{R}}$ is λ^+-c.c. Consequently, as $M^\lambda \subseteq M$, $M[G_0][G_1]^\lambda \subseteq M[G_0][G_1]$. Further, since j is generated by a supercompact ultrafilter over $P_\kappa(\lambda)$, $2^\lambda = \lambda^+$ in V, and $M[G_0][G_1] \vDash$ "$|\mathbb{R}'| = j(\lambda)$", the number of dense open subsets of $\mathbb{R}'$ present in $M[G_0][G_1]$ is $(2^{j(\lambda)})^M = (2^{j(\lambda)})^{M[G_0][G_1]} = j(\lambda^+)$. This is calculated in either V or $V[G_0][G_1]$ as $|\{f \mid f : P_\kappa(\lambda) \to 2^\lambda\}| = |\{f \mid f : \lambda \to 2^\lambda\}| = |\{f \mid f : \lambda \to \lambda^+\}| = [\lambda^+]^\lambda = \lambda^+$. We may therefore let $\langle D_\alpha \mid \alpha < \lambda^+ \rangle \in V[G_0][G_1]$ enumerate the dense open subsets of $\mathbb{R}'$ present in $M[G_0][G_1]$. As $M[G_0][G_1]^\lambda \subseteq M[G_0][G_1]$, by the definition of $\mathbb{P}$, $\mathbb{R}'$ is λ^+-directed closed in both $M[G_0][G_1]$ and $V[G_0][G_1]$. We may hence in $V[G_0][G_1]$ construct an increasing sequence $\langle q_\alpha \mid \alpha < \lambda^+ \rangle$ by letting $q_0 \in D_0$, and for $\alpha > 0$, letting $q_\alpha \in D_\alpha$ extend $\sup(\langle q_\beta \mid \beta < \alpha \rangle)$. Clearly, $G_2 = \{p \in \mathbb{R}' \mid \exists \alpha < \lambda^+ [q_\alpha \geq p]\}$ is $M[G_0][G_1]$-generic over $\mathbb{R}'$. Since by construction, $j''G_0 \subseteq G_0 * G_1 * G_2$, j lifts in $V[G_0][G_1]$ to $j : V[G_0] \to M[G_0][G_1][G_2]$.

Now, since $M[G_0][G_1][G_2]^\lambda \subseteq M[G_0][G_1][G_2]$ in $V[G_0][G_1]$, $V[G_0] \vDash$ "$\mathbb{R}$ is forcing equivalent to a partial ordering having cardinality at most λ", $M[G_0][G_1][G_2] \vDash$ "$j(\mathbb{R})$ is $j(\lambda)$-directed closed", and $j(\lambda) > \lambda^+$, there is a master condition $q \in j(\mathbb{R})$ for $j''G_1$. Further, the number of dense open subsets of $j(\mathbb{R})$ present in $M[G_0][G_1][G_2]$ is at most $(2^{j(\lambda)})^M$. As in the preceding paragraph, since $(2^{j(\lambda)})^M = (2^{j(\lambda)})^{M[G_0][G_1][G_2]}$, this is calculated in either V or $V[G_0][G_1]$ as λ^+. Consequently, we can once again use the same argument as given in the previous paragraph and build in $V[G_0][G_1]$ an $M[G_0][G_1][G_2]$-generic object G_3 over $j(\mathbb{R})$ containing q. Since by construction, $j''(G_0 * G_1) \subseteq G_0 * G_1 * G_2 * G_3$, j now fully lifts to $j : V[G_0][G_1] \to M[G_0][G_1][G_2][G_3]$. Hence, $V[G_0][G_1] \vDash$ "κ is λ supercompact", i.e., $V^{\mathbb{P}} \vDash$ "κ is λ supercompact".

Case 2: $|G_1| = \lambda^+$, i.e., λ is inaccessible, and at stage λ in the definition of $\mathbb{P}$, the partial ordering selected in the lottery by G_1 is $\mathrm{Add}(\lambda, \lambda^+)$. We then have that the part of $\mathbb{R}$ above a suitable condition q_0 is forcing equivalent to a partial ordering of the form $\mathbb{R}^* * \dot{\mathrm{Add}}(\lambda, \lambda^+)$. In analogy to Case 1 above, we will take $\mathbb{R}$ as $\mathbb{R}$ above q_0. This allows us to write $G_1 = H_0 * H_1$, where H_0 is $V[G_0]$-generic over $\mathbb{R}^*$, and H_1 is $V[G_0][H_0]$-generic over $\mathrm{Add}(\lambda, \lambda^+)$. The arguments given in Case 1 above then show that j lifts in $V[G_0][G_1]$ to $j : V[G_0][H_0] \to M[G_0][G_1][G_2][G_3]$, where G_2

and G_3 are constructed as in Case 1. Note it is once again the case that $M[G_0][G_1][G_2][G_3]^\lambda \subseteq M[G_0][G_1][G_2][G_3]$ in $V[G_0][G_1]$.

To lift j fully to $V[G_0][G_1]$, we now use an idea originally due to Magidor [17] but also found in [5, Lemma 2.2] (and elsewhere in the literature as well — readers may consult [5, Lemma 2.2] for additional references). We again feel free to quote verbatim as needed. We will construct in $V[G_0][G_1]$ an $M[G_0][G_1][G_2][G_3]$-generic object over $\mathrm{Add}(j(\lambda), j(\lambda^+))$. For $\alpha \in (\lambda, \lambda^+)$ and $p \in \mathrm{Add}(\lambda, \lambda^+)$, let $p \restriction \alpha = \{\langle\langle \rho, \sigma\rangle, \eta\rangle \in p \mid \sigma < \alpha\}$ and $H_1 \restriction \alpha = \{p \restriction \alpha \mid p \in H_1\}$. Clearly, $V[G_0][G_1] \vDash$ "$|H_1 \restriction \alpha| \le \lambda$ for all $\alpha \in (\lambda, \lambda^+)$". Thus, since $(\mathrm{Add}(j(\lambda), j(\lambda^+))^{M[G_0][G_1][G_2][G_3]}$ is $j(\lambda)$-directed closed and $j(\lambda) > \lambda^+$, $q_\alpha = \bigcup\{j(p) \mid p \in H_1 \restriction \alpha\}$ is well-defined and is an element of $\mathrm{Add}(j(\lambda), j(\lambda^+))^{M[G_0][G_1][G_2][G_3]}$. Further, if $\langle \rho, \sigma\rangle \in \mathrm{dom}(q_\alpha) - \mathrm{dom}(\bigcup_{\beta<\alpha} q_\beta)$ ($\bigcup_{\beta<\alpha} q_\beta$ is well-defined by closure), then $\sigma \in [\bigcup_{\beta<\alpha} j(\beta), j(\alpha))$. To see this, assume to the contrary that $\sigma < \bigcup_{\beta<\alpha} j(\beta)$. Let β be minimal such that $\sigma < j(\beta)$. It must thus be the case that for some $p \in H_1 \restriction \alpha$, $\langle \rho, \sigma\rangle \in \mathrm{dom}(j(p))$. Since by elementarity and the definitions of $H_1 \restriction \beta$ and $H_1 \restriction \alpha$, for $p \restriction \beta = q \in H_1 \restriction \beta$, $j(q) = j(p) \restriction j(\beta) = j(p \restriction \beta)$, it must be the case that $\langle \rho, \sigma\rangle \in \mathrm{dom}(j(q))$. This means $\langle \rho, \sigma\rangle \in \mathrm{dom}(q_\beta)$, a contradiction.

Since $M[G_0][G_1][G_2][G_3] \vDash$ "$2^{j(\lambda)} = j(\lambda^+)$", $M[G_0][G_1][G_2][G_3] \vDash$ "$\mathrm{Add}(j(\lambda), j(\lambda^+))$ is $j(\lambda^+)$-c.c. and has $j(\lambda^+)$ many maximal antichains". This means that if $\mathcal{A} \in M[G_0][G_1][G_2][G_3]$ is a maximal antichain of $\mathrm{Add}(j(\lambda), j(\lambda^+))$, $\mathcal{A} \subseteq \mathrm{Add}(j(\lambda), \beta)$ for some $\beta \in (j(\lambda), j(\lambda^+))$. Thus, since $V \vDash$ "$2^\lambda = \lambda^+$" and the fact j is generated by a supercompact ultrafilter over $P_\kappa(\lambda)$ imply that $V \vDash$ "$|j(\lambda^+)| = \lambda^+$", we can let $\langle \mathcal{A}_\alpha \mid \alpha \in (\lambda, \lambda^+)\rangle \in V[G_0][G_1]$ be an enumeration of all of the maximal antichains of $\mathrm{Add}(j(\lambda), j(\lambda^+))$ present in $M[G_0][G_1][G_2][G_3]$.

Working in $V[G_0][G_1]$, we define now an increasing sequence $\langle r_\alpha \mid \alpha \in (\lambda, \lambda^+)\rangle$ of elements of $\mathrm{Add}(j(\lambda), j(\lambda^+))$ such that $\forall \alpha \in (\lambda, \lambda^+)[r_\alpha \ge q_\alpha$ and $r_\alpha \in \mathrm{Add}(j(\lambda), j(\alpha))]$ and such that $\forall \alpha \in (\lambda, \lambda^+) \exists \beta \in (\lambda, \lambda^+) \exists r \in \mathcal{A}_\alpha [r_\beta \ge r]$. Assuming we have such a sequence, $G_4 = \{p \in \mathrm{Add}(j(\lambda), j(\lambda^+)) \mid \exists \alpha \in (\lambda, \lambda^+)[r_\alpha \ge p]$ is an $M[G_0][G_1][G_2][G_3]$-generic object over $\mathrm{Add}(j(\lambda), j(\lambda^+))$. To define $\langle r_\alpha \mid \alpha \in (\lambda, \lambda^+)\rangle$, if α is a limit, we let $r_\alpha = \bigcup_{\beta \in (\lambda, \alpha)} r_\beta$. By the facts $\langle r_\beta \mid \beta \in (\lambda, \alpha)\rangle$ is (strictly) increasing and $M[G_0][G_1][G_2][G_3]^\lambda \subseteq M[G_0][G_1][G_2][G_3]$ in $V[G_0][G_1]$, this definition is valid. Assuming now r_α has been defined and we wish to define $r_{\alpha+1}$, let $\langle \mathcal{B}_\beta \mid \beta < \eta < \lambda^+ \rangle$ be the subsequence of $\langle \mathcal{A}_\beta \mid \beta \le \alpha + 1 \rangle$ containing each antichain $\mathcal{A}$ such that $\mathcal{A} \subseteq \mathrm{Add}(j(\lambda), j(\alpha+1))$. Since $q_\alpha, r_\alpha \in \mathrm{Add}(j(\lambda), j(\alpha))$, $q_{\alpha+1} \in \mathrm{Add}(j(\lambda), j(\alpha+1))$, and $j(\alpha) < j(\alpha+1)$, the condition $r'_{\alpha+1} = r_\alpha \cup q_{\alpha+1}$ is well-defined, since by our earlier observations, any new elements of $\mathrm{dom}(q_{\alpha+1})$ won't be present in either $\mathrm{dom}(q_\alpha)$ or $\mathrm{dom}(r_\alpha)$. We can thus, using the fact $M[G_0][G_1][G_2][G_3]^\lambda \subseteq M[G_0][G_1][G_2][G_3]$ in $V[G_0][G_1]$, define by induction an increasing sequence $\langle s_\beta \mid \beta < \eta \rangle$ such that $s_0 \ge r'_{\alpha+1}$, $s_\rho = \bigcup_{\beta<\rho} s_\beta$ if ρ is a limit ordinal, and $s_{\beta+1} \ge s_\beta$ is such that $s_{\beta+1}$ extends some element of $\mathcal{B}_\beta$. The just mentioned closure fact implies $r_{\alpha+1} = \bigcup_{\beta<\eta} s_\beta$ is a well-defined condition.

In order to show that G_4 is $M[G_0][G_1][G_2][G_3]$-generic over $\mathrm{Add}(j(\lambda), j(\lambda^+))$, we must show that $\forall \alpha \in (\lambda, \lambda^+) \exists \beta \in (\lambda, \lambda^+) \exists r \in \mathcal{A}_\alpha [r_\beta \ge r]$. To do this, we first note that $\langle j(\alpha) \mid \alpha < \lambda^+ \rangle$ is unbounded in $j(\lambda^+)$. To see this, if $\beta < j(\lambda^+)$ is an ordinal, then for some $f : P_\kappa(\lambda) \to M$ representing β, we can assume that for $\alpha < \lambda$, $f(\alpha) < \lambda^+$. Thus, by the regularity of λ^+ in V, $\beta_0 = \bigcup_{\alpha<\lambda} f(\alpha) < \lambda^+$, and $j(\beta_0) \ge \beta$. This means by our earlier remarks that if $\mathcal{A} \in \langle \mathcal{A}_\alpha \mid \alpha < \lambda^+ \rangle$, $\mathcal{A} = \mathcal{A}_\rho$, then we can let $\beta \in (\lambda, \lambda^+)$ be such that $\mathcal{A} \subseteq \mathrm{Add}(j(\lambda), j(\beta))$. By construction, for $\eta > \max(\beta, \rho)$, there is some $r \in \mathcal{A}$ such that $r_\eta \ge r$. And, as any $p \in \mathrm{Add}(\lambda, \lambda^+)$ is such that for some $\alpha \in (\lambda, \lambda^+)$, $p = p \restriction \alpha$, G_4 is such that if $p \in H_1$, $j(p) \in G_4$. Thus, working in $V[G_0][G_1]$, we have shown that j lifts to $j : V[G_0][H_0][H_1] \to M[G_0][G_1][G_2][G_3][G_4]$, i.e., j lifts to $j : V[G_0][G_1] \to M[G_0][G_1][G_2][G_3][G_4]$. This means that $V[G_0][G_1] \models$ "κ is λ supercompact", i.e., $V^{\mathbb{P}} \models$ "κ is λ supercompact".

Cases 1 and 2 complete the proof of Lemma 2.1.

$\square$

Lemma 2.2 $V^{\mathbb{P}} \models$ "*Level by level equivalence between strong compactness and supercompactness holds*".

Proof: The proof of Lemma 2.2 follows closely the proof of [1, Lemma 1.3]. We again feel free to quote verbatim as needed. Suppose $V^{\mathbb{P}} \models$ "$\kappa < \lambda$ are regular cardinals such that κ is λ strongly compact and κ is not a measurable limit of cardinals which are λ supercompact". By Lemma 2.1, any cardinal which is λ supercompact in V remains λ supercompact in $V^{\mathbb{P}}$. This means that $V \models$ "$\kappa < \lambda$ are regular cardinals such that κ is not a measurable limit of cardinals which are λ supercompact".

Note that it is possible to write $\mathbb{P} = \mathbb{P}^0 * \dot{\mathbb{P}}^1$, where $|\mathbb{P}^0| = \omega$, $\mathbb{P}^0$ is nontrivial, and $\Vdash_{\mathbb{P}^0}$ "$\dot{\mathbb{P}}^1$ is $\aleph_2$-directed closed". Further, by the definition of $\mathbb{P}$, it is easily seen that $\mathbb{P}$ is mild with respect to κ. Therefore, by Theorem 2, $V \models$ "κ is λ strongly compact". Hence, by level by level equivalence between strong compactness and supercompactness in V, $V \models$ "κ is λ supercompact", so another application of Lemma 2.1 yields that $V^{\mathbb{P}} \models$ "κ is λ supercompact". This completes the proof of Lemma 2.2.

□

Lemma 2.3 *Assume $V^{\mathbb{P}} \models$ "$\lambda \geq \kappa$ is inaccessible and κ is λ supercompact". Then $V^{\mathbb{P}} \models$ "The λ supercompactness of κ is indestructible under forcing with either* $\mathrm{Add}(\lambda, 1)$ *or* $\mathrm{Add}(\lambda, \lambda^+)$*".*

Proof: Suppose $V^{\mathbb{P}} \models$ "$\lambda \geq \kappa$ is inaccessible and κ is λ supercompact". As in the proof of Lemma 2.1, write $\mathbb{P} = \mathbb{P}_{\lambda+1} * \dot{\mathbb{S}}$. It is then true that $\Vdash_{\mathbb{P}_{\lambda+1}}$ "$\dot{\mathbb{S}}$ is η-directed closed for η the least strong limit cardinal above λ". Hence, to prove Lemma 2.3, it suffices to show that in $V^{\mathbb{P}_{\lambda+1}}$, κ has its λ supercompactness indestructible under forcing with either $\mathrm{Add}(\lambda, 1)$ or $\mathrm{Add}(\lambda, \lambda^+)$.

To do this, we observe that at each inaccessible cardinal α, if we first force with either $\{\emptyset\}$, $\mathrm{Add}(\alpha, 1)$, or $\mathrm{Add}(\alpha, \alpha^+)$, and then follow this by forcing with $\mathrm{Add}(\alpha, 1)$ or $\mathrm{Add}(\alpha, \alpha^+)$, then this is equivalent to forcing with either $\mathrm{Add}(\alpha, 1)$ or $\mathrm{Add}(\alpha, \alpha^+)$. Therefore, $\mathbb{P}_\lambda * \dot{\mathbb{Q}}_\lambda * \dot{\mathrm{Add}}(\lambda, 1)$ and $\mathbb{P}_\lambda * \dot{\mathbb{Q}}_\lambda * \dot{\mathrm{Add}}(\lambda, \lambda^+)$ are forcing equivalent to $\mathbb{P}_\lambda * \dot{\mathrm{Add}}(\lambda, 1)$ and $\mathbb{P}_\lambda * \dot{\mathrm{Add}}(\lambda, \lambda^+)$ respectively. By the proof of Lemma 2.1, $V^{\mathbb{P}_\lambda * \dot{\mathrm{Add}}(\lambda, 1)} \models$ "κ is λ supercompact" and $V^{\mathbb{P}_\lambda * \dot{\mathrm{Add}}(\lambda, \lambda^+)} \models$ "κ is λ supercompact". Since there are forcing conditions r_0 and s_0 such that the part of $\mathbb{P}_\lambda * \dot{\mathbb{Q}}_\lambda = \mathbb{P}_{\lambda+1}$ above r_0 is forcing equivalent to $\mathbb{P}_\lambda * \dot{\mathrm{Add}}(\lambda, 1)$ and the part of $\mathbb{P}_\lambda * \dot{\mathbb{Q}}_\lambda = \mathbb{P}_{\lambda+1}$ above s_0 is forcing equivalent to $\mathbb{P}_\lambda * \dot{\mathrm{Add}}(\lambda, \lambda^+)$, this completes the proof of Lemma 2.3.

□

Note that by Theorem 2 and the factorization of $\mathbb{P}$ given in Lemma 2.2, if $V^{\mathbb{P}} \models$ "κ is λ supercompact", then $V \models$ "κ is λ supercompact" as well. Lemmas 2.1 – 2.3 therefore complete the proof of Theorem 1.

$\square$

Observe that if $\kappa \in V^{\mathbb{P}}$ is supercompact, then κ is not indestructibly supercompact. This follows from the fact that if it were, then $V^{\mathbb{P} * \dot{\mathrm{A}}\mathrm{dd}(\kappa, \kappa^{++})} \models$ "κ is measurable and $2^\kappa = \kappa^{++}$". Thus, in $V^{\mathbb{P} * \dot{\mathrm{A}}\mathrm{dd}(\kappa, \kappa^{++})}$, $A = \{\delta < \kappa \mid 2^\delta = \delta^{++}\}$ is unbounded in κ. However, since $\mathrm{Add}(\kappa, \kappa^{++})$ is κ-directed closed, A is unbounded in $V^{\mathbb{P}}$ as well. This contradicts the fact that $V^{\mathbb{P}} \models$ GCH.

We feel that Theorem 1 should be viewed as a first step in establishing indestructibility theorems which are compatible with level by level equivalence between strong compactness and supercompactness in which there are no limits on the number of measurable cardinals with indestructibility properties. We consequently conclude by posing the general question of what other theorems along these lines are possible.

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