On Some Questions Concerning Strong Compactness *†

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Abstract

A question of Woodin asks if κ is strongly compact and GCH holds below κ, then must GCH hold everywhere? One variant of this question asks if κ is strongly compact and GCH fails at every regular cardinal $\delta < \kappa$, then must GCH fail at some regular cardinal $\delta \geq \kappa$? Another variant asks if it is possible for GCH to fail at every limit cardinal less than or equal to a strongly compact cardinal κ. We get a negative answer to the first of these questions and positive answers to the second of these questions for a supercompact cardinal κ in the context of the absence of the full Axiom of Choice.

1 Introduction and Preliminaries

In [6, 22.22, page 310], the following question is attributed to Woodin: If κ is strongly compact and GCH holds below κ, then must GCH hold everywhere? Assuming the Axiom of Choice, an easy reflection argument yields that the answer to this question must be yes if κ is supercompact.

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However, when full AC is false, things are very different. Specifically, we have the following theorem from [2], which provides a negative answer to Woodin's question in the context of the absence of AC.

Theorem 1 *Let $V \vDash$ "ZFC + GCH + κ is supercompact". There is then a partial ordering $\mathbb{P} \in V$ and a symmetric inner model N, $V \subseteq N \subseteq V^{\mathbb{P}}$, such that $N \vDash$ "ZF + $\forall \delta < \kappa [DC_\delta]$ + κ is a strong limit cardinal + $\forall \delta < \kappa [2^\delta = \delta^+]$ + κ is supercompact + There is a sequence $\langle A_\alpha \mid \alpha < \kappa^{++} \rangle$ of distinct subsets of κ".*

Woodin's question may be inverted to produce related questions concerning strongly compact cardinals and GCH. In particular, one may ask if κ is strongly compact and GCH fails at every regular cardinal $\delta < \kappa$, then must GCH fail at some regular cardinal $\delta \geq \kappa$? As in Woodin's original question, a simple reflection argument yields that the answer to this question must be yes if κ is supercompact. On the other hand, it is also possible to ask about the possibility of κ being strongly compact and GCH failing at every limit cardinal $\delta \leq \kappa$. Of course, by Solovay's celebrated theorem [10], GCH must always hold at any singular strong limit cardinal above a strongly compact cardinal κ. Consequently, a simple reflection argument now shows that the answer to this question must be no if κ is supercompact.

The purpose of this paper is to provide answers to these questions in the context of the absence of full AC for a supercompact cardinal κ. We show that as in [2], it is possible to get a negative answer to the first of the above questions. On the other hand, it is also possible to get a positive answer to the second of the above questions. Specifically, we prove the following two theorems, where we adopt as our terminology that when AC is false and δ is a cardinal[1], "GCH holds at δ" means that there is an injection $f : \delta^+ \longrightarrow \wp(\delta)$, and for every cardinal $\lambda > \delta^+$, there is no injection $f : \lambda \longrightarrow \wp(\delta)$. Similarly, in a choiceless context, "GCH fails at δ" means that for some cardinal $\lambda > \delta^+$, there is an injection $f : \lambda \longrightarrow \wp(\delta)$.

Theorem 2 *Let $V \vDash$ "ZFC + GCH + κ is supercompact". There is then a partial ordering $\mathbb{P} \in V$ and a symmetric inner model N, $V \subseteq N \subseteq V^{\mathbb{P}}$, such that $N \vDash$ "ZF + $\forall \delta < \kappa [DC_\delta]$ + κ is a strong*

[1]For the purposes of this paper, all cardinals will be well-ordered, i.e., will be alephs.

limit cardinal + κ is supercompact + Every successor cardinal is regular + $\forall \delta < \kappa$[If δ is regular, then $2^\delta = \delta^{++}$, but if δ is singular, then $2^\delta = \delta^+$] + GCH holds at every (regular or singular) cardinal $\delta \geq \kappa$".

Theorem 3 *Let $V \vDash$ "ZFC + GCH + κ is supercompact". There is then a partial ordering $\mathbb{P} \in V$ and a symmetric inner model N, $V \subseteq N \subseteq V^{\mathbb{P}}$, such that $N \vDash$ "ZF + $\neg AC_\omega$ + κ is a limit cardinal + κ is supercompact + Every successor cardinal is regular + GCH fails at every limit cardinal $\delta \leq \kappa$ + GCH holds at every (regular or singular) cardinal $\delta > \kappa$".*

We take this opportunity to make a few brief remarks concerning Theorems 2 and 3. Note that in the absence of full AC, κ being supercompact means that for every cardinal $\lambda \geq \kappa$, $P_\kappa(\lambda)$ carries a κ-additive, fine, normal ultrafilter, and κ being strongly compact means that for every cardinal $\lambda \geq \kappa$, $P_\kappa(\lambda)$ carries a κ-additive, fine (not necessarily normal) ultrafilter. Consequently, the conclusions of Theorems 2 and 3 remain valid, with "κ is strongly compact" replacing "κ is supercompact". Note also that as [5, Example 15.57, pages 259–260] shows, when AC is false, it is possible for successor cardinals to be singular. Thus, the fact that every successor cardinal is regular in the models witnessing the conclusions of Theorems 2 and 3 is especially significant. In addition, in Theorem 2, in direct analogy to Theorem 1, it will literally be the case that "κ is a strong limit cardinal", "For every regular cardinal $\delta < \kappa$, $2^\delta = \delta^{++}$", and "For every singular cardinal $\delta < \kappa$, $2^\delta = \delta^+$" mean the same thing as when AC is true. In Theorem 3, however, this won't be the situation. More specifically, GCH holding and failing will be in the weaker sense described above, although as is the case when AC is true, κ remains a limit cardinal.[2] Finally, as each of our theorems shows, when AC is false, a supercompact cardinal need not possess its full reflection properties.

We mention very briefly some preliminary information. We assume a basic knowledge of set theoretic terminology and large cardinals and forcing, as provided, e.g., by [5]. In particular, when $\mathbb{P}$ is our forcing partial ordering and G is V-generic over $\mathbb{P}$, we will abuse notation somewhat

[2] As [5, Theorem 21.16, pages 404–406] shows, without the Axiom of Choice, it is possible for large cardinals to be successor cardinals.

and use both $V^{\mathbb{P}}$ and $V[G]$ to denote the generic extension by $\mathbb{P}$. We will also frequently abuse notation by writing x instead of $\check{x}$ for ground model sets. We note in addition that for κ a regular cardinal and α an ordinal, $\mathrm{Add}(\kappa, \alpha)$ is the standard partial ordering for adding α many Cohen subsets of κ, i.e., $\mathrm{Add}(\kappa, \alpha) = \{f : \kappa \times \alpha \to \{0, 1\} \mid |\mathrm{dom}(f)| < \kappa\}$, ordered by inclusion. For κ a regular cardinal and $\lambda > \kappa$ an inaccessible cardinal, $\mathrm{Coll}(\kappa, {<}\lambda)$ is the standard Lévy collapse partial ordering for collapsing λ to κ^+, i.e., $\mathrm{Coll}(\kappa, {<}\lambda) = \{f : \kappa \times \lambda \to \lambda \mid |\mathrm{dom}(f)| < \kappa$, and for every $\langle \alpha, \beta \rangle \in \mathrm{dom}(f)$, $f(\langle \alpha, \beta \rangle) < \beta\}$, ordered by inclusion.

2 The Proofs of Theorems 2 and 3

We turn now to the proof of Theorem 2. We will be constructing a symmetric model of "ZF + $\forall \delta < \kappa [\mathrm{DC}_\delta]$" in which κ is supercompact, κ is a strong limit cardinal, every successor cardinal is regular, GCH fails at every regular cardinal $\delta < \kappa$, and GCH holds at all other cardinals.

Proof: The proof of Theorem 2 will be similar to the proof of Theorem 1 found in [2]. We will therefore freely quote (sometimes verbatim when appropriate) from [2].

Let $V \vDash$ "ZFC + GCH + κ is supercompact". Let $\langle \delta_\alpha \mid \alpha < \kappa \rangle$ enumerate in increasing order the regular cardinals less than κ. For each ordinal $\alpha < \kappa$, let $\mathbb{P}_\alpha = \mathrm{Add}(\delta_\alpha, \delta_\alpha^{++})$. The partial ordering $\mathbb{P}$ with which we force is then the Easton support product $\prod_{\alpha < \kappa} \mathbb{P}_\alpha$.

Let G be V-generic over $\mathbb{P}$. The full generic extension $V[G]$ is not our desired model N. In order to define N, we first let G_α for any $\alpha < \kappa$ be the projection of G onto $\prod_{\beta < \alpha} \mathbb{P}_\beta = \mathbb{Q}_\alpha$. By the Product Lemma, G_α is V-generic over $\mathbb{Q}_\alpha$. We can now intuitively describe N as the least model of ZF extending V which contains, for each $\alpha < \kappa$, the set G_α.

In order to define N more formally, let $\mathcal{L}_1$ be the ramified sublanguage of the forcing language $\mathcal{L}$ with respect to $\mathbb{P}$ which contains symbols $\check{v}$ for each $v \in V$, a unary predicate symbol $\check{V}$ (to be interpreted $\check{V}(\check{v})$ iff $v \in V$), and symbols $\dot{G}_\alpha$ for each ordinal $\alpha < \kappa$. N can then be defined inside $V[G]$ as follows.

$$N_0 = \emptyset.$$

$$N_\lambda = \bigcup_{\alpha < \lambda} N_\alpha \text{ if } \lambda \text{ is a limit ordinal.}$$

$$N_{\alpha+1} = \left\{ x \subseteq N_\alpha \;\middle|\; \begin{array}{l} x \text{ is definable over the model } \langle N_\alpha, \in, c\rangle_{c \in N_\alpha} \\ \text{via a term } \tau \in \mathcal{L}_1 \text{ of rank } \leq \alpha \end{array} \right\}.$$
$$N = \bigcup_{\alpha \in \mathrm{Ord}^V} N_\alpha.$$

Standard arguments show $N \vDash \mathrm{ZF}$.

Lemma 2.1 *Let λ be an ordinal. If $x \subseteq \lambda$, $x \in N$, then $x \in V[G_\alpha]$ for some $\alpha < \kappa$.*

Proof: We slightly modify the proof of [2, Lemma 1]. Let τ be a term for x such that $p \Vdash$ "$\tau \subseteq \lambda$". Without loss of generality, by coding if necessary, we can assume τ mentions only one term of the form $\dot{G}_\alpha$ for some $\alpha < \kappa$.

For $q \in \mathbb{P}$, $q = \langle q_\beta \mid \beta < \kappa\rangle$, define $q \restriction \alpha = \langle q^*_\beta \mid \beta < \kappa\rangle$ by $q^*_\beta = q_\beta$ if $\beta < \alpha$ and $q^*_\beta = 0$ (the trivial condition) otherwise. We can now define a term σ by $q \Vdash$ "$\gamma \in \sigma$" iff q extends p and $q \restriction \alpha \Vdash$ "$\gamma \in \tau$". It is clear that $p \Vdash$ "$\sigma \subseteq \tau$". We show in addition that $p \Vdash$ "$\tau \subseteq \sigma$".

To see that this is true, let q extending p, $q = \langle q_\beta \mid \beta < \kappa\rangle$ be such that $q \Vdash$ "$\gamma \in \tau$", and assume towards a contradiction that $q \restriction \alpha \not\Vdash$ "$\gamma \in \tau$". Let r extending $q \restriction \alpha$, $r = \langle r_\beta \mid \beta < \kappa\rangle$ be such that $r \Vdash$ "$\gamma \notin \tau$". If we define $s = \langle s_\beta \mid \beta < \kappa\rangle$ by $s_\beta = r_\beta$ for $\beta < \alpha$ and $s_\beta = q_\beta$ otherwise, then by definition, s extends q and $s \Vdash$ "$\gamma \in \tau$".

Let r_β and s_β be such that r_β and s_β are incompatible. Since $r_\beta, s_\beta \in \mathbb{P}_\beta$ and $\mathbb{P}_\beta = \mathrm{Add}(\delta_\beta, \delta^{++}_\beta)$, there is an automorphism $\psi_\beta : \mathbb{P}_\beta \to \mathbb{P}_\beta$ generated by a permutation of δ_β such that $\psi_\beta(r_\beta)$ is compatible with s_β. (Any $t \in \mathrm{Add}(\delta_\beta, \delta^{++}_\beta)$ is a collection of ordered triples of the form $\langle \xi_0, \xi_1, \xi_2\rangle$, where $\xi_0 < \delta_\beta$, $\xi_1 < \delta^{++}_\beta$, and $\xi_2 \in \{0, 1\}$. This means that we can let t's first domain $\mathrm{dom}_1(t) = \{\xi < \delta_\beta \mid \exists \xi_1 < \delta^{++}_\beta \exists \xi_2 \in \{0,1\}[\langle \xi, \xi_1, \xi_2\rangle \in t]\}$. Let $\eta < \delta_\beta$ be an ordinal greater than $\max(\sup(\mathrm{dom}_1(s_\beta)), \sup(\mathrm{dom}_1(r_\beta)))$. η exists since for any condition $t \in \mathbb{P}_\beta$, $|\mathrm{dom}_1(t)| < \delta_\beta$ and δ_β is a regular cardinal. If $\langle \rho_i \mid i < \zeta\rangle$ enumerates $\mathrm{dom}_1(r_\beta)$ and $\langle \rho'_i \mid i < \zeta\rangle$ enumerates the first ζ ordinals greater than η, then $\psi^*_\beta : \delta_\beta \to \delta_\beta$ given by $\psi^*_\beta(\rho_i) = \rho'_i$, $\psi^*_\beta(\rho'_i) = \rho_i$, and ψ^*_β is the identity otherwise is the desired permutation. The automorphism ψ_β is defined by applying ψ^*_β to each element of a condition's first domain, i.e., for $t \in \mathbb{P}_\beta$, $\psi_\beta(t) = \{\langle \psi^*_\beta(\xi_0), \xi_1, \xi_2\rangle \mid \langle \xi_0, \xi_1, \xi_2\rangle \in t\}$.) Thus, if $\pi = \langle \pi_\beta \mid \beta < \kappa\rangle$ is defined by $\pi_\beta = \psi_\beta$ if r_β and s_β are incompatible and ψ_β is as just described and π_β is the identity otherwise, π generates an automorphism of $\mathbb{P}$ such that $\pi(r)$ is compatible with s.

Note now that π_β is the identity for $\beta < \alpha$. Since terms for ground model sets and terms mentioning only $\dot{G}_\alpha$ can be assumed to be invariant under automorphisms of $\mathbb{P}$ not changing the value of G_α, $\pi(r) \Vdash$ "$\gamma \notin \tau$", $\pi(r)$ is compatible with s, and $s \Vdash$ "$\gamma \in \tau$". This contradiction means that $q \restriction \alpha \Vdash$ "$\gamma \in \tau$", i.e., $p \Vdash$ "$\tau \subseteq \sigma$", i.e., $p \Vdash$ "$\tau = \sigma$". Since σ can clearly be realized in $V[G_\alpha]$, $x \in V[G_\alpha]$. This completes the proof of Lemma 2.1.

$\square$

Lemma 2.2 $N \models$ "$\forall \delta < \kappa[\mathrm{DC}_\delta]$".

Proof: The proof of Lemma 2.2 is identical to the proof of [2, Lemma 2]. For completeness, we present it here. Fix $\delta < \kappa$ a cardinal in N. Recall that DC_δ is the statement that whenever X is a set and $R \subseteq [X]^{<\delta} \times X$ is a relation such that for all $\vec{y} \in [X]^{<\delta}$, there is $z \in X$ such that $\vec{y}\ R\ z$, then there is a δ sequence $\vec{Y}$ such that for all $\alpha < \delta$, $\vec{Y} \restriction \alpha\ R\ Y(\alpha)$. Consequently, working inductively, assume $p \Vdash$ "$\dot{X} \in \dot{N}$ is a set, $\dot{R} \in \dot{N}$, $\dot{R} \subseteq [\dot{X}]^{<\delta} \times \dot{X}$ is a relation, $\langle \tau_\alpha \mid \alpha < \beta < \delta \rangle \in \dot{N}$ is a sequence of elements of $\dot{X}$, and for $\langle \tau_\alpha \mid \alpha < \gamma < \beta \rangle$, $\langle \tau_\alpha \mid \alpha < \gamma \rangle\ \dot{R}\ \tau_\gamma$". We show how to define τ_β. Work in V. Let $\eta = \sup(\{\alpha \mid \exists \gamma < \beta[\dot{G}_\alpha$ occurs in $\tau_\gamma]\})$. Since each τ_γ for $\gamma < \beta$ can be assumed to be an element of $\mathcal{L}_1$, and since κ is a regular limit cardinal, $\eta < \kappa$, so $\langle \tau_\alpha \mid \alpha < \beta \rangle$ can be defined using only $\dot{G}_\eta$ and hence is an element of $\mathcal{L}_1$. By AC in V, since $\mathbb{P}$ is an Easton support product of the appropriate Cohen partial orderings, $\mathbb{P}$ is κ-c.c. Thus, again by AC in V, there is $\mathcal{B}$ with $|\mathcal{B}| < \kappa$, $\mathcal{B} = \{\langle p_\rho, \sigma_\rho \rangle \mid \rho < \gamma^* < \kappa\}$ such that $\mathcal{A} = \{p_\rho \mid \rho < \gamma^*\}$ forms a maximal antichain of conditions extending p and $p_\rho \Vdash$ "$\langle \tau_\alpha \mid \alpha < \beta \rangle\ \dot{R}\ \sigma_\rho$". As before, $\eta^* = \sup(\{\alpha \mid \exists \gamma < \gamma^*[\dot{G}_\alpha$ occurs in $\sigma_\gamma]\})$ is such that $\eta^* < \kappa$, meaning $\mathcal{B}$ can be used to define a term $\tau_\beta \in \mathcal{L}_1$ such that $p \Vdash$ "$\langle \tau_\alpha \mid \alpha < \beta \rangle\ \dot{R}\ \tau_\beta$". Since $\delta < \kappa$, as before, $\langle \tau_\alpha \mid \alpha < \delta \rangle \in \mathcal{L}_1$. By the fact $\langle \tau_\alpha \mid \alpha < \delta \rangle$ can be realized in N, $\langle \tau_\alpha \mid \alpha < \delta \rangle$ will denote in N a DC_δ sequence for $\dot{R}$ and $\dot{X}$. This completes the proof of Lemma 2.2.

$\square$

Lemma 2.3 $N \models$ "*κ is a limit cardinal + Every successor cardinal is regular*".

Proof: Standard arguments (see [5]) in conjunction with the fact that $\mathbb{P}$ is the Easton support product of $\text{Add}(\delta_\alpha, \delta_\alpha^{++})$ where $\alpha < \kappa$ show that V and $V[G]$ have the same cardinals and cofinalities. Therefore, since $V \subseteq N \subseteq V[G]$, N also has the same cardinals and cofinalities as do V and $V[G]$. In particular, $N \vDash$ "κ is a limit cardinal + Every successor cardinal is regular". This completes the proof of Lemma 2.3.

□

Lemma 2.4 *$N \vDash$ "$\forall \delta < \kappa$[If δ is regular, then $2^\delta = \delta^{++}$, but if δ is singular, then $2^\delta = \delta^+$]".*

Proof: Let $\delta < \kappa$ be a (regular or singular) cardinal. Let λ be the least inaccessible cardinal greater than δ. Write $\mathbb{P} = \mathbb{Q}_\lambda \times \mathbb{Q}^\lambda$ and $G = G_\lambda \times G^\lambda$, where $\mathbb{Q}^\lambda = \prod_{\lambda \le \alpha < \kappa} \mathbb{P}_\alpha$ and G^λ is the projection of G onto $\mathbb{Q}^\lambda$. Since $V[G^\lambda]$ and V contain the same bounded subsets of λ and $V[G_\lambda] \subseteq N$, it suffices to show that $V[G_\lambda] \vDash$ "$2^\delta = \delta^{++}$ if δ is regular, but $2^\delta = \delta^+$ if δ is singular". However, once again, standard arguments (see [5]) in conjunction with the fact that $\mathbb{Q}_\lambda$ is the Easton support product of $\text{Add}(\delta_\alpha, \delta_\alpha^{++})$ where $\alpha < \lambda$ yield that $V[G_\lambda] \vDash$ "$2^\delta = \delta^{++}$ if δ is regular, but $2^\delta = \delta^+$ if δ is singular". This completes the proof of Lemma 2.4.

□

We remark that Lemmas 2.3 and 2.4 show $N \vDash$ "κ is a strong limit cardinal". Also, note that $N \vDash \neg \text{AC}_\kappa$. To see this, we follow the remark found after the proof of [2, Lemma 4]. Define in N for each $\alpha < \kappa$ the set $X_\alpha = \{x \subseteq \delta_\alpha^{++} \mid x \text{ codes a } \delta_\alpha^{++} \text{ sequence of subsets of } \delta_\alpha\}$.[3] Although $\langle X_\alpha \mid \alpha < \kappa \rangle \in N$, $(\prod_{\alpha < \kappa} X_\alpha)^N = \emptyset$. This follows since an element y of $(\prod_{\alpha < \kappa} X_\alpha)^N$ may be thought of as a set of ordinals, so by Lemma 2.1, $y \in V[G_\beta]$ for some $\beta < \kappa$. This, however, is impossible, as $|\mathbb{Q}_\beta| < \kappa$, so a final segment of the sequence of regular cardinals below κ satisfies GCH in $V[G_\beta]$.

Lemma 2.5 *$N \vDash$ "GCH holds at every (regular or singular) cardinal $\delta \ge \kappa$".*

[3]By the proof of Lemma 2.3, δ_α is regular in V, N, and $V[G]$.

Proof: Since $|\mathbb{P}| = \kappa$ and $V \models$ GCH, $V[G] \models$ "GCH holds at every (regular or singular) cardinal $\delta \geq \kappa$". The fact that $V \subseteq N \subseteq V[G]$ then immediately implies that $N \models$ "For every (regular or singular) cardinal $\delta \geq \kappa$, there is an injection $f : \delta^+ \to \wp(\delta)$, but for every (regular or singular) cardinal $\delta \geq \kappa$ and every cardinal $\lambda > \delta^+$, there is no injection $f : \lambda \to \wp(\delta)$". This completes the proof of Lemma 2.5.

□

Lemma 2.6 *$N \models$ "κ is supercompact".*

Proof: The proof of Lemma 2.6 is virtually identical to the proof of [2, Lemma 5]. As before, for completeness, we include it here. Fix $\lambda \geq \kappa$ and $\mathcal{U}$ a κ-additive, fine, normal ultrafilter over $P_\kappa(\lambda)$ in V. Working in N, let $\mathcal{U}' = \{x \subseteq (P_\kappa(\lambda))^N \mid \exists y \in \mathcal{U}[y \subseteq x]\}$. We show that $N \models$ "$\mathcal{U}'$ is a κ-additive, fine, normal ultrafilter over $(P_\kappa(\lambda))^N$".

To see this, fix $x \subseteq (P_\kappa(\lambda))^N$, $x \in N$, and let τ be a term for x mentioning only $\dot{G}_\alpha$. Contained in the proof of Lemma 2.1 is the fact that $y = \{p \in (P_\kappa(\lambda))^V \mid p \in x\}$ is actually a set in $V[G_\alpha]$. This follows since the proof of Lemma 2.1 really shows that for a term τ^* as just described and an element $z \in V$, the statement "$z \in \tau^*$" is decidable in $V[G_\alpha]$. Thus, since $|\mathbb{Q}_\alpha| < \kappa$, the Lévy-Solovay arguments [8] show that in $V[G_\alpha] \subseteq N$, either y or $(P_\kappa(\lambda))^V - y$ contains a set in $\mathcal{U}$. Further, if $N \models$ "$\langle x_\beta \mid \beta < \gamma < \kappa \rangle$ is a sequence such that each $x_\beta \in \mathcal{U}'$", then let τ_1 be such that τ_1 denotes $\langle x_\beta \mid \beta < \gamma < \kappa \rangle$ and mentions only $\dot{G}_\alpha$. The methods of [8] yield that for every $\beta < \gamma$, there is a set $y_\beta \in \mathcal{U}$ definable in V such that $\Vdash_{\mathbb{Q}_\alpha}$ "$y_\beta \subseteq \{p \in (P_\kappa(\lambda))^V \mid p \in \dot{x}_\beta\}$". Since $y^* = \bigcap_{\beta<\gamma} y_\beta \in \mathcal{U}$, $N \models$ "$\exists y \in \mathcal{U}[y \subseteq \bigcap_{\beta<\gamma} x_\beta]$". Finally, if $N \models$ "$f : (P_\kappa(\lambda))^N \to \lambda$ is a choice function", then if $\dot{f}$ denotes f and mentions only $\dot{G}_\alpha$, it is possible to define in $V[G_\alpha] \subseteq N$ the function $g = f \restriction (P_\kappa(\lambda))^V$. Once more, the results of [8] show that for some $x \in \mathcal{U}$, $V[G_\alpha] \models$ "g is constant on x". Thus, $N \models$ "$\mathcal{U}'$ is a κ-additive, fine, normal ultrafilter over $(P_\kappa(\lambda))^N$". This completes the proof of Lemma 2.6.

□

Lemmas 2.1 – 2.6 and the intervening remarks complete the proof of Theorem 2.

$\square$

Having completed the proof of Theorem 2, we turn now to the proof of Theorem 3. We will be constructing a symmetric model of "ZF + $\neg\mathrm{AC}_\omega$" in which κ is supercompact, κ is a limit cardinal, every successor cardinal is regular, GCH fails at every limit cardinal $\delta \le \kappa$, and GCH holds at all cardinals above κ.

Proof: Let $V \models$ "ZFC + GCH + κ is supercompact". We define a partial ordering $\overline{\mathbb{Q}}$ such that $V^{\overline{\mathbb{Q}}} = \overline{V} \models$ "ZFC + κ is supercompact + $2^\kappa = \kappa^{++}$ + $2^\delta = \delta^+$ for every cardinal $\delta \ge \kappa^+$ + There is a club $C \subseteq \kappa$ composed of inaccessible cardinals and their limits with $2^\delta = 2^{\delta^+} = \delta^{++}$ for every $\delta \in C$". To obtain $\overline{\mathbb{Q}}$, let $\mathbb{Q}_1$ be Laver's partial ordering of [7] which makes κ's supercompactness indestructible under κ-directed closed forcing. Since $\mathbb{Q}_1$ may be defined so that $|\mathbb{Q}_1| = \kappa$, it is then the case that $V^{\mathbb{Q}_1 * \dot{\mathrm{Add}}(\kappa,\kappa^{++})} = V_2 \models$ "ZFC + κ is supercompact + $2^\kappa = \kappa^{++}$ + $2^\delta = \delta^+$ for every cardinal $\delta \ge \kappa^+$". Let $\mathbb{Q}_3 \in V_2$ be Radin forcing defined over κ using one repeat point (see either [4] or [9] for the precise definition of $\mathbb{Q}_3$). Standard facts about Radin forcing (see [1], [4], and [9]) then show that $V_2^{\mathbb{Q}_3} = V^{\mathbb{Q}_1 * \dot{\mathrm{Add}}(\kappa,\kappa^{++}) * \dot{\mathbb{Q}}_3} = \overline{V} \models$ "ZFC + κ is supercompact + $2^\kappa = \kappa^{++}$ + $2^\delta = \delta^+$ for every cardinal $\delta \ge \kappa^+$ + There is a club $C \subseteq \kappa$ composed of inaccessible cardinals and their limits with $2^\delta = 2^{\delta^+} = \delta^{++}$ for every $\delta \in C$".

With an abuse of notation, we now let $\overline{V} = V$. Let $\langle \kappa_i \mid i < \kappa \rangle \in V$ be the continuous, increasing enumeration of $C \cup \{\omega\}$. For $i < \kappa$, let $\mathbb{P}_i = \mathrm{Coll}(\kappa_i^{++}, {<}\kappa_{i+1})$. The partial ordering $\mathbb{P}$ with which we force is then the Easton support product $\mathbb{P} = \prod_{i<\kappa} \mathbb{P}_i$.

Let G be V-generic over $\mathbb{P}$. $V[G]$, being a model of AC, is once more not our desired model N. In order to define N, we first note that as before, by the Product Lemma, for $i < \kappa$, G_i, the projection of G onto $\mathbb{P}_i$, is V-generic over $\mathbb{P}_i$. Again by the Product Lemma, $G_I = \prod_{i \in I} G_i$ is V-generic over $\mathbb{P}_I = \prod_{i\in I} \mathbb{P}_i$. We can now intuitively describe N as the least model of ZF extending V which contains, for each finite set of ordinals $I \subseteq \kappa$, the set G_I.

In order to define N more formally, let $\mathcal{L}_1$ be the ramified sublanguage of the forcing language $\mathcal{L}$ with respect to $\mathbb{P}$ which contains symbols $\check{v}$ for each $v \in V$, a unary predicate symbol $\check{V}$ (to be interpreted $\check{V}(\check{v})$ iff $v \in V$), and symbols $\dot{G}_I$ for each finite set of ordinals $I \subseteq \kappa$. N can then be defined inside $V[G]$ as follows.

$$N_0 = \emptyset.$$

$$N_\lambda = \bigcup_{\alpha<\lambda} N_\alpha \text{ if } \lambda \text{ is a limit ordinal.}$$

$$N_{\alpha+1} = \left\{ x \subseteq N_\alpha \;\middle|\; \begin{array}{l} x \text{ is definable over the model } \langle N_\alpha, \in, c\rangle_{c \in N_\alpha} \\ \text{via a term } \tau \in \mathcal{L}_1 \text{ of rank} \leq \alpha \end{array} \right\}.$$

$$N = \bigcup_{\alpha \in \mathrm{Ord}^V} N_\alpha.$$

As in the proof of Theorem 2, standard arguments show $N \vDash \mathrm{ZF}$.[4]

Lemma 2.7 *Let λ be an ordinal. If $x \subseteq \lambda$, $x \in N$, then $x \in V[G_I]$ for some finite set of ordinals $I \subseteq \kappa$.*

Proof: Suppose $i < \kappa$. It is a standard fact (see, e.g., [3, Lemma 5.2]) that since $\mathbb{P}_i$ is a version of the Lévy collapse, for any $p, q \in \mathbb{P}_i$, there is an automorphism $\pi_i : \mathbb{P}_i \to \mathbb{P}_i$ such that $\pi_i(p)$ is compatible with q. The proof of Lemma 2.7 is now essentially the same as the proof of Lemma 2.1, with each occurrence of "α" in Lemma 2.1 replaced by an occurrence of "I". This completes the proof of Lemma 2.7.

□

Lemma 2.8 $N \vDash$ *"Every successor cardinal is regular".*

Proof: Suppose first that $N \vDash$ "$\delta > \kappa$ is a successor cardinal". Since $\mathbb{P} = \prod_{i<\kappa} \mathbb{P}_i$ is an Easton support product, $\mathbb{P}$ is κ-c.c. This means that V and $V[G]$ have the same cardinals and cofinalities at and above κ. Therefore, since $V \subseteq N \subseteq V[G]$, V, N, and $V[G]$ all have the same cardinals and cofinalities at and above κ. In particular, $N \vDash$ "δ is regular".

Suppose next that $N \vDash$ "$\delta < \kappa$ is a (successor or limit) cardinal". We claim that either $\delta = \kappa_i$, $\delta = (\kappa_i^+)^V$, or $\delta = (\kappa_i^{++})^V$ for some $i < \kappa$. To see this, since C is club in κ, we can let $k < \kappa$ be such that κ_{k+1} is the least member of C greater than δ. If the claim is false, then because $\delta \neq \kappa_k$, $\delta \neq (\kappa_k^+)^V$, and $\delta \neq (\kappa_k^{++})^V$, $\delta \in ((\kappa_k^{++})^V, \kappa_{k+1})$. However, since G_k is V-generic over

[4] Although defining N using G_i for every $i < \kappa$ is equivalent to our presentation, it is not as useful for the arguments we are about to give.

$\mathrm{Coll}(\kappa_k^{++}, {<}\kappa_{k+1})$, $V[G_k] \models$ "δ is not a cardinal". Consequently, because $V[G_k] \subseteq N$, $N \models$ "δ is not a cardinal", a contradiction to the assumption that $\delta \neq \kappa_k$, $\delta \neq (\kappa_k^+)^V$, and $\delta \neq (\kappa_k^{++})^V$.

We next claim that for any $i < \kappa$, $N \models$ "κ_i, $(\kappa_i^+)^V$, and $(\kappa_i^{++})^V$ are all cardinals". However, since any collapse map f would have to be coded by a set of ordinals, if this were false, then by Lemma 2.7, there would have to be some finite set of ordinals $I \subseteq \kappa$ such that $f \in V[G_I] = V[\prod_{j \in I} G_j]$. Because $\prod_{j \in I} G_j$ is V-generic over $\prod_{j \in I} \mathbb{P}_j = \prod_{j \in I} \mathrm{Coll}(\kappa_j^{++}, {<}\kappa_{j+1})$ and I is finite, this is impossible.

We now know that if $N \models$ "$\delta < \kappa$ is a successor cardinal", then there must be some $i < \kappa$ such that either $\delta = \kappa_i$, $\delta = (\kappa_i^+)^V$, or $\delta = (\kappa_i^{++})^V$. Assume that $\delta = \kappa_i$. It must be the case that i is a successor ordinal. This is since if i were a limit ordinal, then $N \models$ "$\kappa_i = \sup_{k<i} \kappa_k$ and κ_k for $k < i$ is a cardinal", i.e., $N \models$ "κ_i is a limit cardinal". Consequently, because i is a successor ordinal, κ_i is a successor member of the Radin generic club C. This means that $V \models$ "κ_i is inaccessible", so in particular, $V \models$ "κ_i is a regular cardinal". If $N \models$ "κ_i is singular", then let $S \subseteq \kappa_i$, $S \in N$ be a witness to this fact. Again by Lemma 2.7, there must be some finite set of ordinals $I \subseteq \kappa$ such that $f \in V[G_I] = V[\prod_{j \in I} G_j]$. Once more, because $\prod_{j \in I} G_j$ is V-generic over $\prod_{j \in I} \mathbb{P}_j = \prod_{j \in I} \mathrm{Coll}(\kappa_j^{++}, {<}\kappa_{j+1})$ and I is finite, this is impossible. Hence, $N \models$ "δ is a regular cardinal".

Assume finally that either $\delta = (\kappa_i^+)^V$ or $\delta = (\kappa_i^{++})^V$. Clearly, since $V \models$ ZFC, it is also true that $V \models$ "κ_i^+ and κ_i^{++} are regular cardinals". The same contradiction as obtained in the preceding paragraph again yields that $N \models$ "δ is a regular cardinal". This completes the proof of Lemma 2.8.

□

As has just been noted in the the proof of Lemma 2.8, $N \models$ "$\delta < \kappa$ is a cardinal" iff either $\delta = \kappa_i$, $\delta = (\kappa_i^+)^V$, or $\delta = (\kappa_i^{++})^V$ for some $i < \kappa$. Therefore, since $\kappa = \sup_{i<\kappa} \kappa_i$, $N \models$ "κ is a limit cardinal".

Lemma 2.9 $N \models$ "*GCH fails at every limit cardinal* $\delta \leq \kappa$*".*

Proof: Suppose first that $\delta = \kappa$. As noted in the proof of Lemma 2.8, V, N, and $V[G]$ all have the same cardinals and cofinalities at and above κ. In addition, $V \models$ "$2^\kappa = \kappa^{++}$". Therefore, because $V \subseteq N$, $N \models$ "There is an injection $f : \kappa^{++} \to \wp(\kappa)$".

Suppose now that $\delta < \kappa$. By the second paragraph of the proof of Lemma 2.8, there must be some $i < \kappa$ such that either $\delta = \kappa_i$, $\delta = (\kappa_i^+)^V$, or $\delta = (\kappa_i^{++})^V$. Since $V \subseteq N$, it cannot be the case that either $\delta = (\kappa_i^+)^V$ or $\delta = (\kappa_i^{++})^V$. This means we can let $i < \kappa$ be such that $\delta = \kappa_i$. As $\delta \in C$, $V \models$ "$2^\delta = \delta^{++}$". Consequently, since $V \subseteq N$ and the third paragraph of the proof of Lemma 2.8 implies that both $\delta = (\kappa_i^+)^V$ and $\delta = (\kappa_i^{++})^V$ remain cardinals in N, $N \models$ "There is an injection $f : \delta^{++} \to \wp(\delta)$". This completes the proof of Lemma 2.9.

□

Lemma 2.10 $N \models$ "*GCH holds at every (regular or singular) cardinal* $\delta > \kappa$".

Proof: Since V, N, and $V[G]$ all have the same cardinals and cofinalities at and above κ, $|\mathbb{P}| = \kappa$, and $V \models$ "$2^\delta = \delta^+$ for every cardinal $\delta \geq \kappa^+$", $V[G] \models$ "$2^\delta = \delta^+$ for every cardinal $\delta \geq \kappa^+$". Therefore, again because $V \subseteq N \subseteq V[G]$, $N \models$ "For every (regular or singular) cardinal $\delta > \kappa$, there is an injection $f : \delta^+ \to \wp(\delta)$, and for every cardinal $\lambda > \delta^+$, there is no injection $f : \lambda \to \wp(\delta)$". This completes the proof of Lemma 2.10.

□

Lemma 2.11 $N \models \neg AC_\omega$.

Proof: We follow the remarks given after the proofs of [2, Lemma 4] and Lemma 2.4, making the appropriate modifications in proof. Define in N for each $n < \omega$ the set $X_n = \{x \subseteq (\kappa_n^{++})^V \mid x$ codes a well-ordering of $(\kappa_n^{+3})^V$ of order type $(\kappa_n^{++})^V\}$.[5] Although each $X_n \neq \emptyset$ and $\langle X_n \mid n < \omega \rangle \in N$, $(\prod_{n<\omega} X_n)^N = \emptyset$. This follows since an element y of $(\prod_{n<\omega} X_n)^N$ is a set of ordinals, so by Lemma 2.7, $y \in V[G_I] = V[\prod_{i\in I} G_i]$ for some finite set of ordinals $I \subseteq \kappa$. Let m be the maximum integer which is an element of I. Write $I = I_0 \cup I_1$, where $I_0 = \{i \in I \mid i \leq m\}$ and $I_1 = I - I_0 = \{i \in I \mid i >$

[5]Since $(\kappa_n^{++})^V = (\kappa_n^{++})^N$, it is also possible to define X_n in N as $X_n = \{x \subseteq \kappa_n^{++} \mid x$ codes a well-ordering of $(\kappa_n^{+3})^V$ of order type $\kappa_n^{++}\}$.

$m\} = \{i \in I \mid i \geq \omega\}$. By the closure properties of the Lévy collapse, each member of the sequence $\langle (\kappa_n^{+3})^V \mid n < \omega \rangle$ remains a cardinal in $V[\prod_{i \in I_1} G_i]$. Since $\prod_{i \in I_0} \mathbb{P}_i = \prod_{i \in I_0} \mathrm{Coll}(\kappa_i^{++}, {<}\kappa_{i+1})$ and I_0 is finite, there is some $j < \omega$ such that for all $\ell \geq j$, $|\prod_{i \in I_0} \mathbb{P}_i| < \kappa_\ell$. Thus, a final segment of the sequence $\langle (\kappa_n^{+3})^V \mid n < \omega \rangle$ remains a sequence of cardinals in $V[\prod_{i \in I_1} G_i][\prod_{i \in I_0} G_i] = V[G_I]$, which is impossible. This completes the proof of Lemma 2.11.

$\square$

The proof that $N \vDash$ "κ is supercompact" is the same as in Lemma 2.6, with each occurrence of "α" replaced by an occurrence of "I". Lemmas 2.7 – 2.11 and the intervening remarks consequently complete the proof of Theorem 3.

$\square$

The proof of Lemma 2.9 indicates that for any $i < \kappa$, GCH fails at κ_i. Since it is possible to show that any κ_i for i a successor ordinal is a successor cardinal in N (its predecessor in N must be κ_{i-1}^{++}), there are many successor cardinals below κ violating GCH. We remark that by slightly changing the definition of N, it is possible to obtain a model of ZF + $\neg\mathrm{AC}_\omega$ satisfying the conclusions of Theorem 3 in which GCH holds at every successor cardinal $\delta < \kappa$. Specifically, we have the following theorem.

Theorem 4 *Let $V \vDash$ "ZFC + GCH + κ is supercompact". There is then a partial ordering $\mathbb{P} \in V$ and a symmetric inner model N, $V \subseteq N \subseteq V^{\mathbb{P}}$, such that $N \vDash$ "ZF + $\neg AC_\omega$ + κ is a limit cardinal + κ is supercompact + Every successor cardinal is regular + GCH fails at every limit cardinal $\delta \leq \kappa$ + GCH holds at every (regular or singular) cardinal $\delta > \kappa$, as well as at every successor cardinal $\delta < \kappa$".*

Sketch of Proof: We suppose as in the proof of Theorem 3 that $V \vDash$ "ZFC + κ is supercompact + $2^\kappa = \kappa^{++}$ + $2^\delta = \delta^+$ for every cardinal $\delta \geq \kappa^+$ + There is a club $C \subseteq \kappa$ composed of inaccessible cardinals and their limits with $2^\delta = 2^{\delta^+} = \delta^{++}$ for every $\delta \in C$". Once again, let $\langle \kappa_i \mid i < \kappa \rangle \in V$ be the continuous, increasing enumeration of $C \cup \{\omega\}$. Change the definition of $\mathbb{P}_i$ so that $\mathbb{P}_i = \mathrm{Coll}(\kappa_i, {<}\kappa_{i+1})$ if $i < \kappa$ is either 0 or a successor ordinal, but $\mathbb{P}_i = \mathrm{Coll}(\kappa_i^{++}, {<}\kappa_{i+1})$ if

$i < \kappa$ is a limit ordinal. The remainder of the definition of $\mathbb{P}$ is as before, i.e., $\mathbb{P} = \prod_{i<\kappa} \mathbb{P}_i$ with Easton support. Let G be V-generic over $\mathbb{P}$, and for each $i < \kappa$, let G_i be the projection of G onto $\mathbb{P}_i$. N is then constructed as in the proof of Theorem 3.

The same argument as given in the first paragraph of the proof of Lemma 2.8 shows that $N \vDash$ "Every successor cardinal $\delta > \kappa$ is regular". The proofs of the natural analogues of Lemmas 2.6, 2.7, and 2.10 are as before. The proof of the natural analogue of Lemma 2.11 is as before, with the definition of X_n changed to $X_n = \{x \subseteq \kappa_n \mid x$ codes a well-ordering of $(\kappa_n^+)^V$ of order type $\kappa_n\}$. This shows that $N \vDash$ "ZF + $\neg\mathrm{AC}_\omega$ + GCH holds at every (regular or singular) cardinal $\delta > \kappa$ + κ is supercompact". The natural analogue of the argument found in the second paragraph of the proof of Lemma 2.8 shows that if $N \vDash$ "$\delta < \kappa$ is a (successor or limit) cardinal", then either $\delta = \kappa_i$ for some $i < \kappa$, or for some limit ordinal $i < \kappa$, either $\delta = (\kappa_i^+)^V$ or $\delta = (\kappa_i^{++})^V$. (As in the proof of Lemma 2.8, let $k < \kappa$ be such that κ_{k+1} is the least member of C greater than δ. If this is false, then either $\delta \in ((\kappa_k^{++})^V, \kappa_{k+1})$ if k is a limit ordinal, or $\delta \in (\kappa_k, \kappa_{k+1})$ if k is either a successor ordinal or 0. In each case, in $V[G_k] \subseteq N$ and N, δ is not a cardinal.) The same argument as given in the proof of Lemma 2.8 now shows that κ_i for any $i < \kappa$ and both κ_i^+ and κ_i^{++} for $i < \kappa$ a limit ordinal remain cardinals in N. From this, we may infer as in the proofs of Lemmas 2.8 and 2.9 and the intervening remark that $N \vDash$ "κ is a limit cardinal + Every successor cardinal $\delta < \kappa$ is regular + If $i < \kappa$ is a limit ordinal, then κ_i is a limit cardinal + GCH fails at every limit cardinal $\delta \leq \kappa$".

It remains to show that $N \vDash$ "GCH holds at every successor cardinal $\delta < \kappa$". To see this, suppose first that $\delta = \kappa_i$ for some $i < \kappa$. As we have already observed, i must be a successor ordinal. As a consequence, G_i must be V-generic over $\mathrm{Coll}(\kappa_i, {<}\kappa_{i+1})$, so since $V[G_i] \subseteq N$ and $N \vDash$ "κ_{i+1} is a cardinal", $N \vDash$ "$\kappa_{i+1} = \kappa_i^+ = \delta^+$, and there is an injection $f : \delta^+ \to \wp(\delta)$". Because G_{i+1} is V-generic over $\mathrm{Coll}(\kappa_{i+1}, {<}\kappa_{i+2})$, $V[G_{i+1}] \subseteq N$, and $N \vDash$ "κ_{i+2} is a cardinal", $N \vDash$ "$\kappa_{i+2} = \kappa_{i+1}^+ = \delta^{++}$". Assume now that $N \vDash$ "There is an injection $f : \delta^{++} \to \wp(\delta)$", i.e., that $N \vDash$ "There is an injection $f : \kappa_{i+2} \to \wp(\kappa_i)$". Since $N \subseteq V[G]$, it must therefore be the case that $V[G] \vDash$ "There is an injection $f : \kappa_{i+2} \to \wp(\kappa_i)$". By the properties of the Lévy collapse, however, this is impossible.

Suppose finally that either $\delta = (\kappa_i^+)^V$ or $\delta = (\kappa_i^{++})^V$ where $i < \kappa$ is a limit ordinal. If $\delta = (\kappa_i^+)^V$, then since $V \models$ "$2^{\kappa_i^+} = \kappa_i^{++}$" and $(\kappa_i^{++})^V$ remains a cardinal in N, $(\kappa_i^{++})^V = (\delta^+)^N$, and $N \models$ "There is an injection $f : \delta^+ \to \wp(\delta)$". Because G_i is V-generic over $\mathrm{Coll}(\kappa_i^{++}, {<}\kappa_{i+1})$, $V[G_i] \subseteq N$, and $N \models$ "κ_{i+1} is a cardinal", $N \models$ "$\kappa_{i+1} = ((\kappa_i^{++})^V)^+ = \delta^{++}$". If $\delta = (\kappa_i^{++})^V$, then since G_i is V-generic over $\mathrm{Coll}(\kappa_i^{++}, {<}\kappa_{i+1})$, $V[G_i] \models$ "$((\kappa_i^{++})^V)^+ = \kappa_{i+1}$ and $2^{(\kappa_i^{++})^V} = \kappa_{i+1}$", i.e., $V[G_i] \models$ "$2^\delta = \delta^+$". Because $V[G_i] \subseteq N$ and $N \models$ "κ_{i+1} is a cardinal", $N \models$ "There is an injection $f : \delta^+ \to \wp(\delta)$". As G_{i+1} is V-generic over $\mathrm{Coll}(\kappa_{i+1}, {<}\kappa_{i+2})$, $V[G_{i+1}] \subseteq N$, and $N \models$ "κ_{i+2} is a cardinal", $N \models$ "$\kappa_{i+2} = \kappa_{i+1}^+ = \delta^{++}$". In either case, if $N \models$ "There is an injection $f : \delta^{++} \to \wp(\delta)$", then since $N \subseteq V[G]$, we obtain a contradiction as before to the properties of the Lévy collapse. This completes the sketch of the proof of Theorem 4.

□

3 Concluding Remarks

In conclusion to this paper, we note that it is possible to modify Theorem 2 and its proof so that the behavior of the continuum function at regular cardinals below κ is given by a fixed ground model Easton function. We leave it to readers to fill in the details. However, the methods of this paper do not seem to allow us to use a ground model Easton function to control the behavior of the continuum function on all cardinals at and below κ (in either the strong sense of Theorem 2 or the weaker sense of Theorems 3 and 4) while having GCH hold above κ. We ask if this is possible. In particular, is is possible to construct a model analogous to the ones for either Theorem 2 or Theorem 3 in which GCH fails everywhere below κ? Since AC fails completely in the models witnessing the conclusions of Theorems 3 and 4, we ask if it is possible to construct analogues of these models in which some weak version of AC holds. More generally, we finish by asking if it is possible to prove analogues of Theorems 2 – 4, or the generalizations to which we have just alluded, in the context of the full Axiom of Choice.

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