Precisely Controlling Level by Level Behavior *†

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Abstract

We construct four models containing one supercompact cardinal in which level by level equivalence between strong compactness and supercompactness and level by level inequivalence between strong compactness and supercompactness are precisely controlled at each non-supercompact measurable cardinal. In these models, no cardinal κ is $<\kappa'$ supercompact, where κ' is the least inaccessible cardinal greater than κ.

1 Introduction and Preliminaries

Say that a model containing at least one supercompact cardinal satisfies *level by level equivalence between strong compactness and supercompactness* if for every pair of regular cardinals $\kappa < \lambda$, κ is λ strongly compact iff κ is λ supercompact, except possibly if κ is a measurable limit of cardinals δ which are λ supercompact. (The exception is given by a theorem of Menas [15], who showed that if κ is a measurable limit of cardinals δ which are λ strongly compact, then κ is λ

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strongly compact but need not be λ supercompact.) In addition, say that κ satisfies *level by level equivalence between strong compactness and supercompactness* if for every regular cardinal $\lambda > \kappa$, κ is λ strongly compact iff κ is λ supercompact. Note that supercompact cardinals automatically satisfy level by level equivalence between strong compactness and supercompactness. These notions were first introduced and studied by the author and Shelah in [10], where models in which level by level equivalence between strong compactness and supercompactness holds nontrivially were constructed.

As a dual to the concepts mentioned in the preceding paragraph, say that a model containing at least one supercompact cardinal satisfies *level by level inequivalence between strong compactness and supercompactness* if for every non-supercompact measurable cardinal δ, there is some $\gamma > \delta$ such that δ is γ strongly compact yet δ is not γ supercompact. The non-supercompact measurable cardinal δ is then said to satisfy *level by level inequivalence between strong compactness and supercompactness*. These ideas were first introduced and studied in [4]. Models containing exactly one supercompact cardinal in which level by level inequivalence between strong compactness and supercompactness holds may be found in [4, Theorem 2], [6, Theorem 2], [2, Theorem 1], [3, Theorems 1–3], [8, Theorem 32(2)], and [1, Theorem 1.1].

The purpose of this paper is to investigate what sorts of interactions between these two dual notions are possible in the same model of ZFC. Specifically, we consider the following

Question: Is it possible to have a model of ZFC containing at least one supercompact cardinal in which level by level equivalence between strong compactness and supercompactness and level by level inequivalence between strong compactness and supercompactness are precisely controlled at every non-supercompact measurable cardinal?

We provide answers to this question with the following four theorems. We will take as notation that for any cardinal δ, δ' is the least inaccessible cardinal greater that δ. We will also say that δ *is* $<\delta'$ *supercompact (strongly compact)* if δ is γ supercompact (strongly compact) for every $\gamma < \lambda$. In words, this is expressed by saying δ *is supercompact (strongly compact) up to* γ. In addition, we will henceforth, for brevity, write "level by level equivalence" and "level by level inequivalence" instead of "level by level equivalence between strong compactness and supercompactness" and "level by level inequivalence between strong compactness and supercompactness". Further, since the statements of our four theorems are quite technical in nature, we first give intuitive descriptions of what each theorem actually says. Roughly speaking, Theorem 1 provides a model containing exactly one supercompact cardinal κ_0 and no other strongly compact cardinals in which no cardinal is supercompact up to an inaccessible cardinal, level by level *inequivalence* holds at every non-supercompact measurable cardinal which is a limit of measurable cardinals, and level by level equivalence and inequivalence at measurable cardinals which are not limits of measurable cardinals is precisely controlled by a ground model function. Theorem 2 provides a model containing exactly one supercompact cardinal κ_0 and no other strongly compact cardinals in which no cardinal is supercompact up to an inaccessible cardinal, level by level *equivalence* holds at every measurable cardinal which is a limit of measurable cardinals, and level by level equivalence and inequivalence at measurable cardinals which are not limits of measurable cardinals is precisely controlled by a ground model function. Theorems 3 and 4 provide models containing exactly one supercompact cardinal κ and no other strongly compact cardinals in which no cardinal is supercompact up to an inaccessible cardinal and an arbitrary ordinal $\alpha < \kappa$ acts as a "pivot", in the sense that every measurable cardinal $\delta \le \alpha$ either satisfies level by level equivalence or level by level inequivalence, but every non-supercompact measurable cardinal $\delta > \alpha$ satisfies the dual property.

Our four theorems are precisely stated as follows:

Theorem 1 *Suppose $V \models$ "ZFC + GCH + κ is κ' supercompact". Let $A = \{\delta < \kappa \mid \delta$ is $<\delta'$ supercompact$\}$ and $B = \{\delta < \kappa \mid \delta$ is $<\delta'$ supercompact but is not a limit of cardinals λ which are $<\lambda'$ supercompact$\}$, with $f : B \to 2$ a function. There is then a partial ordering $\mathbb{P} \in V$, a submodel $\overline{V} \subseteq V^{\mathbb{P}}$, and a cardinal $\kappa_0 < \kappa$ such that $\overline{V} \models$ "ZFC + GCH + κ_0 is supercompact + The only measurable cardinals are the members of $A \cap (\kappa_0 + 1)$". In $\overline{V}$, level by level inequivalence holds at every non-supercompact measurable cardinal which is a limit of measurable cardinals, no measurable cardinal δ is $<\delta'$ supercompact, the measurable cardinals which are not limits of measurable cardinals are the members of $B \cap \kappa_0$, and κ_0 is the only strongly compact cardinal. Further, $\overline{V} \models$ "If $f(\delta) = 0$,* *then level by level equivalence holds at δ, but if $f(\delta) = 1$, then level by level inequivalence holds at δ".*

Theorem 2 *Suppose $V \models$ "ZFC + GCH + κ is κ' supercompact". Let $A = \{\delta < \kappa \mid \delta$ is $<\delta'$ supercompact$\}$ and $B = \{\delta < \kappa \mid \delta$ is $<\delta'$ supercompact but is not a limit of cardinals λ which are $<\lambda'$ supercompact$\}$, with $f : B \to 2$ a function. There is then a partial ordering $\mathbb{P}^* \in V$, a submodel $V^* \subseteq V^{\mathbb{P}^*}$, and a cardinal $\kappa_0 < \kappa$ such that $V^* \models$ "ZFC + GCH + κ_0 is supercompact + The only measurable cardinals are the members of $A \cap (\kappa_0 + 1)$". In V^*, level by level equivalence holds at every measurable cardinal which is a limit of measurable cardinals, no measurable cardinal δ is $<\delta'$ supercompact, the measurable cardinals which are not limits of measurable cardinals are the members of $B \cap \kappa_0$, and κ_0 is the only strongly compact cardinal. Further, $V^* \models$ "If $f(\delta) = 0$, then level by level equivalence holds at δ, but if $f(\delta) = 1$, then level by level inequivalence holds at δ".*

Theorem 3 *Suppose $V \models$ "ZFC + GCH + κ is supercompact and is the only strongly compact cardinal + No cardinal is supercompact up to an inaccessible cardinal + Level by level inequivalence holds at every non-supercompact measurable cardinal δ (and the least witness to level by level inequivalence is some $\gamma < \delta'$)". Let $\alpha < \kappa$ be a fixed but arbitrary ordinal. There is then a partial ordering $\mathbb{P} \subseteq V$ such that $V^{\mathbb{P}} \models$ "ZFC + GCH + κ is supercompact and is the only strongly compact cardinal + No cardinal is supercompact up to an inaccessible cardinal + Level by level inequivalence holds for all measurable cardinals less than or equal to α + Level by equivalence holds for all measurable cardinals greater than α". In addition, V and $V^{\mathbb{P}}$ contain the same measurable cardinals.*

Theorem 4 *Suppose $V \models$ "ZFC + GCH + κ is supercompact and is the only strongly compact cardinal + No cardinal is supercompact up to an inaccessible cardinal + Level by level inequivalence holds at every non-supercompact measurable cardinal δ (and the least witness to level by level inequivalence is some $\gamma < \delta'$)". Let $\alpha < \kappa$ be a fixed but arbitrary ordinal. There is then a partial ordering $\mathbb{P} \in V$ such that $V^{\mathbb{P}} \models$ "ZFC + GCH + κ is supercompact and is the only strongly compact* *cardinal + No cardinal is supercompact up to an inaccessible cardinal + Level by level equivalence holds for all measurable cardinals less than or equal to* α *+ Level by inequivalence holds for all non-supercompact measurable cardinals greater than* α*". In addition,* V *and* $V^{\mathbb{P}}$ *contain the same measurable cardinals.*

We take this opportunity to make a few additional remarks concerning Theorems 1 – 4. The exact hypotheses used to prove Theorems 3 and 4 are GCH together with the existence of cardinals $\kappa_1 < \kappa_2$ such that κ_1 is supercompact and κ_2 is inaccessible. In particular, we first force to construct the ground model V mentioned in the statements of Theorems 3 and 4 (which will be the model witnessing the conclusions of [3, Theorem 3]), and then force over this model to complete the proofs of these theorems. To avoid excessive technicalities, we have chosen to state these theorems as we did above. We will discuss in greater detail at the end of this paper the exact hypotheses we seem to need to construct models for level by level inequivalence containing exactly one supercompact cardinal in which no measurable cardinal δ is $<\delta'$ supercompact. Also, because a measurable cardinal κ with a normal measure μ concentrating on measurable cardinals will exhibit either level by level equivalence or level by level inequivalence in the ultrapower via μ, this will reflect below κ on a measure 1 set. Hence, we cannot expect to control level by level equivalence or level by level inequivalence at measurable cardinals of nontrivial Mitchell rank as arbitrarily as we do in Theorems 1 and 2 at measurable cardinals which are not limits of measurable cardinals. Further, we note that if κ is supercompact and $\lambda > \kappa$ is inaccessible, then by reflection, $\{\delta < \kappa \mid \delta \text{ is } \delta' \text{ supercompact}\}$ is unbounded in κ. Thus, in the models witnessing the conclusions of Theorems 1 – 4, there cannot be any inaccessible cardinals above the supercompact cardinal in that model.

Before beginning the proof of Theorems 1 – 4, we very briefly mention some preliminary information concerning notation and terminology. When forcing, $q \geq p$ means that q *is stronger than* p. For $\alpha < \beta$ ordinals, $(\alpha, \beta]$ and (α, β) are as in standard interval notation. If G is V-generic over $\mathbb{P}$, we will abuse notation slightly and use both $V[G]$ and $V^{\mathbb{P}}$ to indicate the universe obtained by forcing with $\mathbb{P}$. We will, from time to time, confuse terms with the sets they denote and write x

when we actually mean $\dot{x}$ or $\check{x}$.

Suppose $\kappa > \omega$ is a regular cardinal. The partial ordering $\mathbb{P}$ is *κ-strategically closed* if in the two person game in which the players construct an increasing sequence $\langle p_\alpha \mid \alpha \le \kappa \rangle$, where player I plays odd stages and player II plays even (which of course includes limit) stages (choosing the trivial condition at stage 0), player II has a strategy which ensures the game can always be continued. $\mathbb{P}$ is *$\prec\kappa$-strategically closed* if in the two person game in which the players construct an increasing sequence $\langle p_\alpha \mid \alpha < \kappa \rangle$, where player I plays odd stages and player II plays even stages (again choosing the trivial condition at stage 0), player II has a strategy which ensures the game can always be continued.

An example of a partial ordering which is $\prec\kappa$-strategically closed and which will be used in the proof of Theorem 1 is the partial ordering $\mathbb{P}(\omega, \kappa)$ for adding a nonreflecting stationary set of ordinals of cofinality ω to κ. For additional details and the exact definition, readers are urged to consult [10, second paragraph of Section 1, page 106].

We mention that we are assuming complete familiarity with the notions of measurability, strong compactness, and supercompactness. Interested readers may consult [16] for further details. We note only that all elementary embeddings witnessing the λ supercompactness of κ are presumed to come from some fine, κ-complete, normal ultrafilter $\mathcal{U}$ over $P_\kappa(\lambda) = \{x \subseteq \lambda \mid |x| < \kappa\}$, and all elementary embeddings witnessing the λ strong compactness of κ are presumed to come from some fine, κ-complete ultrafilter $\mathcal{U}$ over $P_\kappa(\lambda)$. An equivalent definition for κ being λ strongly compact is that there is an elementary embedding $j : V \to M$ having critical point κ such that for any $x \subseteq M$ with $|x| \le \lambda$, there is some $y \in M$ such that $x \subseteq y$ and $M \vDash$ "$|y| < j(\kappa)$". A measurable cardinal κ has *nontrivial Mitchell rank* if κ carries a normal measure μ such that κ is measurable in the ultrapower V^κ/μ.

When discussing the proofs of Theorems 2 – 4, we will be assuming some familiarity with the work of [10]. This material, which is rather complicated, will be taken as a "black box", with many definitions and facts only referenced and not given explicitly. However, all relevant details can be found in [10, Sections 1 and 2].

A corollary of Hamkins' work on gap forcing found in [11, 12] will be employed in the proofs of our theorems. We therefore state as a separate theorem what is relevant for this paper, along with some associated terminology, quoting from [11, 12] when appropriate. Suppose $\mathbb{P}$ is a partial ordering which can be written as $\mathbb{Q} * \dot{\mathbb{R}}$, where $|\mathbb{Q}| < \delta$, $\mathbb{Q}$ is nontrivial, and $\Vdash_{\mathbb{Q}}$ "$\dot{\mathbb{R}}$ is δ-strategically closed". In Hamkins' terminology of [11, 12], $\mathbb{P}$ *admits a gap at* δ. In Hamkins' terminology of [11, 12], $\mathbb{P}$ is *mild with respect to a cardinal* κ iff every set of ordinals x in $V^{\mathbb{P}}$ of size less than κ has a "nice" name τ in V of size less than κ, i.e., there is a set y in V, $|y| < \kappa$, such that $\Vdash_{\mathbb{P}}$ "$\tau \subseteq \check{y}$". Also, as in the terminology of [11, 12] and elsewhere, an embedding $j : \overline{V} \to \overline{M}$ is *amenable to* $\overline{V}$ when $j \restriction A \in \overline{V}$ for any $A \in \overline{V}$. The specific corollary of Hamkins' work from [11, 12] we will be using is then the following.

Theorem 5 (Hamkins) *Suppose that $V[G]$ is a generic extension obtained by forcing with $\mathbb{P}$ that admits a gap at some regular $\delta < \kappa$. Suppose further that $j : V[G] \to M[j(G)]$ is an embedding with critical point κ for which $M[j(G)] \subseteq V[G]$ and $M[j(G)]^{\delta} \subseteq M[j(G)]$ in $V[G]$. Then $M \subseteq V$; indeed, $M = V \cap M[j(G)]$. If the full embedding j is amenable to $V[G]$, then the restricted embedding $j \restriction V : V \to M$ is amenable to V. If j is definable from parameters (such as a measure or extender) in $V[G]$, then the restricted embedding $j \restriction V$ is definable from the names of those parameters in V. Finally, if $\mathbb{P}$ is mild with respect to κ and κ is λ strongly compact in $V[G]$ for any $\lambda \geq \kappa$, then κ is λ strongly compact in V.*

Theorem 5 implies that if κ is measurable or supercompact in $V^{\mathbb{P}}$ and $\mathbb{P}$ admits a gap below κ, then κ was measurable or supercompact in V as well. In addition, if κ is strongly compact in $V^{\mathbb{P}}$ and $\mathbb{P}$ is both mild with respect to κ and admits a gap below κ, then κ was also strongly compact in V.

2 The Proofs of Theorems 1 − 4

We turn now to the proofs of our theorems, beginning with the proof of Theorem 1.

Proof: Suppose $V \models$ "ZFC + GCH + κ is κ' supercompact". The partial ordering $\mathbb{P}$ used in the proof of Theorem 1 is the reverse Easton iteration of length κ which begins by adding a Cohen subset of ω and then does nontrivial forcing only at those $\delta < \kappa$ such that either δ is a V-measurable cardinal but $\delta \notin A$, or $\delta = \lambda^+$ where $f(\lambda) = 0$. At these δ, we force with $\mathbb{P}(\omega, \delta)$. Note that by its definition, forcing with $\mathbb{P}$ preserves GCH.

Lemma 2.1 *For $\lambda = (\kappa')^V$, $V^{\mathbb{P}} \models$ "κ is $<\lambda$ supercompact".*

Proof: Let $j : V \to M$ be an elementary embedding witnessing the λ supercompactness of κ generated by a supercompact ultrafilter over $P_\kappa(\lambda)$. Because $M^\lambda \subseteq M$, $M \models$ "$\lambda = \kappa'$ and κ is $<\lambda$ supercompact". By reflection, $\{\delta < \kappa \mid \delta \text{ is } <\delta' \text{ supercompact}\}$ is unbounded in κ in both V and M. From this, we may infer that $\kappa \in j(A \setminus B)$. The last three sentences therefore allow us to conclude that $j(\mathbb{P}) = \mathbb{P} * \dot{\mathbb{Q}}$, where the first ordinal at which $\dot{\mathbb{Q}}$ is forced to do nontrivial forcing is well above λ. The argument given by Laver in [13] now applies and shows that $V^{\mathbb{P}} \models$ "κ is $<\lambda$ supercompact". (In particular, suppose $\delta < \lambda$ is arbitrary. Let $G_0 * G_1$ be V-generic over $\mathbb{P} * \dot{\mathbb{Q}}$. Lift j in $V[G_0][G_1]$ to $j : V[G_0] \to M[G_0][G_1]$, and show by the λ-directed closure of $\mathbb{Q}$ that the supercompactness measure over $(P_\kappa(\delta))^{V[G_0][G_1]}$ generated by j is actually a member of $V[G_0]$.) This completes the proof of Lemma 2.1.

$\square$

Lemma 2.2 *$V^{\mathbb{P}} \models$ "If $\delta < \kappa$ is measurable, then $\delta \in A$".*

Proof: Suppose $V^{\mathbb{P}} \models$ "$\delta < \kappa$ is measurable". Write $\mathbb{P} = \mathbb{P}_0 * \dot{\mathbb{Q}}$, where $|\mathbb{P}_0| = \omega$, $\mathbb{P}_0$ is nontrivial, and $\Vdash_{\mathbb{P}_0}$ "$\dot{\mathbb{Q}}$ is $\aleph_1$-strategically closed". By Theorem 5, because $\mathbb{P}$ admits a gap at $\aleph_1$, $V \models$ "δ is measurable" as well. However, by the definition of $\mathbb{P}$, if $\delta \notin A$, then $V^{\mathbb{P}} \models$ "There is a non-reflecting stationary subset of δ composed of ordinals of cofinality ω", which further implies that $V^{\mathbb{P}} \models$ "δ is not weakly compact". This contradiction completes the proof of Lemma 2.2.

$\square$

Lemma 2.3 *If $\delta \in A$, then $V^{\mathbb{P}_\delta} \models$ "δ is $<\delta'$ strongly compact".*

Proof: Suppose $\delta \in A$. Let $\lambda < \delta'$, $\lambda > \delta$ be a regular cardinal. Take $j : V \to M$ as an elementary embedding witnessing the λ supercompactness of δ generated by a supercompact ultrafilter over $P_\delta(\lambda)$ such that $M \vDash$ "δ is not λ supercompact". Note that since $M^\lambda \subseteq M$, $M \vDash$ "$\lambda < \delta'$". In addition, because GCH implies that $\lambda \geq \delta^+ = 2^\delta$, $M \vDash$ "δ is a measurable cardinal". Therefore, by the definition of $\mathbb{P}$, $j(\mathbb{P}_\delta) = \mathbb{P}_\delta * \dot{\mathbb{P}}(\omega, \delta) * \dot{\mathbb{R}}$, where the first ordinal at which $\dot{\mathbb{R}}$ is forced to do nontrivial forcing is well above λ. The same argument as found in the proofs of [7, Lemma 2.4] and [9, Lemma 2.3] now shows that $V^{\mathbb{P}_\delta} \vDash$ "δ is λ strongly compact".[1] Also, as $|\mathbb{P}_\delta| = \delta$, δ' is the same in both V and $V^{\mathbb{P}_\delta}$. Consequently, since $\lambda > \delta$ was an arbitrary regular cardinal below the least V-inaccessible cardinal above δ, which we know is the same as the least $V^{\mathbb{P}_\delta}$-inaccessible cardinal above δ, the proof of Lemma 2.3 has been completed.

□

Lemma 2.4 *If either $\delta \in A \setminus B$ or $f(\delta) = 1$, then $V^{\mathbb{P}} \vDash$ "δ is $<\delta'$ strongly compact".*

Proof: Suppose δ is as in the hypotheses for Lemma 2.4. By the definition of $\mathbb{P}$, in either case, we can write $\mathbb{P} = \mathbb{P}_\delta * \dot{\mathbb{P}}^\delta$, where $\Vdash_{\mathbb{P}_\delta}$ "$\dot{\mathbb{P}}^\delta$ is δ'-strategically closed". Since by Lemma 2.3, $V^{\mathbb{P}_\delta} \vDash$ "δ is $<\delta'$ strongly compact", it follows that $V^{\mathbb{P}_\delta * \dot{\mathbb{P}}^\delta} = V^{\mathbb{P}} \vDash$ "δ is $<\delta'$ strongly compact" as well. This completes the proof of Lemma 2.4.

□

[1] An outline of this argument, due originally to Magidor but unpublished by him, is as follows. Let $k : M \to N$ be an elementary embedding generated by a normal measure $\mathcal{U} \in M$ over δ such that $N \vDash$ "δ is not measurable". The elementary embedding $i = k \circ j$ witnesses the λ strong compactness of δ in V. This embedding lifts in $V^{\mathbb{P}_\delta}$ to an elementary embedding $i : V^{\mathbb{P}_\delta} \to N^{i(\mathbb{P}_\delta)}$ witnessing the λ strong compactness of δ. To see this, write $i(\mathbb{P}_\delta) = \mathbb{P}_\delta * \dot{\mathbb{Q}}^1 * \dot{\mathbb{Q}}^2$, where $\dot{\mathbb{Q}}^1$ is forced to act nontrivially on ordinals in the interval $(\delta, k(\delta)]$, and $\dot{\mathbb{Q}}^2$ is forced to act nontrivially on ordinals in the interval $(k(\delta), k(j(\delta))) = (k(\delta), i(\delta))$. Next, take G_0 to be V-generic over $\mathbb{P}_\delta$, and build in $V[G_0]$ generic objects G_1 and G_2 for $\mathbb{Q}^1$ and $\mathbb{Q}^2$ respectively. The construction of G_1 uses that by GCH and the fact that k is given by an ultrapower embedding, we may let $\langle D_\alpha \mid \alpha < \delta^+ \rangle$ enumerate in $V[G_0]$ the dense open subsets of $\mathbb{Q}^1$ present in $N[G_0]$. Since $N \vDash$ "δ is not measurable", the first nontrivial stage of forcing in $\mathbb{Q}^1$ occurs well above δ. This implies that $N[G_0] \vDash$ "$\mathbb{Q}^1$ is $\prec\delta^+$-strategically closed". Because $N[G_0]$ remains δ-closed with respect to $V[G_0]$, by the $\prec\delta^+$-strategic closure of $\mathbb{Q}^1$ in both $N[G_0]$ and $V[G_0]$, we may work in $V[G_0]$ and meet each D_α in order to construct G_1. The construction of G_2 first requires building an M-generic object G_2^* for the term forcing partial ordering $\mathbb{T}$ associated with $\dot{\mathbb{R}}$ and defined in M with respect to $\mathbb{P}_\delta * \dot{\mathbb{P}}(\omega, \delta)$. G_2^* is built using the facts that since $M^\lambda \subseteq M$ and the first nontrivial stage of forcing in $\mathbb{T}$ occurs well above λ, $\mathbb{T}$ is $\prec\lambda^+$-strategically closed in both M and V, which means that the diagonalization argument employed in the construction of G_1 may be applied in this situation as well. $k''G_2^*$ now generates an N-generic object G_2^{**} for $k(\mathbb{T})$ and an $N[G_0][G_1]$-generic object G_2 for $\mathbb{Q}^2$ This tells us that i lifts in $V[G_0]$ to $i : V[G_0] \to N[G_0][G_1][G_2]$.

Lemma 2.5 *If $f(\delta) = 0$, then $V^{\mathbb{P}} \models$ "δ is measurable and level by level equivalence holds at δ".*

Proof: Suppose that δ is as in the hypotheses for Lemma 2.5. By the definition of $\mathbb{P}$, we can write $\mathbb{P} = \mathbb{P}_\delta * \dot{\mathbb{P}}(\omega, \delta^+) * \dot{\mathbb{P}}^\delta$. Lemma 2.3 implies that $V^{\mathbb{P}_\delta} \models$ "δ is measurable". Since $\Vdash_{\mathbb{P}_\delta}$ "$\dot{\mathbb{P}}(\omega, \delta^+)$ is δ-strategically closed", $V^{\mathbb{P}_\delta * \dot{\mathbb{P}}(\omega, \delta^+)} \models$ "δ is measurable" as well. As $V^{\mathbb{P}_\delta * \dot{\mathbb{P}}(\omega, \delta^+)} \models$ "There is a nonreflecting stationary subset of δ^+ composed of ordinals of cofinality ω", by [16, Theorem 4.8], $V^{\mathbb{P}_\delta * \dot{\mathbb{P}}(\omega, \delta^+)} \models$ "δ is not δ^+ strongly compact". It therefore follows that $V^{\mathbb{P}_\delta * \dot{\mathbb{P}}(\omega, \delta^+)} \models$ "Level by level equivalence holds at δ". Because $\Vdash_{\mathbb{P}_\delta * \dot{\mathbb{P}}(\omega, \delta^+)}$ "$\dot{\mathbb{P}}^\delta$ is δ'-strategically closed", $V^{\mathbb{P}_\delta * \dot{\mathbb{P}}(\omega, \delta^+) * \dot{\mathbb{P}}^\delta} = V^{\mathbb{P}} \models$ "δ is measurable and δ is not δ^+ strongly compact", i.e., $V^{\mathbb{P}} \models$ "Level by level equivalence holds at δ". This completes the proof of Lemma 2.5.

□

Let now κ_0 be the least cardinal such that $V^{\mathbb{P}} \models$ "κ_0 is $<\kappa_0'$ supercompact". Lemma 2.1 implies that κ_0 exists and $\kappa_0 \le \kappa$. To see that in fact $\kappa_0 < \kappa$, let $\lambda = \kappa'$ and $j : V \to M$ be an elementary embedding witnessing the λ supercompactness of κ generated by a supercompact ultrafilter over $P_\kappa(\lambda)$. As in the proof of Lemma 2.1, $\kappa \in j(A \setminus B)$, and $j(\mathbb{P}) = \mathbb{P} * \dot{\mathbb{Q}} = \mathbb{P}_\kappa * \dot{\mathbb{Q}}$, where the first ordinal at which $\dot{\mathbb{Q}}$ is forced to do nontrivial forcing is well above κ'. Because $M^{\kappa'} \subseteq M$ and $V^{\mathbb{P}} \models$ "κ is $<\kappa'$ supercompact", it follows that in M, both $\Vdash_{\mathbb{P}}$ "κ is $<\kappa'$ supercompact" and $\Vdash_{\mathbb{P}_\kappa * \dot{\mathbb{Q}}}$ "κ is $<\kappa'$ supercompact". In other words, in M, $\Vdash_{j(\mathbb{P})}$ "κ is $<\kappa'$ supercompact". By reflection, $C = \{\delta < \kappa \mid \Vdash_{\mathbb{P}}$ "δ is $<\delta'$ supercompact"$\}$ is unbounded in κ. The smallest member κ_0 of C witnesses that $\kappa_0 < \kappa$.

Define $\overline{V} = (V_{\kappa_0'})^{V^{\mathbb{P}}}$. It follows that $\overline{V}$ is a model of ZFC + GCH containing no measurable cardinals greater than κ_0 and $\overline{V} \models$ "κ_0 is supercompact and no cardinal $\delta < \kappa_0$ is $<\delta'$ supercompact". By Lemmas 2.2, 2.4, and 2.5, the only measurable cardinals in $\overline{V}$ less than κ_0 are the members of $A \cap \kappa_0 = \{\delta < \kappa_0 \mid \delta$ is $<\delta'$ supercompact in $V\}$. Since $B \cap \kappa_0 = \{\delta < \kappa_0 \mid \delta$ is $<\delta'$ supercompact in V but is not a limit of cardinals λ which are $<\lambda'$ supercompact in $V\}$, this means that in $\overline{V}$, the measurable cardinals which are not limits of measurable cardinals are the members of $B \cap \kappa_0$, and the measurable limits of measurable cardinals below κ_0 are the members of $(A \setminus B) \cap \kappa_0$. Therefore, the fact κ_0 is the smallest cardinal such that $V^{\mathbb{P}} \models$ "κ_0 is $<\kappa_0'$ supercompact" and Lemma 2.4

together imply that if either $\delta < \kappa_0$ is a measurable limit of measurable cardinals or $f(\delta) = 1$, then $\overline{V} \models$ "Level by level inequivalence holds at δ (and the least witness to level by level inequivalence is some $\gamma < \delta'$)". Lemma 2.5 implies that $\overline{V} \models$ "If $f(\delta) = 0$, then δ is measurable and level by level equivalence holds at δ". The proof of Theorem 1 is consequently completed by the following lemma.

Lemma 2.6 $\overline{V} \models$ "κ_0 *is the only strongly compact cardinal".*

Proof: By the definition of $\mathbb{P}$, unboundedly many cardinals $\delta < \kappa_0$ (e.g., those δ which were measurable in V but not $<\delta'$ supercompact in V) have nonreflecting stationary subsets composed of ordinals of cofinality ω. By [16, Theorem 4.8] and the succeeding remarks, this means that $\overline{V} \models$ "No cardinal less than κ_0 is strongly compact". This completes the proof of both Lemma 2.6 and Theorem 1.

$\square$

$\square$

Turning now to the proof of Theorem 2, let V be such that $V \models$ "ZFC + GCH + κ is κ' supercompact". Because V satisfies the same hypotheses as the ground model of Theorem 1, we may let $\mathbb{P} \in V$ and $\overline{V} \subseteq V^{\mathbb{P}}$ be such that $\overline{V}$ witnesses the conclusions of Theorem 1. We will take $\overline{V}$ as our ground model and force over $\overline{V}$ with a class partial ordering $\mathbb{Q}$ defined in $\overline{V}$ (which will be a set partial ordering defined in $V^{\mathbb{P}}$). To define $\mathbb{Q}$, let $\mathfrak{D} = \langle \delta_\alpha \mid \alpha \le \kappa_0 \rangle$ enumerate the members of $((A \setminus B) \cap \kappa_0) \cup \{\kappa_0\}$. $\mathbb{Q}$ is then taken as the reverse Easton proper class iteration which begins by adding a Cohen subset of ω and then is the partial ordering P of [10, page 114] defined using the members of $\mathfrak{D}$. By the Lévy-Solovay results [14] and [10, Lemmas 8–11, pages 114–120], $\overline{V}^{\mathbb{Q}} \models$ "ZFC + GCH + Every $\delta \in \mathfrak{D}$ is measurable + Level by level equivalence holds for each $\delta \in \mathfrak{D}$ + κ_0 is supercompact". In addition, as in the proof of Lemma 2.2, we can write $\mathbb{Q} = \mathbb{Q}_0 * \dot{\mathbb{R}}$, where $|\mathbb{Q}_0| = \omega$, $\mathbb{Q}_0$ is nontrivial, and $\Vdash_{\mathbb{Q}_0}$ "$\dot{\mathbb{R}}$ is $\aleph_1$-strategically closed". Further, by its definition, $\mathbb{Q}$ is mild with respect to any $\overline{V}$-measurable cardinal. Hence, by Theorem 5, any cardinal measurable in $\overline{V}^{\mathbb{Q}}$ had to have been measurable in $\overline{V}$, and no cardinal $\delta < \kappa_0$ is either strongly compact or $<\delta'$ supercompact.

The work of the preceding paragraph indicates that the members of $(A \setminus B) \cap \kappa_0$ remain measurable in $\overline{V}^{\mathbb{Q}}$. To see that the members of $B \cap \kappa_0$ remain measurable in $\overline{V}^{\mathbb{Q}}$ as well, suppose $\delta \in B \cap \kappa_0$. By the definition of $\mathbb{Q}$, we may write $\mathbb{Q} = \mathbb{Q}_\delta * \dot{\mathbb{Q}}^\delta$, where $|\mathbb{Q}_\delta| < \delta$ and $\Vdash_{\mathbb{Q}_\delta}$ "$\dot{\mathbb{Q}}^\delta$ is δ' strategically closed". By the results of [14], $\overline{V}^{\mathbb{Q}_\delta} \models$ "δ is measurable", from which it immediately follows that $\overline{V}^{\mathbb{Q}_\delta * \dot{\mathbb{Q}}^\delta} = \overline{V}^{\mathbb{Q}} \models$ "δ is measurable" as well. Further, if $f(\delta) = 0$, then since $|\mathbb{Q}_\delta| < \delta$ and $\Vdash_{\mathbb{Q}_\delta}$ "$\dot{\mathbb{Q}}^\delta$ is δ' strategically closed", the fact that $\overline{V} \models$ "Level by level equivalence holds at δ and δ is not δ^+ strongly compact" is preserved to both $\overline{V}^{\mathbb{Q}_\delta}$ and $\overline{V}^{\mathbb{Q}_\delta * \dot{\mathbb{Q}}^\delta} = \overline{V}^{\mathbb{Q}}$. If $f(\delta) = 1$, then reasoning similar to that used in the preceding sentence shows that $\overline{V} \models$ "Level by level inequivalence holds at δ (and the least witness to level by level inequivalence is some $\gamma < \delta'$)" is also preserved to both $\overline{V}^{\mathbb{Q}_\delta}$ and $\overline{V}^{\mathbb{Q}_\delta * \dot{\mathbb{Q}}^\delta} = \overline{V}^{\mathbb{Q}}$.

Let now $\mathbb{P}^* = \mathbb{P} * \dot{\mathbb{Q}}$ and $V^* = \overline{V}^{\mathbb{Q}} = (V^{\mathbb{P} * \dot{\mathbb{Q}}})_{\kappa_0'}$. The preceding two paragraphs tell us the facts that the only measurable cardinals less than κ_0 are the members of $A \cap \kappa_0$, the measurable cardinals which are not limits of measurable cardinals are the members of $B \cap \kappa_0$, and the measurable limits of measurable cardinals are the members of $(A \setminus B) \cap \kappa_0$ are preserved from $\overline{V}$ to $\overline{V}^{\mathbb{Q}}$. Thus, V^* is our desired model. This completes the proof of Theorem 2.

□

To prove Theorem 3, suppose $V \models$ "ZFC + GCH + κ is supercompact + No cardinal is supercompact up to an inaccessible cardinal + Level by level inequivalence holds at every non-supercompact measurable cardinal δ (and the least witness to level by level inequivalence is some $\gamma < \delta'$)". Let $\alpha < \kappa$ be fixed but arbitrary. The partial ordering $\mathbb{P}$ used in the proof of Theorem 3 is once again the reverse Easton proper class iteration which begins by adding a Cohen subset of ω and then is the partial ordering P of [10, page 114] defined using $\mathfrak{D} = \{\delta \mid \delta$ is a measurable cardinal greater than $\alpha\}$.

By the definition of $\mathbb{P}$, we may write $\mathbb{P} = \mathbb{P}_0 * \dot{\mathbb{Q}}$, where $|\mathbb{P}_0| = \omega$, $\mathbb{P}_0$ is nontrivial, and $\Vdash_{\mathbb{P}_0}$ "$\dot{\mathbb{Q}}$ is α'-strategically closed". Consequently, as in the proof of Theorem 2, Theorem 5 implies that every cardinal measurable in $V^{\mathbb{P}}$ had to have been measurable in V. In addition, exactly as in the proof of Theorem 2, $V^{\mathbb{P}} \vDash$ "ZFC + GCH + Every V-measurable cardinal $\delta > \alpha$ remains measurable and satisfies level by level equivalence + No cardinal is supercompact up to an inaccessible cardinal + κ is supercompact and is the only strongly compact cardinal". The above factorization of $\mathbb{P}$, the results of [14], and the fact that every non-supercompact V-measurable cardinal δ witnesses level by level inequivalence in V via some $\gamma < \delta'$ then allow us to infer that in both $V^{\mathbb{P}_0}$ and $V^{\mathbb{P}_0 * \dot{\mathbb{Q}}} = V^{\mathbb{P}}$, the measurable cardinals less than or equal to α are the same as in V and satisfy level by level inequivalence. In particular, we now know that V and $V^{\mathbb{P}}$ contain the same measurable cardinals. This completes the proof of Theorem 3.

$\square$

Finally, to prove Theorem 4, suppose $V \vDash$ "ZFC + GCH + κ is supercompact + No cardinal is supercompact up to an inaccessible cardinal + Level by level inequivalence holds at every non-supercompact measurable cardinal δ (and the least witness to level by level inequivalence is some $\gamma < \delta'$)". Let $\alpha < \kappa$ be fixed but arbitrary. In analogy to the proof of Theorem 3, the partial ordering $\mathbb{P}$ used in the proof of Theorem 4 is the reverse Easton iteration which begins by adding a Cohen subset of ω and then is the partial ordering P of [10, page 114] defined using $\mathfrak{D} = \{\delta \mid \delta$ is a measurable cardinal less than or equal to $\alpha\}$. Note that unlike the proof of Theorem 3, the current partial ordering is a set, not a proper class.

Once again, by the definition of $\mathbb{P}$, we may write $\mathbb{P} = \mathbb{P}_0 * \dot{\mathbb{Q}}$, where $|\mathbb{P}_0| = \omega$, $\mathbb{P}_0$ is nontrivial, and $\Vdash_{\mathbb{P}_0}$ "$\dot{\mathbb{Q}}$ is $\aleph_1$-strategically closed". Consequently, as before, Theorem 5 implies that every cardinal measurable in $V^{\mathbb{P}}$ had to have been measurable in V. In addition, in analogy to the proofs of Theorems 2 and 3, $V^{\mathbb{P}} \vDash$ "ZFC + GCH + Every V-measurable cardinal $\delta \leq \alpha$ remains measurable and satisfies level by level equivalence + No cardinal is supercompact up to an inaccessible cardinal + κ is supercompact and is the only strongly compact cardinal". The above factorization of $\mathbb{P}$ consequently tells us that the measurable cardinals less than or equal to α in V, $V^{\mathbb{P}_0}$, and $V^{\mathbb{P}_0 * \dot{\mathbb{Q}}} = V^{\mathbb{P}}$ are exactly the same. The fact that every non-supercompact V-measurable cardinal δ witnesses level by level inequivalence in V via some $\gamma < \delta'$ and the definition of $\mathbb{P}$ then allow us to infer that for α^* the least V-measurable cardinal greater than α, $|\mathbb{P}| < \alpha^*$. Thus, in both $V^{\mathbb{P}_0}$ and $V^{\mathbb{P}_0 * \dot{\mathbb{Q}}} = V^{\mathbb{P}}$, the non-supercompact measurable cardinals greater than or equal to α^* are the same as in V and satisfy level by level inequivalence. In particular, we now know that V and $V^{\mathbb{P}}$ contain the same measurable cardinals. This completes the proof of Theorem 4.

$\square$

In conclusion to this paper, we note that the exact consistency strength of level by level inequivalence is discussed in [5]. The current evidence seems to suggest that hypotheses slightly stronger than the existence of a supercompact cardinal are needed to establish Theorems 1 – 4. We therefore ask if Theorems 1 – 4 can be established using only one supercompact cardinal. Finally, the techniques available at the present time do not seem to allow results analogous to Theorems 1 – 4 in which the models constructed contain more than one supercompact cardinal. We end by asking if this is indeed possible.

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