Controlling the Number of Normal Measures at Successor Cardinals

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Abstract

We examine the number of normal measures a successor cardinal can carry, in universes in which the Axiom of Choice is false. When considering successors of singular cardinals, we establish relative consistency results assuming instances of supercompactness, together with the Ultrapower Axiom UA (introduced by Goldberg in [9]). When considering successors of regular cardinals, we establish relative consistency results only assuming the existence of one measurable cardinal. This allows for equiconsistencies.

1 Introduction and Preliminaries

In [2] and [3], the following theorems were proven.

Theorem 1 ([2, Theorem 1]) Let $V^* \models \text{"ZFC + GCH + } \kappa < \lambda \text{ are such that } \kappa \text{ is supercompact and } \lambda \text{ is the least measurable cardinal greater than } \kappa + \tau > \lambda^+ \text{ is a fixed but arbitrary regular}$

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There is then a generic extension $V$ of $V^*$, a partial ordering $\mathbb{P} \in V$, and a symmetric submodel $N \subseteq V^\mathbb{P}$ such that $N \models \text{"ZF + DC_{\aleph_\omega} + } \lambda = \aleph_{\omega+1} \text{ is a measurable cardinal".}$. In $N$, the cardinal and cofinality structure at and above $\lambda$ is the same as in $V$ (which has the same cardinal and cofinality structure at and above $\lambda$ as $V^*$), and $\aleph_{\omega+1}$ carries exactly $\tau$ normal measures.

**Theorem 2 ([2, Theorem 2])** Let $V^* \models \text{"ZFC + GCH + } \kappa < \lambda \text{ are such that } \kappa \text{ is supercompact and } \lambda \text{ is the least measurable cardinal greater than } \kappa \text{".}$. There is then a generic extension $V$ of $V^*$, a partial ordering $\mathbb{P} \in V$, and a symmetric submodel $N \subseteq V^\mathbb{P}$ such that $N \models \text{"ZF + DC_{\aleph_\omega} + } \lambda = \aleph_{\omega+1} \text{ is a measurable cardinal".}$. In $N$, $\aleph_{\omega+2}$ is regular, and $\aleph_{\omega+1}$ carries exactly $\aleph_{\omega+2}$ normal measures.

**Theorem 3 ([3, Theorem 1])** Suppose $V^* \models \text{"ZFC + GCH + } \kappa < \lambda \text{ are such that } \kappa \text{ is supercompact and } \lambda \text{ is the least measurable cardinal greater than } \kappa + \tau > \lambda^+ \text{ is a fixed but arbitrary regular cardinal".}$. There is then a generic extension $V$ of $V^*$, a partial ordering $\mathbb{P} \in V$, and a symmetric submodel $N \subseteq V^\mathbb{P}$ such that $N \models \text{"ZF + } \neg \text{AC}_{\aleph_\omega} + \aleph_1 \text{ is a regular cardinal + } \kappa = \aleph_{\omega_1} + \text{ For every limit ordinal } \nu < \aleph_1, \aleph_{\nu+1} \text{ is a measurable cardinal + } \lambda = \kappa^+ = \aleph_{\omega_1+1} \text{ is a measurable cardinal".}$. In $N$, every successor cardinal less than $\aleph_{\omega_1}$ is regular, the cardinal and cofinality structure at and above $\lambda$ is the same as in $V$ (which has the same cardinal and cofinality structure at and above $\lambda$ as $V^*$), and $\aleph_{\omega_1+1}$ carries exactly $\tau$ normal measures.

**Theorem 4 ([3, Theorem 2])** Suppose $V^* \models \text{"ZFC + GCH + } \kappa < \lambda \text{ are such that } \kappa \text{ is supercompact and } \lambda \text{ is the least measurable cardinal greater than } \kappa \text{".}$. There is then a generic extension $V$ of $V^*$, a partial ordering $\mathbb{P} \in V$, and a symmetric submodel $N \subseteq V^\mathbb{P}$ such that $N \models \text{"ZF + } \neg \text{AC}_{\aleph_\omega} + \aleph_1 \text{ is a regular cardinal + } \kappa = \aleph_{\omega_1} + \text{ For every limit ordinal } \nu < \aleph_1, \aleph_{\nu+1} \text{ is a measurable cardinal + } \lambda = \kappa^+ = \aleph_{\omega_1+1} \text{ is a measurable cardinal".}$. In $N$, every successor cardinal less than $\aleph_{\omega_1}$ is regular, $\aleph_{\omega_1+2}$ is regular, and $\aleph_{\omega_1+1}$ carries exactly $\aleph_{\omega_1+2}$ normal measures.

Fix $\eta$ as either $\omega$ or $\omega_1$. Taken in aggregate, Theorems 1 – 4 show that relative to the appropriate hypotheses, it is consistent for $\aleph_{\eta+1}$ to carry exactly $\gamma$ normal measures, where $\gamma \geq \aleph_{\eta+2}$ is an arbitrary regular cardinal. This, however, leaves open the following
Question: Is it possible for $\aleph_{\eta+1}$ to carry exactly $\gamma$ normal measures, where $\gamma \leq \aleph_{\eta+1}$ is an arbitrary cardinal (regular or singular)?

The aforementioned Question motivates this paper. We will begin by showing that the Ultra-power Axiom UA (introduced by Goldberg in [9]), together with the appropriate large cardinal assumptions, can provide positive answers. Specifically, we will prove the following theorem.

**Theorem 5** Fix $\eta$ as either $\omega$ or $\omega_1$. Suppose $V \models "ZFC + UA + \kappa < \lambda"$ are such that $\kappa$ is supercompact and $\lambda$ is the least measurable cardinal greater than $\delta$ with $o(\lambda) = \delta$ for some ordinal $\delta \leq \lambda$". There is then a partial ordering $\mathbb{P} \in V$ and a symmetric submodel $N \subseteq V^\mathbb{P}$ such that $N \models "ZF + \lambda = \aleph_{\eta+1} \text{ is measurable and carries exactly } \gamma = |\delta| \text{ normal measures}"$.

For successors of regular cardinals, however, it is possible to establish analogous results to Theorems 1 – 5, assuming only the existence of one measurable cardinal. In particular, we have the following theorem.

**Theorem 6** Suppose $V \models "ZFC + \kappa < \lambda"$ are such that $\kappa$ is regular and $\lambda$ is measurable”. Fix $\delta \leq \kappa$ an arbitrary cardinal (including 0) or $\delta \geq \lambda$ an arbitrary cardinal of cofinality greater than $\lambda^+$. There is then a partial ordering $\mathbb{P} \in V$ and a symmetric submodel $N \subseteq V^\mathbb{P}$ such that $N \models "ZF + \lambda = \kappa^+ + \lambda \text{ is measurable and carries exactly } \delta \text{ normal measures}"$. In $N$, the cardinal and cofinality structure at and above $\lambda$ is the same as in $V$.

We note that unlike Theorems 1 and 2, no assumption of GCH is required in the proof of Theorem 5. This is since the use of GCH in the proofs of those theorems was to allow the number of normal measures carried by either $\aleph_{\omega+1}$ or $\aleph_{\omega_1+1}$ to be an arbitrarily large regular cardinal, rather than to contract the number of normal measures carried by either of these cardinals. Also, examples of easy corollaries to Theorems 5 and 6 include the existence of models $N$ of ZF in which either $\aleph_{\omega+1}$ or $\aleph_{\omega_1+1}$ is measurable and carries, e.g., exactly $\gamma = 1, 2, 75, \omega, \aleph_{57}, \aleph_{\omega}, \aleph_{\omega+1}, \aleph_{\omega_1+1}$, etc. normal measures, $\aleph_{\omega_5+8}$ is measurable and carries exactly $\aleph_{36}$ normal measures, $\aleph_{77}$ is measurable and carries exactly $\aleph_{85}$ normal measures, $\aleph_1$ is measurable and carries no normal measures, etc.
Further, the method of proof used to establish Theorem 6 shows that in Theorems 1 and 3, \( \tau \) can be an arbitrary cardinal of cofinality greater than \( \lambda^+ \). In addition, core model theory tells us that unlike for Theorem 6, strong hypotheses beyond the existence of one measurable cardinal (i.e., at least a Woodin cardinal) must be used in order to establish Theorem 5. We will come back to this issue again after having completed the proof of Theorem 5.

We now take this opportunity to mention some preliminary material and background information. We will feel free to quote verbatim from [2, 3, 4] for both expository material and proofs when appropriate. We note that the *Ultrapower Axiom UA*, introduced by Goldberg in [9, Definitions 2.1 – 2.3], says the following:

Suppose \( V \models \text{ZFC} \) and \( U_0, U_1 \in V \) are countably complete ultrafilters over \( x_0 \in V, x_1 \in V \) respectively with \( j_{U_0} : V \rightarrow M_{U_0} \) and \( j_{U_1} : V \rightarrow M_{U_1} \) the associated elementary embeddings. Then there exist \( W_0 \in M_{U_0} \) an \( M_{U_0} \)-countably complete ultrafilter over \( y_0 \in M_{U_0} \) and \( W_1 \in M_{U_1} \) an \( M_{U_1} \)-countably complete ultrafilter over \( y_1 \in M_{U_1} \) such that:

1. For \( j_{W_0} : M_{U_0} \rightarrow M_{W_0} \) and \( j_{W_1} : M_{U_1} \rightarrow M_{W_1} \) the associated elementary embeddings, 
   \[ M_{W_0} = M_{W_1} = M. \]
2. \( j_{W_0} \circ j_{U_0} = j_{W_1} \circ j_{U_1}. \)

The following commutative diagram illustrates UA pictorially:

\[
\begin{array}{ccc}
V & \xrightarrow{j_{U_0}} & M_{U_0} \\
\downarrow{j_{U_1}} & & \downarrow{j_{W_0}} \\
M_{U_1} & \xrightarrow{j_{W_1}} & M
\end{array}
\]

It is the case that currently, UA is not known to hold in any model of ZFC containing very large cardinals, including models of ZFC with supercompact cardinals. However, as Goldberg has pointed out in [9, Section 1], UA is true in the usual inner models constructed at lower levels of the large cardinal hierarchy. UA has also been verified in canonical inner models built by Woodin and Neeman-Steel for finite levels of supercompactness using iteration hypotheses,
thereby suggesting that UA may be expected to hold in canonical inner models containing (fully) supercompact cardinals (if they indeed exist). Further, by [9, Theorem 2.5], UA implies that the Mitchell ordering of normal measures over a measurable cardinal is linear (which is likely far weaker than the full UA and is the only consequence of UA used in the proof of Theorem 5). These facts make UA, as well as its implication concerning the linearity of the Mitchell ordering, plausible when assumed in models of ZFC containing very large cardinals.

The Mitchell ordering on normal measures over a measurable cardinal $\kappa$, introduced by Mitchell in [11], is defined by $U_0 < U_1$ for normal measures $U_0, U_1$ over $\kappa$ iff $U_0 \in V^\kappa/U_1$. By [10, Lemma 19.32], the Mitchell ordering is well-founded. As in [10, Definition 19.33], the Mitchell order of the normal measure $U$, $o(U)$, is the rank of $U$ in $<_\triangleleft$, and the Mitchell order of $\kappa$, $o(\kappa)$, is the height of $<_\triangleleft$. Assuming GCH, the maximal value of $o(\kappa)$ is $\kappa^{++}$. Further information on the Mitchell ordering may be found in [10, pages 357 – 360], as well as [11].

As is customary, for $\alpha < \beta$ ordinals, $[\alpha, \beta)$, $(\alpha, \beta)$, $(\alpha, \beta]$ and $(\alpha, \beta]$ are as in the usual interval notation. For $\kappa < \lambda$ such that $\kappa$ is a regular cardinal and $\lambda$ is inaccessible, $\text{Coll}(\kappa, < \lambda)$ is the standard Lévy collapse partial ordering for collapsing every cardinal $\delta \in (\kappa, \lambda)$ to $\kappa$, i.e., $\text{Coll}(\kappa, < \lambda) = \{f | f : \kappa \times \lambda \to \lambda \text{ is a function such that } |\text{dom}(f)| < \kappa \text{ and } f((\alpha, \beta)) < \beta\}$, ordered by inclusion. For $\delta \in (\kappa, \lambda)$ and any $S \subseteq \text{Coll}(\kappa, < \lambda)$, we define $S \upharpoonright \delta = \{p \in S | \text{dom}(p) \subseteq \kappa \times \delta\}$.

It is well-known that if $G$ is $V$-generic over $\text{Coll}(\kappa, < \lambda)$ and $\delta \in (\kappa, \lambda)$ is inaccessible, then $G \upharpoonright \delta$ is $V$-generic over $\text{Coll}(\kappa, < \lambda) \upharpoonright \delta$.

2 The Proofs of Theorems 5 and 6

We turn now to the proofs of Theorems 5 and 6, starting with the proof of Theorem 5.

**Proof:** Suppose $V \models \text{“ZFC + UA + } \kappa < \lambda \text{ are such that } \kappa \text{ is supercompact and } \lambda \text{ is the least measurable cardinal greater than } \delta \text{ with } o(\lambda) = \delta \text{ for some ordinal } \delta \leq \lambda \text{”}. We begin with the following simple fact, which will be key to the proof of Theorem 5.

**Proposition 1 ([4, Proposition 1])** Assume UA. Let $\gamma = |\delta|$. If $\lambda$ is a measurable cardinal such that $o(\lambda) = \delta$, then the number of normal measures $\lambda$ carries is $\gamma$.  


Proof: As we have already stated in Section 1, because UA holds, the Mitchell ordering over any measurable cardinal must be linear (and in fact, must be a well-ordering, since the Mitchell ordering is well-founded). Let $U$ be the set of normal measures over $\lambda$. The function $f: \delta \to U$ given by

$$f(\alpha) = \text{The unique normal measure over } \lambda \text{ of Mitchell order } \alpha$$

therefore is well-defined and is a bijection between $\delta$ and $U$. Thus, the number of normal measures $\lambda$ carries is $\gamma$. This completes the proof of Proposition 1. 

$\square$

Continuing with the proof of Theorem 5, suppose first that $\eta = \omega$. The inner model $N$ witnessing the conclusions of Theorem 5 will be built analogously to the models $N$ witnessing the conclusions of [2, Theorems 1 and 2]. The forcing conditions $P$ used and inner model $N$ constructed are described in detail in [2, pages 62 – 63], with the current $\kappa$ and $\lambda$ replacing the $\kappa$ and $\lambda$ of [2]. Intuitively, $P$ is a version of supercompact Prikry forcing with interleaved collapses which simultaneously changes $\kappa$’s cofinality to $\omega$, collapses $\kappa$ to $\aleph_\omega$, and collapses all cardinals in the half-open interval $(\kappa, \lambda]$ to $\kappa$. $N$ is then essentially the least model of ZF extending $V$ which contains, for every $V$-inaccessible cardinal $\delta \in [\kappa, \lambda)$, the generic $\omega$-sequence collapsing all cardinals in the half-open interval $(\kappa, \delta]$ to $\kappa$, together with the generic $\omega$-sequence of functions collapsing $\kappa$ to $\aleph_\omega$. As in [2], the construction allows us to infer that $N \models \text{DC}_{\aleph_\omega}$.

Similarly, when $\eta = \omega_1$, the inner model $N$ witnessing the conclusions of Theorem 5 will be built analogously to the models $N$ witnessing the conclusions of [3, Theorems 1 and 2]. The forcing conditions $P$ used and inner model $N$ constructed are described in detail in [3, pages 165 – 168], with the current $\kappa$ and $\lambda$ replacing the $\kappa$ and $\lambda$ of [3]. Intuitively, $P$ is an analogue of Gitik’s forcing of [8], i.e., a product of a version of supercompact Radin forcing and Lévy collapses which simultaneously changes $\kappa$’s cofinality to $\omega_1$, collapses $\kappa$ to $\aleph_{\omega_1}$, and collapses all cardinals in the half-open interval $(\kappa, \lambda]$ to $\kappa$. $N$ is then essentially the least model of ZF extending $V$ which contains, for every $V$-inaccessible cardinal $\delta \in [\kappa, \lambda)$, the generic $\omega_1$-sequence collapsing all
cardinals in the half-open interval \((\kappa, \delta]\) to \(\kappa\), together with the generic \(\omega_1\)-sequence of functions generated by the cofinal \(\omega_1\) sequence through \(\kappa\) and product of Lévy collapses collapsing \(\kappa\) to \(\aleph_{\omega_1}\). As in [3], as opposed to when \(\eta = \omega\), we will have that \(N \models \neg \text{AC}_\omega\).

Since the exact details of either construction of \(N\) referenced in the preceding two paragraphs are quite complicated, we do not give them here, but refer readers to [2, 3] for the missing details. We do note that regardless of the value of \(\eta\), the arguments of [2, 3] show that \(N \models \text{"ZF } + \kappa = \aleph_{\eta+1} + \lambda = \aleph_{\eta+1} + \aleph_{\eta+1}\) is measurable and carries normal measures”. In addition, in both cases, we have the following critical fact.

**Lemma 2.1 ([2, Lemma 2.2], [3, Lemma 3])** Suppose \(U^* \in N\) is a normal measure over \(\lambda\). Then for some normal measure \(U \in V\) over \(\lambda\), \(U^* = \{x \subseteq \lambda \mid \exists y \subseteq x(y \in U)\}\).

We may now complete the proof of Theorem 5. Let \(U\) be the set of normal measures over \(\lambda\) present in \(V\) and \(U^*\) be the set of normal measures over \(\lambda\) present in \(N\). Since \(V \models \text{"UA } + \alpha(\lambda) = \delta"\), by Proposition 1, \(V \models \text{"}\|U\| = \|\delta\|"\). By Lemma 2.1, because \(V \subseteq N\), the function \(f : U^* \to U\) given by

\[
f(U^*) = \text{The unique } U \in U \text{ such that } U^* = \{x \subseteq \lambda \mid \exists y \subseteq x(y \in U)\}
\]

is a well-defined member of \(N\) which is a bijection. This means that in \(N\), the number of normal measures \(\lambda = \aleph_{\eta+1}\) carries is precisely \(\|\delta\|^N\). This completes the proof of Theorem 5.

\(\square\)

We remark that the methods just described are applicable to the successors of other singular cardinals. In particular, \(\aleph_{\omega_1+1}\) can be replaced by \(\aleph_{\omega + \omega_1+1}\), as well as different successors of singular cardinals of cofinality \(\omega\). In a similar vein, \(\aleph_{\omega_1+1}\) can be replaced by \(\aleph_{\omega_1+\omega_1+1}, \aleph_{\omega_2+1}\), or additional successors of singular cardinals of any uncountable cofinality.

As we have previously mentioned, strong hypotheses beyond the existence of one measurable cardinal are required to construct models in which the successor of a singular cardinal is measurable and carries normal measures. To see this, suppose \(\kappa\) is singular and \(\kappa^+\) is measurable. By doing
Prikry forcing using one of the normal measures \(\kappa^+\) carries, we may change the cofinality of \(\kappa^+\) to \(\omega\) without adding bounded subsets of \(\kappa^+\).\(^1\) This allows us to obtain a model in which both \(\kappa\) and \(\kappa^+\) are singular. By [12, Theorem 1], this means there must be an inner model containing a Woodin cardinal.

Having completed the proof of Theorem 5, we turn now to the proof of Theorem 6.

**Proof:** Suppose \(V \models \text{ZFC + } \kappa < \lambda\) are such that \(\kappa\) is regular and \(\lambda\) is measurable. Fix \(\delta \leq \kappa\) an arbitrary cardinal (including 0) or \(\delta \geq \lambda\) an arbitrary cardinal of cofinality greater than \(\lambda^+\). As in [7], we assume without loss of generality that \(V = L[U]\) for some normal measure \(U\).

We consider two cases.

Case 1: \(\delta \neq 0\). By [7, Theorem 1] (if \(\delta \leq \kappa\) or \(\lambda \leq \delta \leq \lambda^{++}\) is an arbitrary cardinal) or [7, Theorem 11] (if \(\delta \geq \lambda^{+++}\) is a cardinal of cofinality greater than \(\lambda^+\)), we may assume that \(V\) has been generically extended to a model \(V^*\) having the same cardinals and cofinalities as \(V\) such that \(V^* \models \text{ZFC + } \lambda\) is measurable and carries exactly \(\delta\) normal measures”. We then force over \(V^*\) with \(Q = \text{Coll}(\kappa, <\lambda)\) and let \(N \subseteq (V^*)^Q\) be the symmetric inner model in which \(\kappa^+ = \lambda\) and \(\lambda\) is measurable.

We note that the construction of \(N\) is described in detail in the proof of [10, Theorem 21.16], with \(\omega\) replaced by \(\kappa\), or [6, Theorem 3.1], with \(\mathcal{P}'(A)\) replaced by trivial forcing \(\{0\}\) and \(\aleph_1\) replaced by \(\kappa\). It therefore follows that \(V, V^*, N\) have the same cardinal and cofinality structure at and above \(\lambda\). Intuitively, \(N\) is the least model of \(ZF\) extending \(V^*\) which contains, for every cardinal \(\delta \in (\kappa, \lambda)\), the Lévy-generic function collapsing \(\delta\) to \(\kappa\). The proof of [6, Theorem 3.1(d)], which tells us that any measures \(\lambda\) carries in \(N\) (normal or otherwise) are generated as in Lemma 2.1, then allows us to argue as in the proofs of [6, Theorem 3.1(d)] and Theorem 5 and infer that in \(N\), \(\kappa^+ = \lambda\) carries exactly \(\delta\) normal measures. This completes the proof of Case 1. \(\square\)

Case 2: \(\delta = 0\). Let \(\mathbb{R}\) be the forcing of [5] used by Bilinsky and Gitik to construct a model \(M \subseteq V^{\mathbb{R}}\) such that \(M \models \text{ZF + } \lambda\) is a limit cardinal + \(\lambda\) is measurable and doesn’t carry any normal measures”. It thus follows that \(V\) and \(M\) have the same cardinal and cofinality structure

\(^1\)Readers are referred to [1, Lemmas 1.1 – 1.3] for a discussion of Prikry forcing in a choiceless context.
at and above $\lambda$. By the proof of [5, Theorem 1.1], we may assume that $R$ has been defined so that its first nontrivial stage occurs at an inaccessible cardinal $\zeta \in (\kappa, \lambda)$. This means that forcing with $R$ adds no new $\gamma$-sequences of ordinals for any $\gamma < \zeta$. Also, for $\zeta \in (\kappa, \lambda]$ an inaccessible cardinal, $(Q_\zeta)^V = (\text{Coll}(\kappa, <\zeta))^V$ (i.e., $\text{Coll}(\kappa, <\zeta)$ as defined in $V$) may be coded as a set of ordinals of length less than $\kappa$. Hence, because $V$ and $V^R$ contain the same $\kappa$-sequences of ordinals and $M \subseteq V^R$, $(Q_\zeta)^V = (Q_\zeta)^M$. Further, because $V \subseteq M$, what we may now write unambiguously as $Q_\zeta$ has a well-ordering in $V$ which is a member of $M$. We may therefore force over $M$ with $Q_\lambda$ and build $N \subseteq M^{Q_\lambda}$ as in Case 1 and infer that $N$ has the same properties as though we were forcing over a model of AC. In particular, $N \vDash \text{"}\lambda = \kappa^+ \text{ is measurable"}$, and $V$, $M$, and $N$ all have the same cardinal and cofinality structure at and above $\lambda$. In addition, as in Case 1, we have that if $U^* \in N$ is a measure over $\lambda$, then for some measure $U \in M$ over $\lambda$, $U^* = \{x \subseteq \lambda \mid \exists y \subseteq x[y \in U]\}$.

We may now conclude that $N \vDash \text{"There are no normal measures over $\lambda$"}$. To see this, suppose towards a contradiction that $N \vDash \text{"$U^*$ is a normal measure over $\lambda$"}$. From the last sentence of the preceding paragraph, it follows that for some (non-normal) measure $U \in M$ over $\lambda$, $U^* = \{x \subseteq \lambda \mid \exists y \subseteq x[y \in U]\}$. Because $M \vDash \text{"There are no normal measures over $\lambda$"}$, for some $f \in M$ with $M \vDash \text{"$f : \lambda \to \lambda$ is such that $\{\alpha < \lambda \mid f(\alpha) < \alpha\} \in U^*$\"}$, $M \vDash \text{"There is no set in $U$ on which $f$ is constant"}$. Since $M \subseteq N$ and any set in $U^*$ must contain a subset which is in $U$, $N \vDash \text{"$f : \lambda \to \lambda$ is such that $\{\alpha < \lambda \mid f(\alpha) < \alpha\} \in U^*$ yet there is no set in $U^*$ on which $f$ is constant"}$. This contradiction completes the proof of Case 2.

Cases 1 and 2 complete the proof of Theorem 6.

We remark that as opposed to the situation with Theorem 5, the methods used to prove Theorem 6 allow for equiconsistencies to be established.\footnote{The exact definitions of $R$ and $M$, as well as the intuition behind these definitions, are quite complicated and will not be given here. We emphasize that the definition of $R$ only requires the use of an arbitrary final segment of inaccessible cardinals below $\lambda$.} For instance, suppose $T_1$ is the theory

\begin{itemize}
\item \text{\textit{Note that if $\kappa = \omega$, then since $\text{Coll}(\omega, <\zeta)$ for $\zeta \in (\kappa, \lambda]$ an inaccessible cardinal is absolute and can be canonically well-ordered, the exact way in which $M$ has been defined is irrelevant.}}
\item \text{\textit{Prior to the work of [7], it would have been necessary to collapse measurable cardinals of high Mitchell order to establish some of the consequences of Theorem 6, which would not have allowed for equiconsistencies.}}
\end{itemize}
"ZFC + There exists a measurable cardinal", $T_2$ is the theory "ZF + $\aleph_1$ is measurable and carries exactly 75 normal measures", and $T_3$ is the theory "ZF + $\aleph_{\omega+8}$ is measurable and carries no normal measures". Some examples would then include "$T_1$ and $T_2$ are equiconsistent" and "$T_1$ and $T_3$ are equiconsistent" (from which it of course follows that "$T_2$ and $T_3$ are equiconsistent" is easily established). The proof of Theorem 6 shows that $\text{Con}(T_1) \Rightarrow \text{Con}(T_2)$ and $\text{Con}(T_1) \Rightarrow \text{Con}(T_3)$. For the reverse direction in each equiconsistency, simply take a measure $U$ over the measurable cardinal in question and build $L[U]$.

3 Concluding Remarks

For the results of Theorem 5, UA is key to the proof. In addition, if the number of normal measures $\gamma$ at the successor of the singular cardinal $\kappa$ in question is to be such that $1 \leq \gamma \leq \kappa^+$, the proofs require that the measurable cardinal $\lambda$ collapsed to $\kappa^+$ be such that $o(\lambda) = \delta$ for the appropriate $\delta$. These lead us to asking whether UA can be removed as a hypothesis, and if in the proof of Theorem 5, the requirement that $o(\lambda) = \delta$ can be removed and somehow be replaced by an argument in the style of [7] to control the number of normal measures $\lambda$ carries. We conjecture that failing the existence of inner models for supercompactness hypotheses that also satisfy the appropriate fine structural properties, the answer to both of these questions is no.

In conclusion, we ask whether it is possible to force and obtain a model in which the successor of a singular cardinal is measurable and carries no normal measures. Because we can ensure that the relevant Lévy collapse in a model of AC retains the properties necessary for the construction of the desired witnessing model when forcing over a model in which AC is false, which seems to be difficult if not impossible for the appropriate versions of either supercompact Prikry forcing or supercompact Radin forcing, the methods used in the proof of Case 2 of Theorem 6 do not seem to be applicable.
References


