Normal Measures and Strongly Compact Cardinals *†

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Abstract

We prove four theorems concerning the number of normal measures a non-$(\kappa + 2)$-strong strongly compact cardinal κ can carry.

1 Introduction and Preliminaries

We consider in this paper the number of normal measures a non-$(\kappa + 2)$-strong strongly compact cardinal κ can carry. It follows from a theorem of Solovay [7, Corollary 20.20(i)] that if κ is $(\kappa+2)$-

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strong, then κ is a measurable limit of measurable cardinals with 2^{2^κ} many normal measures over κ, the maximal number of normal measures a measurable cardinal can have. It is known, however, that there can be strongly compact cardinals κ which are not $(\kappa+2)$-strong. A result of Menas [13, Theorem 2.21] shows that if κ is a measurable limit of strongly compact cardinals (which might or might not also be supercompact), then κ itself must be strongly compact. By the arguments of [13, Theorem 2.22], the smallest such κ cannot be $(\kappa+2)$-strong. In addition, Magidor's famous theorem of [11] establishes that it is consistent, relative to the existence of a strongly compact cardinal, for the least strongly compact cardinal κ to be the least measurable cardinal. Under these circumstances, by the previously mentioned theorem of Solovay, κ also cannot be $(\kappa+2)$-strong. The work of Menas and Magidor therefore raises the following

Question: Suppose κ is a strongly compact cardinal which is not $(\kappa+2)$-strong. How many normal measures is it consistent for κ to carry?

In trying to provide answers to this question, we will begin by examining what occurs when the strongly compact cardinals being considered are either the least measurable limit of supercompact cardinals or the least measurable cardinal. Specifically, we will prove the following two theorems.

Theorem 1 *Suppose V is a model of "ZFC + GCH" in which κ is the least measurable limit of supercompact cardinals and $\lambda \geq \kappa^{++}$ is a regular cardinal. There is then a partial ordering $\mathbb{P} \subseteq V$ such that $V^{\mathbb{P}}$ is a model of ZFC in which the following hold:*

- *κ is the least measurable limit of supercompact cardinals.*
- $2^\kappa = \kappa^+$.
- $2^{\kappa^+} = 2^{2^\kappa} = \lambda$.
- *κ carries 2^{2^κ} many normal measures.*

Theorem 2 *Suppose V is a model of "ZFC + GCH" in which κ is the least supercompact cardinal and $\lambda \geq \kappa^{++}$ is a regular cardinal. There is then a partial ordering $\mathbb{P} \subseteq V$ such that $V^{\mathbb{P}}$ is a model of ZFC in which the following hold:*

- *κ is both the least measurable and least strongly compact cardinal.*

- $2^\kappa = \kappa^+$.

- $2^{\kappa^+} = 2^{2^\kappa} = \lambda$.

- *κ carries 2^{2^κ} many normal measures.*

Theorems 1 and 2 handle the case where the non-$(\kappa+2)$-strong strongly compact cardinal κ in question carries 2^{2^κ} many normal measures and $2^\kappa = \kappa^+$. We may also ask if it is possible to have $2^\kappa > \kappa^+$. The next two theorems take care of this situation for certain non-$(\kappa+2)$-strong strongly compact cardinals κ. Specifically, we will also prove the following two theorems.

Theorem 3 *Suppose V is a model of ZFC in which there is a supercompact limit of supercompact cardinals. There is then a model of ZFC containing a strongly compact cardinal κ such that κ satisfies the following properties:*

- *κ is a measurable limit of strongly compact cardinals but is not the least measurable limit of strongly compact cardinals.*

- *κ is not $(\kappa+2)$-strong.*

- $2^\kappa = \kappa^{+17}$.

- $2^{\kappa^{+17}} = 2^{2^\kappa} = \kappa^{+95}$.

- *κ carries 2^{2^κ} many normal measures.*

Theorem 4 *Suppose V is a model of ZFC in which there is a supercompact limit of supercompact cardinals. There is then a model of ZFC containing a strongly compact cardinal κ such that κ satisfies the following properties:*

- *κ is both the least strongly compact cardinal and a measurable limit of strongly compact cardinals.*

- *κ is not $(\kappa + 2)$-strong.*

- $2^\kappa = \kappa^{+17}$.

- $2^{\kappa^{+17}} = 2^{2^\kappa} = \kappa^{+95}$.

- *κ carries 2^{2^κ} many normal measures.*

In Theorems 3 and 4, there is nothing special about "κ^{+17}" and "κ^{+95}". The values for both 2^κ and 2^{2^κ} can be produced by an Easton function F as in [12, Theorem, Section 18, pages 83–88]. In particular, in order to be able both to preserve all supercompact cardinals and control the size of each of 2^δ and 2^{2^δ} for δ an inaccessible cardinal, we require F to have the following properties:

- F's domain is the class of regular cardinals.

- F is definable over V by a Δ_2 formula.

- For all regular cardinals $\delta_1 \le \delta_2$, $F(\delta_1) \le F(\delta_2)$.

- For every regular cardinal δ, $\mathrm{cof}(F(\delta)) > \delta$. In fact:

- For every regular cardinal δ, $F(\delta)$ is regular (something not required in [12], but necessary for our purposes, since we need to be able to control the value of $2^{F(\delta)}$).

However, for comprehensibility and ease of presentation, Theorems 3 and 4 have been stated as written. Also, Theorems 3 and 4 are in some ways "weaker" than Theorems 1 and 2. This is in the sense that in Theorem 3, unlike in Theorem 1, κ is not the least measurable limit of supercompact cardinals. In Theorem 4, unlike in Theorem 2, κ is not the least measurable cardinal. We will discuss this further towards the end of the paper.

Before beginning the proofs of our theorems, we briefly discuss some preliminary information. Essentially, our notation and terminology are standard. When exceptions occur, these will be clearly noted. In particular, when forcing, $q \ge p$ means that *q is stronger than p*. If $\mathbb{P}$ is a notion of forcing for the ground model V and G is V-generic over $\mathbb{P}$, then we will abuse notation somewhat by using both $V[G]$ and $V^{\mathbb{P}}$ to denote the generic extension when forcing with $\mathbb{P}$. We will also occasionally abuse notation by writing x when we actually mean $\dot{x}$ or $\check{x}$.

Suppose κ is a regular cardinal. As in [9], we will say that $\mathbb{P}$ is *κ-directed closed* if every directed subset of $\mathbb{P}$ of size less than κ has an upper bound. For λ any ordinal, the standard partial ordering for adding λ many Cohen subsets of κ will be written as $\mathrm{Add}(\kappa, \lambda)$. It is defined as $\{f \mid f : \kappa \times \lambda \to \{0, 1\}$ is a function such that $|\mathrm{dom}(f)| < \kappa\}$, ordered by $q \geq p$ iff $q \supseteq p$. Note that $\mathrm{Add}(\kappa, \lambda)$ is κ-directed closed.

We presume a basic knowledge and understanding of large cardinals and forcing, as found in, e.g., [7] (see also [8] for additional material on large cardinals), to which we refer readers for further details. We do mention that the cardinal κ is *λ-strongly compact* for $\lambda \geq \kappa$ a cardinal if $P_\kappa(\lambda) = \{x \subseteq \lambda \mid |x| < \kappa\}$ carries a κ-additive, *fine ultrafilter* $\mathcal{U}$ (where $\mathcal{U}$ being *fine* means that for every $\alpha < \lambda$, $\{p \in P_\kappa(\lambda) \mid \alpha \in p\} \in \mathcal{U}$). If $\mathcal{U}$ is in addition *normal* (i.e., if every $f : P_\kappa(\lambda) \to \lambda$ is constant on a $\mathcal{U}$ measure 1 set), then κ is *λ-supercompact*. An equivalent definition for κ being λ-strongly compact is that there is an elementary embedding $j : V \to M$ having critical point κ such that for any $x \subseteq M$, $x \in V$ with $|x| \leq \lambda$, there is some $y \in M$ having the properties that $x \subseteq y$ and $M \vDash$ "$|y| < j(\kappa)$". An equivalent definition for κ being λ-supercompact is that there is an elementary embedding $j : V \to M$ having critical point κ such that $j(\kappa) > \lambda$ and $M^\lambda \subseteq M$ (i.e., every $f : \lambda \to M$ with $f \in V$ is such that $f \in M$). κ is *strongly compact* (*supercompact*) if κ is λ-strongly compact (λ-supercompact) for every cardinal $\lambda \geq \kappa$. Also, κ is *$(\kappa + 2)$-strong* if there is an elementary embedding $j : V \to M$ having critical point κ such that $V_{\kappa+2} \subseteq M$. It is the case that if κ is supercompact, then κ is $(\kappa + 2)$-strong (and much more).

Because Magidor's notion of iterated Prikry forcing from [11] will be used in the proofs of Theorems 2 and 4, we take this opportunity to briefly review its definition and some of its properties. We follow the conventions of [11, pages 39 – 40] and take the liberty to quote nearly verbatim from [11] when appropriate. Suppose that A is a set of measurable cardinals. Let $\Omega = \sup(A)$. For each $\kappa \in A$, let $\mathcal{U}_\kappa$ be a normal measure over κ giving measure 0 to the set of measurable cardinals below κ. For $\kappa \in A$ or $\kappa = \Omega$, we will define inductively a notion of forcing $\mathbb{P}_\kappa$ which changes the cofinality of every member of $A \cap \kappa$ to ω. If κ_0 is the smallest member of A, $\mathbb{P}_{\kappa_0}$ is trivial forcing, and $\dot{\mathcal{U}}_{\kappa_0}$ is a term in the forcing language with respect to $\mathbb{P}_{\kappa_0}$ denoting $\mathcal{U}_{\kappa_0}$. Then, for $\kappa > \kappa_0$, $\kappa \in A$ or $\kappa = \Omega$, $\mathbb{P}_\kappa$ is defined as the set of all sequences of the form $\langle p_\alpha, \dot{B}_\alpha \rangle_{\alpha \in A \cap \kappa}$, where p_α is a finite increasing sequence of members of α, p_α is different from the empty sequence for only finitely many α s, and $\dot{\mathcal{U}}_\alpha$ and $\dot{B}_\alpha$ are terms with $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathcal{U}}_\alpha$ is a normal measure over α such that $\dot{\mathcal{U}}_\alpha \supseteq \mathcal{U}_\alpha$, $\dot{B}_\alpha \in \dot{\mathcal{U}}_\alpha$, and $\sup(p_\alpha) < \inf(\dot{B}_\alpha)$". Let $E = A \cap \kappa$. The ordering on $\mathbb{P}_\kappa$ is $\langle q_\alpha, \dot{C}_\alpha \rangle_{\alpha \in E} \geq \langle p_\alpha, \dot{B}_\alpha \rangle_{\alpha \in E}$ iff q_α extends p_α as a finite sequence, $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{C}_\alpha \subseteq \dot{B}_\alpha$", and if $\beta \in q_\alpha - p_\alpha$, then $\langle q_\gamma, \dot{C}_\gamma \rangle_{\gamma \in E \cap \alpha} \Vdash_{\mathbb{P}_\alpha}$ "$\beta \in \dot{B}_\alpha$". Intuitively, $\mathbb{P}_\kappa$ may be thought of as the iteration of Prikry forcing which has finite support in the stems, full support in the measure 1 sets, and changes the cofinality of every member of A to ω. The work of [11] shows that $\mathbb{P}_\kappa$ is well-defined (which is certainly not obvious from the definitions given above). Also, if $\kappa < \sup(A)$, the preceding definition of $\mathbb{P}_\kappa$ makes sense even if κ is a measurable cardinal and $\kappa \notin A$.

Assume now that κ is a measurable cardinal which is a limit of measurable cardinals. Assume also that in the above definition, A is composed of an unbounded subset of the measurable cardinals below κ. By [11, Lemma 4.4(i)], forcing with $\mathbb{P}_\kappa$ preserves all cardinals. In addition, by the definition of $\mathbb{P}_\kappa$ just given, $\mathbb{P}_\kappa$ is κ^+-c.c., and $|\mathbb{P}_\kappa| = 2^\kappa$. It therefore follows that forcing with $\mathbb{P}_\kappa$ preserves the value of $|2^\delta|$ for all cardinals $\delta \geq \kappa$. Further, [11, Theorem 2.5] tells us that if $A \notin \mathcal{U}$ for some normal measure $\mathcal{U}$ over κ, then $\mathcal{U}$ extends to a normal measure $\overline{\mathcal{U}}$ after forcing with $\mathbb{P}_\kappa$.

A corollary of Hamkins' work on gap forcing found in [5, 6] will be employed in the proof of Theorem 1. We therefore state as a separate theorem what is relevant for this paper, along with some associated terminology, quoting from [5, 6] when appropriate. Suppose $\mathbb{P}$ is a partial ordering which can be written as $\mathbb{Q} * \dot{\mathbb{R}}$, where $|\mathbb{Q}| < \delta$, $\mathbb{Q}$ is nontrivial, and $\Vdash_{\mathbb{Q}}$ "$\dot{\mathbb{R}}$ is δ^+-directed closed". In Hamkins' terminology of [5, 6], $\mathbb{P}$ *admits a gap at* δ. Also, as in the terminology of [5, 6] and elsewhere, an embedding $j : \overline{V} \to \overline{M}$ is *amenable to* $\overline{V}$ when $j \restriction A \in \overline{V}$ for any $A \in \overline{V}$. The specific corollary of Hamkins' work from [5, 6] we will be using is then the following.

Theorem 5 (Hamkins) *Suppose that $V[G]$ is a generic extension obtained by forcing with $\mathbb{P}$ that admits a gap at some regular $\delta < \kappa$. Suppose further that $j : V[G] \to M[j(G)]$ is an elementary* *embedding with critical point κ for which $M[j(G)] \subseteq V[G]$ and $M[j(G)]^\delta \subseteq M[j(G)]$ in $V[G]$. Then $M \subseteq V$; indeed, $M = V \cap M[j(G)]$. If the full embedding j is amenable to $V[G]$, then the restricted embedding $j \restriction V : V \to M$ is amenable to V. If j is definable from parameters (such as a measure or extender) in $V[G]$, then the restricted embedding $j \restriction V$ is definable from the names of those parameters in V.*

A consequence of Theorem 5 is that if $\mathbb{P}$ admits a gap at some regular $\delta < \kappa$ and κ is either supercompact or measurable in $V^{\mathbb{P}}$, then κ is supercompact or measurable in V as well.

2 The Proofs of Theorems 1 – 4

We turn now to the proofs of our theorems.

Proof: To prove Theorem 1, let V be a model of "ZFC + GCH" in which κ is the least measurable limit of supercompact cardinals. Without loss of generality, by doing a preliminary forcing as in [1] if necessary, we assume in addition that in V, every supercompact cardinal $\delta < \kappa$ has its supercompactness indestructible under δ-directed closed forcing, and GCH holds at and above κ.

Let $V_1 = V^{\text{Add}(\kappa^+, \lambda)}$. Because $\text{Add}(\kappa^+, \lambda)$ is κ^+-directed closed, V_1 and V contain the same subsets of κ. Thus, $V_1 \vDash$ "κ is measurable". In addition, standard arguments show that in V_1, $2^\kappa = \kappa^+$, and $2^{\kappa^+} = 2^{2^\kappa} = \lambda$. Further, if $V \vDash$ "$\delta < \kappa$ is supercompact", then because $V \vDash$ "δ has its supercompactness indestructible under δ-directed closed forcing", $V_1 \vDash$ "δ is supercompact". We may consequently infer that $V_1 \vDash$ "κ is a measurable limit of supercompact cardinals".

To show κ is in fact the least measurable limit of supercompact cardinals in V_1, observe that the closure properties of $\text{Add}(\kappa^+, \lambda)$ tell us forcing with $\text{Add}(\kappa^+, \lambda)$ creates no new measurable cardinals below κ. To see that forcing with $\text{Add}(\kappa^+, \lambda)$ creates no new supercompact cardinals below κ, let $\delta < \kappa$ be such that $V \vDash$ "δ is not supercompact". Let $\rho > \kappa$ be large enough so that $V \vDash$ "δ is not ρ-supercompact". If $V_1 \vDash$ "δ is supercompact", then again by the closure properties of $\text{Add}(\kappa^+, \lambda)$, it must be the case that $V \vDash$ "δ is η-supercompact for every $\eta < \kappa$". Since $V \vDash$ "κ is supercompact", we may now argue in analogy to [3, page 31, paragraph 4] to see that $V \vDash$ "δ is ρ-supercompact". In particular, if $\ell : V \to M$ is an elementary embedding witnessing the ρ'-

supercompactness of κ for some strong limit cardinal $\rho' > \rho > \kappa$, then as $V \vDash$ "δ is η-supercompact for every $\eta < \kappa$", $M \vDash$ "$\ell(\delta) = \delta$ is η-supercompact for every $\eta < \ell(\kappa)$". Since $\ell(\kappa) > \rho'$ and ρ' is a strong limit cardinal, in both M and V, δ is ρ-supercompact, a contradiction.

We now know that in V_1, κ is the least measurable limit of supercompact cardinals, $2^\kappa = \kappa^+$, and $2^{\kappa^+} = 2^{2^\kappa} = \lambda$. Working in V_1, let $\mathbb{P}^*$ be the (possibly proper class) reverse Easton iteration which forces nontrivially only at inaccessible cardinals δ which are not limits of inaccessible cardinals, where the forcing done is $\mathrm{Add}(\delta, 1)$.[1] A variant of Laver's original argument from [9] shows that every V_1-supercompact cardinal is preserved to $V_2 = V_1^{\mathbb{P}^*}$. Specifically, suppose $V_1 \vDash$ "δ is supercompact". Let $\lambda > \delta$ be a fixed but arbitrary regular cardinal, with $\gamma = 2^{[\lambda]^{<\delta}}$. Take $j : V_1 \to M$ as an elementary embedding witnessing the γ-supercompactness of δ. Let $\dot{\mathbb{Q}}^*$ be a term in the forcing language with respect to $\mathbb{P}^*_\delta$ for the portion of $\mathbb{P}^*$ acting on ordinals in the open interval (δ, γ). By the definition of $\mathbb{P}^*$, $j(\mathbb{P}^*_\delta * \dot{\mathbb{Q}}^*) = \mathbb{P}^*_\delta * \dot{\mathbb{Q}}^* * \dot{\mathbb{R}}^* * j(\dot{\mathbb{Q}}^*)$, where $\dot{\mathbb{R}}^*$ is a term for the portion of $j(\mathbb{P}^*_\delta * \dot{\mathbb{Q}}^*)$ acting on ordinals in the open interval $(\gamma, j(\delta))$. Let $G_0 * G_1 * G_2$ be V_1-generic over $\mathbb{P}^*_\delta * \dot{\mathbb{Q}}^* * \dot{\mathbb{R}}^*$. As in [9], we may lift j in $V_1[G_0][G_1][G_2]$ to $j : V_1[G_0] \to M[G_0][G_1][G_2]$, take a master condition p for $j''G_1$ and a $V_1[G_0][G_1][G_2]$-generic object G_3 over $j(\mathbb{Q}^*)$ containing p, lift j again in $V_1[G_0][G_1][G_2][G_3]$ to $j : V_1[G_0][G_1] \to M[G_0][G_1][G_2][G_3]$, and show by the γ^+-directed closure of $\mathbb{R}^* * j(\dot{\mathbb{Q}}^*)$ in both $M[G_0][G_1]$ and $V_1[G_0][G_1]$ that the supercompactness measure over $(P_\delta(\lambda))^{V_1[G_0][G_1]}$ generated by j is actually a member of $V_1[G_0][G_1]$. Write $\mathbb{P}^* = \mathbb{P}^*_\delta * \dot{\mathbb{Q}}^* * \dot{\mathbb{S}}$. Since λ was arbitrary, and since $\Vdash_{\mathbb{P}^*_\delta * \dot{\mathbb{Q}}^*}$ "$\dot{\mathbb{S}}$ is γ^+-directed closed", $V_2 = V_1^{\mathbb{P}^*} \vDash$ "δ is supercompact".

Since forcing with $\mathbb{P}^*$ does not change either cofinalities or the size of power sets, in V_2, $2^\kappa = \kappa^+$, and $2^{\kappa^+} = 2^{2^\kappa} = \lambda$. In addition, if we write $\mathbb{P}^* = \mathbb{P}_\kappa * \dot{\mathbb{Q}}$, it is the case that $\Vdash_{\mathbb{P}_\kappa}$ "$\dot{\mathbb{Q}}$ is (at least) $(2^\kappa)^+$-directed closed". This means that the set of normal measures κ carries, and hence κ's total number of normal measures, is the same in both $V_2 = V_1^{\mathbb{P}^*} = V_1^{\mathbb{P}_\kappa * \dot{\mathbb{Q}}}$ and $V_1^{\mathbb{P}_\kappa}$.

We can now argue as in [2, Lemma 1.1] to infer that in both $V_1^{\mathbb{P}_\kappa}$ and V_2, κ is a measurable cardinal carrying 2^{2^κ} many normal measures. Explicitly, let $j : V_1 \to M$ be an elementary embedding generated by a normal measure over κ present in V_1 such that $M^\kappa \subseteq M$ and $M \vDash$ "κ

[1] $\mathbb{P}^*$ is a proper class if there are class many inaccessible cardinals, but is a set otherwise. The standard Easton arguments show that $V_2 = V_1^{\mathbb{P}^*} \vDash \mathrm{ZFC}$ if $\mathbb{P}^*$ is a proper class.

is not measurable". Let G be V_1-generic over $\mathbb{P}_\kappa$, and write $j(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \dot{\mathbb{R}}$. Note that since $V_1 \models$ "$|\mathbb{P}_\kappa| = \kappa$", by elementarity, we also have that in M, $\Vdash_{\mathbb{P}_\kappa}$ "$|\dot{\mathbb{R}}| = j(\kappa)$". Therefore, in M, $\Vdash_{\mathbb{P}_\kappa}$ "The number of dense open subsets of $\dot{\mathbb{R}}$ is at most $2^{j(\kappa)}$". In addition, since $V_1[G] \models$ "$2^\kappa = \kappa^+$", $V_1[G] \models$ "$|2^{j(\kappa)}| = |j(2^\kappa)| = |\{f \mid f : \kappa \to 2^\kappa \text{ is a function}\}| = |[2^\kappa]^\kappa| = 2^\kappa = \kappa^+$". This means we can let $\langle D_\alpha \mid \alpha < \kappa^+ \rangle$ be an enumeration in $V_1[G]$ of the dense open subsets of $\mathbb{R}$ present in $M[G]$.

By the fact $\mathbb{P}_\kappa$ is κ-c.c., $M[G]^\kappa \subseteq M[G]$. Further, by the definition of $\mathbb{P}_\kappa$, the first ordinal on which $\mathbb{R}$ acts in $M[G]$ is above κ. This means that $\mathbb{R}$ is κ^+-directed closed in both $M[G]$ and $V_1[G]$. Thus, we can build in $V_1[G]$ a tree $\mathcal{T}$ of height κ^+ such that:

1. The root of $\mathcal{T}$ is the empty condition.

2. If p is an element at level $\alpha < \kappa^+$ of $\mathcal{T}$, then the successors of p at level $\alpha + 1$ are a maximal incompatible subset of D_α extending p. By the definition of $\mathbb{R}$, there will be at least two successors of p at level $\alpha + 1$.

3. If $\lambda < \kappa^+$ is a limit ordinal, then the elements of $\mathcal{T}$ at height λ are upper bounds to any path through $\mathcal{T}$ of height λ.

We observe that any path H of height κ^+ through $\mathcal{T}$ generates an $M[G]$-generic object over $\mathbb{R}$. Therefore, since there are 2^{κ^+} many paths of height κ^+ through $\mathcal{T}$, there are $2^{\kappa^+} = 2^{2^\kappa}$ many different $M[G]$-generic objects over $\mathbb{R}$.

For H an $M[G]$-generic object over $\mathbb{R}$ generated as above, since $j''G \subseteq G * H$, $j : V_1 \to M$ lifts to $j : V_1[G] \to M[G][H]$. This means that if j_1 is the lift generated by H_1 and j_2 is the lift generated by H_2, $j_1(G) = \langle G, H_1 \rangle$ and $j_2(G) = \langle G, H_2 \rangle$, i.e., there are 2^{2^κ} many different lifts of j after forcing with $\mathbb{P}$. Since [4, Lemma 1] tells us that any $k : V_1[G] \to M[G][H]$ is generated by the normal measure over κ given by $\mathcal{U} = \{x \subseteq \kappa \mid \kappa \in k(x)\}$, there are 2^{2^κ} many different normal measures over κ in $V_1[G]$ and V_2 as well.

The proof of Theorem 1 will consequently be finished if we can show that in V_2, κ is the least measurable limit of supercompact cardinals. To do this, note that by its definition, $\mathbb{P}^*$ is δ_0-directed closed, where δ_0 is the least inaccessible cardinal (in either V or V_1). Thus, δ_0 is the least inaccessible cardinal in V_2 as well, so that there are no measurable or supercompact cardinals below δ_0 in either V, V_1, or V_2. In addition, it is possible to write $\mathbb{P}^* = \mathrm{Add}(\delta_0, 1) * \dot{\mathbb{S}}$, where $\Vdash_{\mathrm{Add}(\delta_0,1)}$ "$\dot{\mathbb{S}}$ is (at least) δ_0^{++}-directed closed". By Theorem 5, this means that forcing with $\mathbb{P}^*$ over V_1 creates no new measurable or supercompact cardinals greater than δ_0, and hence creates no new measurable or supercompact cardinals. Since our earlier work yields that $V_2 \vDash$ "κ is a measurable limit of supercompact cardinals" and $V_1 \vDash$ "κ is the least measurable limit of supercompact cardinals", it consequently follows that $V_2 \vDash$ "κ is the least measurable limit of supercompact cardinals" as well. By taking $\mathbb{P} = \mathrm{Add}(\kappa^+, \lambda) * \dot{\mathbb{P}}^*$, the proof of Theorem 1 has been completed.

$\square$

It is actually the case that the above shows that in V_2, κ carries 2^{2^κ} many normal measures not concentrating on measurable cardinals. To see this, observe that by the proof just given, each of the 2^{2^κ} many normal measures corresponding to the 2^{2^κ} many lifted embeddings must concentrate on the set $A = \{\delta < \kappa \mid \delta \text{ is non-measurable in } V_1\}$. Since as we have just seen, Theorem 5 implies that no $\delta \in A$ is measurable in V_2, A consists of non-measurable cardinals in V_2 as well. Although this fact is not important for the proof of Theorem 1, it will be key in the proof of Theorem 2.

We can now turn our attention to the proof of Theorem 2, which is proven similarly. Let V be a model of "ZFC + GCH" in which κ is supercompact. Without loss of generality, by first doing the forcing of [9], we assume in addition that in V, κ's supercompactness is indestructible under κ-directed closed forcing, and GCH holds at and above κ. If we as before let $V_1 = V^{\mathrm{Add}(\kappa^+, \lambda)}$, as in the proof of Theorem 1, it is then the case that in V_1, κ is supercompact, $2^\kappa = \kappa^+$, and $2^{\kappa^+} = 2^{2^\kappa} = \lambda$. Working in V_1, again as in the proof of Theorem 1, let $\mathbb{P}^*$ be the (possibly proper class) reverse Easton iteration which forces nontrivially only at inaccessible cardinals δ which are not limits of inaccessible cardinals, where the forcing done is $\mathrm{Add}(\delta, 1)$. By the same arguments as in the proof of Theorem 1 and the paragraph immediately following, in $V_2 = V_1^{\mathbb{P}^*}$, it is the case that κ is supercompact, $2^\kappa = \kappa^+$, $2^{\kappa^+} = 2^{2^\kappa} = \lambda$, and κ carries 2^{2^κ} many normal measures not concentrating on measurable cardinals. In addition, as in the proof of Theorem 1, if $\mathbb{P}^*$ is a proper class, then $V_2 \vDash$ ZFC.

Now, working in V_2, let $\mathbb{Q}$ be the Magidor iteration of Prikry forcing [11] which adds a cofinal ω sequence to each measurable cardinal below κ, with $V_3 = V_2^{\mathbb{Q}}$. As in [11], $V_3 \vDash$ "κ is both the least strongly compact and least measurable cardinal". Since as we have noted in Section 1, forcing with $\mathbb{Q}$ neither collapses any cardinals nor changes the value of $|2^\delta|$ for cardinals $\delta \geq \kappa$, in V_3, $2^\kappa = \kappa^+$, and $2^{\kappa^+} = 2^{2^\kappa} = \lambda$. We have just mentioned that $V_2 \vDash$ "κ carries 2^{2^κ} many normal measures not concentrating on measurable cardinals". Therefore, since as we observed in Section 1 in the paragraph immediately prior to the discussion of Theorem 5, any normal measure over κ not concentrating on $A = \{\delta < \kappa \mid \delta$ is a measurable cardinal$\}$ extends after forcing with $\mathbb{Q}$ and forcing with $\mathbb{Q}$ collapses no cardinals, $V_3 \vDash$ "κ carries 2^{2^κ} many normal measures" as well. V_3 is thus as desired. By taking $\mathbb{P} = \text{Add}(\kappa^+, \lambda) * \dot{\mathbb{P}}^* * \dot{\mathbb{Q}}$, the proof of Theorem 2 has been completed.

$\square$

To prove Theorem 3, let $V \vDash$ "κ' is the least supercompact limit of supercompact cardinals". Assume without loss of generality that a reverse Easton iteration has been done as in [12, Theorem, Section 18, pages 83–88] so that in addition, $V \vDash$ "For every inaccessible cardinal δ, $2^\delta = \delta^{+17}$ and $2^{\delta^{+17}} = 2^{2^\delta} = \delta^{+95}$". Let $k : V \to M'$ be an elementary embedding witnessing the $2^{\kappa'}$-supercompactness of κ'. By [3, Lemma 2.1] and the succeeding remarks, $M' \vDash$ "κ' is a strong cardinal". Since κ' is the critical point of k, if $V \vDash$ "$\delta < \kappa'$ is supercompact", $M' \vDash$ "$k(\delta) = \delta < \kappa'$ is supercompact". This means that $M' \vDash$ "κ' is a strong cardinal which is a limit of supercompact cardinals", so by reflection, $\{\delta < \kappa' \mid \delta$ is a strong cardinal which is a limit of supercompact cardinals$\}$ is unbounded in κ' in V. Thus, we can let $\kappa < \kappa'$ be the least cardinal in V which is both $(\kappa + 2)$-strong and is a limit of supercompact cardinals.

Now, let $j : V \to M$ be an elementary embedding witnessing the $(\kappa + 2)$-strongness of κ. As in the preceding paragraph, $M \vDash$ "κ is a limit of supercompact cardinals". Because $V \vDash$ "κ is the least cardinal which is both $(\kappa + 2)$-strong and a limit of supercompact cardinals", $M \vDash$ "$j(\kappa) > \kappa$ is the least cardinal which is both $(j(\kappa) + 2)$-strong and a limit of supercompact cardinals". Hence, $M \vDash$ "κ is not $(\kappa + 2)$-strong". We will show that in M, κ is our desired strongly compact cardinal.

To do this, because $V_{\kappa+2} \subseteq M$ and a measure over κ is a member of $V_{\kappa+2}$, M contains every (normal or non-normal) measure over κ. Further, by the fact that κ is $(\kappa+2)$-strong in V, $V \vDash$ "κ carries 2^{2^κ} many normal measures". It consequently follows that $M \vDash$ "κ is measurable and carries 2^{2^κ} many normal measures". Because V and M are elementarily equivalent, $M \vDash$ "$2^\kappa = \kappa^{+17}$ and $2^{\kappa^{+17}} = 2^{2^\kappa} = \kappa^{+95}$". Since $M \vDash$ "κ is a measurable limit of supercompact cardinals", by Menas' theorem of [13], $M \vDash$ "κ is strongly compact". Putting the above together, we now have that in M, κ is a strongly compact cardinal which is a measurable limit of supercompact cardinals, κ is not $(\kappa+2)$-strong, $2^\kappa = \kappa^{+17}$, $2^{\kappa^{+17}} = 2^{2^\kappa} = \kappa^{+95}$, and κ carries 2^{2^κ} many normal measures. By reflection, $A = \{\delta < \kappa \mid \delta$ is a measurable limit of supercompact cardinals which is not $(\delta+2)$-strong, $2^\delta = \delta^{+17}$, $2^{\delta^{+17}} = 2^{2^\delta} = \delta^{+95}$, and δ carries 2^{2^δ} many normal measures$\}$ is unbounded in κ in V. Since for any $\delta \in A$, $M \vDash$ "$j(\delta) = \delta$ is a measurable limit of supercompact cardinals", κ is not the least measurable limit of supercompact cardinals in either V or M. This completes the proof of Theorem 3.

□

Turning now to the proof of Theorem 4, we will use the cardinal κ and the model M witnessing the conclusions of Theorem 3 in our proof. First, let us observe that in M, since $\kappa < j(\kappa)$ and $M \vDash$ "$j(\kappa)$ is the least cardinal which is both $(j(\kappa)+2)$-strong and a limit of supercompact cardinals", κ is below the least supercompact limit of supercompact cardinals. Keeping this in mind, we take M as our ground model. Let $\mathbb{P}$ be the Magidor iteration of Prikry forcing [11] which adds a cofinal ω sequence to each supercompact cardinal below κ. By [11, Theorem 3.4], $M \vDash$ "κ is strongly compact". Because V and M are elementarily equivalent, $M \vDash$ "For every inaccessible cardinal δ, $2^\delta = \delta^{+17}$". Consequently, since the supercompact cardinals are unbounded in κ in M, as in the proof of [11, Theorem 4.5], $M^{\mathbb{P}} \vDash$ "There are unboundedly in κ many singular strong limit cardinals violating GCH". By Solovay's theorem of [14], this means we may now infer that $M^{\mathbb{P}} \vDash$ "No cardinal $\delta < \kappa$ is strongly compact", i.e., $M^{\mathbb{P}} \vDash$ "κ is the least strongly compact cardinal".

For any M-measurable cardinal δ, by the definition of the Magidor iteration of Prikry forcing given in Section 1, it is possible to write $\mathbb{P} = \mathbb{P}_\delta * \dot{\mathbb{R}}$. If δ is not supercompact, then because forcing with $\mathbb{P}$ does not add a cofinal ω sequence to δ, the definition of $\mathbb{P}$ and [11, Lemma 2.4] yield that $\Vdash_{\mathbb{P}_\delta}$ "Forcing with $\dot{\mathbb{R}}$ adds no new subsets of the least inaccessible cardinal above δ (and so in particular adds no new subsets of 2^δ)". If $|\mathbb{P}_\delta| < \delta$, then by the Lévy-Solovay results [10], $M^{\mathbb{P}_\delta} \models$ "δ is measurable". If $|\mathbb{P}_\delta| = \delta$, then by [11, Theorem 2.5], it again follows that $M^{\mathbb{P}_\delta} \models$ "δ is measurable". It is thus the case that $M^{\mathbb{P}_\delta * \dot{\mathbb{R}}} = M^{\mathbb{P}} \models$ "δ is measurable" as well. In addition, because κ is in M a limit of supercompact cardinals, there are in M unboundedly many in κ measurable cardinals which are not supercompact. Consequently, we may now infer that $M^{\mathbb{P}} \models$ "κ is a limit of measurable cardinals".

Because κ is below the least supercompact limit of supercompact cardinals, no normal measure over κ concentrates on $A = \{\delta < \kappa \mid \delta \text{ is supercompact}\}$.[2] Hence, as in the last paragraph of the proof of Theorem 2, we may now infer that in $M^{\mathbb{P}}$, κ carries 2^{2^κ} many normal measures, $2^\kappa = \kappa^{+17}$, and $2^{\kappa^{+17}} = 2^{2^\kappa} = \kappa^{+95}$. We have therefore completed the proof of Theorem 4, unless it also happens to be true that $M^{\mathbb{P}} \models$ "κ is $(\kappa+2)$-strong". If this is the case, then $M \models \varphi(\kappa)$, where $\varphi(x)$ is the formula in one free variable in the language of ZFC which says "x is a limit of supercompact cardinals, x is not $(x+2)$-strong, x is measurable and carries 2^{2^x} many normal measures, $2^x = x^{+17}$, $2^{x^{+17}} = 2^{2^x} = x^{+95}$, and forcing with the Magidor iteration of Prikry forcing which changes the cofinality of each supercompact cardinal below x to ω makes x into an $(x+2)$-strong cardinal". Without loss of generality, assume κ is least such that $M \models \varphi(\kappa)$. Let $j : M^{\mathbb{P}} \to N^{j(\mathbb{P})}$ be an elementary embedding witnessing the $(\kappa+2)$-strongness of κ. We will show that in $N^{\mathbb{P}}$, κ is our desired strongly compact cardinal.

We first show that $N \models$ "κ is a measurable limit of supercompact cardinals" (and hence is strongly compact in N, by Menas' theorem from [13]). To do this, consider $j \restriction M : M \to N$, which is still an elementary embedding having critical point κ. As before, for any $\delta < \kappa$ such that $M \models$ "δ is supercompact", $N \models$ "$j(\delta) = \delta$ is supercompact". Thus, since $M \models$ "κ is a

[2]This is since otherwise, if μ were a normal measure over κ concentrating on supercompact cardinals, with $j_\mu : M \to N$ the associated elementary embedding, then $N \models$ "κ is supercompact". Further, as j_μ has critical point κ, for any $\delta < \kappa$ such that $M \models$ "δ is supercompact", $N \models$ "$j_\mu(\delta) = \delta$ is supercompact". Since the supercompact cardinals are unbounded in κ in M, this means that $N \models$ "κ is a supercompact limit of supercompact cardinals". By reflection, the set of supercompact limits of supercompact cardinals is unbounded below κ in M. This contradicts that in M, κ is below the least supercompact limit of supercompact cardinals.

limit of supercompact cardinals", $N \models$ "κ is a limit of supercompact cardinals". Further, because $M \models$ "κ is below the least supercompact limit of supercompact cardinals", $N \models$ "$j(\kappa) > \kappa$ is below the least supercompact limit of supercompact cardinals". Hence, $N \models$ "κ is below the least supercompact limit of supercompact cardinals" as well. It therefore immediately follows that $N \models$ "κ is not supercompact". This means that by the definition of $\mathbb{P}$ given in Section 1, $j(\mathbb{P})$ factors as $\mathbb{P}_\kappa * \dot{\mathbb{Q}} = \mathbb{P} * \dot{\mathbb{Q}}$, where the first ordinal to which $\dot{\mathbb{Q}}$ is forced to add a cofinal ω sequence is well above κ. Consequently, as in the second paragraph of the proof of this theorem, $\Vdash_{\mathbb{P}}$ "Forcing with $\dot{\mathbb{Q}}$ adds no new subsets of the least inaccessible cardinal above κ (and so in particular adds no new subsets of 2^κ)". Also, exactly as in the proof of Theorem 3, because $M^{\mathbb{P}} \models$ "κ is a measurable cardinal carrying 2^{2^κ} many normal measures" and j is an elementary embedding witnessing that κ is $(\kappa+2)$-strong in $M^{\mathbb{P}}$, $N^{j(\mathbb{P})} \models$ "κ is a measurable cardinal carrying 2^{2^κ} many normal measures" as well. The preceding two sentences now allow us to infer that $N^{\mathbb{P}} \models$ "κ is a measurable cardinal carrying 2^{2^κ} many normal measures". Therefore, since by [11, Theorem 3.1], forcing with $\mathbb{P}$ creates no new measurable cardinals, $N \models$ "κ is a measurable cardinal", i.e., $N \models$ "κ is a measurable limit of supercompact cardinals".

We now know that $N \models$ "κ is strongly compact and is a limit of supercompact cardinals". In addition, by elementarity, it is again the case that $N \models$ "For every inaccessible cardinal δ, $2^\delta = \delta^{+17}$". This means that as in the first two paragraphs of the proof of this theorem, we may infer that $N^{\mathbb{P}} \models$ "κ is the least strongly compact cardinal and is a limit of measurable cardinals". Because M and N are elementarily equivalent and as we noted in Section 1, forcing with $\mathbb{P}$ neither collapses any cardinals nor changes the value of $|2^\delta|$ for cardinals $\delta \geq \kappa$, $N^{\mathbb{P}} \models$ "$2^\kappa = \kappa^{+17}$ and $2^{\kappa^{+17}} = 2^{2^\kappa} = \kappa^{+95}$". Also, because $M \models$ "κ is the least cardinal such that $\varphi(\kappa)$ is true", by elementarity, $N \models$ "$j(\kappa) > \kappa$ is the least cardinal such that $\varphi(j(\kappa))$ is true". By our work above, it therefore follows that $N^{\mathbb{P}} \models$ "κ is not $(\kappa+2)$-strong". Since we have already seen that $N^{\mathbb{P}} \models$ "κ carries 2^{2^κ} many normal measures", this completes the proof of Theorem 4.

$\square$

3 Concluding Remarks

We conclude with a few observations. As we remarked in Section 1, the non-$(\kappa+2)$-strong strongly compact cardinals κ witnessing the conclusions of Theorems 3 and 4 are neither the least measurable limit of supercompact cardinals nor the least measurable cardinal. This is since the methods used in the proofs of Theorems 1 and 2 do not seem to be adaptable to the situation where $2^\kappa > \kappa^+$. The reason is that in the proofs of Theorems 1 and 2, we need to know the partial ordering $\mathbb{P}_\kappa$ of Theorem 1 increases the number of normal measures over the strongly compact cardinal κ in question not concentrating on measurable cardinals to 2^{2^κ}. In order to show that this is indeed the case, as the proof of Theorem 1 indicates, we have to be able to construct 2^{2^κ} many generic objects for a certain κ^+-directed closed partial ordering $\mathbb{R}$ by meeting all of the dense open subsets of $\mathbb{R}$ present in a generic extension $M[G]$ of a κ-closed inner model M of the ground model V_1 (where we adopt the same notation as in the proof of Theorem 1). If $2^\kappa = \kappa^+$, then this is not a problem, as we have already seen. However, if $2^\kappa > \kappa^+$, then the calculation given in the proof of Theorem 1 for the number of dense open subsets of $\mathbb{R}$ present in $M[G]$ yields some $\lambda \geq \kappa^{++}$. Building the generic object for $\mathbb{R}$ via the induction given previously does not work, as there are λ many dense open subsets which must be met. The construction will break down at stage κ^+, because $\mathbb{R}$ is only κ^+-directed closed. It is not at all clear at the moment how to overcome this obstacle.

Theorems 1 – 4 only barely scratch the surface of what we feel is possible for non-$(\kappa+2)$-strong strongly compact cardinals κ. We finish by making this precise via the following

Conjecture: For any non-$(\kappa+2)$-strong strongly compact cardinal κ (such as the ones considered earlier), it is relatively consistent for κ to carry exactly δ many normal measures. Here, $1 \leq \delta \leq 2^{2^\kappa}$ is any cardinal, and the values of both 2^κ and 2^{2^κ} can be freely manipulated in a way compatible with the value of δ. In particular, it is relatively consistent to have a non-$(\kappa+2)$-strong strongly compact cardinal κ which carries exactly 1, 2, 3, 98, $\aleph_{64}$, δ for δ the least inaccessible cardinal, κ^{+99}, etc. many normal measures, with arbitrary values for either 2^κ or 2^{2^κ} which are compatible with δ many normal measures over κ.

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