

# The Enhanced Levinski Property and the Class of Supercompact Cardinals <sup>\*†</sup>

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## Abstract

We define a generalization of a property originally due to Levinski [13], show its relative consistency, and investigate some of its possible interactions with the class of supercompact cardinals.

## 1 Introduction and Preliminaries

We begin with some terminology and notational conventions. As in [2], call an ordinal  $\alpha > 0$  *good* if  $\alpha$  is definable and is also such that for any cardinal  $\delta$ ,  $\delta^{+\alpha}$  is a regular cardinal below the least inaccessible cardinal above  $\delta$ .<sup>1</sup> Say that a measurable cardinal  $\kappa$  satisfies the *Levinski property*

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<sup>1</sup>This represents a slight abuse of notation. Strictly speaking,  $\alpha$  is a formula in the language of set theory with parameter  $\delta$ , which we will suppress. This means that there are only countably many good ordinals. Examples of good ordinals include 2, 3,  $\aleph_5$ ,  $\omega + 1$ , the ordinal successor of the least beth fixed point above  $\delta$ , etc. This is since 2, 3,  $\aleph_5$ ,  $\omega + 1$ , and the ordinal successor of the least beth fixed point above  $\delta$  are all definable with parameter  $\delta$ . Further, for  $\alpha$  any of these values,  $\delta^{+\alpha}$  is regular since it is a successor cardinal.  $\delta^{+\alpha}$  is also below the least inaccessible cardinal above  $\delta$ , which must be a (regular) limit cardinal.

$LP(\kappa, \alpha)$  for a fixed but arbitrary good ordinal  $\alpha$  if  $2^\kappa = \kappa^+$ , yet there exists a normal measure  $\mathcal{U}_\alpha$  over  $\kappa$  such that  $\{\delta < \kappa \mid \delta \text{ is inaccessible and } 2^\delta = \delta^{+\alpha}\} \in \mathcal{U}_\alpha$  (so in particular, assuming  $\alpha > 1$ , GCH holds at  $\kappa$  in the universe  $V$ , yet fails at  $\kappa$  in the ultrapower  $V^\kappa/\mathcal{U}_\alpha$ ). The normal measure  $\mathcal{U}_\alpha$  is then said to *witness*  $LP(\kappa, \alpha)$ . Extend the preceding by saying that a measurable cardinal  $\kappa$  satisfies the *enhanced Levinski property*  $ELP(\kappa)$  if  $2^\kappa = \kappa^+$ , yet for every good ordinal  $\alpha$ , there exists a normal measure  $\mathcal{U}_\alpha$  over  $\kappa$  such that  $\{\delta < \kappa \mid \delta \text{ is inaccessible and } 2^\delta = \delta^{+\alpha}\} \in \mathcal{U}_\alpha$ . Both  $LP(\kappa, \alpha)$  and  $ELP(\kappa)$  are variants of a property first studied by Levinski in [13].

Suppose  $V$  is a model of ZFC in which for all regular cardinals  $\kappa < \lambda$ ,  $\kappa$  is  $\lambda$  strongly compact iff  $\kappa$  is  $\lambda$  supercompact, except possibly if  $\kappa$  is a measurable limit of cardinals  $\delta$  which are  $\lambda$  supercompact. Such a universe will be said to witness *level by level equivalence between strong compactness and supercompactness*. For brevity, we will henceforth abbreviate this as just *level by level equivalence*. The exception is provided by a theorem of Menas [16], who showed that if  $\kappa$  is a measurable limit of cardinals  $\delta$  which are  $\lambda$  strongly compact, then  $\kappa$  is  $\lambda$  strongly compact but need not be  $\lambda$  supercompact. Any model of ZFC with this property also witnesses the Kimchi-Magidor property [12] that the classes of strongly compact and supercompact cardinals coincide precisely, except at measurable limit points. Models in which GCH and level by level equivalence between strong compactness and supercompactness hold nontrivially were first constructed in [6].

The purpose of this paper is first to establish the consistency of  $ELP(\kappa)$  relative to the existence of a measurable cardinal  $\kappa$ , and then investigate some of its possible interactions with the class of supercompact cardinals. Specifically, we will prove the following theorems.

**Theorem 1** *a) Con(ZFC + GCH +  $\kappa$  is a measurable cardinal +  $\lambda \geq \kappa^{++}$  is a regular cardinal)  $\implies$  Con(ZFC +  $2^{\kappa^+} = 2^{2^\kappa} = \lambda$  +  $ELP(\kappa)$  + For every good ordinal  $\alpha$ ,  $\kappa$  carries  $2^{2^\kappa}$  many normal measures witnessing  $LP(\kappa, \alpha)$ ).*

*b) Con(ZFC + GCH +  $\kappa$  is a measurable cardinal)  $\implies$  Con(ZFC +  $2^{\kappa^+} = 2^{2^\kappa} = \kappa^+$  +  $ELP(\kappa)$  +  $\kappa$  carries  $\kappa^+$  many normal measures + For every good ordinal  $\alpha$ ,  $\kappa$  carries  $\kappa^+$  many normal measures witnessing  $LP(\kappa, \alpha)$ ).*

**Theorem 2** *Suppose  $V \models \text{“ZFC} + \text{GCH} + \kappa \text{ is supercompact”}$ . Assume in addition that in  $V$ , no cardinal is supercompact up to an inaccessible cardinal, and level by level equivalence holds. There is then a partial ordering  $\mathbb{P} \in V$  such that  $V^{\mathbb{P}} \models \text{“ZFC} + \text{For every measurable cardinal } \delta, 2^\delta = \delta^+ \text{ and } 2^{\delta^+} = 2^{2^\delta} = \delta^{++} + \kappa \text{ is supercompact”}$ . In  $V^{\mathbb{P}}$ , no cardinal is supercompact up to an inaccessible cardinal, and level by level equivalence holds. Further, in  $V^{\mathbb{P}}$ , every measurable cardinal  $\delta$  witnesses  $ELP(\delta)$ , and for every good ordinal  $\alpha$  and every measurable cardinal  $\delta$ ,  $LP(\delta, \alpha)$  holds with respect to  $2^{2^\delta} = \delta^{++}$  many normal measures.*

**Theorem 3** *Suppose  $V \models \text{“ZFC} + \text{GCH} + \kappa \text{ is supercompact”}$ . Assume in addition that in  $V$ , no cardinal is supercompact up to an inaccessible cardinal, and level by level equivalence holds. There is then a partial ordering  $\mathbb{P} \in V$  such that  $V^{\mathbb{P}} \models \text{“ZFC} + \text{For every measurable cardinal } \delta, 2^\delta = \delta^+ \text{ and } 2^{\delta^+} = 2^{2^\delta} = \delta^{++} + \kappa \text{ is supercompact”}$ . In  $V^{\mathbb{P}}$ , no cardinal is supercompact up to an inaccessible cardinal, and level by level equivalence holds. Further, in  $V^{\mathbb{P}}$ , every measurable cardinal  $\delta$  witnesses  $ELP(\delta)$ . In addition, for every good ordinal  $\alpha$  and every measurable cardinal  $\delta$ ,  $LP(\delta, \alpha)$  holds with respect to  $2^{2^\delta} = \delta^{++}$  many normal measures if  $\delta$  is a limit of measurable cardinals, but  $LP(\delta, \alpha)$  holds with respect to  $\delta^+$  many normal measures if  $\delta$  is not a limit of measurable cardinals. Finally, every measurable cardinal  $\delta$  which is not a limit of measurable cardinals carries only  $\delta^+$  many normal measures.*

**Theorem 4** *Suppose  $V \models \text{“ZFC} + \text{GCH} + \mathcal{K} \neq \emptyset \text{ is the class of supercompact cardinals”}$ . There is then a partial ordering  $\mathbb{P} \subseteq V$  such that  $V^{\mathbb{P}} \models \text{“ZFC} + \mathcal{K} \text{ is the class of supercompact cardinals} + \text{For every measurable cardinal } \delta, 2^\delta = \delta^+ \text{ and } 2^{\delta^+} = 2^{2^\delta} = \delta^{++}\text{”}$ . In  $V^{\mathbb{P}}$ ,  $\kappa$  is supercompact iff  $\kappa$  is strongly compact, except possibly if  $\kappa$  is a measurable limit of supercompact cardinals. Further, in  $V^{\mathbb{P}}$ , every measurable cardinal  $\delta$  witnesses  $ELP(\delta)$ , and for every good ordinal  $\alpha$  and every measurable cardinal  $\delta$ ,  $LP(\delta, \alpha)$  holds with respect to  $2^{2^\delta} = \delta^{++}$  many normal measures.*

**Theorem 5** *Suppose  $V \models \text{“ZFC} + \text{GCH} + \mathcal{K} \neq \emptyset \text{ is the class of supercompact cardinals”}$ . There is then a partial ordering  $\mathbb{P} \subseteq V$  such that  $V^{\mathbb{P}} \models \text{“ZFC} + \mathcal{K} \text{ is the class of supercompact cardinals} + \text{For every measurable cardinal } \delta, 2^\delta = \delta^+ \text{ and } 2^{\delta^+} = 2^{2^\delta} = \delta^{++}\text{”}$ . In  $V^{\mathbb{P}}$ ,  $\kappa$  is supercompact iff  $\kappa$*

is strongly compact, except possibly if  $\kappa$  is a measurable limit of supercompact cardinals. Further, in  $V^{\mathbb{P}}$ , every measurable cardinal  $\delta$  witnesses  $ELP(\delta)$ . In addition, for every good ordinal  $\alpha$  and every measurable cardinal  $\delta$ ,  $LP(\delta, \alpha)$  holds with respect to  $2^{2^\delta} = \delta^{++}$  many normal measures if  $\delta$  is a limit of measurable cardinals, but  $LP(\delta, \alpha)$  holds with respect to  $\delta^+$  many normal measures if  $\delta$  is not a limit of measurable cardinals. Finally, every measurable cardinal  $\delta$  which is not a limit of measurable cardinals carries only  $\delta^+$  many normal measures.

We take this opportunity to make a few remarks concerning Theorems 1 – 5. First, we mention that our definition of “good ordinal” is arbitrary, in the sense that it can be changed to allow  $\delta^{+\alpha}$  to be larger (e.g.,  $\alpha$  could be taken as a definable ordinal such that  $\delta^{+\alpha}$  is a regular cardinal below, say, the least Ramsey cardinal). This would require certain relatively minor changes in the definitions of our forcing iterations. We note in addition that in Theorem 1a),  $\kappa$  will carry the maximum number of normal measures possible (namely  $2^{2^\kappa}$ ), whereas in Theorem 1b),  $\kappa$  carries fewer than the maximum number of normal measures (namely  $\kappa^+$ ). Also, in Theorems 2 and 3,  $\kappa$  is the only supercompact cardinal. This is because in each case, in both  $V$  and  $V^{\mathbb{P}}$ , no cardinal is supercompact up to an inaccessible cardinal. This is in sharp contrast, however, to Theorems 4 and 5, where the class of supercompact cardinals can be arbitrary. (If the class of supercompact cardinals contains at least two members  $\kappa_0 < \kappa_1$ , then  $\kappa_0$  is supercompact up to the inaccessible cardinal  $\kappa_1$  (and much more)). Finally, Theorems 2 and 3 are significant generalizations of [1, Theorem 5], whose proof is only very briefly sketched in [1]. In particular, [1, Theorem 5] constructs a model for level by level equivalence with the same limited number of large cardinals as in Theorems 2 and 3 in which it is the case that for only one fixed good ordinal  $\alpha$ ,  $LP(\delta, \alpha)$  holds for each measurable cardinal  $\delta$ . Thus, not only does  $ELP(\delta)$  fail in this model, but no consideration is given either to the number of normal measures witnessing  $LP(\delta, \alpha)$  or to varying the number of normal measures witnessing  $LP(\delta, \alpha)$  depending on whether  $\delta$  is a measurable cardinal which is a limit of measurable cardinals.

We conclude Section 1 with a very brief discussion of some additional preliminary material. When forcing,  $q \geq p$  means that  $q$  is stronger than  $p$ . We will have some slight abuses of notation.

In particular, when  $G$  is  $V$ -generic over  $\mathbb{P}$ , we take both  $V[G]$  and  $V^{\mathbb{P}}$  as being the generic extension of  $V$  by  $\mathbb{P}$ . We will also, from time to time, confuse terms with the sets they denote and write  $x$  when we actually mean  $\dot{x}$  or  $\check{x}$ . For  $\alpha < \beta$  ordinals,  $[\alpha, \beta)$ ,  $(\alpha, \beta]$ ,  $[\alpha, \beta]$ , and  $(\alpha, \beta)$  are as in standard interval notation. For any ordinal  $\alpha$ ,  $\alpha'$  is the least inaccessible cardinal above  $\alpha$ . For  $\kappa < \lambda$  regular cardinals,  $\text{Coll}(\kappa, \lambda)$  is the standard Lévy collapse of all cardinals in the half-open interval  $(\kappa, \lambda]$  to  $\kappa$ . For  $\kappa$  a regular cardinal and  $\lambda$  an ordinal,  $\text{Add}(\kappa, \lambda)$  is the standard partial ordering for adding  $\lambda$  many Cohen subsets of  $\kappa$ . The partial ordering  $\mathbb{P}$  is  $\kappa$ -directed closed if every directed set of conditions of size less than  $\kappa$  has an upper bound.

We assume familiarity with the large cardinal notions of measurability, strong compactness, and supercompactness. Readers are urged to consult [11] for further details. We do note, however, that we will say  $\kappa$  is supercompact up to the inaccessible cardinal  $\lambda$  if  $\kappa$  is  $\delta$  supercompact for every  $\delta < \lambda$ .

We recall for the benefit of readers the definition given by Hamkins in [10, Section 3] of the lottery sum of a collection of partial orderings. If  $\mathfrak{A}$  is a collection of partial orderings, then the lottery sum is the partial ordering  $\oplus\mathfrak{A} = \{\langle \mathbb{P}, p \rangle \mid \mathbb{P} \in \mathfrak{A} \text{ and } p \in \mathbb{P}\} \cup \{0\}$ , ordered with 0 below everything and  $\langle \mathbb{P}, p \rangle \leq \langle \mathbb{P}', p' \rangle$  iff  $\mathbb{P} = \mathbb{P}'$  and  $p \leq p'$ . Intuitively, if  $G$  is  $V$ -generic over  $\oplus\mathfrak{A}$ , then  $G$  first selects an element of  $\mathfrak{A}$  (or as Hamkins says in [10], “holds a lottery among the posets in  $\mathfrak{A}$ ”) and then forces with it.<sup>2</sup>

A corollary of Hamkins’ work on gap forcing found in [8, 9] will be employed in the proof of Theorems 2 – 5. We therefore state as a separate theorem what is relevant for this paper, along with some associated terminology, quoting from [8, 9] when appropriate. Suppose  $\mathbb{P}$  is a partial ordering which can be written as  $\mathbb{Q} * \dot{\mathbb{R}}$ , where  $|\mathbb{Q}| < \delta$ ,  $\mathbb{Q}$  is nontrivial, and  $\Vdash_{\mathbb{Q}} \dot{\mathbb{R}}$  is  $\delta^+$ -directed closed”. In Hamkins’ terminology of [8, 9],  $\mathbb{P}$  admits a gap at  $\delta$ . In Hamkins’ terminology of [8, 9],  $\mathbb{P}$  is mild with respect to a cardinal  $\kappa$  iff every set of ordinals  $x$  in  $V^{\mathbb{P}}$  of size below  $\kappa$  has a “nice” name  $\tau$  in  $V$  of size below  $\kappa$ , i.e., there is a set  $y$  in  $V$ ,  $|y| < \kappa$ , such that any ordinal forced by a condition in  $\mathbb{P}$  to be in  $\tau$  is an element of  $y$ . Also, as in the terminology of [8, 9] and elsewhere, an

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<sup>2</sup>The terminology “lottery sum” is due to Hamkins, although the concept of the lottery sum of partial orderings has been around for quite some time and has been referred to at different junctures via the names “disjoint sum of partial orderings,” “side-by-side forcing,” and “choosing which partial ordering to force with generically.”

embedding  $j : V \rightarrow \overline{M}$  is *amenable to  $V$*  when  $j \upharpoonright A \in V$  for any  $A \in V$ . The specific corollary of Hamkins' work from [8, 9] we will be using is then the following.

**Theorem 6 (Hamkins)** *Suppose that  $V[G]$  is a generic extension obtained by forcing with  $\mathbb{P}$  that admits a gap at some regular  $\delta < \kappa$ . Suppose further that  $j : V[G] \rightarrow M[j(G)]$  is an elementary embedding with critical point  $\kappa$  for which  $M[j(G)] \subseteq V[G]$  and  $M[j(G)]^\delta \subseteq M[j(G)]$  in  $V[G]$ . Then  $M \subseteq V$ ; indeed,  $M = V \cap M[j(G)]$ . If the full embedding  $j$  is amenable to  $V[G]$ , then the restricted embedding  $j \upharpoonright V : V \rightarrow M$  is amenable to  $V$ . If  $j$  is definable from parameters (such as a measure or extender) in  $V[G]$ , then the restricted embedding  $j \upharpoonright V$  is definable from the names of those parameters in  $V$ . Finally, if  $\mathbb{P}$  is mild with respect to  $\kappa$  and  $\kappa$  is  $\lambda$  strongly compact in  $V[G]$  for any  $\lambda \geq \kappa$ , then  $\kappa$  is  $\lambda$  strongly compact in  $V$ .*

A consequence of Theorem 6 is that if  $\mathbb{P}$  admits a gap at  $\aleph_1$  and  $V^{\mathbb{P}} \models \text{“}\varphi(\kappa)\text{”}$  where  $\varphi(\kappa)$  is either the formula which says “ $\kappa$  is supercompact” or the formula which says “ $\kappa$  is measurable”, then  $V \models \text{“}\varphi(\kappa)\text{”}$  as well. In addition, it follows from Theorem 6 that if  $\mathbb{P}$  admits a gap at  $\aleph_1$ ,  $\mathbb{P}$  is mild with respect to  $\kappa$ , and  $V^{\mathbb{P}} \models \text{“}\kappa \text{ is strongly compact”}$ , then it is also true that  $V \models \text{“}\kappa \text{ is strongly compact”}$ .

To conclude Section 1, suppose  $\kappa \leq \lambda$  are such that  $V \models \text{“}\kappa \text{ is } \lambda \text{ supercompact”}$ . We mention that during the course of this paper, we will be referring to the “standard lifting arguments” for lifting a  $\lambda$  supercompactness embedding  $j : V \rightarrow M$  generated by a supercompactness measure over  $P_\kappa(\lambda)$  to a generic extension given by a suitably defined Easton support iteration. Although there are numerous references to this in the literature, we will use the proof found in [4, Theorem 4] as the basis for the sketch we are about to present. (Readers may also consult the last paragraph of [1, Section 1], which we are quoting with certain minor modifications.) Very briefly, this argument assumes the following.

1.  $V \models \text{“}2^\kappa = \kappa^+ \text{ and there is a cardinal } \gamma \in (\kappa, \lambda] \text{ such that } 2^\delta = \delta^+ \text{ for every } \delta \in [\gamma, \lambda]\text{”}$ .
2.  $\lambda$  is a regular cardinal.

3.  $\mathbb{P} * \dot{\mathbb{Q}} = \langle \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle \mid \alpha \leq \kappa \rangle$  is an Easton support iteration having length  $\kappa + 1$  such that  $|\mathbb{P}| \leq \kappa$ .
4. For any inaccessible cardinal  $\delta < \kappa$ ,  $\Vdash_{\mathbb{P}_\delta} “|\dot{\mathbb{Q}}_\delta| < \kappa”$ .
5. For any inaccessible cardinal  $\delta < \kappa$ ,  $\Vdash_{\mathbb{P}_\delta} “\dot{\mathbb{Q}}_\delta$  is  $\delta$ -directed closed”.
6.  $G_0 * G_1$  is  $V$ -generic over  $\mathbb{P} * \dot{\mathbb{Q}}$ .
7.  $\Vdash_{\mathbb{P}} “|\dot{\mathbb{Q}}| \leq \lambda$  and  $\dot{\mathbb{Q}}$  is  $\kappa$ -directed closed”.
8.  $j(\mathbb{P} * \dot{\mathbb{Q}}) = \mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}} * j(\dot{\mathbb{Q}})$ .
9. In  $M$ ,  $\Vdash_{\mathbb{P} * \dot{\mathbb{Q}}} “\dot{\mathbb{R}}$  is  $\lambda^+$ -directed closed”.

Since  $V \models “2^\kappa = \kappa^+”$  and there is a cardinal  $\gamma \in (\kappa, \lambda]$  such that  $2^\delta = \delta^+$  for every  $\delta \in [\gamma, \lambda]$ ,  $|P_\kappa(\lambda)| = \lambda$ . Because  $M[G_0][G_1] \models “|\mathbb{R}| = j(\kappa)”$  and  $V \models “|j(\kappa^+)| = |j(2^\kappa)| = |\{f \mid f : P_\kappa(\lambda) \rightarrow \kappa^+\}| = |\{f \mid f : \lambda \rightarrow \kappa^+\}| = |\{f \mid f : \lambda \rightarrow \lambda\}| = 2^\lambda = \lambda^+”$ ,  $V[G_0][G_1] \models “There are (at most)  $\lambda^+ = 2^\lambda = |j(\kappa^+)| = |j(2^\kappa)|$  many dense open subsets of  $\mathbb{R}$  present in  $M[G_0][G_1]$ ”.$

Because  $M[G_0][G_1]$  remains  $\lambda$ -closed with respect to  $V[G_0][G_1]$  and  $\mathbb{R}$  is  $\lambda^+$ -directed closed in both  $M[G_0][G_1]$  and  $V[G_0][G_1]$ , working in  $V[G_0][G_1]$ , it is possible to build an  $M[G_0][G_1]$ -generic object  $G_2$  over  $\mathbb{R}$  such that  $j''G_0 \subseteq G_0 * G_1 * G_2$ . Still working in  $V[G_0][G_1]$ , one then lifts  $j$  to  $j : V[G_0] \rightarrow M[G_0][G_1][G_2]$ . Since  $M[G_0][G_1][G_2]$  remains  $\lambda$ -closed with respect to  $V[G_0][G_1]$  and  $V[G_0] \models “|\mathbb{Q}| \leq \lambda”$ , there is a master condition  $q \in V[G_0][G_1]$  for  $\{j(p) \mid p \in G_1\}$ . Because  $V \models “|j(\lambda^+)| = |j(2^\lambda)| = |\{f \mid f : P_\kappa(\lambda) \rightarrow \lambda^+\}| = |\{f \mid f : \lambda \rightarrow \lambda^+\}| = |[\lambda^+]^\lambda| = \lambda^+”$  and  $M[G_0][G_1][G_2] \models “|j(\mathbb{Q})| \leq j(\lambda)”$ , there are (at most)  $\lambda^+$  many dense open subsets of  $j(\mathbb{Q})$  present in  $V[G_0][G_1]$ . We may thus build in  $V[G_0][G_1]$  an  $M[G_0][G_1][G_2]$ -generic object  $G_3$  for  $j(\mathbb{Q})$  containing  $q$ . It is then the case that  $j''(G_0 * G_1) \subseteq G_0 * G_1 * G_2 * G_3$ , so we may fully lift  $j$  in  $V[G_0][G_1]$  to a  $\lambda$  supercompactness embedding  $j : V[G_0][G_1] \rightarrow M[G_0][G_1][G_2][G_3]$ . This argument remains valid (and in fact becomes even simpler) if no forcing is done at stage  $\kappa$  in  $V$ , i.e., if  $\dot{\mathbb{Q}}$  is a term for trivial forcing. The argument also remains valid if  $j$  is an elementary embedding generated by

a normal measure over  $\kappa$  (which can be canonically identified with a normal measure over  $P_\kappa(\kappa)$ ) and  $\dot{\mathbb{Q}}$  is a term for trivial forcing.

## 2 The Proofs of Theorems 1 – 5

We turn now to the proofs of our theorems, beginning with the proof of Theorem 1.

**Proof:** Suppose  $V^* \models$  “ZFC + GCH +  $\kappa$  is a measurable cardinal +  $\lambda \geq \kappa^{++}$  is a regular cardinal”.

We consider two cases. If  $\lambda = \kappa^{++}$ , let  $V = V^*$ . If  $\lambda > \kappa^{++}$ , let  $V = (V^*)^{\text{Add}(\kappa^+, \lambda)}$ . Regardless of which case holds, because  $\text{Add}(\kappa^+, \lambda)$  is  $\kappa^+$ -directed closed, we have that  $V \models$  “ $\kappa$  is a measurable cardinal +  $2^\delta = \delta^+$  for every cardinal  $\delta \leq \kappa + 2^{\kappa^+} = 2^{2^\kappa} = \lambda$ ”. The partial ordering used in the proof of Theorem 1a) is defined as  $\mathbb{P} = \mathbb{P}_{\kappa+1} = \langle \langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta \rangle \mid \beta \leq \kappa \rangle$ , the Easton support iteration of length  $\kappa + 1$  which begins by forcing with  $\text{Add}(\omega, 1)$  and then does nontrivial forcing only at  $V$ -inaccessible cardinals  $\delta \leq \kappa$ . At such a stage  $\delta$ , if  $\delta$  isn’t measurable,  $\dot{\mathbb{Q}}_\delta$  is a term for the lottery sum  $\oplus \{ \text{Add}(\delta, \delta^{+\alpha}) \mid \alpha \geq 2 \text{ is a good ordinal} \}$ . If, however,  $\delta$  is measurable,  $\dot{\mathbb{Q}}_\delta$  is a term for  $\text{Add}(\delta, \delta^+)$ .

It now follows that  $V^\mathbb{P} \models$  “ $2^\kappa = \kappa^+$  and  $2^{\kappa^+} = 2^{2^\kappa} = \lambda$ ”. This of course is the case if  $\lambda = \kappa^{++}$ , since under those circumstances,  $V = V^*$ . If, however,  $\lambda > \kappa^{++}$ , then because  $\kappa$  remains inaccessible after forcing over  $V^*$  with  $\text{Add}(\kappa^+, \lambda)$ , by [11, Lemma 15.4, page 227],  $\text{Add}(\kappa, \kappa^+)$  (or indeed,  $\text{Add}(\kappa, \gamma)$  for any ordinal  $\gamma$ ) is still  $\kappa^+$ -c.c. Thus, no cardinals are collapsed when forcing with  $\text{Add}(\kappa, \kappa^+)$ . This allows us to infer that  $V^\mathbb{P} \models$  “ $2^\kappa = \kappa^+$  and  $2^{\kappa^+} = 2^{2^\kappa} = \lambda$ ”. Also, although the proof of Theorem 1 does not require that the definition of  $\mathbb{P}$  begin by forcing with  $\text{Add}(\omega, 1)$ , it is useful to do this so that Theorem 6 may be applied in the proofs of Theorems 2 – 5.

**Lemma 2.1**  $V^\mathbb{P} \models \text{ELP}(\kappa)$ .

**Proof:** Let  $\alpha \geq 2$  be a fixed but arbitrary good ordinal. Take  $j : V \rightarrow M$  to be an elementary embedding witnessing the measurability of  $\kappa$  in  $V$  generated by a normal measure over  $\kappa$  such that  $M \models$  “ $\kappa$  isn’t measurable”. In particular,  $M^\kappa \subseteq M$ . We combine several ideas (including a standard lifting argument, an idea due to Levinski [13], and an idea due to Magidor [15]) to show that  $j$  lifts in  $V^{\mathbb{P}_{\kappa^*} \text{Add}(\kappa, \kappa^+)} = V^\mathbb{P}$  to  $j : V^{\mathbb{P}_{\kappa^*} \text{Add}(\kappa, \kappa^+)} \rightarrow M^{j(\mathbb{P}_{\kappa^*} \text{Add}(\kappa, \kappa^+))}$ . We also follow to a certain



extent the proof of [2, Theorem 5], quoting verbatim when appropriate. Specifically, let  $G_0$  be  $V$ -generic over  $\mathbb{P}_\kappa$ , and let  $G_1$  be  $V[G_0]$ -generic over  $\text{Add}(\kappa, \kappa^+)$ . Observe that  $j(\mathbb{P}_\kappa * \text{Add}(\kappa, \kappa^+)) = \mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa * \dot{\mathbb{Q}} * \text{Add}(j(\kappa), j(\kappa^+))$ , where  $\dot{\mathbb{Q}}_\kappa$  is a term for the lottery sum  $\oplus\{\text{Add}(\kappa, \kappa^{+\beta}) \mid \beta \geq 2 \text{ is a good ordinal}\}$ . By forcing above the appropriate condition  $p_0$  which opts for  $\text{Add}(\kappa, \kappa^{+\alpha})$  in the stage  $\kappa$  lottery held in  $M^{\mathbb{P}_\kappa}$  in the definition of  $j(\mathbb{P}_\kappa * \text{Add}(\kappa, \kappa^+))$ , we may in fact assume that  $j(\mathbb{P}_\kappa * \text{Add}(\kappa, \kappa^+))$  is forcing equivalent to  $\mathbb{P}_\kappa * \text{Add}(\kappa, \kappa^{+\alpha}) * \dot{\mathbb{Q}} * \text{Add}(j(\kappa), j(\kappa^+))$ . For the remainder of the proof of Lemma 2.1, we assume that we are forcing above  $p_0$ .

Working in  $V[G_0]$ , we first note that since  $\mathbb{P}_\kappa$  is  $\kappa$ -c.c.,  $M[G_0]$  remains  $\kappa$ -closed with respect to  $V[G_0]$ . Next, we use Levinski's ideas of [13] to show that it is possible to rearrange  $G_1$  to form an  $M[G_0]$ -generic object  $H_1$  over  $(\text{Add}(\kappa, \kappa^{+\alpha}))^{M[G_0]}$  in  $V[G_0][G_1]$ . Since  $V \models "2^\kappa = \kappa^+"$  and  $j$  is generated by a normal measure over  $\kappa$ ,  $(\kappa^+)^V = (\kappa^+)^M$  and  $(\kappa^+)^V < j(\kappa) < (\kappa^{++})^V$ . In particular, any  $\gamma \in ((\kappa^+)^V, j(\kappa))$  which  $M[G_0]$  believes to be a cardinal actually is an ordinal of cardinality  $\kappa^+$  in either  $V$ ,  $V[G_0]$ , or  $V[G_0][G_1]$ . Hence,  $V[G_0][G_1] \models "|(\kappa^{+\alpha})^{M[G_0][H_0]}| = \kappa^+"$ . Let  $(\kappa^{+\alpha})^{M[G_0][H_0]} = \rho$ . Working in  $V[G_0][G_1]$ , we may therefore let  $f : \kappa^+ \rightarrow \rho$  be a bijection. For any  $p \in \text{Add}(\kappa, \kappa^+)$ ,  $g(p) = \{\langle \langle \sigma, f(\beta) \rangle, \gamma \rangle \mid \langle \langle \sigma, \beta \rangle, \gamma \rangle \in p\} \in (\text{Add}(\kappa, \rho))^{M[G_0]}$ . As can be easily checked (see [13]),  $H_1 = \{g(p) \mid p \in G_1\}$  is an  $M[G_0]$ -generic object over  $(\text{Add}(\kappa, \rho))^{M[G_0]}$ . Our construction guarantees that  $j''G_0 \subseteq G_0 * H_1$ , so  $j$  lifts in  $V[G_0][G_1]$  to  $j : V[G_0] \rightarrow M[G_0][H_1]$ .

We now use a version of the standard lifting argument mentioned at the end of Section 1 to build in  $V[G_0][G_1]$  an  $M[G_0][H_1]$ -generic object  $H_2$  over  $\mathbb{Q}$ . At the risk of redundancy, we repeat some of the ideas and details mentioned earlier, since they will be relevant in the proof of Lemma 2.2. In  $V$ , since  $M$  is given via an ultrapower by a normal measure over  $\kappa$ ,  $|j(\kappa)|$  and  $|j(\kappa^+)|$  may be calculated as  $|\{f \mid f : \kappa \rightarrow \kappa\}| = 2^\kappa = \kappa^+$  and  $|\{f \mid f : \kappa \rightarrow \kappa^+\}| = |[\kappa^+]^\kappa| = \kappa^+$  respectively. Also, by elementarity, since  $V \models "2^\kappa = \kappa^+"$ ,  $M \models "2^{j(\kappa)} = (j(\kappa))^+ = j(\kappa^+)"$ . Because  $(\text{Add}(\kappa, \kappa^+))^{V[G_0]}$  is  $\kappa^+$ -c.c. in  $V[G_0]$ ,  $M[G_0][H_1]$  remains  $\kappa$ -closed with respect to  $V[G_0][G_1]$ . In addition, since  $M[G_0][H_1] \models "\mathbb{Q}$  is an Easton support iteration of length  $j(\kappa)"$ ,  $M[G_0][H_1] \models "|\mathbb{Q}| = j(\kappa)$  and  $2^{j(\kappa)} = (j(\kappa))^+ = j(\kappa^+)"$ . This means the number of dense open subsets of  $\mathbb{Q}$  present in  $M[G_0][H_1]$  is  $j(\kappa^+)$ . Further, as  $M[G_0][H_1] \models "\mathbb{Q}$  is  $\kappa^+$ -directed closed" and  $M[G_0][H_1]$  is  $\kappa$ -closed with

respect to  $V[G_0][G_1]$ ,  $\mathbb{Q}$  is  $\kappa^+$ -directed closed in  $V[G_0][G_1]$  as well. Since  $\kappa^+$  is preserved from  $V$  to  $V[G_0][G_1]$ , we may let  $\langle D_\beta \mid \beta < \kappa^+ \rangle \in V[G_0][G_1]$  enumerate the dense open subsets of  $\mathbb{Q}$  present in  $M[G_0][H_1]$ . We may now use the fact that  $\mathbb{Q}$  is  $\kappa^+$ -directed closed in  $V[G_0][G_1]$  to meet each  $D_\beta$  and thereby construct in  $V[G_0][G_1]$  an  $M[G_0][H_1]$ -generic object  $H_2$  over  $\mathbb{Q}$ . Our construction guarantees that  $j''G_0 \subseteq G_0 * H_1 * H_2$ , so  $j$  lifts in  $V[G_0][G_1]$  to  $j : V[G_0] \rightarrow M[G_0][H_1][H_2]$ . By the fact that  $\mathbb{Q}$  is  $\kappa^+$ -directed closed in  $M[G_0][H_1]$ ,  $M[G_0][H_1][H_2]$  remains  $\kappa$ -closed with respect to  $V[G_0][G_1][H_2] = V[G_0][G_1]$ .

We now use arguments originally due to Magidor [15], which are also given in [6, pages 119–120] and are found other places in the literature as well, to construct in  $V[G_0][G_1]$  an  $M[G_0][H_1][H_2]$ -generic object  $H_3$  over  $(\text{Add}(j(\kappa), j(\kappa^+)))^{M[G_0][H_1][H_2]}$  such that  $j''(G_0 * G_1) \subseteq G_0 * H_1 * H_2 * H_3$ . For the convenience of readers, we present these arguments below.

For  $\zeta \in (\kappa, \kappa^+)$  and  $p \in \text{Add}(\kappa, \kappa^+)$ , let  $p \upharpoonright \zeta = \{ \langle \rho, \sigma \rangle, \eta \rangle \in p \mid \sigma < \zeta \}$  and  $G_1 \upharpoonright \zeta = \{ p \upharpoonright \zeta \mid p \in G_1 \}$ . Clearly,  $V[G_0][G_1] \models \text{“}|G_1 \upharpoonright \zeta| \leq \kappa \text{ for all } \zeta \in (\kappa, \kappa^+) \text{”}$ . Thus, since  $\text{Add}(j(\kappa), j(\kappa^+))^{M[G_0][H_1][H_2]}$  is  $j(\kappa)$ -directed closed and  $j(\kappa) > \kappa^+$ ,  $q_\zeta = \bigcup \{ j(p) \mid p \in G_1 \upharpoonright \zeta \}$  is well-defined and is an element of  $\text{Add}(j(\kappa), j(\kappa^+))^{M[G_0][H_1][H_2]}$ . Further, if  $\langle \rho, \sigma \rangle \in \text{dom}(q_\zeta) - \text{dom}(\bigcup_{\beta < \zeta} q_\beta)$  ( $\bigcup_{\beta < \zeta} q_\beta$  is well-defined by closure), then  $\sigma \in [\bigcup_{\beta < \zeta} j(\beta), j(\zeta))$ . To see this, assume to the contrary that  $\sigma < \bigcup_{\beta < \zeta} j(\beta)$ . Let  $\beta$  be minimal such that  $\sigma < j(\beta)$ . It must thus be the case that for some  $p \in G_1 \upharpoonright \zeta$ ,  $\langle \rho, \sigma \rangle \in \text{dom}(j(p))$ . Since by elementarity and the definitions of  $G_1 \upharpoonright \beta$  and  $G_1 \upharpoonright \zeta$ , for  $p \upharpoonright \beta = q \in G_1 \upharpoonright \beta$ ,  $j(q) = j(p) \upharpoonright j(\beta) = j(p \upharpoonright \beta)$ , it must be the case that  $\langle \rho, \sigma \rangle \in \text{dom}(j(q))$ . This means  $\langle \rho, \sigma \rangle \in \text{dom}(q_\beta)$ , a contradiction.

Since  $M[G_0][H_1][H_2] \models \text{“}j(\kappa) \text{ is inaccessible and } 2^{j(\kappa)} = j(\kappa^+) \text{”}$ , an application of [11, Lemma 15.4, page 227] shows that  $M[G_0][H_1][H_2] \models \text{“}\text{Add}(j(\kappa), j(\kappa^+)) \text{ is } j(\kappa^+)\text{-c.c. and has } j(\kappa^+) \text{ many maximal antichains”}$ . This means that if  $\mathcal{A} \in M[G_0][H_1][H_2]$  is a maximal antichain of  $\text{Add}(j(\kappa), j(\kappa^+))$ ,  $\mathcal{A} \subseteq \text{Add}(j(\kappa), \beta)$  for some  $\beta \in (j(\kappa), j(\kappa^+))$ . Thus, since  $V \models \text{“}|j(\kappa^+)| = \kappa^+ \text{”}$ , we can let  $\langle \mathcal{A}_\zeta \mid \zeta \in (\kappa, \kappa^+) \rangle \in V[G_0][G_1]$  be an enumeration of all of the maximal antichains of  $\text{Add}(j(\kappa), j(\kappa^+))$  present in  $M[G_0][H_1][H_2]$ .

Working in  $V[G_0][G_1]$ , we define now an increasing sequence  $\langle r_\zeta \mid \zeta \in (\kappa, \kappa^+) \rangle$  of elements

of  $\text{Add}(j(\kappa), j(\kappa^+))$  such that  $\forall \zeta \in (\kappa, \kappa^+)[r_\zeta \geq q_\zeta$  and  $r_\zeta \in \text{Add}(j(\kappa), j(\zeta))]$  and such that  $\forall \mathcal{A} \in \langle \mathcal{A}_\zeta \mid \zeta \in (\kappa, \kappa^+) \rangle \exists \beta \in (\kappa, \kappa^+) \exists r \in \mathcal{A}[r_\beta \geq r]$ . Assuming we have such a sequence,  $H_3 = \{p \in \text{Add}(j(\kappa), j(\kappa^+)) \mid \exists r \in \langle r_\zeta \mid \zeta \in (\kappa, \kappa^+) \rangle [r \geq p]\}$  is an  $M[G_0][H_1][H_2]$ -generic object over  $\text{Add}(j(\kappa), j(\kappa^+))$ . To define  $\langle r_\zeta \mid \zeta \in (\kappa, \kappa^+) \rangle$ , if  $\zeta$  is a limit, we let  $r_\zeta = \bigcup_{\beta \in (\kappa, \zeta)} r_\beta$ . By the facts  $\langle r_\beta \mid \beta \in (\kappa, \zeta) \rangle$  is (strictly) increasing and  $M[G_0][H_1][H_2]$  is  $\kappa$ -closed with respect to  $V[G_0][G_1]$ , this definition is valid. Assuming now  $r_\zeta$  has been defined and we wish to define  $r_{\zeta+1}$ , let  $\langle \mathcal{B}_\beta \mid \beta < \eta \leq \kappa \rangle$  be the subsequence of  $\langle \mathcal{A}_\beta \mid \beta \leq \zeta + 1 \rangle$  containing each antichain  $\mathcal{A}$  such that  $\mathcal{A} \subseteq \text{Add}(j(\kappa), j(\zeta + 1))$ . Because  $q_\zeta, r_\zeta \in \text{Add}(j(\kappa), j(\zeta))$ ,  $q_{\zeta+1} \in \text{Add}(j(\kappa), j(\zeta + 1))$ , and  $j(\zeta) < j(\zeta + 1)$ , the condition  $r'_{\zeta+1} = r_\zeta \cup q_{\zeta+1}$  is well-defined. This is since by our earlier observations, any new elements of  $\text{dom}(q_{\zeta+1})$  won't be present in either  $\text{dom}(q_\zeta)$  or  $\text{dom}(r_\zeta)$ . We can thus, using the fact  $M[G_0][H_1][H_2]$  is  $\kappa$ -closed with respect to  $V[G_0][G_1]$ , define by induction an increasing sequence  $\langle s_\beta \mid \beta < \eta \rangle$  such that  $s_0 \geq r'_{\zeta+1}$ ,  $s_\rho = \bigcup_{\beta < \rho} s_\beta$  if  $\rho$  is a limit ordinal, and  $s_{\beta+1} \geq s_\beta$  is such that  $s_{\beta+1}$  extends some element of  $\mathcal{B}_\beta$ . The just mentioned closure fact implies  $r_{\zeta+1} = \bigcup_{\beta < \eta} s_\beta$  is a well-defined condition.

In order to show that  $H_3$  is  $M[G_0][H_1][H_2]$ -generic over  $\text{Add}(j(\kappa), j(\kappa^+))$ , we must show that  $\forall \mathcal{A} \in \langle \mathcal{A}_\zeta \mid \zeta \in (\kappa, \kappa^+) \rangle \exists \beta \in (\kappa, \kappa^+) \exists r \in \mathcal{A}[r_\beta \geq r]$ . To do this, we first note that  $\langle j(\zeta) \mid \zeta < \kappa^+ \rangle$  is unbounded in  $j(\kappa^+)$ . To see this, if  $\beta < j(\kappa^+)$  is an ordinal, then for some  $f : \kappa \rightarrow M$  representing  $\beta$ , we can assume that for  $\rho < \kappa$ ,  $f(\rho) < \kappa^+$ . Thus, by the regularity of  $\kappa^+$  in  $V$ ,  $\beta_0 = \bigcup_{\rho < \kappa} f(\rho) < \kappa^+$ , and  $j(\beta_0) > \beta$ . This means by our earlier remarks that if  $\mathcal{A} \in \langle \mathcal{A}_\zeta \mid \zeta < \kappa^+ \rangle$ ,  $\mathcal{A} = \mathcal{A}_\rho$ , then we can let  $\beta \in (\kappa, \kappa^+)$  be such that  $\mathcal{A} \subseteq \text{Add}(j(\kappa), j(\beta))$ . By construction, for  $\eta > \max(\beta, \rho)$ , there is some  $r \in \mathcal{A}$  such that  $r_\eta \geq r$ . And, as any  $p \in \text{Add}(\kappa, \kappa^+)$  is such that for some  $\zeta \in (\kappa, \kappa^+)$ ,  $p = p \upharpoonright \zeta$ ,  $H_3$  is such that if  $p \in G_1$ ,  $j(p) \in H_3$ . Thus, working in  $V[G_0][G_1]$ , we have shown that because  $j''(G_0 * G_1) \subseteq G_0 * H_1 * H_2 * H_3$ ,  $j$  lifts to  $j : V[G_0][G_1] \rightarrow M[G_0][H_1][H_2][H_3]$ , i.e.,  $V[G_0][G_1] \models \text{“}\kappa \text{ is measurable”}$ . Since  $M[G_0][H_1] \models \text{“}\mathbb{Q} * \text{Add}(j(\kappa), j(\kappa^+)) \text{ is } \kappa'\text{-directed closed”}$  and  $(\kappa^{+\alpha})^{M[G_0][H_1]} < (\kappa')^{M[G_0][H_1]}$ ,  $M[G_0][H_1][H_2][H_3] \models \text{“}2^\kappa = \kappa^{+\alpha}\text{”}$ . Consequently,  $\kappa \in \{\delta < j(\kappa) \mid 2^\delta = \delta^{+\alpha}\} = j(\{\delta < \kappa \mid 2^\delta = \delta^{+\alpha}\})$ . Hence,  $A_\alpha =_{\text{df}} \{\delta < \kappa \mid 2^\delta = \delta^{+\alpha}\}$  has measure 1 with respect to the normal measure  $\mathcal{U}_\alpha \in V[G_0][G_1]$  over  $\kappa$  generated by  $j$ . As  $\alpha$  was arbitrary and

$V^{\mathbb{P}} \models "2^\kappa = \kappa^+", V^{\mathbb{P}} \models \text{ELP}(\kappa)$ . This completes the proof of Lemma 2.1. □

**Lemma 2.2**  $V^{\mathbb{P}} \models "For every good ordinal  $\alpha$ ,  $\kappa$  carries  $2^{2^\kappa} = \lambda$  many normal measures witnessing  $LP(\kappa, \alpha)$ ".$

**Proof:** The proof of Lemma 2.1 shows that for every good ordinal  $\alpha$ ,  $V^{\mathbb{P}} \models LP(\kappa, \alpha)$ . However, we can now argue as in [3, Lemma 1.1] to infer that in  $V^{\mathbb{P}}$ , there are  $2^{2^\kappa}$  many normal measures witnessing  $LP(\kappa, \alpha)$ . Specifically, we can show via a folklore argument that there are  $2^{2^\kappa}$  many different ways of constructing the generic object  $H_2$  over  $\mathbb{Q}$ . To do this, as in the proof of Lemma 2.1, let  $\langle D_\beta \mid \beta < \kappa^+ \rangle \in V[G_0][G_1]$  enumerate the dense open subsets of  $\mathbb{Q}$  present in  $M[G_0][H_1]$ . Since  $V[G_0][G_1] \models "Q$  is  $\kappa^+$ -directed closed", we can build in  $V[G_0][G_1]$  a tree  $\mathcal{T}$  of height  $\kappa^+$  such that:

1. The root of  $\mathcal{T}$  is the empty condition.
2. If  $p$  is an element at level  $\beta < \kappa^+$  of  $\mathcal{T}$ , then the successors of  $p$  at level  $\beta + 1$  are a maximal incompatible subset of  $D_\beta$  extending  $p$ . By the definition of  $\mathbb{Q}$ , there will be at least two incompatible successors of  $p$  at level  $\beta + 1$ .
3. If  $\lambda < \kappa^+$  is a limit ordinal, then the elements of  $\mathcal{T}$  at height  $\lambda$  are upper bounds to any path through  $\mathcal{T}$  of height  $\lambda$ .

We observe that any path  $I_2$  of height  $\kappa^+$  through  $\mathcal{T}$  generates an  $M[G_0][H_1]$ -generic object  $I_2^*$  over  $\mathbb{Q}$ . Therefore, since there are  $2^{\kappa^+}$  many paths of height  $\kappa^+$  through  $\mathcal{T}$ , there are  $2^{\kappa^+} = 2^{2^\kappa} = \lambda$  many different  $M[G_0][H_1]$ -generic objects over  $\mathbb{Q}$  present in  $V[G_0][G_1]$ .

Suppose  $H_2^* \neq H_2^{**}$  are any two distinct  $M[G_0][H_1]$ -generic objects over  $\mathbb{Q}$  generated as above. If  $j^*$  is the lift of  $j$  associated with  $H_2^*$  and  $j^{**}$  is the lift of  $j$  associated with  $H_2^{**}$ , it will be the case that  $j^*(\langle G_0, G_1 \rangle) = \langle G_0, H_1, H_2^*, H_3 \rangle$  and  $j^{**}(\langle G_0, G_1 \rangle) = \langle G_0, H_1, H_2^{**}, H_3 \rangle$ . Since [7, Lemma 1] tells us that any  $k : V[G_0][G_1] \rightarrow N$  witnessing the measurability of  $\kappa$  which is a lift of  $j$  is

generated by the normal measure over  $\kappa$  given by  $\mathcal{U} = \{x \subseteq \kappa \mid \kappa \in k(x)\}$ , there are  $2^{2^\kappa} = \lambda$  many different normal measures over  $\kappa$  in  $V[G_0][G_1]$  witnessing  $\text{LP}(\kappa, \alpha)$ . This completes the proof of Lemma 2.2. □

Lemmas 2.1 and 2.2 complete the proof of Theorem 1a). To prove Theorem 1b) and thereby complete the proof of Theorem 1, we force over the model  $V^{\mathbb{P}}$  witnessing the conclusions of Theorem 1a) with  $\text{Add}(\omega, 1) * \text{Coll}(\kappa^+, 2^{2^\kappa}) = \text{Add}(\omega, 1) * \text{Coll}(\kappa^+, \lambda)$  to obtain the model  $\bar{V}$ . Because  $V^{\mathbb{P}} \models "2^\kappa = \kappa^+ \text{ and } 2^{\kappa^+} = 2^{2^\kappa} = \lambda"$ ,  $\bar{V} \models "2^\kappa = \kappa^+ \text{ and } 2^{\kappa^+} = 2^{2^\kappa} = \kappa^{++}"$ . By the proof of [5, Theorem 1], since  $|\text{Add}(\omega, 1)| < \kappa$  and  $\text{Add}(\omega, 1)$  is nontrivial, in  $\bar{V}$ ,  $\kappa$  carries exactly  $|(2^{2^\kappa})^{V^{\mathbb{P}}}| = (\kappa^+)^{\bar{V}}$  many normal measures. Suppose  $\alpha \geq 2$  is a fixed but arbitrary good ordinal. The proof of Theorem 1b) will therefore be complete if we can show that each of the  $(2^{2^\kappa})^{V^{\mathbb{P}}}$  many normal measures  $\mathcal{U}_\alpha$  witnessing  $\text{LP}(\kappa, \alpha)$  in  $V^{\mathbb{P}}$  has an extension  $\mathcal{U}_\alpha^* \supseteq \mathcal{U}_\alpha$  witnessing  $\text{LP}(\kappa, \alpha)$  in  $\bar{V}$ . To do this, recall that by the Lévy-Solovay results [14],  $\mathcal{U}_\alpha^* =_{\text{df}} \{x \subseteq \kappa \mid \exists y \in \mathcal{U}_\alpha[y \subseteq x]\}$  is a normal measure over  $\kappa$  in  $V^{\mathbb{P} * \text{Add}(\omega, 1)}$ . Since forcing with  $\text{Add}(\omega, 1)$  preserves cardinals and cofinalities and does not change the size of power sets,  $\mathcal{U}_\alpha^*$  witnesses  $\text{LP}(\kappa, \alpha)$  in  $V^{\mathbb{P} * \text{Add}(\omega, 1)}$ . Because forcing with  $\text{Coll}(\kappa^+, 2^{2^\kappa})$  adds no subsets of  $\kappa$  as  $\text{Coll}(\kappa^+, 2^{2^\kappa})$  is  $\kappa^+$ -directed closed,  $\mathcal{U}_\alpha^*$  witnesses  $\text{LP}(\kappa, \alpha)$  in  $V^{\mathbb{P} * \text{Add}(\omega, 1) * \text{Coll}(\kappa^+, 2^{2^\kappa})} = \bar{V}$  as well. This completes the proof of both Theorem 1b) and Theorem 1. □

Turning to the proof of Theorem 2, suppose  $V \models \text{"ZFC + GCH + } \kappa \text{ is supercompact"}$ . Assume in addition that in  $V$ , no cardinal is supercompact up to an inaccessible cardinal, and level by level equivalence holds. It must be true that  $V \models \text{"No cardinal above } \kappa \text{ is inaccessible"}$  (so in particular,  $V \models \text{"No cardinal above } \kappa \text{ is measurable"}$ ). The partial ordering  $\mathbb{P}$  used in the proof of Theorem 2 may now be taken as the partial ordering  $\mathbb{P} = \mathbb{P}_{\kappa+1}$  of Theorem 1a) defined assuming  $\lambda = \kappa^{++}$ . Note that by its definition, forcing with  $\mathbb{P}$  preserves all cardinals, cofinalities, and the fact  $2^\gamma = \gamma^+$  whenever  $\delta$  is a  $V$ -measurable cardinal and  $\gamma \in [\delta, \delta')$ . It is of course also the case that  $V^{\mathbb{P}} \models \text{"No cardinal above } \kappa \text{ is inaccessible"}$ .

**Lemma 2.3** *Suppose  $\delta$  is a measurable cardinal in  $V^{\mathbb{P}}$ . Then  $ELP(\delta)$  holds, and for every good ordinal  $\alpha$ ,  $LP(\delta, \alpha)$  holds with respect to  $2^{2^\delta} = \delta^{++}$  many normal measures.*

**Proof:** As we have just observed,  $V \models$  “No cardinal above  $\kappa$  is measurable”, i.e.,  $V \models$  “If  $\delta$  is a measurable cardinal,  $\delta \leq \kappa$ ”. Therefore, by the proof of Theorem 1a), if  $V \models$  “ $\delta$  is a measurable cardinal”, then in  $V^{\mathbb{P}_{\delta+1}}$ ,  $ELP(\delta)$  holds, and for every good ordinal  $\alpha$ ,  $LP(\delta, \alpha)$  holds with respect to  $2^{2^\delta} = \delta^{++}$  many normal measures. Write  $\mathbb{P} = \mathbb{P}_{\delta+1} * \dot{\mathbb{Q}}$ . By its definition,  $\mathbb{P}$  acts nontrivially only on inaccessible cardinals. It consequently follows that  $\Vdash_{\mathbb{P}_{\delta+1}}$  “ $\dot{\mathbb{Q}}$  is  $\delta'$ -directed closed”, so in  $V^{\mathbb{P}_{\delta+1} * \dot{\mathbb{Q}}} = V^{\mathbb{P}}$ ,  $ELP(\delta)$  holds, and for every good ordinal  $\alpha$ ,  $LP(\delta, \alpha)$  holds with respect to  $2^{2^\delta} = \delta^{++}$  many normal measures. The proof of Lemma 2.3 will therefore be finished once we have shown that if  $V^{\mathbb{P}} \models$  “ $\delta$  is a measurable cardinal”, then  $V \models$  “ $\delta$  is a measurable cardinal” as well. To do this, write  $\mathbb{P} = \text{Add}(\omega, 1) * \dot{\mathbb{R}}$ . Since  $|\text{Add}(\omega, 1)| = \omega$ ,  $\text{Add}(\omega, 1)$  is nontrivial, and  $\Vdash_{\text{Add}(\omega, 1)}$  “ $\dot{\mathbb{R}}$  is  $\aleph_2$ -directed closed”, as we observed immediately after the statement of Theorem 6,  $\delta$  must be measurable in  $V$  as well. This completes the proof of Lemma 2.3. □

As we have just noted, any cardinal measurable in  $V^{\mathbb{P}}$  must also have been measurable in  $V$ . Therefore, by our remarks in the paragraph immediately preceding Lemma 2.3, it must be the case that if  $\delta$  is measurable in  $V^{\mathbb{P}}$ , then  $2^\gamma = \gamma^+$  for all  $\gamma \in [\delta, \delta')$ .

**Lemma 2.4** *If  $V \models$  “ $\delta < \lambda$  are such that  $\delta$  is  $\lambda$  supercompact and  $\lambda$  is regular”, then  $V^{\mathbb{P}} \models$  “ $\delta$  is  $\lambda$  supercompact”.*

**Proof:** Suppose  $\delta < \lambda$  are as in the hypotheses for Lemma 2.4. Because  $V \models$  “No cardinal is supercompact up to an inaccessible cardinal”, it must be the case that  $\lambda \geq \delta^+$  is a successor cardinal and  $\lambda < \delta'$ . As in the proof of Lemma 2.3, write  $\mathbb{P} = \mathbb{P}_{\delta+1} * \dot{\mathbb{Q}}$ . Suppose  $j : V \rightarrow M$  is an elementary embedding witnessing the  $\lambda$  supercompactness of  $\delta$  which is generated by a supercompact ultrafilter over  $P_\delta(\lambda)$ . Note that since  $2^\delta = \delta^+ \leq \lambda$  and  $M^\lambda \subseteq M$ ,  $M \models$  “ $\delta$  is a measurable cardinal”. In addition,  $(\lambda^+)^M = (\lambda^+)^V$ . Hence, by the definition of  $\mathbb{P}$ , the nine criteria necessary to apply the standard lifting arguments mentioned in Section 1 when forcing with  $\mathbb{P}_{\delta+1}$

may all be easily verified. This means that  $V^{\mathbb{P}_{\delta+1}} \models$  “ $\delta$  is  $\lambda$  supercompact”. Since as we have already observed in the proof of Lemma 2.3,  $\Vdash_{\mathbb{P}_{\delta+1}} \dot{Q}$  is  $\delta'$ -directed closed”.  $V^{\mathbb{P}_{\delta+1} * \dot{Q}} = V^{\mathbb{P}} \models$  “ $\delta$  is  $\lambda$  supercompact”. This completes the proof of Lemma 2.4.

□

**Lemma 2.5**  $V^{\mathbb{P}} \models$  “*Level by level equivalence holds*”.

**Proof:** Suppose  $V^{\mathbb{P}} \models$  “ $\delta < \lambda$  are such that  $\lambda$  is regular and  $\delta$  is  $\lambda$  strongly compact”. Because  $V^{\mathbb{P}} \models$  “No cardinal above  $\kappa$  is inaccessible”, it must be the case that  $\delta \leq \kappa$ . Since it is an immediate corollary of Lemma 2.4 that  $V^{\mathbb{P}} \models$  “ $\kappa$  is supercompact”, we may assume without loss of generality that  $\delta < \kappa$ .

Note that by its definition,  $\mathbb{P}$  is mild with respect to  $\delta$ . Thus, using the factorization of  $\mathbb{P}$  given in the proof of Lemma 2.3 and Theorem 6,  $V \models$  “ $\delta$  is  $\lambda$  strongly compact”. As  $V \models$  “No cardinal is supercompact up to an inaccessible cardinal”,  $\delta$  cannot be a measurable limit of cardinals  $\gamma$  which are  $\lambda$  supercompact. Hence, because level by level equivalence holds in  $V$ ,  $V \models$  “ $\delta$  is  $\lambda$  supercompact” as well. By Lemma 2.4,  $V^{\mathbb{P}} \models$  “ $\delta$  is  $\lambda$  supercompact”. This completes the proof of Lemma 2.5.

□

**Lemma 2.6**  $V^{\mathbb{P}} \models$  “*No cardinal is supercompact up to an inaccessible cardinal*”.

**Proof:** Suppose  $V^{\mathbb{P}} \models$  “ $\delta < \lambda$  are such that  $\lambda$  is inaccessible and  $\delta$  is  $\alpha$  supercompact for every  $\alpha < \lambda$ ”. By the factorization of  $\mathbb{P}$  given in the proof of Lemma 2.3 and Theorem 6,  $V \models$  “ $\delta < \lambda$  are such that  $\lambda$  is inaccessible and  $\delta$  is  $\alpha$  supercompact for every  $\alpha < \lambda$ ” also. This, however, contradicts our hypotheses on  $V$  and completes the proof of Lemma 2.6.

□

Lemmas 2.3 – 2.6 and the intervening remarks complete the proof of Theorem 2.

□

To prove Theorem 3, suppose  $V \models \text{“ZFC} + \text{GCH} + \kappa \text{ is supercompact”}$ . Assume in addition that in  $V$ , no cardinal is supercompact up to an inaccessible cardinal, and level by level equivalence holds. We first force with the partial ordering used in the proof of Theorem 2 and then abuse notation by calling the resulting generic extension  $V$  as well. In this way, we may also assume that what we will take as our ground model  $V$  satisfies the conclusions of Theorem 2 (which of course means that  $V \models \text{“No cardinal above } \kappa \text{ is inaccessible”}$ ). The partial ordering used in the proof of Theorem 3 is now defined as  $\mathbb{P} = \mathbb{P}_\kappa = \langle \langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta \rangle \mid \beta < \kappa \rangle$ . This is the Easton support iteration of length  $\kappa$  which begins by forcing with  $\text{Add}(\omega, 1)$  and then does nontrivial forcing only at cardinals  $\delta < \kappa$  which are  $V$ -measurable cardinals which are not limits of  $V$ -measurable cardinals. At such a stage  $\delta$ ,  $\dot{\mathbb{Q}}_\delta$  is a term for  $\text{Coll}(\delta^+, \delta^{++}) = \text{Coll}(\delta^+, 2^{2^\delta})$ .

Once again, by its definition, forcing with  $\mathbb{P}$  preserves  $2^\gamma = \gamma^+$  whenever  $\delta$  is a  $V$ -measurable cardinal and  $\gamma \in [\delta, \delta')$ . As in the proof of Theorem 2, we may now infer that if  $\delta$  is measurable in  $V^\mathbb{P}$ , then  $2^\gamma = \gamma^+$  for all  $\gamma \in [\delta, \delta')$ . It is also true that by its definition, for every  $V$ -measurable cardinal which is a limit of  $V$ -measurable cardinals, forcing with  $\mathbb{P}$  preserves all cardinals and cofinalities in the half-open interval  $[\delta, \delta')$ .

**Lemma 2.7** *Assume  $V^\mathbb{P} \models \text{“}\delta \text{ is a measurable cardinal which is not a limit of measurable cardinals”}$ . Then  $V^\mathbb{P} \models \text{“ELP}(\delta) \text{ holds} + \delta \text{ carries } \delta^+ \text{ many normal measures} + \text{For every good ordinal } \alpha, \delta \text{ carries } \delta^+ \text{ many normal measures witnessing LP}(\delta, \alpha)\text{”}$ .*

**Proof:** Suppose  $V \models \text{“}\delta \text{ is a measurable cardinal which is not a limit of measurable cardinals”}$ . We show that since we have assumed  $V$  satisfies the conclusions of Theorem 2, forcing with  $\mathbb{P}$  preserves  $\text{ELP}(\delta)$ . We also show that after forcing with  $\mathbb{P}$ ,  $\delta$  carries  $\delta^+$  many normal measures, and for every good ordinal  $\alpha$ ,  $\delta$  carries  $\delta^+$  many normal measures witnessing  $\text{LP}(\delta, \alpha)$ . To do this, note that by its definition, we may write  $\mathbb{P} = \mathbb{Q} * \dot{\text{Coll}}(\delta^+, \delta^{++}) * \dot{\mathbb{R}}^*$ , where  $\Vdash_{\mathbb{Q} * \dot{\text{Coll}}(\delta^+, \delta^{++})} \text{“}\dot{\mathbb{R}}^* \text{ is } \delta'\text{-directed closed”}$ . It thus suffices to show that the preceding three facts hold after forcing with  $\mathbb{Q} * \dot{\text{Coll}}(\delta^+, \delta^{++})$ . However, since  $|\mathbb{Q}| < \delta$  and  $\mathbb{Q}$  is nontrivial, this follows as in the proof of Theorem 1b) by using the proof of [5, Theorem 1].

To finish the proof of Lemma 2.7, it suffices to show that if  $V^\mathbb{P} \models \text{“}\gamma \text{ is a measurable cardinal}$



which is not a limit of measurable cardinals”, then  $V \models$  “ $\gamma$  is a measurable cardinal which is not a limit of measurable cardinals” as well. As in the proof of Theorem 2, we may write  $\mathbb{P} = \text{Add}(\omega, 1) * \dot{\mathbb{R}}$ . Since  $|\text{Add}(\omega, 1)| = \omega$ ,  $\text{Add}(\omega, 1)$  is nontrivial, and as before,  $\Vdash_{\text{Add}(\omega, 1)} \dot{\mathbb{R}}$  is  $\aleph_2$ -directed closed”, it is once more true that Theorem 6 implies  $\gamma$  must be measurable in  $V$ . If  $V \models$  “ $\gamma$  is not a limit of measurable cardinals”, then we are done. If not, then  $V \models$  “ $\gamma$  is a limit of measurable cardinals”, so  $V \models$  “ $\gamma$  is a limit of measurable cardinals  $\rho$  which are not themselves limits of measurable cardinals”. By our work in the first paragraph, all such  $\rho$  remain measurable cardinals in  $V^{\mathbb{P}}$ , so  $V^{\mathbb{P}} \models$  “ $\gamma$  is a limit of measurable cardinals”. This contradiction completes the proof of Lemma 2.7.

□

**Lemma 2.8** *Assume  $V^{\mathbb{P}} \models$  “ $\delta$  is a measurable cardinal which is a limit of measurable cardinals”. Then  $V^{\mathbb{P}} \models$  “ELP( $\delta$ ) holds + For every good ordinal  $\alpha$ ,  $\delta$  carries  $2^{2^\delta} = \delta^{++}$  many normal measures witnessing LP( $\delta, \alpha$ )”.*

**Proof:** Suppose  $V \models$  “ $\delta$  is a measurable cardinal which is a limit of measurable cardinals”. We first show that since we have assumed  $V$  satisfies the conclusions of Theorem 2, forcing with  $\mathbb{P}$  preserves ELP( $\delta$ ), and after forcing with  $\mathbb{P}$ , for every good ordinal  $\alpha$ ,  $\delta$  carries  $2^{2^\delta} = \delta^{++}$  many normal measures witnessing LP( $\delta, \alpha$ ). To do this, note that by its definition, we may write  $\mathbb{P} = \mathbb{P}_\delta * \dot{\mathbb{R}}$ , where  $\Vdash_{\mathbb{P}_\delta} \dot{\mathbb{R}}$  is  $\delta'$ -directed closed”.<sup>3</sup> Since  $V^{\mathbb{P}} \models$  “ $2^\delta = \delta^+$ ”,  $V^{\mathbb{P}_\delta} \models$  “ $2^\delta = \delta^+$ ”. It thus suffices to show that the preceding two facts hold after forcing with  $\mathbb{P}_\delta$ .

Towards this end, let  $\alpha$  be a fixed but arbitrary good ordinal. Take  $j : V \rightarrow M$  to be an elementary embedding witnessing the measurability of  $\delta$  in  $V$  generated by a normal measure  $\mathcal{U}$  over  $\delta$  witnessing LP( $\delta, \alpha$ ). It is then the case that  $A = \{\gamma < \delta \mid 2^\gamma = \gamma^{+\alpha}\} \in \mathcal{U}$ ,  $\delta \in j(A)$ , and  $M \models$  “ $2^\delta = \delta^{+\alpha}$ ”.

Note that by the definition of  $\mathbb{P}$ , since  $V \models$  “ $\delta$  is a measurable cardinal which is a limit of measurable cardinals”,  $j(\mathbb{P}_\delta) = \mathbb{P}_\delta * \dot{\mathbb{Q}}_\delta * \dot{\mathbb{Q}}$ , where  $\dot{\mathbb{Q}}_\delta$  is a term for trivial forcing. Hence, the nine

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<sup>3</sup>Note that if  $\delta = \kappa$ , then  $\dot{\mathbb{R}}$  is a term for trivial forcing.

criteria necessary to apply the standard lifting arguments mentioned in Section 1 may all once again be easily verified to show that  $j$  lifts in  $V^{\mathbb{P}_\delta}$  to  $\bar{j} : V^{\mathbb{P}_\delta} \rightarrow M^{\bar{j}(\mathbb{P}_\delta)}$ . Further, in  $M$ ,  $\Vdash_{\mathbb{P}_\delta * \dot{\mathbb{Q}}_\delta}$  “ $\dot{\mathbb{Q}}$  is  $\delta'$ -directed closed”. Therefore, since by its definition,  $\delta^{+\alpha} < \delta'$ , the fact  $\bar{j}(\mathbb{P}_\delta) = j(\mathbb{P}_\delta) = \mathbb{P}_\delta * \dot{\mathbb{Q}}_\delta * \dot{\mathbb{Q}}$  allows us to infer that  $M^{\bar{j}(\mathbb{P}_\delta)} \models “2^\delta = \delta^{+\alpha}”$ . This means that  $\delta \in \bar{j}(A)$ , from which it is now possible to infer that the normal measure  $\bar{\mathcal{U}} = \{x \subseteq \delta \mid \delta \in \bar{j}(x)\} \supseteq \mathcal{U}$  over  $\delta$  witnesses  $\text{LP}(\delta, \alpha)$  in  $V^{\mathbb{P}_\delta}$ . As  $\alpha$  was arbitrary,  $V^{\mathbb{P}_\delta} \models \text{ELP}(\delta)$ . Also, what has really just been shown is that for every good ordinal  $\alpha$ , any normal measure witnessing  $\text{LP}(\delta, \alpha)$  in  $V$  extends to a normal measure witnessing  $\text{LP}(\delta, \alpha)$  in  $V^{\mathbb{P}_\delta}$ . Consequently, because  $V$  contains  $2^{2^\delta} = \delta^{++}$  many normal measures witnessing  $\text{LP}(\delta, \alpha)$ , and  $\delta^+$  and  $\delta^{++}$  are not collapsed in  $V^{\mathbb{P}_\delta}$ ,  $V^{\mathbb{P}_\delta}$  contains  $2^{2^\delta} = \delta^{++}$  many normal measures witnessing  $\text{LP}(\delta, \alpha)$  as well.

As in the proof of Lemma 2.7, we complete the proof of Lemma 2.8 by showing that if  $V^{\mathbb{P}} \models “\gamma$  is a measurable cardinal which is a limit of measurable cardinals”, then  $V \models “\gamma$  is a measurable cardinal which is a limit of measurable cardinals” as well. However, this follows by the factorization of  $\mathbb{P}$  given in the proof of Lemma 2.7 and another application of Theorem 6. This completes the proof of Lemma 2.8.

□

Suppose  $V \models “\delta < \lambda$  are such that  $\delta$  is  $\lambda$  supercompact and  $\lambda$  is regular”. Since  $\lambda \geq \delta^+$  and  $V \models “2^\delta = \delta^+”$ ,  $\delta$  is (at least)  $2^\delta$  supercompact. Thus,  $\delta$  is a measurable cardinal which is a limit of measurable cardinals, so as we observed in the proof of Lemma 2.8,  $j(\mathbb{P}_\delta) = \mathbb{P}_\delta * \dot{\mathbb{Q}}_\delta * \dot{\mathbb{Q}}$ , where  $\dot{\mathbb{Q}}_\delta$  is a term for trivial forcing. In addition, because  $V \models “\text{No cardinal is supercompact up to an inaccessible cardinal}”$ ,  $\lambda < \delta'$ . Also, we know that in  $V$ ,  $2^\gamma = \gamma^+$  for all  $\gamma \in [\delta, \delta')$ . Hence, the proofs of Lemmas 2.4 – 2.6 show that in  $V^{\mathbb{P}}$ , level by level equivalence holds,  $\kappa$  is supercompact, and no cardinal is supercompact up to an inaccessible cardinal. Since  $V^{\mathbb{P}} \models “\text{No cardinal above } \kappa \text{ is inaccessible}”$ , this completes the proof of Theorem 3.

□

To prove Theorem 4, suppose  $V \models “\text{ZFC} + \text{GCH} + \mathcal{K} \neq \emptyset$  is the class of supercompact cardinals”. Without loss of generality, by the work of [6], we may also assume that in  $V$ ,  $\kappa$  is

supercompact iff  $\kappa$  is strongly compact, except possibly if  $\kappa$  is a measurable limit of supercompact cardinals.

We now define the partial ordering  $\mathbb{P}$  used in the proof of Theorem 4.  $\mathbb{P} = \langle \langle \mathbb{P}_\delta, \dot{\mathbb{Q}}_\delta \rangle \mid \delta < \Omega \rangle$  (where  $\Omega$  is either the ordinal length of the iteration or the class of all ordinals) is the (possibly proper class) Easton support iteration which begins by forcing with  $\text{Add}(\omega, 1)$  and then does nontrivial forcing only at  $V$ -inaccessible cardinals. At such a stage  $\delta$ , if  $\delta$  isn't measurable,  $\dot{\mathbb{Q}}_\delta$  is a term for the lottery sum  $\oplus \{ \text{Add}(\delta, \delta^{+\alpha}) \mid \alpha \geq 2 \text{ is a good ordinal} \}$ . If, however,  $\delta$  is measurable,  $\dot{\mathbb{Q}}_\delta$  is a term for  $\text{Add}(\delta, \delta^+)$ . By its definition, regardless if  $\mathbb{P}$  is a set or a proper class,  $V^{\mathbb{P}} \models \text{ZFC}$ .

The proof of Lemma 2.3 and our remarks in the paragraphs immediately preceding and following Lemma 2.3 show that in  $V^{\mathbb{P}}$ , every measurable cardinal  $\delta$  is such that  $2^\delta = \delta^+$  and  $2^{\delta^+} = 2^{2^\delta} = \delta^{++}$ , every measurable cardinal  $\delta$  witnesses  $\text{ELP}(\delta)$ , and for every good ordinal  $\alpha$  and every measurable cardinal  $\delta$ ,  $\text{LP}(\delta, \alpha)$  holds with respect to  $2^{2^\delta} = \delta^{++}$  many normal measures. To complete the proof of Theorem 4, it therefore remains to show that  $V^{\mathbb{P}} \models \text{“}\mathcal{K} \text{ is the class of supercompact cardinals + } \kappa \text{ is supercompact iff } \kappa \text{ is strongly compact, except possibly if } \kappa \text{ is a measurable limit of supercompact cardinals”}$ .

**Lemma 2.9**  $V^{\mathbb{P}} \models \text{“}\mathcal{K} \text{ is the class of supercompact cardinals”}$ .

**Proof:** We first show that if  $V \models \text{“}\kappa \text{ is supercompact”}$ , then  $V^{\mathbb{P}} \models \text{“}\kappa \text{ is supercompact”}$  as well. To do this, let  $\sigma > \kappa$  be a regular cardinal. Define  $\rho = \sup(\{ \delta^{+\alpha} \mid \delta \in [\kappa, \sigma] \text{ is a cardinal and } \alpha \text{ is a good ordinal} \})$ . Fix  $\delta \in [\kappa, \sigma]$  a cardinal. Because  $\alpha$  is a good ordinal,  $\delta^{+\alpha} < \delta'$  (if  $\delta'$  exists).<sup>4</sup> Hence, as  $\delta \leq \sigma$ ,  $\delta^{+\alpha} < \sigma'$  (if  $\sigma'$  exists). Since there are only countably many good ordinals and at most  $\sigma$  many cardinals in the closed interval  $[\kappa, \sigma]$ ,  $\rho$  is a singular cardinal below  $\sigma'$  (if  $\sigma'$  exists). Let  $\lambda = \rho^+$ . We prove that  $V^{\mathbb{P}} \models \text{“}\kappa \text{ is } \lambda \text{ supercompact”}$ . Since  $\lambda$  may be made arbitrarily large, this will show that  $V^{\mathbb{P}} \models \text{“}\kappa \text{ is supercompact”}$ .

Towards this end, take  $j : V \rightarrow M$  to be an elementary embedding witnessing the  $\lambda$  supercompactness of  $\kappa$  generated by a supercompactness measure over  $P_\kappa(\lambda)$ . Write  $\mathbb{P} = \mathbb{P}_\kappa * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ , where  $\dot{\mathbb{Q}}$  is a term for the portion of  $\mathbb{P}$  acting on cardinals in the closed interval  $[\kappa, \lambda]$ , and  $\dot{\mathbb{R}}$  is a term for

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<sup>4</sup>If  $\kappa$  is the largest inaccessible cardinal in  $V$ , then  $\delta'$  doesn't exist.

the rest of  $\mathbb{P}$ . Since  $\Vdash_{\mathbb{P}_\kappa * \dot{\mathbb{Q}}} \text{“}\dot{\mathbb{R}} \text{ is } (2^{[\lambda]^{<\kappa}})^+\text{-directed closed”}$ , it suffices to show that  $V^{\mathbb{P}_\kappa * \dot{\mathbb{Q}}} \models \text{“}\kappa \text{ is } \lambda \text{ supercompact”}$ . However, by the definitions of  $\lambda$  and  $\mathbb{P}_\kappa * \dot{\mathbb{Q}}$ , the nine criteria necessary to apply the standard lifting arguments mentioned in Section 1 may all be easily verified. This means that  $V^{\mathbb{P}_\kappa * \dot{\mathbb{Q}}} \models \text{“}\kappa \text{ is } \lambda \text{ supercompact”}$ , i.e.,  $V^{\mathbb{P}} \models \text{“}\kappa \text{ is supercompact”}$ .

The proof of Lemma 2.9 will be finished if we can show that if  $V^{\mathbb{P}} \models \text{“}\kappa \text{ is supercompact”}$ , then  $V \models \text{“}\kappa \text{ is supercompact”}$  as well. Note that we can write  $\mathbb{P} = \text{Add}(\omega, 1) * \dot{\mathbb{Q}}'$ . Since  $|\text{Add}(\omega, 1)| = \omega$ ,  $\text{Add}(\omega, 1)$  is nontrivial, and as before,  $\Vdash_{\text{Add}(\omega, 1)} \text{“}\dot{\mathbb{Q}}' \text{ is } \aleph_2\text{-directed closed”}$ , by our remarks immediately following the statement of Theorem 6,  $\kappa$  must be supercompact in  $V$ . This completes the proof of Lemma 2.9.

□

**Lemma 2.10**  $V^{\mathbb{P}} \models \text{“}\kappa \text{ is supercompact iff } \kappa \text{ is strongly compact, except possibly if } \kappa \text{ is a measurable limit of supercompact cardinals”}$ .

**Proof:** Suppose  $V^{\mathbb{P}} \models \text{“}\kappa \text{ is strongly compact”}$ . Note that by its definition, since it must be the case that  $V \models \text{“}\kappa \text{ is inaccessible”}$ ,  $\mathbb{P}$  is mild with respect to  $\kappa$ . Therefore, by the factorization of  $\mathbb{P}$  given in Lemma 2.9 and our remarks immediately following the statement of Theorem 6,  $V \models \text{“}\kappa \text{ is strongly compact”}$  as well. By our assumptions on  $V$ , it must then be the case that  $V \models \text{“} \text{Either } \kappa \text{ is supercompact, or } \kappa \text{ is a measurable limit of supercompact cardinals”}$ . Since by the proof of Lemma 2.9, forcing with  $\mathbb{P}$  preserves all  $V$ -supercompact cardinals, we may now infer that  $V^{\mathbb{P}} \models \text{“} \text{Either } \kappa \text{ is supercompact, or } \kappa \text{ is a measurable limit of supercompact cardinals”}$ . This completes the proof of Lemma 2.10.

□

Lemmas 2.9 and 2.10 complete the proof of Theorem 4.

□

Finally, to prove Theorem 5, suppose  $V \models \text{“ZFC + GCH + } \mathcal{K} \neq \emptyset \text{ is the class of supercompact cardinals”}$ . As in the proof of Theorem 4, by the work of [6], we may first assume that in  $V$ ,  $\kappa$  is

supercompact iff  $\kappa$  is strongly compact, except possibly if  $\kappa$  is a measurable limit of supercompact cardinals. We then assume, by forcing over  $V$  with the partial ordering used in the proof of Theorem 4, that  $V$  satisfies the conclusions of Theorem 4. We again abuse notation somewhat by relabelling the resulting generic extension as  $V$  and taking this as our ground model.

We are now in a position to define the partial ordering  $\mathbb{P}$  used in the proof of Theorem 5.  $\mathbb{P} = \langle \langle \mathbb{P}_\delta, \dot{\mathbb{Q}}_\delta \rangle \mid \delta < \Omega \rangle$  (where  $\Omega$  is as before either the ordinal length of the iteration or the class of all ordinals) is the (possibly proper class) Easton support iteration which begins by forcing with  $\text{Add}(\omega, 1)$  and then does nontrivial forcing only at cardinals  $\delta$  which are in  $V$  measurable cardinals which are not limits of measurable cardinals. At such a stage  $\delta$ ,  $\dot{\mathbb{Q}}_\delta$  is a term for  $\text{Coll}(\delta^+, \delta^{++}) = \text{Coll}(\delta^+, 2^{2^\delta})$ . As before, by its definition, regardless if  $\mathbb{P}$  is a set or a proper class,  $V^\mathbb{P} \models \text{ZFC}$ .

The proof of Theorem 3 shows that  $V^\mathbb{P} \models$  “For every measurable cardinal  $\delta$ ,  $2^\delta = \delta^+$  and  $2^{\delta^+} = 2^{2^\delta} = \delta^{++}$ ”. Assume now that  $V^\mathbb{P} \models$  “ $\delta$  is a measurable cardinal which is not a limit of measurable cardinals”. The proof of Lemma 2.7 shows that  $\delta$  witnesses  $\text{ELP}(\delta)$ ,  $\delta$  carries  $\delta^+$  many normal measures, and for every good ordinal  $\alpha$ ,  $\text{LP}(\delta, \alpha)$  holds with respect to  $\delta^+$  many normal measures. Assume next that  $V^\mathbb{P} \models$  “ $\delta$  is a measurable cardinal which is a limit of measurable cardinals”. The proof of Lemma 2.8 shows that  $\delta$  witnesses  $\text{ELP}(\delta)$ , and for every good ordinal  $\alpha$  and every measurable cardinal  $\delta$ ,  $\text{LP}(\delta, \alpha)$  holds with respect to  $2^{2^\delta} = \delta^{++}$  many normal measures. To complete the proof of Theorem 5, it therefore once again remains to show that  $V^\mathbb{P} \models$  “ $\mathcal{K}$  is the class of supercompact cardinals +  $\kappa$  is supercompact iff  $\kappa$  is strongly compact, except possibly if  $\kappa$  is a measurable limit of supercompact cardinals”.

**Lemma 2.11**  $V^\mathbb{P} \models$  “ $\mathcal{K}$  is the class of supercompact cardinals”.

**Proof:** As in the proof of Lemma 2.9, we first show that if  $V \models$  “ $\kappa$  is supercompact”, then  $V^\mathbb{P} \models$  “ $\kappa$  is supercompact” as well. To do this, we let  $A = \{\delta > \kappa \mid \delta \text{ is an inaccessible cardinal}\}$ . Suppose  $A$  is a set. Because  $V$  is obtained by forcing over a model  $\bar{V}$  of GCH with a partial ordering  $\mathbb{Q}$  which preserves all inaccessible cardinals, the collection of inaccessible cardinals in  $\bar{V}$  is a set and not a proper class. Thus,  $\mathbb{Q}$  is also a set and not a proper class, and so forcing with  $\mathbb{Q}$  preserves GCH on a final segment of cardinals. This means we can let  $\gamma > \sup(A)$  be a regular cardinal such

that for every cardinal  $\delta \geq \gamma$ ,  $2^\delta = \delta^+$  and take  $\lambda > \gamma$  to be an arbitrary regular cardinal.

If  $A$  is a proper class, then let  $\delta > \kappa$  be a non-measurable inaccessible cardinal. As in the preceding paragraph, let  $\mathbb{Q}$  be the forcing used to obtain  $V$  from  $\bar{V}$ . Although  $\mathbb{Q}$  is now a proper class and not a set, by its definition, since  $\bar{V} \models \text{GCH}$ , there must be some cardinal  $\gamma \in (\delta, \delta')$  such that  $2^\sigma = \sigma^+$  for all cardinals  $\sigma \in [\gamma, \delta')$ . Fix  $\lambda \in (\gamma, \delta')$  as such a regular cardinal. Regardless of whether  $A$  is a set or a proper class, the definition of  $\lambda$ , combined with the fact that  $\lambda$  may be made arbitrarily large, now allow us to argue as in the proof of Lemma 2.9 to show that  $V^{\mathbb{P}} \models$  “ $\kappa$  is supercompact” and then infer that  $V^{\mathbb{P}} \models$  “ $\mathcal{K}$  is the class of supercompact cardinals”. This completes the proof of Lemma 2.11.

□

Since the same proof as given in Lemma 2.10 shows that  $V^{\mathbb{P}} \models$  “ $\kappa$  is supercompact iff  $\kappa$  is strongly compact, except possibly if  $\kappa$  is a measurable limit of supercompact cardinals”, the proof of Theorem 5 is now complete.

□

In conclusion, we ask whether it is possible to establish analogues of Theorems 2 and 3 in which the large cardinal structure of both our ground models and generic extensions is richer. In particular, can there be models witnessing the conclusions of Theorems 2 and 3 in which the class of supercompact cardinals is arbitrary, or must there be some restrictions of necessity? Finally, we ask whether it is possible to establish analogues of each of our theorems in which for measurable cardinals  $\delta$  and good ordinals  $\alpha$ , the cardinality of the number of normal measures witnessing  $\text{LP}(\delta, \alpha)$  is different from either  $\delta^+$  or  $2^{2^\delta}$ .

## References

- [1] A. Apter, “Inaccessible Cardinals, Failures of GCH, and Level by Level Equivalence”, *Notre Dame Journal of Formal Logic* 55, 2014, 431–444.

- [2] A. Apter, “Indestructibility and the Levinski Property”, *Sarajevo Journal of Mathematics* 12, 2016, 3–16.
- [3] A. Apter, “Some Remarks on Normal Measures and Measurable Cardinals”, *Mathematical Logic Quarterly* 47, 2001, 35–44.
- [4] A. Apter, “Some Structural Results Concerning Supercompact Cardinals”, *Journal of Symbolic Logic* 66, 2001, 1919–1927.
- [5] A. Apter, J. Cummings, J. D. Hamkins, “Large Cardinals with Few Measures”, *Proceedings of the American Mathematical Society* 135, 2007, 2291–2300.
- [6] A. Apter, S. Shelah, “On the Strong Equality between Supercompactness and Strong Compactness”, *Transactions of the American Mathematical Society* 349, 1997, 103–128.
- [7] J. Cummings, “Possible Behaviours for the Mitchell Ordering”, *Annals of Pure and Applied Logic* 65, 1993, 107–123.
- [8] J. D. Hamkins, “Gap Forcing”, *Israel Journal of Mathematics* 125, 2001, 237–252.
- [9] J. D. Hamkins, “Gap Forcing: Generalizing the Lévy-Solovay Theorem”, *Bulletin of Symbolic Logic* 5, 1999, 264–272.
- [10] J. D. Hamkins, “The Lottery Preparation”, *Annals of Pure and Applied Logic* 101, 2000, 103–146.
- [11] T. Jech, *Set Theory: The Third Millennium Edition, Revised and Expanded*, Springer-Verlag, Berlin and New York, 2003.
- [12] Y. Kimchi, M. Magidor, “The Independence between the Concepts of Compactness and Supercompactness”, circulated manuscript.
- [13] J. -P. Levinski, “Filters and Large Cardinals”, *Annals of Pure and Applied Logic* 72, 1995, 177–212.

- [14] A. Lévy, R. Solovay, “Measurable Cardinals and the Continuum Hypothesis”, *Israel Journal of Mathematics* 5, 1967, 234–248.
- [15] M. Magidor, “On the Existence of Nonregular Ultrafilters and the Cardinality of Ultrapowers”, *Transactions of the American Mathematical Society* 249, 1979, 97–111.
- [16] T. Menas, “On Strong Compactness and Supercompactness”, *Annals of Mathematical Logic* 7, 1974, 327–359.