How Many Normal Measures Can $\aleph_{\omega_1+1}$ Carry? [*†]

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Abstract

Relative to the existence of a supercompact cardinal with a measurable cardinal above it, we show that it is consistent for $\aleph_1$ to be regular and for $\aleph_{\omega_1+1}$ to be measurable and to carry precisely τ normal measures, where $\tau \geq \aleph_{\omega_1+2}$ is any regular cardinal. This extends the work of [2], in which the analogous result was obtained for $\aleph_{\omega+1}$ using the same hypotheses.

In [2], models were constructed for the theory "ZF + $\mathrm{DC}_{\aleph_\omega}$ + $\aleph_{\omega+1}$ is a measurable cardinal" in which $\aleph_{\omega+1}$ carries exactly τ normal measures, where $\tau \geq \aleph_{\omega+2}$ is an arbitrary regular cardinal. The proof given easily generalizes to handling successors of other singular cardinals of cofinality ω, such as $\aleph_{\omega+\omega+1}$, $\aleph_{\omega^2+1}$, etc. No attempt was made in that paper, however, to handle successors of singular cardinals of uncountable cofinality, such as $\aleph_{\omega_1+1}$.

The purpose of this paper is to rectify this situation and establish a similar result for $\aleph_{\omega_1+1}$. Specifically, in analogy to [2], we prove the following two theorems.

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Theorem 1 *Suppose $V^* \vDash$ "ZFC + GCH + $\kappa < \lambda$ are such that κ is supercompact and λ is the least measurable cardinal greater than κ + $\tau > \lambda^+$ is a fixed but arbitrary regular cardinal". There is then a generic extension V of V^*, a partial ordering $\mathbb{P} \in V$, and a symmetric submodel $N \subseteq V^{\mathbb{P}}$ such that $N \vDash$ "ZF + $\neg AC_\omega$ + $\aleph_1$ is a regular cardinal + $\kappa = \aleph_{\omega_1}$ + For every limit ordinal $\nu < \aleph_1$, $\aleph_{\nu+1}$ is a measurable cardinal + $\lambda = \kappa^+ = \aleph_{\omega_1+1}$ is a measurable cardinal". In N, every successor cardinal less than $\aleph_{\omega_1}$ is regular, the cardinal and cofinality structure at and above λ is the same as in V (which has the same cardinal and cofinality structure at and above λ as V^*), and $\aleph_{\omega_1+1}$ carries exactly τ normal measures.*

Theorem 2 *Suppose $V^* \vDash$ "ZFC + GCH + $\kappa < \lambda$ are such that κ is supercompact and λ is the least measurable cardinal greater than κ". There is then a generic extension V of V^*, a partial ordering $\mathbb{P} \in V$, and a symmetric submodel $N \subseteq V^{\mathbb{P}}$ such that $N \vDash$ "ZF + $\neg AC_\omega$ + $\aleph_1$ is a regular cardinal + $\kappa = \aleph_{\omega_1}$ + For every limit ordinal $\nu < \aleph_1$, $\aleph_{\nu+1}$ is a measurable cardinal + $\lambda = \kappa^+ = \aleph_{\omega_1+1}$ is a measurable cardinal". In N, every successor cardinal less than $\aleph_{\omega_1}$ is regular, $\aleph_{\omega_1+2}$ is regular, and $\aleph_{\omega_1+1}$ carries exactly $\aleph_{\omega_1+2}$ normal measures.*

As in [2], Theorems 1 and 2 provide our desired results. Taken together, they show that relative to the appropriate assumptions, it is consistent for $\aleph_{\omega_1+1}$ to be measurable and to carry precisely τ normal measures, where $\tau \geq \aleph_{\omega_1+2}$ is any regular cardinal.

To prove Theorems 1 and 2, let $V^* \vDash$ "ZFC + GCH + $\kappa < \lambda$ are such that κ is supercompact and λ is the least measurable cardinal greater than κ". As in [2], we give a uniform proof of Theorems 1 and 2. In particular, as in the proof of [2, Theorem 1], for Theorem 1 of this paper and τ as in the statement of this theorem, we may assume in addition that V^* has been generically extended to a model V with the same cardinal and cofinality structure as V^* at and above λ such that $V \vDash$ "λ carries exactly τ normal measures". For Theorem 2 of this paper, as in the proof of [2, Theorem 2], we may assume in addition that V^* has been generically extended to a model V such that $V \vDash$ "λ carries exactly λ^+ normal measures". For the exact manner in which V is constructed from V^*, we refer readers to [2]. We note only that for Theorem 1, $V \vDash$ "$2^\lambda = \lambda^+$ + $2^{\lambda^+} = \tau$", and for Theorem 2, $V \vDash$ "$2^\lambda = \lambda^+$ + $2^{\lambda^+} = \lambda^{++}$".

Our proof combines Gitik's techniques of [11] with the methods of [2]. Our presentation of Gitik's techniques is based on the one given in [3], but also follows the ones given in [5], [1], and [6]. All of these rely heavily on [11]. As the necessary facts about Radin forcing are distributed throughout the literature, our bibliographical citations will reflect this.

Our witnessing model N for Theorems 1 and 2 is the specific version of the model N_A of [3] described at the end of that paper, only constructed using a Radin sequence of measures of length $\aleph_1$ instead of κ^+ and not truncating the universe at κ. We explicitly give the construction below. Let $j : V \to M$ be an elementary embedding witnessing the 2^λ supercompactness of κ. Our first step is to define a Radin sequence of measures $\mu_{<\aleph_1} = \langle \mu_\alpha \mid \alpha < \aleph_1 \rangle$ appropriate for supercompact Radin forcing over $P_\kappa(\lambda)$. Specifically, if $\alpha = 0$, μ_α is defined by $X \in \mu_\alpha$ iff $\langle j(\beta) \mid \beta < \lambda \rangle \in j(X)$, and if $\alpha > 0$, $\alpha < \aleph_1$, μ_α is defined by $X \in \mu_\alpha$ iff $\langle \mu_\beta \mid \beta < \alpha \rangle =_{\rm df} \mu_{<\alpha} \in j(X)$.

Next, we let $\mathbb{R}_{<\aleph_1}$ be supercompact Radin forcing over $P_\kappa(\lambda)$ defined using $\mu_{<\aleph_1}$. The particulars of the definition are virtually identical to the ones found in [5], [3], [1], and [6], but for clarity, we repeat them here. $\mathbb{R}_{<\aleph_1}$ is composed of all finite sequences of the form $\langle \langle p_0, u_0, C_0, \rangle, \ldots, \langle p_n, u_n, C_n \rangle, \langle \mu_{<\aleph_1}, C \rangle \rangle$ such that the following conditions hold.

1. For $0 \le i < j \le n$, $p_i \precneqq p_j$, where for $p, q \in P_\kappa(\lambda)$, $p \precneqq q$ means $p \subseteq q$ and $\text{otp}(p) < q \cap \kappa$.

2. For $0 \le i \le n$, $p_i \cap \kappa$ is a measurable cardinal.

3. $\text{otp}(p_i)$ is the least measurable cardinal greater than $p_i \cap \kappa$. In analogy to the notation of [11], [5], [3], [1], and [6], we write $\text{otp}(p_i) = (p_i \cap \kappa)^*$. By extension of this notation, $\lambda = \kappa^*$.

4. For $0 \le i \le n$, u_i is a Radin sequence of measures appropriate for supercompact Radin forcing over $P_{p_i \cap \kappa}(\text{otp}(p_i))$ with $(u_i)_0$, the 0th coordinate of u_i, a supercompact measure over $P_{p_i \cap \kappa}(\text{otp}(p_i))$.

5. C_i is a sequence of measure 1 sets for u_i.

6. C is a sequence of measure 1 sets for $\mu_{<\aleph_1}$.

7. For each $p \in (C)_0$, where $(C)_0$ is the coordinate of C such that $(C)_0 \in \mu_0$, $\bigcup_{i \in \{0,\ldots,n\}} p_i \precneqq p$.

8. For each $p \in (C)_0$, $\mathrm{otp}(p) = (p \cap \kappa)^*$ and $p \cap \kappa$ is a measurable cardinal.

Conditions (5) and (6) are both standard to any definition of Radin forcing. Conditions (1), (2), (4), and (7) are all standard to any definition of *supercompact* Radin forcing. Conditions (3) and (8) are used because of our ultimate aim of constructing a model in which $\aleph_{\omega_1+1}$ is measurable and carries the desired number of normal measures. That they may be included and have the Radin forcing attain its desired goals follows by the fact that $V \vDash$ "κ is supercompact and λ is the least measurable cardinal greater than κ". Thus, by closure, $M \vDash$ "κ is measurable and λ is the least measurable cardinal greater than κ". This means that by reflection, $\{p \in P_\kappa(\lambda) \mid p \cap \kappa$ is a measurable cardinal and $\mathrm{otp}(p)$ is the least measurable cardinal greater than $p \cap \kappa\} \in \mu_0$. This will ensure that the Radin sequence of cardinals eventually produced can be used in our final symmetric inner model N.

For completeness of exposition, we recall now the definition of the ordering on $\mathbb{R}_{<\aleph_1}$. If $\pi_0 = \langle\langle p_0, u_0, C_0\rangle, \ldots, \langle p_n, u_n, C_n\rangle, \langle \mu_{<\aleph_1}, C\rangle\rangle$ and $\pi_1 = \langle\langle q_0, v_0, D_0\rangle, \ldots, \langle q_m, v_m, D_m\rangle, \langle \mu_{<\aleph_1}, D\rangle\rangle$, then π_1 extends π_0 iff the following conditions hold.

1. For each $\langle p_j, u_j, C_j\rangle$ which appears in π_0, there is a $\langle q_i, v_i, D_i\rangle$ which appears in π_1 such that $\langle q_i, v_i\rangle = \langle p_j, u_j\rangle$ and $D_i \subseteq C_j$, i.e., for each coordinate $(D_i)_\alpha$ and $(C_j)_\alpha$, $(D_i)_\alpha \subseteq (C_j)_\alpha$.

2. $D \subseteq C$.

3. $n \le m$.

4. If $\langle q_i, v_i, D_i\rangle$ does not appear in π_0, let $\langle p_j, u_j, C_j\rangle$ (or $\langle \mu_{<\aleph_1}, C\rangle$) be the first element of π_0 such that $p_j \cap \kappa > q_i \cap \kappa$. Then

   (a) q_i is order isomorphic to some $q \in (C_j)_0$.

   (b) There exists an $\alpha < \alpha_0$, where α_0 is the length of u_j, such that v_i is isomorphic "in a natural way" to an ultrafilter sequence $v \in (C_j)_\alpha$.

(c) For β_0 the length of v_i, there is a function $f : \beta_0 \to \alpha_0$ such that for $\beta < \beta_0$, $(D_i)_\beta$ is a set of ultrafilter sequences such that for some subset $(D_i)'_\beta$ of $(C_j)_{f(\beta)}$, each ultrafilter sequence in $(D_i)_\beta$ is isomorphic "in a natural way" to an ultrafilter sequence in $(D_i)'_\beta$.

For further information on the definition of the ordering on $\mathbb{R}_{<\aleph_1}$ (including the meaning of "in a natural way") and more facts about Radin forcing in general, readers are referred to [5], [3], [1], [6], [8], [9], [11], [10], and [13].

We are now ready to define the partial ordering $\mathbb{P}$ used in the proof of Theorem 1. It is given by the finite support product ordered componentwise

$$\prod_{\{\langle \alpha,\beta\rangle \mid \aleph_1 \le \alpha < \beta < \kappa \text{ are regular cardinals}\}} \mathrm{Coll}(\alpha, <\beta) \times \mathbb{R}_{<\aleph_1},$$

where $\mathrm{Coll}(\alpha, <\beta)$ is the Lévy collapse of all cardinals of size less than β to α.

Let G be V-generic over $\mathbb{P}$, and let G_0 be the projection of G onto $\mathbb{R}_{<\aleph_1}$. For any condition $\pi \in \mathbb{R}_{<\aleph_1}$, call $\langle p_0, \ldots p_n \rangle$ *the p-part of* π. Let $R = \{p \mid \exists \pi \in G_0[p \in \mathrm{p-part}(\pi)]\}$, and let $R_\ell = \{p \mid p \in R$ and p is a limit point of $R\}$. Define three sets E_0, E_1, and E_2 by $E_0 = \{\alpha \mid$ For some $\pi \in G_0$ and some $p \in \mathrm{p-part}(\pi)$, $p \cap \kappa = \alpha\}$, $E_1 = \{\alpha \le \kappa \mid \alpha$ is a limit point of $E_0\}$, and $E_2 = E_1 \cup \{(\aleph_1)^V\} \cup \{\beta \mid \exists \alpha \in E_1[\beta = \alpha^*]\}$. By a simple density argument, forcing with $\mathbb{R}_{<\aleph_1}$ changes the cofinality of κ to $(\aleph_1)^V$. We can therefore let $\langle \alpha_\nu \mid \nu < (\aleph_1)^V \rangle$ be the continuous, increasing enumeration of $E_2 - \{\kappa, \lambda\}$, and also let $\alpha_{(\aleph_1)^V} = \kappa$ and $\alpha_{(\aleph_1)^V+1} = \lambda$. Note that the sequence $\langle \alpha_\nu \mid \nu < (\aleph_1)^V \rangle$ is cofinal in κ. Let $\nu = \nu' + n$ for some $n \in \omega$, where $\nu' \le (\aleph_1)^V$ is either a limit ordinal or 0. For β where $\nu < (\aleph_1)^V$ and $\beta \in [\alpha_\nu, \alpha_{\nu+1})$ in the first case, $\nu = (\aleph_1)^V$ and $\beta \in [\kappa, \lambda)$ in the second case, and $\beta = \alpha_{\nu+1}$ and $\nu < (\aleph_1)^V$ in the last two cases, define sets $C_i(\alpha_\nu, \beta)$ for $i = 1, \ldots, 4$ according to specific conditions on ν and ν' in the following manner:

1. $\nu = \nu' \neq 0$, $\nu < (\aleph_1)^V$, and $n = 0$, i.e., $\nu < (\aleph_1)^V$ is a limit ordinal. Let $p(\alpha_\nu)$ be the element p of R such that $p \cap \kappa = \alpha_\nu$, and let $h_{p(\alpha_\nu)} : p(\alpha_\nu) \to \mathrm{otp}(p(\alpha_\nu))$ be the order isomorphism between $p(\alpha_\nu)$ and $\mathrm{otp}(p(\alpha_\nu))$. Then $C_1(\alpha_\nu, \beta) = \{h_{p(\alpha_\nu)}{}'' p \cap \beta \mid p \in R_\ell,\ p \subseteq p(\alpha_\nu)$, and $h^{-1}_{p(\alpha_\nu)}(\beta) \in p\}$.

2. $\nu = (\aleph_1)^V$. Then $C_2(\kappa, \beta) = C_2(\alpha_{(\aleph_1)^V}, \beta) = \{p \cap \beta \mid p \in R_\ell$ and $\beta \in p\}$.

3. ($\nu = \nu' + n$, $\nu' > 0$, $\nu' < (\aleph_1)^V$, and $n \geq 2$) or ($\nu' = 0$ and $n \in \omega$), i.e., ν is neither a limit ordinal nor the successor of a limit ordinal less than $(\aleph_1)^V$. Let $H(\alpha_\nu, \alpha_{\nu+1})$ be the projection of G onto $\mathrm{Coll}(\alpha_\nu, {<}\alpha_{\nu+1})$. Then $C_3(\alpha_\nu, \alpha_{\nu+1}) = H(\alpha_\nu, \alpha_{\nu+1})$.

4. $\nu = \nu' + 1$ for $\nu' > 0$, $\nu' < (\aleph_1)^V$, i.e., ν is the successor of a limit ordinal less than $(\aleph_1)^V$. Let $H(\alpha_\nu^+, \alpha_{\nu+1})$ be the projection of G onto $\mathrm{Coll}(\alpha_\nu^+, {<}\alpha_{\nu+1})$. Then $C_4(\alpha_\nu^+, \alpha_{\nu+1}) = H(\alpha_\nu^+, \alpha_{\nu+1})$.

$C_1(\alpha_\nu, \beta)$ and $C_2(\kappa, \beta)$ are used to collapse β to α_ν when $\nu \leq (\aleph_1)^V$ is a limit ordinal, and are also used to generate the closed, cofinal sequence $\langle \alpha_\gamma \mid \gamma < \nu \rangle$. $C_3(\alpha_\nu, \alpha_{\nu+1})$ is used to collapse $\alpha_{\nu+1}$ to be the successor of α_ν when $\nu < (\aleph_1)^V$ is neither a limit ordinal nor the successor of a limit ordinal, and $C_4(\alpha_\nu^+, \alpha_{\nu+1})$ is used to collapse $\alpha_{\nu+1}$ to be the successor of α_ν^+ when $\nu < (\aleph_1)^V$ is the successor of a limit ordinal. The C_i have been chosen so as to ensure that all successor cardinals less than $\aleph_{\omega_1}$ are regular and that the successor of every limit cardinal less than or equal to $\aleph_{\omega_1}$ is measurable. Intuitively, the symmetric inner model $N \subseteq V[G]$ witnessing the conclusions of Theorem 1 is the least model of ZF extending V which contains $C_1(\alpha_\nu, \beta)$ if $\nu < (\aleph_1)^V$ is a limit ordinal and $\beta \in [\alpha_\nu, \alpha_{\nu+1})$, $C_2(\kappa, \beta)$ if $\beta \in [\kappa, \lambda)$, $C_3(\alpha_\nu, \alpha_{\nu+1})$ if $\nu < (\aleph_1)^V$ is neither a limit ordinal nor the successor of a limit ordinal, and $C_4(\alpha_\nu^+, \alpha_{\nu+1})$ if $\nu < (\aleph_1)^V$ is the successor of a limit ordinal.

To define N more precisely, it is necessary to define canonical names $\underline{\alpha_\nu}$ for the α_ν's when $\nu < (\aleph_1)^V$ and canonical names $\underline{C_i(\nu, \beta)}$ for $i = 1, 2$ and $\underline{C_i(\nu, \nu+1)}$ for $i = 3, 4$. Recall that when $\nu < (\aleph_1)^V$ it is possible to decide $p(\alpha_\nu)$ (and hence $\mathrm{otp}(p(\alpha_\nu))$) by writing $\omega \cdot \nu = \omega^{\sigma_0} \cdot n_0 + \omega^{\sigma_1} \cdot n_1 + \cdots + \omega^{\sigma_m} \cdot n_m$ (where $\sigma_0 > \sigma_1 > \cdots > \sigma_m$ are ordinals, $n_0, \ldots, n_m > 0$ are integers, and $+$, $\cdot$, and exponentiation are the ordinal arithmetical operations), letting $\pi = \langle \langle p_{ij_i}, u_{ij_i}, C_{ij_i} \rangle_{i \leq m, 1 \leq j_i \leq n_i}, \langle \mu_{<\aleph_1}, C \rangle \rangle$ be such that $\min(p_{i1} \cap \kappa, \omega^{\mathrm{length}(u_{i1})}) = \sigma_i$ and $\mathrm{length}(u_{ij_i}) = \min(p_{i1} \cap \kappa, \mathrm{length}(u_{i1}))$ for $1 \leq j_i \leq n_i$, and letting $p(\alpha_\nu)$ be p_{mn_m}. Further, $D_\nu = \{r \in \mathbb{P} \mid r \restriction \mathbb{R}_{<\aleph_1}$ extends a condition π of the above form$\}$ is a dense open subset of $\mathbb{P}$. $\underline{\alpha_\nu}$ is the name of the α_ν determined by any element of $D_\nu \cap G$; in the notation of [11], [5], [3], [1], and [6], $\underline{\alpha_\nu} = \{\langle r, \check{\alpha}_\nu(r) \rangle \mid r \in D_\nu\}$, where $\alpha_\nu(r)$ is the α_ν determined by the condition r.

The canonical names $\underline{C_i(\nu, \beta)}$ for $i = 1, 2$ and $\underline{C_i(\nu, \nu+1)}$ for $i = 3, 4$ are defined in a manner so as to be invariant under the appropriate group of automorphisms. Specifically, there are four cases to consider. We again write $\nu = \nu' + n$, where $n \in \omega$ and $\nu' \leq (\aleph_1)^V$ is either a limit ordinal or 0, and let β be as before. We also assume without loss of generality that as in [11], [5], [3], [1], and [6], $\alpha_{\nu+1}$ is determined by D_ν when $\nu < (\aleph_1)^V$. Further, we adopt throughout each of the four cases the notation of [11], [5], [3], [1], and [6].

1. $\nu' = \nu \neq 0$, $\nu < (\aleph_1)^V$, and $n = 0$, i.e., $\nu < (\aleph_1)^V$ is a limit ordinal. $\underline{C_1(\nu, \beta)} = \{\langle r, (\check{r} \restriction \mathbb{R}_{<\aleph_1}) \restriction (\alpha_\nu(r), \beta)\rangle \mid r \in D_\nu\}$, where for $r \in \mathbb{P}$, $\pi = r \restriction \mathbb{R}_{<\aleph_1}$, $\pi \restriction (\alpha_\nu(r), \beta) = \{h_{p(\alpha_\nu)(r)}{}'' p \cap \beta \mid p \in \text{p-part}(\pi),\ p \subseteq p(\alpha_\nu)(r),\ p \in R_\ell \restriction \pi, \text{ and } h^{-1}_{p(\alpha_\nu)(r)}(\beta) \in p\}$.

2. $\nu = (\aleph_1)^V$. $\underline{C_2(\nu, \beta)} = \{\langle r, (\check{r} \restriction \mathbb{R}_{<\aleph_1}) \restriction (\kappa, \beta)\rangle\}$, where for $r \in \mathbb{P}$, $\pi = r \restriction \mathbb{R}_{<\aleph_1}$, $\pi \restriction (\kappa, \beta) = \{p \cap \beta \mid p \in \text{p-part}(\pi),\ p \in R_\ell \restriction \pi, \text{ and } \beta \in p\}$.

3. ($\nu = \nu' + n$, $\nu' > 0$, $\nu' < (\aleph_1)^V$, and $n \geq 2$) or ($\nu' = 0$ and $n \in \omega$), i.e., ν is neither a limit ordinal nor the successor of a limit ordinal less than $(\aleph_1)^V$. $\underline{C_3(\nu, \nu+1)} = \{\langle r, (\check{r} \restriction \text{Coll}(\alpha_\nu(r), {<}\alpha_{\nu+1}(r)))\rangle \mid r \in D_\nu\}$.

4. $\nu = \nu' + 1$ for $\nu' > 0$, $\nu' < (\aleph_1)^V$, i.e., ν is the successor of a limit ordinal less than $(\aleph_1)^V$. $\underline{C_4(\nu, \nu+1)} = \{\langle r, (\check{r} \restriction \text{Coll}(\alpha^+_\nu(r), {<}\alpha_{\nu+1}(r)))\rangle \mid r \in D_\nu\}$.

As in [11], [5], [3], [1], and [6], since for any $r, r' \in D_\nu \cap G$, $p(\alpha_\nu)(r) = p(\alpha_\nu)(r')$, each of the definitions just given is unambiguous.

Let $\mathcal{G}$ be the group of automorphisms of [11], and let $\underline{C(G)} = \{\psi(\underline{C_1(\nu, \beta)}) \mid \psi \in \mathcal{G},\ 0 \leq \nu < (\aleph_1)^V, \text{ and } \beta \in [\nu, \kappa) \text{ is a cardinal}\} \cup \{\psi(\underline{C_2(\nu, \beta)}) \mid \psi \in \mathcal{G},\ \nu = (\aleph_1)^V, \text{ and } \beta \in [\kappa, \lambda) \text{ is a cardinal}\} \cup \bigcup_{i=3,4} \{\psi(\underline{C_i(\nu, \nu+1)}) \mid \psi \in \mathcal{G} \text{ and } 0 \leq \nu < (\aleph_1)^V\}$. $C(G) = \{i_G(\psi(\underline{C_1(\nu, \beta)})) \mid \psi \in \mathcal{G},\ 0 \leq \nu < (\aleph_1)^V, \text{ and } \beta \in [\nu, \kappa) \text{ is a cardinal}\} \cup \{i_G(\psi(\underline{C_2(\nu, \beta)})) \mid \psi \in \mathcal{G},\ \nu = (\aleph_1)^V, \text{ and } \beta \in [\kappa, \lambda) \text{ is a cardinal}\} \cup \bigcup_{i=3,4} \{i_G(\psi(\underline{C_i(\nu, \nu+1)})) \mid \psi \in \mathcal{G} \text{ and } 0 \leq \nu < (\aleph_1)^V\} = i_G(\underline{C(G)})$. N is then the set of all sets which are hereditarily V definable from $C(G)$, i.e., $N = \text{HVD}(C(G))$.

Let $\langle \delta_\nu \mid \nu \leq (\aleph_1)^V + 1\rangle$ be the continuous, increasing enumeration of $\{\alpha_\nu \mid \nu \leq (\aleph_1)^V + 1\} \cup \{(\alpha^+_\nu)^V \mid \nu = \gamma + 1 \text{ and } \gamma < (\aleph_1)^V \text{ is a limit ordinal}\}$. The arguments of [11] and [3] allow us to conclude that $N \models$ "ZF + $\neg\mathrm{AC}_\omega$ + $\aleph_1 = (\aleph_1)^V$ is a regular cardinal + $\langle \aleph_\nu \mid \nu < \aleph_1 \rangle = \langle \delta_\nu \mid \nu < \aleph_1 \rangle$ + For every limit ordinal $\nu < \aleph_1$, $\aleph_{\nu+1}$ is a measurable cardinal + $\kappa = \aleph_{\omega_1}$ + Every successor cardinal less than $\aleph_{\omega_1}$ is regular + $\kappa^+ = \lambda = \aleph_{\omega_1+1}$ is a measurable cardinal". In addition, we know that for any ordinal γ and any set $x \subseteq \gamma$, $x \in N$, $x = \{\alpha < \gamma \mid V[G] \models \phi(\alpha, i_G(\psi_1(\underline{C_{i_1}(\nu_1, \beta_1)})), \ldots, i_G(\psi_n(\underline{C_{i_n}(\nu_n, \beta_n)})), C(G))\}$, where i_j is an integer, $1 \leq j \leq n$, $1 \leq i_j \leq 4$, each $\psi_i \in \mathcal{G}$, each β_i is an appropriate ordinal for ν_i, and $\phi(x_0, \ldots, x_{n+1})$ is a formula which may also contain some parameters from V which we shall suppress.

Let

$$\overline{\mathbb{P}} = \prod_{i_j=3, j \leq n} \mathrm{Coll}(\alpha_{\nu_j}, {<}\alpha_{\nu_j+1}) \times \prod_{i_j=4, j\leq n} \mathrm{Coll}(\alpha^+_{\nu_j}, {<}\alpha_{\nu_j+1}) \times \mathbb{R}_{<\aleph_1}.$$

For $\pi \in \mathbb{R}_{<\aleph_1}$, let $\pi \restriction \gamma = \{\langle q, u, C\rangle \in \pi \mid q \cap \kappa \leq \gamma\}$. For $p \in \overline{\mathbb{P}}$, $p = \langle p_1, \ldots, p_m, \pi\rangle$, $m \leq n$, $\pi \in \mathbb{R}_{<\aleph_1}$, let $p \restriction \gamma = \langle q_1, \ldots, q_m, \pi \restriction \gamma\rangle$, where $q_j = p_j$ if $\alpha_{\nu_j} \leq \gamma$ and $q_j = \emptyset$ otherwise. In other words, $p \restriction \gamma$ is the part of p below or at γ. Without loss of generality, we ignore the empty coordinates and let $\overline{\mathbb{P}} \restriction \gamma = \{p \restriction \gamma \mid p \in \overline{\mathbb{P}}\}$. Let $G \restriction \gamma$ be the projection of G onto $\overline{\mathbb{P}} \restriction \gamma$. An analogous fact to [11, Theorem 3.2.11] holds, using the same proof as in [11], namely $x \in V[G \restriction \gamma]$. In addition, the elements of $\overline{\mathbb{P}} \restriction \gamma$ can be partitioned into equivalence classes (the "almost similar" equivalence classes of [11]) with respect to $\underline{C_{i_1}(\nu_1, \beta_1)}, \ldots, \underline{C_{i_n}(\nu_n, \beta_n)}$ via an equivalence relation to be called $\sim$ such that if φ is any formula mentioning only (terms for ground model sets and) $\underline{C_{i_1}(\nu_1, \beta_1)}, \ldots, \underline{C_{i_n}(\nu_n, \beta_n)}$, and $\underline{C(G)}$, $p \parallel \varphi$ (i.e., p *decides* φ), and $q \sim p$, then $q \parallel \varphi$ in the same way that p does. It thus follows as an immediate corollary of the work of [11] that if we define $G_\gamma^{\langle\langle \nu_1, \beta_1\rangle, \ldots, \langle \nu_n, \beta_n\rangle\rangle} = \{[p]_\sim \mid p \in G \restriction \gamma\}$, then $x \in V[G_\gamma^{\langle\langle \nu_1, \beta_1\rangle, \ldots, \langle \nu_n, \beta_n\rangle\rangle}]$ and $V[G_\gamma^{\langle\langle \nu_1, \beta_1\rangle, \ldots, \langle \nu_n, \beta_n\rangle\rangle}] \subseteq N$. Further, suppose $\nu \leq (\aleph_1)^V$ is a limit ordinal. The work of [11] also tells us that for $\gamma = \alpha_{\nu+1} = \delta_{\nu+1} = (\aleph_{\nu+1})^N$, since x is a set of ordinals, we may assume that for $\beta = \max(\{\beta_i \mid i \leq n\})$, $\beta \in [\alpha_\nu, \alpha_{\nu+1})$. Because $\beta \in [\alpha_\nu, \alpha_{\nu+1})$, it follows that $G_\gamma^{\langle\langle \nu_1, \beta_1\rangle, \ldots, \langle \nu_n, \beta_n\rangle\rangle}$ is V-generic over a partial ordering forcing equivalent to a partial ordering $\mathbb{Q}_\gamma^{\langle\langle \nu_1, \beta_1\rangle, \ldots, \langle \nu_n, \beta_n\rangle\rangle}$ such that $|\mathbb{Q}_\gamma^{\langle\langle \nu_1, \beta_1\rangle, \ldots, \langle \nu_n, \beta_n\rangle\rangle}| < \gamma$. And, if $\gamma \geq \lambda$ is a V-cardinal, it is the case that $G_\gamma^{\langle\langle \nu_1, \beta_1\rangle, \ldots, \langle \nu_n, \beta_n\rangle\rangle}$ is V-generic over a partial ordering forcing equivalent to a partial ordering $\mathbb{Q}_\gamma^{\langle\langle \nu_1, \beta_1\rangle, \ldots, \langle \nu_n, \beta_n\rangle\rangle}$ such that $|\mathbb{Q}_\gamma^{\langle\langle \nu_1, \beta_1\rangle, \ldots, \langle \nu_n, \beta_n\rangle\rangle}| < \lambda$. The facts in the next to last sentence and [3, Lemma 2], in tandem with the way in which N is defined, provide a proof that if $\mathcal{U} \in V$ is a normal measure over γ, then $N \models$ "$\mathcal{U}' = \{x \subseteq \gamma \mid \exists y \subseteq x[y \in \mathcal{U}]\}$ is a normal measure over γ". From this, we will be able to infer that $\aleph_{\omega_1+1}$ carries the desired number of normal measures in N. In particular, the following two lemmas complete the proof of Theorem 1.

Lemma 1.1 *Suppose $\mathcal{U}^* \in N$ is a normal measure over λ. Then for some normal measure $\mathcal{U} \in V$ over λ, $\mathcal{U}^* = \{x \subseteq \lambda \mid \exists y \subseteq x[y \in \mathcal{U}]\}$.*

Proof: Our proof is very similar to the proofs of [2, Lemma 2.2] and [4, Lemma 1.1], both of which use ideas from the proof of [7, Theorem 2.3(e)]. Let σ be a term for $\mathcal{U}^*$. Since $\mathcal{U}^* \in N$, we may choose ordinals $\nu_1, \ldots, \nu_n, \beta_1, \ldots, \beta_n$ and terms $\underline{C_{i_1}(\nu_1, \beta_1)}, \ldots, \underline{C_{i_n}(\nu_n, \beta_n)}$ such that σ mentions only $\underline{C_{i_1}(\nu_1, \beta_1)}, \ldots, \underline{C_{i_n}(\nu_n, \beta_n)}$, $\underline{C(G)}$, and canonical terms for sets in V. We may assume (by "padding" if necessary) that for $\beta = \max(\{\beta_i \mid i \leq n\})$, $\beta \in [\kappa, \lambda)$. This means by our remarks in the last paragraph that $G_\lambda^{\langle\langle \nu_1, \beta_1\rangle, \ldots, \langle \nu_n, \beta_n\rangle\rangle}$ is V-generic over a partial ordering forcing equivalent to one having size less than λ. Again by what was mentioned in the preceding paragraph, the set $\mathcal{U}^{**} = \mathcal{U}^* \cap V[G_\lambda^{\langle\langle \nu_1, \beta_1\rangle, \ldots, \langle \nu_n, \beta_n\rangle\rangle}] \in V[G_\lambda^{\langle\langle \nu_1, \beta_1\rangle, \ldots, \langle \nu_n, \beta_n\rangle\rangle}]$, which immediately implies that $\mathcal{U}^{**}$ is in $V[G_\lambda^{\langle\langle \nu_1, \beta_1\rangle, \ldots, \langle \nu_n, \beta_n\rangle\rangle}]$ a normal measure over λ. Once more by what was stated in the last paragraph and the Lévy-Solovay results [12], it must consequently be the case that for some $\mathcal{U} \in V$ a normal measure over λ, $\mathcal{U}^{**}$ is definable in $V[G_\lambda^{\langle\langle \nu_1, \beta_1\rangle, \ldots, \langle \nu_n, \beta_n\rangle\rangle}]$ as $\{x \subseteq \lambda \mid \exists y \subseteq x[y \in \mathcal{U}]\}$. Therefore, since $\mathcal{U} \subseteq \mathcal{U}^{**} \subseteq \mathcal{U}^*$ and $\mathcal{U}' = \{x \subseteq \lambda \mid \exists y \subseteq x[y \in \mathcal{U}]\}$ as defined in N is an ultrafilter over λ, $\mathcal{U}' = \mathcal{U}^*$. This completes the proof of Lemma 1.1.

□

Lemma 1.2 *In N, the cardinal and cofinality structure above λ is the same as in V.*

Proof: We follow the proof of [2, Lemma 2.3]. Let β and γ be arbitrary ordinals, and suppose that $N \models$ "$f : \beta \to \gamma$ is a function". Since f may be coded by a set of ordinals, by our remarks in the paragraph immediately preceding the statement of Lemma 1.1, f is a member of a generic extension of V via a partial ordering having cardinality less than λ. Thus, f cannot witness that any V-cardinal greater than or equal to λ has a different cardinality or cofinality. This completes the proof of Lemma 1.2.

$\square$

By Lemmas 1.1 and 1.2 and our earlier exposition, if V^* is as in Theorem 1, then N witnesses the conclusions of Theorem 1. Similarly, Lemmas 1.1 and 1.2 and our earlier exposition imply that if V^* is as in Theorem 2, then N witnesses the conclusions of Theorem 2. This completes the proofs of Theorems 1 and 2.

$\square$

As with the proof found in [2], the proof that has just been given easily generalizes to handling successors of different singular cardinals of uncountable cofinality, such as $\aleph_{\omega_1+\omega_1+1}$, $\aleph_{\omega_2+1}$, etc. It is only necessary to change the length of the Radin sequence of measures accordingly, i.e., to $\omega_1 + \omega_1$, ω_2, etc. As in [2], though, it is unknown if it is possible for $\aleph_{\omega_1+1}$ to carry precisely τ normal measures, where $\tau \le \aleph_{\omega_1+1}$ is an arbitrary infinite or finite cardinal.

It is natural to wonder about the possibility of extending Theorems 1 and 2 so as also to control the number of normal measures over $\aleph_{\nu+1}$, where $\nu < \aleph_1$ is a limit ordinal. The methods of this paper do not seem to allow this to be done. In particular, the proof of Lemma 1.1 breaks down. To see this, let γ be such that $N \vDash$ "$\gamma = \aleph_{\nu+1}$, where $\nu < \aleph_1$ is a limit ordinal". We will not always be able to infer that $\mathcal{U}^{**} \in V[G_\gamma^{\langle\langle\nu_1,\beta_1\rangle,\ldots,\langle\nu_n,\beta_n\rangle\rangle}]$. This is since $\mathcal{U}^{**}$ is not a set of ordinals, so we cannot necessarily assume that $\beta \in [\alpha_\nu, \alpha_{\nu+1})$, where as before, $\beta = \max(\{\beta_i \mid i \le n\})$. We therefore conclude by asking whether it is possible to extend Theorems 1 and 2 so that the successor of each singular cardinal less than or equal to $\aleph_{\omega_1}$, or even more generally, the successor of each singular cardinal, is both measurable and has the number of normal measures it carries exactly controlled.

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