A Note on Tall Cardinals and Level by Level Equivalence *†

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Abstract

Starting from a model $V \vDash$ "ZFC + GCH + κ is supercompact + No cardinal is supercompact up to a measurable cardinal", we force and construct a model $V^{\mathbb{P}}$ such that $V^{\mathbb{P}} \vDash$ "ZFC + κ is supercompact + No cardinal is supercompact up to a measurable cardinal + δ is measurable iff δ is tall" in which level by level equivalence between strong compactness and supercompactness holds. This extends and generalizes both [4, Theorem 1] and the results of [5].

1 Introduction and Preliminaries

We begin with some definitions and terminology. Suppose κ is a cardinal and $\lambda \geq \kappa$ is an arbitrary ordinal. κ is λ *tall* if there is an elementary embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \lambda$ and $M^\kappa \subseteq M$. κ is *tall* if κ is λ tall for every ordinal λ. Hamkins made a systematic study of tall cardinals in [8].

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Suppose V is a model of ZFC containing at least one supercompact cardinal in which for all regular cardinals $\kappa \le \lambda$, κ is λ strongly compact iff κ is λ supercompact. Such a model will be said to witness *level by level equivalence between strong compactness and supercompactness.* We will also say that κ *is a witness to level by level equivalence between strong compactness and supercompactness* iff for every regular cardinal $\lambda \ge \kappa$, κ is λ strongly compact iff κ is λ supercompact. Models in which level by level equivalence between strong compactness and supercompactness holds nontrivially were first constructed in [5]. For brevity, we will henceforth write only *level by level equivalence.*

Turning now to the main narrative, in [4], the following theorem was proven.

Theorem 1 ([4, Theorem 1]) *Suppose $V \vDash$ "ZFC + GCH + κ is supercompact + No cardinal $\lambda > \kappa$ is measurable". There is then a partial ordering $\mathbb{P} \in V$ such that $V^{\mathbb{P}} \vDash$ "ZFC + κ is supercompact + No cardinal $\lambda > \kappa$ is measurable + δ is measurable iff δ is tall".*

Theorem 1 extends and generalizes [8, Corollary 4.3].

The purpose of this note is to show that it is possible to combine the results of Theorem 1 with the property of level by level equivalence. Specifically, we will prove the following theorem.

Theorem 2 *Suppose $V \vDash$ "ZFC + GCH + κ is supercompact + No cardinal is supercompact up to a measurable cardinal". There is then a partial ordering $\mathbb{P} \in V$ such that $V^{\mathbb{P}} \vDash$ "ZFC + κ is supercompact + No cardinal is supercompact up to a measurable cardinal + δ is measurable iff δ is tall". In $V^{\mathbb{P}}$, level by level equivalence holds.*

We remark that the hypotheses of Theorem 2 automatically imply the hypotheses of Theorem 1. This is since otherwise, if there were a measurable cardinal above the supercompact cardinal κ, κ and by reflection, unboundedly in κ many cardinals below it, would be supercompact up to a measurable cardinal. Also, Theorem 2 may be thought of as a very strong "identity crisis" type of result, the study of which was first initiated by Magidor in [13]. In the model witnessing the conclusions of Theorem 2, not only do the measurable and tall cardinals coincide precisely, but in addition, the partially strongly compact and partially supercompact cardinals also coincide precisely, as long as the coincidence is at regular levels. Further, the unique supercompact cardinal κ in the model witnessing the conclusions of Theorem 2 is also the unique strongly compact cardinal. This is since any strongly compact cardinal $\delta < \kappa$ would of necessity have to witness a failure of level by level equivalence, and there are no measurable cardinals above κ.

Before beginning the proofs of our theorems, we briefly mention some preliminary information and terminology. Essentially, our notation and terminology are standard, and when this is not the case, this will be clearly noted. For $\alpha < \beta$ ordinals, $[\alpha, \beta]$, $[\alpha, \beta)$, $(\alpha, \beta]$, and (α, β) are as in the usual interval notation. When forcing, $q \geq p$ will mean that q *is stronger than* p. If G is V-generic over $\mathbb{P}$, we will abuse notation slightly and use both $V[G]$ and $V^{\mathbb{P}}$ to indicate the universe obtained by forcing with $\mathbb{P}$. If $x \in V[G]$, then $\dot{x}$ will be a term in V for x. We may, from time to time, confuse terms with the sets they denote and write x when we actually mean $\dot{x}$ or $\check{x}$, especially when x is some variant of the generic set G, or x is in the ground model V. The abuse of notation mentioned above will be compounded by writing $x \in V^{\mathbb{P}}$ instead of $\dot{x} \in V^{\mathbb{P}}$. If φ is a formula in the forcing language with respect to $\mathbb{P}$ and $p \in \mathbb{P}$, then $p \parallel \varphi$ means that p *decides* φ.

As in [7], we will say that the partial ordering $\mathbb{P}$ is κ^+*-weakly closed and satisfies the Prikry property* if it meets the following criteria.

1. $\mathbb{P}$ has two partial orderings $\leq$ and $\leq^*$ with $\leq^* \subseteq \leq$.

2. For every $p \in \mathbb{P}$ and every statement φ in the forcing language with respect to $\mathbb{P}$, there is some $q \in \mathbb{P}$ such that $p \leq^* q$ and $q \parallel \varphi$.

3. The partial ordering $\leq^*$ is κ-closed, i.e., there is an upper bound for every increasing chain of conditions having length κ.

Note that if $\mathbb{P}$ is κ^+-weakly closed and satisfies the Prikry property, then in analogy to Prikry forcing, V and $V^{\mathbb{P}}$ contain the same subsets of κ.

From time to time within the course of our discussion, we will refer to partial orderings $\mathbb{P}$ as being *Gitik iterations of forcings satisfying the Prikry property.* By this we will mean an Easton support iteration as first given by Gitik in [6], to which we refer readers for a discussion of the basic properties of and terminology associated with such an iteration.

Key to the proofs of Theorems 1 and 2 is the following result due to Gitik and Shelah [7]. It is a corollary of the work of [7, Section 2] and is an analogue to [4, Theorem 9].

Theorem 3 *Suppose $V \models$ "ZFC + GCH + κ is a strong cardinal greater than δ". Let δ^* be the least measurable cardinal greater than δ. There is then a $(\delta^*)^+$-weakly closed partial ordering $\mathbb{I}(\delta, \kappa)$ satisfying the Prikry property having cardinality κ such that $V^{\mathbb{I}(\delta,\kappa)} \models$ "κ is a strong cardinal whose strongness is indestructible under κ^+-weakly closed partial orderings satisfying the Prikry property".*

We mention that we are assuming some familiarity with the large cardinal notions of measurability, tallness, strongness, strong compactness, and supercompactness. Interested readers may consult [10] or [8]. We note only that we will say κ *is supercompact (strongly compact) up to the cardinal* λ if κ is δ supercompact (strongly compact) for every $\delta < \kappa$.

2 The Proof of Theorem 2

We turn now to the proof of Theorem 2.

Proof: Suppose $V \models$ "ZFC + GCH + κ is supercompact + No cardinal is supercompact up to a measurable cardinal". Without loss of generality, by first doing the forcing of [5], we abuse notation and also assume that level by level equivalence holds in V.

Let $\mathcal{C} = \{\delta < \kappa \mid \delta$ is a strong cardinal which is not a limit of strong cardinals$\}$. $\mathbb{I}$ will be the partial ordering of [4, Lemma 2.2] defined using Theorem 3 which makes each $\delta \in \mathcal{C}$ a strong cardinal indestructible under δ^+-weakly closed partial orderings satisfying the Prikry property. We explicitly give the definition of $\mathbb{I}$ now, quoting almost verbatim from [4]. Let $\langle \delta_\alpha \mid \alpha < \kappa \rangle$ enumerate in increasing order the members of $\mathcal{C}$. For every $\alpha < \kappa$, let $\gamma_\alpha = \sup_{\beta < \alpha} \delta_\beta$, where $\gamma_0 = \omega$. $\mathbb{I} = \langle \langle \mathbb{I}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle \mid \alpha < \kappa \rangle$ is taken as the Gitik iteration of forcings satisfying the Prikry property of length κ such that $\mathbb{I}_0 = \{\emptyset\}$. For every $\alpha < \kappa$, $\mathbb{I}_{\alpha+1} = \mathbb{I}_\alpha * \dot{\mathbb{Q}}_\alpha$, where $\dot{\mathbb{Q}}_\alpha$ is a term for the partial ordering $\mathbb{I}(\gamma_\alpha, \delta_\alpha)$ of Theorem 3 as defined in $V^{\mathbb{I}_\alpha}$. Note that this makes sense, since inductively, it is the case that $|\mathbb{I}_\alpha| < \delta_\alpha$. By the Hamkins-Woodin results [9], $V^{\mathbb{I}_\alpha} \models$ "δ_α is a strong cardinal", meaning that $\mathbb{I}_{\alpha+1}$ as just given is valid.

Lemma 2.1 *Suppose that $V^{\mathbb{I}} \models$ "δ is a measurable limit of members of $\mathcal{C}$". Then $V^{\mathbb{I}} \models$ "Level by level equivalence holds at δ".*

Proof: Let δ be as in the hypotheses for Lemma 2.1. Write $\mathbb{I} = \mathbb{I}_\delta * \dot{\mathbb{I}}^\delta$. Because $V^{\mathbb{I}} \models$ "δ is a measurable limit of members of $\mathcal{C}$", δ is Mahlo in $V^{\mathbb{I}}$. Since forcing can't create a new Mahlo cardinal, δ is Mahlo in V. Thus, by the definition of $\mathbb{I}$, $\mathbb{I}_\delta$ is the direct limit of $\langle \mathbb{I}_\alpha \mid \alpha < \delta \rangle$. Hence, $\mathbb{I}_\delta$ is δ-c.c., $|\mathbb{I}_\delta| = \delta$, $\gamma_\delta = \delta$, and $\Vdash_{\mathbb{I}_\delta}$ " Forcing with $\dot{\mathbb{I}}^\delta$ adds no bounded subsets of $(\delta^*)^V = (\delta^*)^{V^{\mathbb{I}_\delta}}$".

Suppose now $\lambda \geq \delta$ is such that $V^{\mathbb{I}} \models$ "δ is λ strongly compact and λ is regular". We first assume that $\lambda \in [\delta, (\delta^*)^V)$. Since $\Vdash_{\mathbb{I}_\delta}$ "$\dot{\mathbb{I}}^\delta$ adds no bounded subsets of $(\delta^*)^V$", it follows that $\Vdash_{\mathbb{I}_\delta}$ "δ is λ strongly compact". Note that since δ is measurable in $V^{\mathbb{I}_\delta}$, δ is once again Mahlo in $V^{\mathbb{I}_\delta}$. This means that since $\mathbb{I}_\delta$ satisfies δ-c.c. in $V^{\mathbb{I}_\delta}$ as well (this follows because δ is Mahlo in $V^{\mathbb{I}_\delta}$ and $\mathbb{I}_\delta$ is a subordering of the direct limit of $\langle \mathbb{I}_\alpha \mid \alpha < \delta \rangle$ as calculated in $V^{\mathbb{I}_\delta}$), (the proof of) [1, Lemma 8] (see in particular the argument found starting in [1, third paragraph of page 111]) or (the proof of) [2, Lemma 3] tells us that every δ-additive uniform ultrafilter over a regular cardinal $\beta \geq \delta$ present in $V^{\mathbb{I}_\delta}$ must be an extension of a δ-additive uniform ultrafilter over β in V. Therefore, since the λ strong compactness of δ in $V^{\mathbb{I}_\delta}$ implies that every $V^{\mathbb{I}_\delta}$-regular cardinal $\beta \in [\delta, \lambda]$ carries a δ-additive uniform ultrafilter in $V^{\mathbb{I}_\delta}$, and since the fact $\mathbb{I}_\delta$ is δ-c.c. tells us the regular cardinals at or above δ in $V^{\mathbb{I}_\delta}$ are the same as those in V, the preceding sentence implies that every V-regular cardinal $\beta \in [\delta, \lambda]$ carries a δ-additive uniform ultrafilter in V. Ketonen's theorem of [11] then implies that δ is λ strongly compact in V. By level by level equivalence, $V \models$ "δ is λ supercompact" as well.

Let G be V-generic over $\mathbb{I}_\delta$. Take $j : V \to M$ as an elementary embedding witnessing the λ supercompactness of δ generated by a supercompact ultrafilter over $P_\delta(\lambda)$. Write $j(\mathbb{I}_\delta) = \mathbb{I}_\delta * \dot{\mathbb{I}}'$. Because $\mathrm{cp}(j) = \delta$, $j(\mathcal{C}) \cap \delta = \mathcal{C} \cap \delta$. Further, if $V \models$ "γ is a member of $\mathcal{C} \cap \delta$", $M \models$ "$j(\gamma) = \gamma$ is a member of $j(\mathcal{C}) \cap \delta = \mathcal{C} \cap \delta$". Thus, since $V \models$ "δ is a Mahlo limit of members of $\mathcal{C} \cap \delta$" and $M^\lambda \subseteq M$, $M \models$ "δ is a Mahlo limit of members of $j(\mathcal{C}) \cap \delta = \mathcal{C} \cap \delta$". By the definition of $\mathbb{I}$ and $\mathbb{I}_\delta$, this means that in M, $\Vdash_{\mathbb{I}_\delta}$ "$\dot{\mathbb{I}}'$ is $((\delta^*)^+)^M$-weakly closed and satisfies the Prikry property". Also, as $\lambda < (\delta^*)^V$, $\lambda < (\delta^*)^M$. This is since otherwise, if $(\delta^*)^M \leq \lambda$, then because $M^\lambda \subseteq M$, $(\delta^*)^M$ is measurable in both V and M. Again using the fact $M^\lambda \subseteq M$, it follows that in V as well as M, δ is supercompact up to the measurable cardinal $(\delta^*)^M$. This is a contradiction to our hypotheses on V. In particular, in both V and M, as $\lambda < (\delta^*)^M$, $\Vdash_{\mathbb{I}_\delta}$ "$\dot{\mathbb{I}}'$ is λ^+-weakly closed and satisfies the Prikry property".

As in the proof of [4, Lemma 2.2], we may now apply the argument of [6, Lemma 1.5]. We again feel free to quote almost verbatim from [4] as appropriate. Specifically, since GCH in V implies that $V \models$ "$2^\lambda = \lambda^+$", we may let $\langle \dot{x}_\alpha \mid \alpha < \lambda^+ \rangle$ be an enumeration in V of all of the canonical $\mathbb{I}_\delta$-names of subsets of $(P_\delta(\lambda))^{V[G]}$. Because $\mathbb{I}_\delta$ is δ-c.c. and $M^\lambda \subseteq M$, $M[G]^\lambda \subseteq M[G]$. By [6, Lemmas 1.4 and 1.2], we may therefore define in $V[G]$ an increasing sequence $\langle p_\alpha \mid \alpha < \lambda^+ \rangle$ of elements of $j(\mathbb{I}_\delta)/G$ such that if $\alpha < \beta < \lambda^+$, p_β is an Easton extension of p_α,[1] every initial segment of the sequence is in $M[G]$, and for every $\alpha < \lambda^+$, $p_{\alpha+1} \parallel$ "$\langle j(\beta) \mid \beta < \lambda \rangle \in j(\dot{x}_\alpha)$". The remainder of the argument of [6, Lemma 1.5] remains valid and shows that a supercompact ultrafilter $\mathcal{U}$ over $(P_\delta(\lambda))^{V[G]}$ may be defined in $V[G]$ by $x \in \mathcal{U}$ iff $x \subseteq (P_\delta(\lambda))^{V[G]}$ and for some $\alpha < \lambda^+$ and some $\mathbb{I}_\delta$-name $\dot{x}$ of x, in $M[G]$, $p_\alpha \Vdash_{j(\mathbb{I}_\delta)/G}$ "$\langle j(\beta) \mid \beta < \lambda \rangle \in j(\dot{x})$". (The fact that $j''G = G$ tells us $\mathcal{U}$ is well-defined.) Thus, $\Vdash_{\mathbb{I}_\delta}$ "δ is λ supercompact", so since $\Vdash_{\mathbb{I}_\delta}$ "Forcing with $\dot{\mathbb{I}}^\delta$ adds no bounded subsets of $(\delta^*)^V$", $V^{\mathbb{I}_\delta * \dot{\mathbb{I}}^\delta} = V^{\mathbb{I}} \models$ "δ is λ supercompact".

The work of the preceding paragraph shows that if $\lambda \geq \delta$ is such that $\lambda \in [\delta, (\delta^*)^V)$ and $V^{\mathbb{I}} \models$ "δ is λ strongly compact and λ is regular", then $V^{\mathbb{I}} \models$ "δ is λ supercompact". The proof of Lemma 2.1 will therefore be completed by showing that for no $V^{\mathbb{I}}$-regular cardinal $\lambda \geq (\delta^*)^V$ is it true that $V^{\mathbb{I}} \models$ "δ is λ strongly compact". Assume to the contrary that λ is a counterexample. Since $V^{\mathbb{I}} \models$ "δ is λ strongly compact", $\lambda \geq (\delta^*)^V$, and $\Vdash_{\mathbb{I}_\delta}$ "Forcing with $\dot{\mathbb{I}}^\delta$ adds no bounded subsets of $(\delta^*)^V$", $V^{\mathbb{I}_\delta} \models$ "δ is strongly compact up to $(\delta^*)^V$". Because the regular cardinals at and above δ are the same in both V and $V^{\mathbb{I}_\delta}$, this means that for any (V or $V^{\mathbb{I}_\delta}$)-regular cardinal $\lambda \in (\delta, (\delta^*)^V)$, $V^{\mathbb{I}_\delta} \models$ "δ is λ strongly compact". By our arguments in the second paragraph of the proof of this lemma,

[1] Roughly speaking, this means that p_β extends p_α as in a usual reverse Easton iteration, except that at coordinates at which, e.g., Prikry forcing or some variant or generalization thereof occurs in p_α, measure 1 sets are shrunk and stems are not extended. For a more precise definition, readers are urged to consult [6]. We do note, however, that $\leq^*$ here is Easton extension.

$V \models$ "δ is λ supercompact". This, however, is a contradiction to our assumption that $V \models$ "No cardinal is supercompact up to a measurable cardinal", since $(\delta^*)^V$ is the least measurable cardinal greater than δ in V. This contradiction completes the proof of Lemma 2.1.

$\square$

Lemma 2.2 *If $V^{\mathbb{I}} \models$ "δ is a measurable limit of members of $\mathcal{C}$", then $V^{\mathbb{I}} \models$ "δ is not supercompact up to a measurable cardinal".*

Proof: Suppose to the contrary $\gamma \geq \delta$ is such that $V^{\mathbb{I}} \models$ "δ is supercompact up to γ and γ is measurable". We have that $\Vdash_{\mathbb{I}_\delta}$ "Forcing with $\dot{\mathbb{I}}^\delta$ adds no bounded subsets of $(\delta^*)^V = (\delta^*)^{V^{\mathbb{I}_\delta}}$". Consequently, $V^{\mathbb{I}} \models$ "$\gamma \geq (\delta^*)^V$", and $V^{\mathbb{I}_\delta} \models$ "δ is supercompact (and hence also strongly compact) up to $(\delta^*)^V$". The last paragraph of the proof of Lemma 2.1 shows that this is a contradiction. This completes the proof of Lemma 2.2.

$\square$

We continue as in the proof of [4, Theorem 1]. We assume now that our ground model is $V^{\mathbb{I}}$. Given this, and adopting the notation of [4, Theorem 1], let $\mathbb{P}(\gamma_\alpha, \delta_\alpha)$ for every $\alpha < \kappa$ be the Magidor iteration of Prikry forcing from [13] which adds a Prikry sequence to every measurable cardinal in the open interval $(\gamma_\alpha, \delta_\alpha)$. The partial ordering $\mathbb{P} = \langle\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha\rangle \mid \alpha < \kappa\rangle$ with which we force is defined as the Gitik iteration of forcings satisfying the Prikry property of length κ such that $\mathbb{P}_0 = \{\emptyset\}$. For every $\alpha < \kappa$, $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$, where $\dot{\mathbb{Q}}_\alpha$ is a term for the partial ordering $\mathbb{P}(\gamma_\alpha, \delta_\alpha)$ as defined in $V^{\mathbb{I}}$ (and *not as defined in* $V^{\mathbb{I} * \dot{\mathbb{P}}_\alpha}$, which means that $\mathbb{P}$ may be equivalently defined in $V^{\mathbb{I}}$ as the Easton support product $\prod_{\alpha<\kappa} \mathbb{P}(\gamma_\alpha, \delta_\alpha)$ — see [4, line 3 of the proof of Lemma 2.3]). By [4, Lemmas 2.1 – 2.4] and the intervening remarks, $V^{\mathbb{I} * \dot{\mathbb{P}}} \models$ "κ is supercompact + $\delta < \kappa$ is measurable iff δ is tall iff either $\delta \in \mathcal{C}$ or δ is a measurable limit of members of $\mathcal{C}$". Because $V \models$ "No cardinal is supercompact up to a measurable cardinal + κ is supercompact", $V \models$ "No cardinal above κ is measurable". Since by its definition, $|\mathbb{I} * \dot{\mathbb{P}}| = \kappa$, by the Lévy-Solovay results [12], $V^{\mathbb{I} * \dot{\mathbb{P}}} \models$ "No cardinal above κ is measurable". Hence, $V^{\mathbb{I} * \dot{\mathbb{P}}} \models$ "δ is measurable iff δ is tall iff either $\delta \in \mathcal{C}$ or δ is a measurable limit of members of $\mathcal{C}$". The proof of Theorem 2 will therefore be complete once we have shown the following.

Lemma 2.3 *In $V^{\mathbb{I} * \dot{\mathbb{P}}}$, no cardinal is supercompact up to a measurable cardinal, and level by level equivalence holds.*

Proof: We first note that these facts are true if $V^{\mathbb{I} * \dot{\mathbb{P}}} \models$ "δ is a measurable limit of members of $\mathcal{C}$". To see this, working in $V^{\mathbb{I}}$, write $\mathbb{P} = \mathbb{P}_\delta * \dot{\mathbb{P}}^\delta$. Because δ is measurable and hence Mahlo in $V^{\mathbb{I} * \dot{\mathbb{P}}}$ and forcing can't create a new Mahlo cardinal, δ is Mahlo in $V^{\mathbb{I}}$. Therefore, by the definition of $\mathbb{P}$, $\mathbb{P}_\delta$ is the direct limit of $\langle \mathbb{P}_\alpha \mid \alpha < \delta \rangle$, so as in the proof of Lemma 2.1, $\mathbb{P}_\delta$ satisfies δ-c.c. in both $V^{\mathbb{I}}$ and $V^{\mathbb{I} * \dot{\mathbb{P}}_\delta}$. Further, by [4, page 351, paragraph immediately prior to Lemma 2.3], $\Vdash_{\mathbb{P}_\delta}$ "Forcing with $\dot{\mathbb{P}}^\delta$ adds no bounded subsets of $(\delta^*)^{V^{\mathbb{I} * \dot{\mathbb{P}}_\delta}} = (\delta^*)^{V^{\mathbb{I}}}$". This means that the arguments of Lemma 2.1 remain valid with $\mathbb{I}$ in V replaced by $\mathbb{P}$ in $V^{\mathbb{I}}$ and show that if $V^{\mathbb{I} * \dot{\mathbb{P}}} \models$ "δ is a measurable limit of members of $\mathcal{C}$", then level by level equivalence holds at δ in $V^{\mathbb{I} * \dot{\mathbb{P}}}$. The arguments of Lemma 2.2 also remain valid and show that if $V^{\mathbb{I} * \dot{\mathbb{P}}} \models$ "δ is a measurable limit of members of $\mathcal{C}$", then δ is not supercompact up to a measurable cardinal in $V^{\mathbb{I} * \dot{\mathbb{P}}}$.

We consider now what happens if $\delta \in \mathcal{C}$. Under these circumstances, we will show that $V^{\mathbb{I} * \dot{\mathbb{P}}} \models$ "δ is not δ^+ strongly compact". This automatically implies that level by level equivalence holds at δ, since δ is measurable iff δ is δ strongly compact iff δ is δ supercompact.

To see this, assume towards a contradiction that $V^{\mathbb{I} * \dot{\mathbb{P}}} \models$ "δ is δ^+ strongly compact". By the fact $\mathbb{P}$ may be equivalently defined as the Easton support product $\prod_{\alpha < \kappa} \mathbb{P}(\gamma_\alpha, \delta_\alpha)$, it is possible in $V^{\mathbb{I}}$ to write $\mathbb{P} = \mathbb{Q}^0 \times \mathbb{Q}^1 \times \mathbb{Q}^2$. Here, $|\mathbb{Q}^0| < \delta$, $\mathbb{Q}^1 = \mathbb{P}(\gamma, \delta)$, where $\gamma = \gamma_\beta$ for the ordinal β such that $\delta = \delta_\beta$, and $\mathbb{Q}^2 = \prod_{\alpha \geq \beta + 1} \mathbb{P}(\gamma_\alpha, \delta_\alpha)$. Since $|\mathbb{Q}^0| < \delta$, the results of [12] tell us that $V^{\mathbb{I} * (\dot{\mathbb{Q}}^1 \times \dot{\mathbb{Q}}^2)} \models$ "δ is δ^+ strongly compact". Again by [4, page 351, paragraph immediately prior to Lemma 2.3], $\Vdash_{\mathbb{I}}$ "Forcing with $\dot{\mathbb{Q}}^2$ adds no bounded subsets of $(\delta^*_\beta)^{V^{\mathbb{I}}}$". Thus, $\mathbb{Q}^1$ has the same definition in both $V^{\mathbb{I}}$ and $V^{\mathbb{I} * \dot{\mathbb{Q}}^2}$, and $V^{\mathbb{I} * \dot{\mathbb{Q}}^1} \models$ "δ is δ^+ strongly compact". Hence, both δ and δ^+ carry δ-additive, uniform ultrafilters in $V^{\mathbb{I} * \dot{\mathbb{Q}}^1}$. However, because the Magidor iteration of Prikry forcing from [13] does not collapse any cardinals, by [13, Theorem 3.1] and its proof, both δ and δ^+ carry δ-additive, uniform ultrafilters in $V^{\mathbb{I}}$. Ketonen's theorem of [11] therefore again implies that $V^{\mathbb{I}} \models$ "δ is δ^+ strongly compact".

Write $\mathbb{I} = \mathbb{I}^1 * \dot{\mathbb{I}}^2$, where $\mathbb{I}^1$ is the portion of $\mathbb{I}$ having length δ which forces each $\gamma \in \mathcal{C}$, $\gamma \le \delta$ to be a strong cardinal whose strongness is indestructible under γ^+-weakly closed partial orderings satisfying the Prikry property. Because each $\mathbb{I}(\gamma_\alpha, \delta_\alpha)$ for $\alpha < \kappa$ is an Easton support iteration of length δ_α, by the definition of $\mathbb{I}$, $\mathbb{I}^1$ is an Easton support iteration of length δ which is the direct limit of its components. Therefore, by the proof of Lemma 2.1, $V \models$ "δ is δ^+ supercompact". Since $V \models$ GCH and $\delta \in \mathcal{C}$, $V \models$ "δ is 2^δ supercompact and strong". Hence, by [3, Lemma 2.1], $V \models$ "δ is a limit of strong cardinals", which contradicts that $\delta \in \mathcal{C}$ and in V, $\mathcal{C} = \{\delta < \kappa \mid \delta$ is a strong cardinal which is not a limit of strong cardinals$\}$. Consequently, $V^{\mathbb{I} * \dot{\mathbb{P}}} \models$ "δ is not δ^+ strongly compact", $V^{\mathbb{I} * \dot{\mathbb{P}}} \models$ "Level by level equivalence holds at δ", and $V^{\mathbb{I} * \dot{\mathbb{P}}} \models$ "δ is not supercompact up to a measurable cardinal". This completes the proof of Lemma 2.3.

$\square$

Lemmas 2.1 – 2.3 and the intervening remarks complete the proof of Theorem 2.

$\square$

We conclude by asking whether it is possible to prove analogues to either Theorems 1 or 2 for universes in which there are no restrictions on the large cardinal structure. As our methods indicate, this would require rather different proofs.

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