Tall, Strong, and Strongly Compact Cardinals *†

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August 12, 2017

Abstract

We construct three models in which there are different relationships among the classes of strongly compact, strong, and non-strong tall cardinals. In the first two of these models, the strongly compact and strong cardinals coincide precisely, and every strongly compact/strong cardinal is a limit of non-strong tall cardinals. In the remaining model, the strongly compact cardinals are precisely characterized as the measurable limits of strong cardinals, and every strongly compact cardinal is a limit of non-strong tall cardinals. These results extend and generalize those of of [3] and [1].

1 Introduction and Preliminaries

We begin with some definitions. Suppose κ is a cardinal and $\lambda \geq \kappa$ is an arbitrary ordinal. κ is λ *tall* if there is an elementary embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \lambda$ and $M^\kappa \subseteq M$. κ is *tall* if κ is λ tall for every ordinal λ. Hamkins made a systematic study of tall cardinals in [10]. In particular, among many other results, he showed that every cardinal which is

*2010 Mathematics Subject Classifications: 03E35, 03E55.

†Keywords: Supercompact cardinal, strongly compact cardinal, strong cardinal, tall cardinal, non-reflecting stationary set of ordinals, indestructibility.

‡The author wishes to thank Stamatis Dimopoulos for helpful email correspondence on the subject matter of this paper.

either strong or strongly compact is in addition tall, and also produced models of ZFC with many different varieties of non-strong tall cardinals.

Turning now to the main narrative, in [3] and [1], the following theorems were proven.

Theorem 1 ([3, Theorem 1]) *Con(ZFC + There is a proper class of supercompact cardinals)* $\Longrightarrow$ *Con(ZFC + There is a proper class of strongly compact cardinals + No strongly compact cardinal* κ *is* $2^\kappa = \kappa^+$ *supercompact +* $\forall\kappa$*[*κ *is strongly compact iff* κ *is a strong cardinal]).*

Theorem 2 ([1, Theorem 1]) *Suppose* $V \models$ *"ZFC +* $\mathcal{K}$ *is the proper class of supercompact cardinals". There is then a partial ordering* $\mathbb{P} \subseteq V$ *such that* $V^{\mathbb{P}} \models$ *"ZFC +* κ *is strongly compact iff* κ *is a measurable limit of strong cardinals + The strongly compact cardinals are the elements of* $\mathcal{K}$ *together with their measurable limit points". Further, in* $V^{\mathbb{P}}$*, any* $\kappa \in \mathcal{K}$ *which was a supercompact limit of supercompact cardinals in* V *remains supercompact.*

Since Theorems 1 and 2 were proven prior to Hamkins' research leading to his paper [10], the issue of tall cardinals was not considered in either [3] or [1]. In particular, these theorems do not address the question of whether it is possible to construct models of ZFC witnessing the same conclusions in which each strongly compact cardinal is also a limit of non-strong tall cardinals.

The purpose of this paper is to produce such universes. Specifically, we will prove the following three theorems.

Theorem 3 *Con(ZFC + There is a proper class of supercompact cardinals)* $\Longrightarrow$ *Con(ZFC + There is a proper class of strongly compact cardinals + No strongly compact cardinal* κ *is* $2^\kappa = \kappa^+$ *supercompact +* $\forall\kappa$*[*κ *is strongly compact iff* κ *is a strong cardinal] + Every strongly compact cardinal is a limit of (non-strong) tall cardinals).*

Theorem 4 *Suppose* $V \models$ *"ZFC + GCH +* κ *is supercompact + No cardinal* $\lambda > \kappa$ *is measurable". Then there is a partial ordering* $\mathbb{P} \in V$ *such that* $V^{\mathbb{P}} \models$ *"ZFC +* κ *is both the only strong and only strongly compact cardinal +* κ *is not* $2^\kappa = \kappa^+$ *supercompact + Every measurable cardinal is tall + No cardinal* $\lambda > \kappa$ *is measurable".*

Theorem 5 *Suppose $V \models$ "ZFC + $\mathcal{K}$ is the proper class of supercompact cardinals". There is then a partial ordering $\mathbb{P} \subseteq V$ such that $V^{\mathbb{P}} \models$ "ZFC + κ is strongly compact iff κ is a measurable limit of strong cardinals + The strongly compact cardinals are the elements of $\mathcal{K}$ together with their measurable limit points". Further, in $V^{\mathbb{P}}$, every strongly compact cardinal is a limit of non-strong tall cardinals. Finally, in $V^{\mathbb{P}}$, any $\kappa \in \mathcal{K}$ which was a supercompact limit of supercompact cardinals in V remains supercompact.*

Thus, the models witnessing the conclusions of Theorems 3 and 5 have the same characterizations of the strongly compact cardinals as do the models witnessing the conclusions of Theorems 1 and 2, except that each strongly compact cardinal is in addition a limit of non-strong tall cardinals. The model witnessing the conclusions of Theorem 4 is an analogue of the model witnessing the conclusions of Theorem 1, except in a universe with a restricted number of large cardinals. However, it has the additional feature that each measurable cardinal is also tall. Further, as in [3] and [1], we will concentrate on the proper class versions of Theorems 3 and 5, and not discuss the (easier) analogues of these theorems when the class of supercompact cardinals is actually a set.

The structure of this paper is as follows. Section 1 contains our introductory comments and preliminary information concerning notation and terminology. Section 2 contains the proofs of Theorems 3 – 5. Section 3 contains our concluding remarks.

Before beginning the proofs of our theorems, we briefly mention some preliminary information and terminology. Essentially, our notation and terminology are standard, and when this is not the case, this will be clearly noted. For $\alpha < \beta$ ordinals, $[\alpha, \beta]$, $[\alpha, \beta)$, $(\alpha, \beta]$, and (α, β) are as in the usual interval notation. If $\kappa \geq \omega$ is a regular cardinal and λ is an arbitrary ordinal, then $\mathrm{Add}(\kappa, \lambda)$ is the standard partial ordering for adding λ Cohen subsets of κ. When forcing, $q \geq p$ will mean that *q is stronger than p*. If G is V-generic over $\mathbb{P}$, we will abuse notation slightly and use both $V[G]$ and $V^{\mathbb{P}}$ to indicate the universe obtained by forcing with $\mathbb{P}$. If $x \in V[G]$, then $\dot{x}$ will be a term in V for x. We may, from time to time, confuse terms with the sets they denote and write x when we actually mean $\dot{x}$ or $\check{x}$, especially when x is some variant of the generic set G, or x is in the ground model V. The abuse of notation mentioned above will be compounded by writing

$x \in V^{\mathbb{P}}$ instead of $\dot{x} \in V^{\mathbb{P}}$.

The partial ordering $\mathbb{P}$ is *κ-directed closed* if every directed set of conditions of size less than κ has an upper bound. $\mathbb{P}$ is *κ-strategically closed* if in the two person game in which the players construct an increasing sequence $\langle p_\alpha \mid \alpha \leq \kappa \rangle$, where player I plays odd stages and player II plays even stages (choosing the trivial condition at stage 0), player II has a strategy which ensures the game can always be continued. $\mathbb{P}$ is *$\prec\kappa$-strategically closed* if in the two person game in which the players construct an increasing sequence $\langle p_\alpha \mid \alpha < \kappa \rangle$, where player I plays odd stages and player II plays even stages (choosing the trivial condition at stage 0), player II has a strategy which ensures the game can always be continued. $\mathbb{P}$ is *(κ, ∞)-distributive* if given a sequence $\langle D_\alpha \mid \alpha < \kappa \rangle$ of dense open subsets of $\mathbb{P}$, $\bigcap_{\alpha<\kappa} D_\alpha$ is dense open as well. Note that if $\mathbb{P}$ is κ-strategically closed, then $\mathbb{P}$ is (κ, ∞)-distributive. Further, if $\mathbb{P}$ is (κ, ∞)-distributive and $f : \kappa \to V$ is a function in $V^{\mathbb{P}}$, then $f \in V$.

Suppose that $\kappa < \lambda$ are regular cardinals. A partial ordering $\mathbb{P}(\kappa, \lambda)$ that will be used throughout the course of this paper is the partial ordering for adding a non-reflecting stationary set of ordinals of cofinality κ to λ. Specifically, $\mathbb{P}(\kappa, \lambda)$ is defined as $\{p \mid$ For some $\alpha < \lambda$, $p : \alpha \to \{0, 1\}$ is a characteristic function of S_p, a subset of α not stationary at its supremum nor having any initial segment which is stationary at its supremum, such that $\beta \in S_p$ implies $\beta > \kappa$ and $\text{cof}(\beta) = \kappa\}$, ordered by $q \geq p$ iff $q \supseteq p$ and $S_p = S_q \cap \sup(S_p)$, i.e., S_q is an end extension of S_p. It is well-known that for G V-generic over $\mathbb{P}(\kappa, \lambda)$ (see [5]), in $V[G]$, if we assume λ is inaccessible in V, a non-reflecting stationary set $S = S[G] = \bigcup\{S_p \mid p \in G\} \subseteq \lambda$ of ordinals of cofinality κ has been introduced, the bounded subsets of λ are the same as those in V, and cofinalities have been preserved. It is also virtually immediate that $\mathbb{P}(\kappa, \lambda)$ is κ-directed closed, and it can be shown (see [5]) that $\mathbb{P}(\kappa, \lambda)$ is $\prec\lambda$-strategically closed.

A corollary of Hamkins' work on gap forcing found in [7, 8] will be employed in the proof of our theorems. We therefore state as a separate theorem what is relevant for this paper, along with some associated terminology, quoting from [7, 8] when appropriate. Suppose $\mathbb{P}$ is a partial ordering which can be written as $\mathbb{Q} * \dot{\mathbb{R}}$, where $|\mathbb{Q}| < \delta$, $\mathbb{Q}$ is nontrivial, and $\Vdash_{\mathbb{Q}}$ "$\dot{\mathbb{R}}$ is δ-strategically closed".

In Hamkins' terminology of [7, 8], $\mathbb{P}$ *admits a gap at* δ. Also, as in the terminology of [7, 8] and elsewhere, an embedding $j : \overline{V} \to \overline{M}$ is *amenable to* $\overline{V}$ when $j \restriction A \in \overline{V}$ for any $A \in \overline{V}$. The specific corollary of Hamkins' work from [7, 8] we will be using is then the following.

Theorem 6 (Hamkins) *Suppose that $V[G]$ is a generic extension obtained by forcing that admits a gap at some regular $\delta < \kappa$. Suppose further that $j : V[G] \to M[j(G)]$ is an embedding with critical point κ for which $M[j(G)] \subseteq V[G]$ and $M[j(G)]^\delta \subseteq M[j(G)]$ in $V[G]$. Then $M \subseteq V$; indeed, $M = V \cap M[j(G)]$. If the full embedding j is amenable to $V[G]$, then the restricted embedding $j \restriction V : V \to M$ is amenable to V. If j is definable from parameters (such as a measure or extender) in $V[G]$, then the restricted embedding $j \restriction V$ is definable from the names of those parameters in V.*

An immediate corollary of Theorem 6 is that forcing with a partial ordering $\mathbb{P}$ admitting a gap at some regular cardinal δ creates no new measurable, tall, strong, or supercompact cardinals above δ. In particular, if $\delta = \omega$, then forcing with $\mathbb{P}$ creates no new measurable, tall, strong, or supercompact cardinals. In addition, by [7, Corollary 13], if κ is λ strong in $V^{\mathbb{P}}$ via j^* where $\mathbb{P}$ admits a gap at some regular cardinal $\delta < \kappa$ and λ is either a successor ordinal or has cofinality greater than δ, then κ was λ strong in the ground model as witnessed by $j^* \restriction V$.

We mention that we are assuming some familiarity with the large cardinal notions of measurability, tallness, strongness, strong compactness, and supercompactness. Interested readers may consult [14] or [10].

2 The Proofs of Theorems 3 – 5

We turn now to the proofs of Theorem 3 – 5, starting with the proof of Theorem 3.

Proof: In analogy to [3], we first prove Theorem 3 for one cardinal. In particular, starting from a model for "ZFC + κ is supercompact", we will force and construct a model where κ is both the least strong and least strongly compact cardinal in which κ is also a limit of (non-strong) tall cardinals. In this model, it will in addition be the case that κ is not $2^\kappa = \kappa^+$ supercompact.

Before beginning the proof, however, we give some intuition and motivation for the definition of our forcing conditions. In [3], in order to construct the requisite models, it was only necessary to add non-reflecting stationary sets of ordinals of the appropriate cofinality to rid ourselves of each strong cardinal $\delta < \kappa$. This is not sufficient in the current situation, since the forcing just described will not ensure that there are non-strong tall cardinals below κ. We will therefore destroy all ground model strong cardinals which are themselves limits of ground model strong cardinals, after first adding a Cohen subset of ω to create a gap at $\aleph_1$. This will guarantee by Theorem 6 that all strong cardinals below κ have been eliminated, and that κ has become a limit of non-strong tall cardinals.

Getting specific, suppose $V \vDash$ "ZFC + κ is supercompact". Without loss of generality, by first doing a preliminary forcing if necessary, we assume in addition that $V \vDash$ GCH. By [3, Lemma 2.1] and the succeeding remarks, the V-strong cardinals below κ which are limits of V-strong cardinals are unbounded in κ. We may therefore let $A = \langle \delta_\alpha \mid \alpha < \kappa \rangle$ be an enumeration of this set. The partial ordering $\mathbb{P}^\kappa$ we use in the proof of Theorem 1 for one cardinal is defined analogously as in [3]. It is the Easton support iteration $\langle \langle \mathbb{P}^\kappa_\alpha, \dot{\mathbb{Q}}^\kappa_\alpha \rangle \mid \alpha < \kappa \rangle$, where $\mathbb{P}^\kappa_0 = \text{Add}(\omega, 1)$ and $\Vdash_{\mathbb{P}^\kappa_\alpha}$ "$\dot{\mathbb{Q}}^\kappa_\alpha = \dot{\mathbb{P}}(\omega, \delta_\alpha)$". Since by its definition, $|\mathbb{P}^\kappa| = \kappa$, $V^{\mathbb{P}^\kappa} \vDash$ "$2^\delta = \delta^+$ for every cardinal $\delta \geq \kappa$".

Lemma 2.1 $V^{\mathbb{P}^\kappa} \vDash$ "*No cardinal* $\delta < \kappa$ *is a strong cardinal*".

Proof: The proof is quite different from and subtler than [3, Lemma 2.2], its analogue in [3]. It is motivated by ideas due to Hamkins found in [9] and [12].

By its definition, we may write $\mathbb{P}^\kappa = \text{Add}(\omega, 1) * \dot{\mathbb{Q}}$, where $\Vdash_{\text{Add}(\omega,1)}$ "$\dot{\mathbb{Q}}$ is $\aleph_1$-strategically closed". By our remarks immediately following Theorem 6, we may consequently infer that if $V^{\mathbb{P}^\kappa} \vDash$ "δ is a strong cardinal", then $V \vDash$ "δ is a strong cardinal" as well. Therefore, to prove Lemma 2.1, it suffices to show that if $V \vDash$ "$\delta < \kappa$ is strong", then $V^{\mathbb{P}^\kappa} \vDash$ "δ is not a strong cardinal". This is clearly true if $V \vDash$ "δ is a strong cardinal which is a limit of strong cardinals". This is since under these circumstances, by the definition of $\mathbb{P}^\kappa$, $V^{\mathbb{P}^\kappa} \vDash$ "There is $S \subseteq \delta$ which is a non-reflecting stationary set of ordinals of cofinality ω and thus δ is not weakly compact". Hence, to complete the proof of Lemma 2.1, we must show that if $V \vDash$ "δ is a strong cardinal which is not a limit of strong cardinals", then $V^{\mathbb{P}^\kappa} \vDash$ "δ is not a strong cardinal".

To do this, suppose to the contrary that $V^{\mathbb{P}^\kappa} \vDash$ "δ is a strong cardinal". Because $V \vDash$ "δ is not a limit of strong cardinals", we may write $\mathbb{P}^\kappa = \mathbb{R}^* * \dot{\mathbb{R}}^{**} * \dot{\mathbb{R}}$, where $|\mathbb{R}^*| < \delta$, $\mathbb{R}^*$ adds a Cohen subset of ω and also adds non-reflecting stationary sets of ordinals of cofinality ω to each cardinal below δ which is a V-strong limit of V-strong cardinals, $\dot{\mathbb{R}}^{**}$ is a term for the partial ordering adding a non-reflecting stationary set of ordinals of cofinality ω to the least V-strong cardinal $\delta' > \delta$ which is a limit of V-strong cardinals, and $\dot{\mathbb{R}}$ is a term for the rest of $\mathbb{P}^\kappa$. Since $\Vdash_{\mathbb{R}^* * \dot{\mathbb{R}}^{**}}$ "$\dot{\mathbb{R}}$ is σ-strategically closed for σ the least inaccessible cardinal above δ'", it is the case that $V^{\mathbb{P}^\kappa} = V^{\mathbb{R}^* * \dot{\mathbb{R}}^{**} * \dot{\mathbb{R}}} \vDash$ "δ is $\delta' + 2$ strong" iff $V^{\mathbb{R}^* * \dot{\mathbb{R}}^{**}} \vDash$ "δ is $\delta' + 2$ strong". The proof of Lemma 2.1 will therefore be complete if we can show that $V^{\mathbb{R}^* * \dot{\mathbb{R}}^{**}} \vDash$ "δ is not $\delta' + 2$ strong".

Towards this end, let G^* be V-generic over $\mathbb{R}^*$ and G^{**} be $V[G^*]$-generic over $\mathbb{R}^{**}$. Since $V[G^*][G^{**}] \vDash$ "δ is $\delta' + 2$ strong", we may let $j^* : V[G^*][G^{**}] \to M[j^*(G^*)][j^*(G^{**})]$ be an elementary embedding having critical point δ which witnesses the $\delta' + 2$ strongness of δ such that $M[j^*(G^*)][j^*(G^{**}] \subseteq V[G^*][G^{**}]$, $M[j^*(G^*)][j^*(G^{**})]^\delta \subseteq M[j^*(G^*)][j^*(G^{**})]$ in $V[G^*][G^{**}]$, and $(V_{\delta'+2})^{V[G^*][G^{**}]} \in M[j^*(G^*)][j^*(G^{**}]$. Observe that it is also possible to write $\mathbb{R}^* * \dot{\mathbb{R}}^{**} = \mathrm{Add}(\omega, 1) * \dot{\mathbb{S}}$, where $\Vdash_{\mathrm{Add}(\omega,1)}$ "$\dot{\mathbb{S}}$ is $\aleph_1$-strategically closed". Thus, by Theorem 6 and the succeeding remarks, j^* must lift some elementary embedding $j : V \to M$ witnessing the $\delta' + 2$ strongness of δ in V, where $M \subseteq V$, $V_{\delta'+2} \in M$, and $j(\delta) > \delta' + 2$. Further, as $|\mathbb{R}^*| < \delta$ and δ is the critical point of both j and j^*, $j(\mathbb{R}^*) = \mathbb{R}^*$ and $j^*(G^*) = G^*$, i.e., $j^* : V[G^*][G^{**}] \to M[G^*][j^*(G^{**})]$.

Because $V_{\delta'+2} \subseteq M$, $(V_{\delta'+1})^{V[G^*]} = (V_{\delta'+1})^{M[G^*]}$. Thus, as $\mathbb{R}^{**} \in (V_{\delta'+1})^{V[G^*]}$, $\mathbb{R}^{**} \in (V_{\delta'+1})^{M[G^*]}$. Therefore, since $M[G^*] \subseteq V[G^*]$, G^{**} is also $M[G^*]$-generic over $\mathbb{R}^{**}$, so that in particular, G^{**} is not a member of either $V[G^*]$ or $M[G^*]$. However, because $(V_{\delta'+2})^{V[G^*][G^{**}]} \in M[G^*][[j^*(G^{**})]$ and $G^{**} \in (V_{\delta'+2})^{V[G^*][G^{**}]}$, $G^{**} \in M[G^*][[j^*(G^{**})]$.

Note that by elementarity, as $\Vdash_{\mathbb{R}^*}$ "$\dot{\mathbb{R}}^{**}$ adds a non-reflecting stationary set of ordinals of cofinality ω to a measurable cardinal $\delta' > \delta$", in M, $\Vdash_{\mathbb{R}^*}$ "$j(\dot{\mathbb{R}}^{**})$ adds a non-reflecting stationary set of ordinals of cofinality ω to a measurable cardinal $j(\delta') > j(\delta) > \delta' + 2 > \delta'$". Hence, in

M, $\Vdash_{\mathbb{R}^*}$ "$j(\dot{\mathbb{R}}^{**})$ is σ-strategically closed for σ the least inaccessible cardinal above δ'". Therefore, because $j^*(G^{**})$ is $M[G^*]$-generic over $j(\mathbb{R}^{**})$, $G^{**} \in M[G^*]$. This contradiction completes the proof of Lemma 2.1.

$\square$

Lemma 2.2 $V^{\mathbb{P}^\kappa} \vDash$ "*κ is a limit of non-strong tall cardinals*".

Proof: Since the set A defined above is unbounded in κ, the set $B = \{\delta < \kappa \mid \delta$ is a V-strong cardinal which is not a limit of V-strong cardinals$\}$ is unbounded in κ as well. We show that $V^{\mathbb{P}^\kappa} \vDash$ "Every $\delta \in B$ is a tall cardinal". This will suffice, since by Lemma 2.1, $V^{\mathbb{P}^\kappa} \vDash$ "No $\delta \in B$ is a strong cardinal".

Towards this end, fix $\delta \in B$. With the same meaning as in the proof of Lemma 2.1, write $\mathbb{P}^\kappa = \mathbb{R}^* * \dot{\mathbb{R}}^{**} * \dot{\mathbb{R}}$. Since $|\mathbb{R}^*| < \delta$, by the Hamkins-Woodin results [13], $V^{\mathbb{R}^*} \vDash$ "δ is a strong cardinal". As we have already noted, it then immediately follows that $V^{\mathbb{R}^*} \vDash$ "δ is a tall cardinal". By [10, Theorem 3.1], δ's tallness is indestructible under (δ, ∞)-distributive forcing. Because by its definition, $\Vdash_{\mathbb{R}^*}$ "$\dot{\mathbb{R}}^{**} * \dot{\mathbb{R}}$ is (δ, ∞)-distributive", $V^{\mathbb{R}^* * \dot{\mathbb{R}}^{**} * \dot{\mathbb{R}}} = V^{\mathbb{P}^\kappa} \vDash$ "δ is a tall cardinal". This completes the proof of Lemma 2.2.

$\square$

Lemma 2.3 $V^{\mathbb{P}^\kappa} \vDash$ "*No cardinal $\delta < \kappa$ is strongly compact*".

Proof: The proof of Lemma 2.3 is essentially the same as the proof of [3, Lemma 2.3]. Since it is relatively brief, we include it for completeness. Specifically, by the definition of $\mathbb{P}^\kappa$ and the fact A is unbounded in κ, $V^{\mathbb{P}^\kappa} \vDash$ "There are unboundedly in κ many cardinals $\delta < \kappa$ containing a non-reflecting stationary set of ordinals of cofinality ω". Hence, by [17, Theorem 4.8] and the succeeding remarks, $V^{\mathbb{P}^\kappa} \vDash$ "No cardinal $\delta < \kappa$ is strongly compact". This completes the proof of Lemma 2.3.

$\square$

Lemma 2.4 $V^{\mathbb{P}^\kappa} \vDash$ "*κ is strongly compact*".

Proof: The proof of Lemma 2.4 is essentially the same as the proof of [3, Lemma 2.4]. The argument is originally due to Magidor, but was unpublished by him. For completeness and ease of presentation, we provide a sketch, and refer readers to [3] for any missing details. Specifically, let $\lambda > \kappa$ be an arbitrary regular cardinal, with $j : V \to M$ an elementary embedding witnessing the λ supercompactness of κ generated by a supercompact ultrafilter over $P_\kappa(\lambda)$ such that $V \vDash$ "κ is not λ supercompact". Since $\lambda \geq \kappa^+ = 2^\kappa$, we know that $M \vDash$ "κ is measurable". We may therefore let $k : M \to N$ be an elementary embedding generated by a normal measure $\mathcal{U} \in M$ over κ such that $N \vDash$ "κ is not measurable". The elementary embedding $i = k \circ j$ witnesses the λ strong compactness of κ in V. It will follow that i lifts in $V^{\mathbb{P}^\kappa}$ to an elementary embedding $i : V^{\mathbb{P}^\kappa} \to N^{i(\mathbb{P}^\kappa)}$ witnessing the λ strong compactness of κ.

To see this, we begin with a few observations. First, note that by [3, Lemma 2.1] and the succeeding remarks, in both V and M, κ is a strong cardinal which is a limit of strong cardinals. Further, as was observed in [3, proof of Lemma 2.4, page 31, fourth paragraph], since $M \vDash$ "κ is not λ supercompact", $M \vDash$ "No cardinal in the half-open interval $(\kappa, \lambda]$ is a strong cardinal". The previous two sentences consequently imply that by the definition of $\mathbb{P}$, $j(\mathbb{P}^\kappa) = \mathbb{P}^\kappa * \dot{\mathbb{P}}(\omega, \kappa) * \dot{\mathbb{R}}$, where the first ordinal at which $\dot{\mathbb{R}}$ is forced to do nontrivial forcing is above λ. This means we may write $i(\mathbb{P}^\kappa) = \mathbb{P}^\kappa * \dot{\mathbb{Q}}^1 * \dot{\mathbb{Q}}^2$, where $\dot{\mathbb{Q}}^1$ is forced to act nontrivially on ordinals in the interval $(\kappa, k(\kappa)]$, and $\dot{\mathbb{Q}}^2$ is forced to act nontrivially on ordinals in the interval $(k(\kappa), k(j(\kappa))) = (k(\kappa), i(\kappa))$.

Now, take G_0 to be V-generic over $\mathbb{P}^\kappa$, and build in $V[G_0]$ generic objects G_1 and G_2 for $\mathbb{Q}^1$ and $\mathbb{Q}^2$ respectively. The construction of G_1 uses that by GCH and the fact that k is given by an ultrapower embedding, we may let $\langle D_\alpha \mid \alpha < \kappa^+ \rangle$ enumerate in $V[G_0]$ the dense open subsets of $\mathbb{Q}^1$ present in $N[G_0]$. Since $N \vDash$ "κ is not measurable", the first nontrivial stage of forcing in $\mathbb{Q}^1$ occurs above κ. This implies that $N[G_0] \vDash$ "$\mathbb{Q}^1$ is $\prec\kappa^+$-strategically closed". Because $N[G_0]$ remains κ-closed with respect to $V[G_0]$, by the $\prec\kappa^+$-strategic closure of $\mathbb{Q}^1$ in both $N[G_0]$ and $V[G_0]$, we may work in $V[G_0]$ and meet each D_α in order to construct G_1. The construction of G_2 first requires building an M-generic object G_2^* for the term forcing partial ordering $\mathbb{T}$ associated with $\dot{\mathbb{R}}$ and defined in M with respect to $\mathbb{P}^\kappa * \dot{\mathbb{P}}(\omega, \kappa)$. G_2^* is built using the facts that since $M^\lambda \subseteq M$ and the first nontrivial stage of forcing in $\mathbb{T}$ occurs above λ, $\mathbb{T}$ is $\prec\lambda^+$-strategically closed in both M and V, which means that the diagonalization argument employed in the construction of G_1 may be applied in this situation as well. $k''G_2^*$ now generates an N-generic object G_2^{**} for $k(\mathbb{T})$ and an $N[G_0][G_1]$-generic object G_2 for $\mathbb{Q}^2$ This tells us that i lifts in $V[G_0]$ to $i : V[G_0] \to N[G_0][G_1][G_2]$, i.e., $V^{\mathbb{P}^\kappa} \models$ "κ is λ strongly compact". Since λ was arbitrary, this completes the proof sketch of Lemma 2.4.

□

Lemma 2.5 $V^{\mathbb{P}^\kappa} \models$ "*κ is a strong cardinal*".

Proof: Let $\lambda > \kappa$ be a singular strong limit cardinal whose cofinality is at least κ, with $j : V \to M$ an elementary embedding witnessing the λ strongness of κ generated by a (κ, λ)-extender such that $M \models$ "κ is not a strong cardinal". Since $M \models$ "κ is not a strong cardinal", it follows that κ is a trivial stage of forcing in the definition of $j(\mathbb{P}^\kappa)$ in M.

The proof that the embedding j lifts to an embedding $j : V^{\mathbb{P}^\kappa} \to M^{j(\mathbb{P}^\kappa)}$ witnessing the λ strongness of κ is a modification of the one given in [11, Theorem 4.10] and [2, Lemma 4.2] which takes into account that only trivial forcing occurs at stage κ in M in the definition of $j(\mathbb{P}^\kappa)$.[1] We will take the liberty to quote freely from the proof of [2, Lemma 4.2] as appropriate. We may assume that $M = \{j(f)(a) \mid a \in [\lambda]^{<\omega},\ f \in V, \text{ and } \mathrm{dom}(f) = [\kappa]^{|a|}\}$. Further, as in [3, proof of Lemma 2.5, page 32, second paragraph], $M \models$ "There are no strong cardinals in the half-open interval $(\kappa, \lambda]$". Consequently, $j(\mathbb{P}^\kappa) = \mathbb{P}^\kappa * \dot{\mathbb{R}}$, where the first ordinal on which $\dot{\mathbb{R}}$ is forced to act nontrivially is above λ. Suppose G_0 is V-generic over $\mathbb{P}^\kappa$. Since $\mathbb{P}^\kappa$ is an Easton support iteration having length κ, $\mathbb{P}^\kappa$ is κ-c.c. Thus, as we may assume that $M^\kappa \subseteq M$, $M[G_0]^\kappa \subseteq M[G_0]$ in $V[G_0]$. Therefore, $\mathbb{R}$ is $\prec\kappa^+$-strategically closed in both $V[G_0]$ and $M[G_0]$, and $\mathbb{R}$ is λ-strategically closed in $M[G_0]$.

[1]Note that it is also possible to use the argument found in [3, Lemma 2.5] to prove this lemma. However, since the proof found here is a bit shorter and more direct, we give it instead.

Let $N = \{i_{G_0}(\dot{z}) \mid \dot{z} = j(f)(\kappa, \lambda)$ for some function $f \in V\}$. As in [11, Theorem 4.10] and [2, Lemma 4.2], one may verify that $N \prec M[G_0]$, that N is closed under κ sequences in $V[G_0]$, and that κ, λ, and $\mathbb{R}$ are all elements of N. Further, since $\mathbb{R}$ is $j(\kappa)$-c.c. in $M[G_0]$ and there are only $2^\kappa = \kappa^+$ many functions $f : [\kappa]^2 \to V_\kappa$ in V, there are at most κ^+ many dense open subsets of $\mathbb{R}$ in N. Therefore, since $\mathbb{R}$ is $\prec\kappa^+$-strategically closed in both $M[G_0]$ and $V[G_0]$, we can build an N-generic object G_1 over $\mathbb{R}$ in $V[G_0]$ as in the construction of the generic object G_1 found in the proof sketch of Lemma 2.4.

We show now that G_1 is actually $M[G_0]$-generic over $\mathbb{R}$. If D is a dense open subset of $\mathbb{R}$ in $M[G_0]$, then $D = i_{G_0}(\dot{D})$ for some name $\dot{D} \in M$. Consequently, $\dot{D} = j(f)(\kappa, \kappa_1, \ldots, \kappa_n)$ for some function $f \in V$ and $\kappa < \kappa_1 < \cdots < \kappa_n < \lambda$. Let $\overline{D}$ be a name for the intersection of all $i_{G_0}(j(f)(\kappa, \alpha_1, \ldots, \alpha_n))$, where $\kappa < \alpha_1 < \cdots < \alpha_n < \lambda$ is such that $j(f)(\kappa, \alpha_1, \ldots, \alpha_n)$ yields a name for a dense open subset of $\mathbb{R}$ in $M[G_0]$. Since this name can be given in M and $\mathbb{R}$ is λ-strategically closed in $M[G_0]$ and therefore λ-distributive in $M[G_0]$, $\overline{D}$ is a name for a dense open subset of $\mathbb{R}$ in $M[G_0]$ which is definable without the parameters $\kappa_1, \ldots, \kappa_n$. Hence, by its definition, $i_{G_0}(\overline{D}) \in N$. Thus, since G_1 meets every dense open subset of $\mathbb{R}$ present in N, $G_1 \cap i_{G_0}(\overline{D}) \neq \emptyset$, so since $\overline{D}$ is forced to be a subset of $\dot{D}$, $G_1 \cap i_{G_0}(\dot{D}) \neq \emptyset$. This means G_1 is $M[G_0]$-generic over $\mathbb{R}$, so in $V[G_0]$, j lifts to $j : V[G_0] \to M[G_0][G_1]$. The lifted version of j is an embedding witnessing the λ strongness of κ in $V[G_0]$. This is since $V_\lambda \subseteq M$, meaning $(V_\lambda)^{V[G_0]} \subseteq M[G_0] \subseteq M[G_0]G_1]$. As a consequence, $V[G_0] \models$ "κ is λ strong". Since λ was arbitrary, this completes the proof of Lemma 2.5.

$\square$

Lemma 2.6 $V^{\mathbb{P}_\kappa} \models$ "*κ is not $2^\kappa = \kappa^+$ supercompact*".

Proof: The proof of Lemma 2.6 is essentially the same as the proof of [3, Lemma 2.6]. Once again, since it is relatively brief, we include it for completeness. By Lemmas 2.1 and 2.5, $V^{\mathbb{P}_\kappa} \models$ "κ is a strong cardinal such that no cardinal $\delta < \kappa$ is a strong cardinal". Thus, by [3, Lemma 2.1] and the succeeding remarks, $V^{\mathbb{P}_\kappa} \models$ "κ is not 2^κ supercompact". Since as we have already observed,

$V^{\mathbb{P}^\kappa} \models$ "$2^\kappa = \kappa^+$", this completes the proof of Lemma 2.6.

$\square$

Lemmas 2.1 – 2.6 complete the proof of Theorem 3 for one cardinal.

$\square$

To prove Theorem 3 in the general case, i.e., when there is a proper class of supercompact cardinals, we will modify the proof given in [3, Section 3], quoting verbatim where appropriate. Suppose $V \models$ "ZFC + There is a proper class of supercompact cardinals". Let $\langle \kappa_\alpha \mid \alpha \in \mathrm{Ord} \rangle$ enumerate the supercompact cardinals in increasing order. Without loss of generality, we assume in addition that $V \models$ "For every ordinal α, each κ_α has its supercompactness Laver indestructible [15] under κ_α-directed closed forcing + For every ordinal α, $2^{\kappa_\alpha} = \kappa_\alpha^+$" and that by "cutting off" the universe if necessary at the least inaccessible limit of supercompact cardinals, for $\gamma_0 = \omega$ and $\gamma_\alpha = \bigcup_{\beta<\alpha} \kappa_\beta$ for $\alpha > 0$, $\gamma_\alpha < \kappa_\alpha$ is singular if α is a limit ordinal.

For each ordinal α, let $\langle \delta^\alpha_\beta \mid \beta < \kappa_\alpha \rangle$ be an enumeration of the V-strong cardinals which are limits of V-strong cardinals in the interval $(\gamma_\alpha, \kappa_\alpha)$, and let $\mathbb{P}^{\kappa_\alpha} = \langle \langle \mathbb{P}^{\kappa_\alpha}_\beta, \dot{\mathbb{Q}}^{\kappa_\alpha}_\beta \rangle \mid \beta < \kappa_\alpha \rangle$ be the Easton support iteration where $\mathbb{P}^{\kappa_\alpha}_0 = \mathrm{Add}(\gamma^+_\alpha, 1)$ and $\Vdash_{\mathbb{P}^{\kappa_\alpha}_\beta}$ "$\dot{\mathbb{Q}}^{\kappa_\alpha}_\beta = \dot{\mathbb{P}}(\gamma^+_\alpha, \delta^\alpha_\beta)$". We define $\mathbb{P}$ as the Easton support product $\prod_{\alpha \in \mathrm{Ord}} \mathbb{P}^{\kappa_\alpha}$. The definition of each $\mathbb{P}^{\kappa_\alpha}$ together with the standard Easton arguments show $V^{\mathbb{P}} \models$ "ZFC + For every ordinal α, $2^{\kappa_\alpha} = \kappa^+_\alpha$".

For each ordinal α, write $\mathbb{P} = \mathbb{P}_{<\alpha} \times \mathbb{P}^{\kappa_\alpha} \times \mathbb{P}^{>\alpha}$, where $\mathbb{P}_{<\alpha} = \prod_{\beta<\alpha} \mathbb{P}^{\kappa_\beta}$ and $\mathbb{P}^{>\alpha}$ is the remainder of $\mathbb{P}$. By the definition of $\mathbb{P}$ and the fact the supercompactness of κ_α is indestructible under set or class forcing, $V_1^{\mathbb{P}^{>\alpha}} \models$ "For every ordinal β, $2^{\kappa_\beta} = \kappa^+_\beta$ + κ_α is supercompact". Further, the cardinals in the open interval $(\gamma_\alpha, \kappa_\alpha)$ which are strong in $V^{\mathbb{P}^{>\alpha}}$ are precisely the same as the cardinals in the open interval $(\gamma_\alpha, \kappa_\alpha)$ which are strong in V. To see this, suppose $\delta \in (\gamma_\alpha, \kappa_\alpha)$ is such that $V^{\mathbb{P}^{>\alpha}} \models$ "δ is a strong cardinal". Since $\mathbb{P}^{>\alpha}$ is κ_α-directed closed, $V \models$ "δ is λ strong for every $\lambda < \kappa_\alpha$". Because $V \models$ "κ_α is supercompact and hence strong", it follows from [3, proof of Lemma 2.5, page 32, second paragraph] that $V \models$ "δ is a strong cardinal". Now, if $\delta \in (\gamma_\alpha, \kappa_\alpha)$ is such that $V \models$ "δ is a strong cardinal", then again because $\mathbb{P}^{>\alpha}$ is κ_α-directed closed, $V^{\mathbb{P}^{>\alpha}} \models$ "δ is λ strong for every $\lambda < \kappa_\alpha$". As $V^{\mathbb{P}^{>\alpha}} \models$ "κ_α is supercompact and therefore strong", as we just noted,

$V^{\mathbb{P}^{>\alpha}} \models$ "δ is a strong cardinal".

By the facts that $\mathbb{P}^{>\alpha}$ is κ_α-directed closed and the cardinals in the open interval $(\gamma_\alpha, \kappa_\alpha)$ which are strong in $V^{\mathbb{P}^{>\alpha}}$ are precisely the same as the cardinals in the open interval $(\gamma_\alpha, \kappa_\alpha)$ which are strong in V, the definition of $\mathbb{P}^{\kappa_\alpha}$ is the same in either V or $V^{\mathbb{P}^{>\alpha}}$. This means that we can apply the results used in the proof of Theorem 3 for one cardinal to show that $V^{\mathbb{P}^{>\alpha} \times \mathbb{P}^{\kappa_\alpha}} \models$ "κ_α is both strongly compact and strong, there are no strongly compact or strong cardinals in the interval $(\gamma_\alpha, \kappa_\alpha)$, κ_α is a limit of non-strong tall cardinals, and κ_α is not $2^{\kappa_\alpha} = \kappa_\alpha^+$ supercompact". Since $V \models$ "$|\mathbb{P}_{<\alpha}| < 2^{\gamma_\alpha^+}$", the Lévy-Solovay results [16] show that $V_1^{\mathbb{P}^{>\alpha} \times \mathbb{P}^{\kappa_\alpha} \times \mathbb{P}_{<\alpha}} = V^{\mathbb{P}} \models$ "κ_α is both strongly compact and strong, there are no strongly compact or strong cardinals in the interval $(\gamma_\alpha, \kappa_\alpha)$, κ_α is a limit of non-strong tall cardinals, and κ_α is not $2^{\kappa_\alpha} = \kappa_\alpha^+$ supercompact". Therefore, since any cardinal δ which is strongly compact or strong and is not a κ_α would have to be such that $\delta \in (\gamma_\alpha, \kappa_\alpha)$, $V^{\mathbb{P}}$ is our desired model. This proves Theorem 3 for a proper class of cardinals.

$\square$

Turning now to the proof of Theorem 4, suppose $V \models$ "ZFC + GCH + κ is supercompact + No cardinal $\lambda > \kappa$ is measurable". By first forcing with the partial ordering $\mathbb{Q}$ used in the proof of [4, Theorem 1], we obtain a model $V^{\mathbb{Q}}$ such that $V^{\mathbb{Q}} \models$ "ZFC + κ is supercompact + No cardinal $\lambda > \kappa$ is measurable + δ is measurable iff δ is tall". By the definition of $\mathbb{Q}$, we may assume in addition that $V^{\mathbb{Q}} \models$ "$2^\delta = \delta^+$ for every $\delta \geq \kappa$".

Let $V_0 = V^{\mathbb{Q}}$. Suppose $\mathbb{P}^\kappa$ is defined in V_0 as in the proof of Theorem 3. Since $|\mathbb{P}^\kappa| = \kappa$, by the results of [16], $V_0^{\mathbb{P}^\kappa} \models$ "No cardinal $\lambda > \kappa$ is measurable". It also follows that $V_0^{\mathbb{P}^\kappa} \models$ "$2^\delta = \delta^+$ for every $\delta \geq \kappa$". In addition, because $V_0 \models$ "$2^\kappa = \kappa^+$", the same arguments used in the proof of Theorem 3 tell us that $V_0^{\mathbb{P}^\kappa} \models$ "κ is both the least strong and least strongly compact cardinal". We may consequently immediately infer that $V_0^{\mathbb{P}^\kappa} \models$ "κ is both the only strong and only strongly compact cardinal". Therefore, again as in the proof of Theorem 3, $V_0^{\mathbb{P}^\kappa} \models$ "κ is not $2^\kappa = \kappa^+$ supercompact". The proof of Theorem 4 is therefore completed by the following lemma.

Lemma 2.7 $V_0^{\mathbb{P}^\kappa} \models$ "*Every measurable cardinal is tall*".

Proof: Suppose $V_0^{\mathbb{P}^\kappa} \models$ "δ is measurable". Since $V_0^{\mathbb{P}^\kappa} \models$ "No cardinal $\lambda > \kappa$ is measurable and κ is a strong cardinal", we may assume without loss of generality that $\delta < \kappa$. Further, by the factorization of $\mathbb{P}^\kappa$ given in the first paragraph of the proof of Lemma 2.1 and the remarks immediately following Theorem 6, it follows that in addition, $V_0 = V^{\mathbb{Q}} \models$ "δ is measurable". Since $\mathbb{Q}$ is the partial ordering of [4, Theorem 1], we know that $V_0 \models$ "δ is tall" as well.

We consider now the following two cases.

Case 1: δ is not a limit of V_0-strong cardinals which are limits of V_0-strong cardinals. We also know, by the definition of $\mathbb{P}^\kappa$, that $V_0 \models$ "δ is not a strong cardinal which is a limit of strong cardinals". We may therefore use the factorization $\mathbb{P}^\kappa = \mathbb{R}^* * \dot{\mathbb{R}}^{**} * \dot{\mathbb{R}}$ given in the proofs of Lemmas 2.1 and 2.2 and employ the same argument found in the second paragraph of the proof of Lemma 2.2 (with the slight modification that by the arguments of [16], since $|\mathbb{R}^*| < \delta$, forcing with $\mathbb{R}^*$ preserves the fact that δ is a tall cardinal) to infer that $V_0^{\mathbb{P}^\kappa} \models$ "δ is a tall cardinal".

Case 2: δ is a limit of V_0-strong cardinals which are limits of V_0-strong cardinals. It then immediately follows that δ must be a limit of V_0-strong cardinals which are themselves not limits of V_0-strong cardinals. For any such γ, by Lemma 2.2, $V_0^{\mathbb{P}^\kappa} \models$ "γ is a tall cardinal". Since $V_0^{\mathbb{P}^\kappa} \models$ "δ is a measurable limit of tall cardinals", by [10, Corollary 2.7], $V_0^{\mathbb{P}^\kappa} \models$ "δ is a tall cardinal".

Cases 1 and 2 complete the proof of Lemma 2.7.

$\square$

Lemma 2.7 and setting $\mathbb{P} = \mathbb{Q} * \dot{\mathbb{P}}^\kappa$ complete the proof of Theorem 4.

$\square$

Turning now to the proof of Theorem 5, suppose $V \models$ "ZFC + $\mathcal{K}$ is the proper class of supercompact cardinals". As in [1] and the proof of Theorem 3 in the general case, we also assume without loss of generality that $V \models$ "Every $\kappa \in \mathcal{K}$ has its supercompactness Laver indestructible under κ-directed closed forcing + $2^\kappa = \kappa^+$". In analogy to the proof of [1, Theorem], we start by defining the building blocks used in the definition of the partial ordering $\mathbb{P}$ witnessing the conclusions of Theorem 5.

Before beginning the proof, however, as we did in the above discussion of Theorem 3, we first give some intuition and motivation for the definition of our forcing conditions. Suppose $\kappa \in \mathcal{K}$ is not a limit of supercompact cardinals. In [1], in order to construct the requisite model, it was only necessary to force to add non-reflecting stationary sets of ordinals of the appropriate cofinality to rid ourselves of each ground measurable limit of strong cardinals $\delta < \kappa$, while also forcing to preserve each ground model strong cardinal $\delta < \kappa$. As before, this is not sufficient in the current situation, since the partial orderings from [1] will not ensure that there are non-strong tall cardinals below κ. We will therefore force to preserve the strongness of some, although not all, of the ground model strong cardinals below κ which are not limits of ground model strong cardinals, while also adding non-reflecting stationary sets of ordinals of the appropriate cofinality to each ground model measurable limit of strong cardinals $\delta < \kappa$. We will leave alone the remaining ground model strong cardinals $\delta < \kappa$ which are not limits of ground model strong cardinals. Such δ will become tall but not strong. This will guarantee both that all measurable limits of strong cardinals below κ have been eliminated, and that κ has become a limit of non-strong tall cardinals.

We continue in analogy to [1], quoting verbatim when appropriate, and working under the assumption that all computations and notions found in this paragraph are given in V. Fix $\kappa \in \mathcal{K}$ which is not a limit of supercompact cardinals. Let ξ be either the successor of the supremum of the supercompact cardinals below κ or ω if κ is the least supercompact cardinal, and let η be the least strong cardinal above ξ in V. By [3, Lemma 2.1] and the succeeding remarks, $\eta \in (\xi, \kappa)$. Let $\langle \delta_\alpha \mid \alpha < \kappa \rangle$ be the continuous, increasing enumeration of the cardinals in the interval (η, κ) which are either strong cardinals or measurable limits of strong cardinals. For α an arbitrary ordinal, define α^- as the immediate ordinal predecessor of α if α is a successor ordinal, and 0 if α is either a limit ordinal or 0. For each $\alpha < \kappa$, let $\gamma_\alpha = (\bigcup_{\beta < \alpha} \delta_\beta)^+$, where if $\alpha = 0$, $\gamma_\alpha = (2^{\eta^+})^+$. Also, for each $\alpha < \kappa$, define θ_α as the least cardinal such that $V \models$ "δ_α is not θ_α supercompact". As in [1], θ_α is well-defined for every $\alpha < \kappa$. Further, it must be the case that $\theta_\alpha < \delta_{\alpha+1}$. This is since it follows from the argument found in [3, proof of Lemma 2.4, page 31, fourth paragraph] that if δ is γ supercompact for every $\gamma < \delta'$ and δ' is strong, then δ is supercompact. Thus, $\theta_\alpha < \delta_{\alpha+1}$ is true because $\delta_{\alpha+1}$ is a strong cardinal.

We now define the partial ordering $\mathbb{P}_\kappa = \langle\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha\rangle \mid \alpha < \kappa\rangle$ as the Easton support iteration of length κ satisfying the following properties:

1. $\mathbb{P}_0 = \mathrm{Add}(\eta^+, 1)$.

2. Suppose δ_α is not a measurable limit of strong cardinals in V and α is either a limit ordinal or a successor ordinal of the form $\alpha' + 2n + 1$ for $n \in \omega$. Here, α' is either a limit ordinal or 0. Then $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$, where $\dot{\mathbb{Q}}_\alpha$ is a term for a Gitik-Shelah partial ordering of [6] for the cardinal δ_α. We stipulate that this partial ordering be defined using only components that are at least ρ_α-strategically closed and ξ-directed closed, and that the first nontrivial forcing in the definition be $\mathrm{Add}(\lambda_\alpha, 1)$ for λ_α the least measurable cardinal above $\rho_\alpha = \max(\theta_{\alpha^-}, \gamma_\alpha, \xi)$.[2] Under these restrictions, the realization of $\dot{\mathbb{Q}}_\alpha$ makes the strongness of δ_α indestructible under forcing with δ_α-strategically closed partial orderings which are at least ξ-directed closed.

3. Suppose δ_α is not a measurable limit of strong cardinals in V and α is a successor ordinal of the form $\alpha' + 2n + 2$ for $n \in \omega$. Here, α' is either a limit ordinal or 0. Then $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$, where $\dot{\mathbb{Q}}_\alpha$ is a term for trivial forcing.

4. Suppose δ_α is a measurable limit of strong cardinals in V. Then $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$, where $\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{P}}(\xi, \delta_\alpha)$.

Let $A = \{\kappa \in \mathcal{K} \mid \kappa$ is not a limit of supercompact cardinals$\}$. We then define the partial ordering $\mathbb{P}$ used in the proof of Theorem 5 as the Easton support product $\prod_{\kappa \in A} \mathbb{P}_\kappa$.

We note that the definition of $\mathbb{P}$ is almost the same as the one found in [1], with the only key difference that the strongness of δ_α for δ_α as in (3) above is not preserved by the iteration $\mathbb{P}_\kappa$. Further, for any $\kappa \in A$, there are unboundedly in κ many δ_α as in (2) above, and as in [1], the strongness of such δ_α is preserved by both $\mathbb{P}_\kappa$ and $\mathbb{P}$. Therefore, the exact same arguments as

[2]It is also possible to let $\dot{\mathbb{Q}}_\alpha$ be a term for Hamkins' partial ordering of [11, Theorem 4.10] for the cardinal δ_α, assuming the same restrictions on components as just mentioned and that the fast function forcing employed in Hamkins' definition is ξ-directed closed.

in [1] virtually unchanged show that $V^{\mathbb{P}} \models$ "ZFC + κ is strongly compact iff κ is a measurable limit of strong cardinals + The strongly compact cardinals are the elements of $\mathcal{K}$ together with their measurable limit points" and that in $V^{\mathbb{P}}$, any $\kappa \in \mathcal{K}$ which was a supercompact limit of supercompact cardinals in $V^{\mathbb{P}}$ remains supercompact. This means that the proof of Theorem 5 is completed by the following lemma.

Lemma 2.8 $V^{\mathbb{P}} \models$ *"If $\kappa \in \mathcal{K}$, then κ is a limit of non-strong tall cardinals".*

Proof: It suffices to prove Lemma 2.8 for $\kappa \in A$. This is since if $\lambda \in \mathcal{K} \setminus A$, then λ is a limit of members of A (which are V-supercompact cardinals each of which is not a limit of V-supercompact cardinals), and so λ is a limit of non-strong tall cardinals.

To do this, in analogy to the proof of Theorem 3 in the general case and in analogy to the proof of [1, Theorem], we can write $\mathbb{P} = \mathbb{P}_{<\kappa} \times \mathbb{P}_\kappa \times \mathbb{P}^\kappa$, where $\mathbb{P}_{<\kappa} = \prod_{\delta<\kappa, \delta \in A} \mathbb{P}_\delta$, $\mathbb{P}^\kappa = \prod_{\delta>\kappa, \delta \in A} \mathbb{P}_\delta$, and all products have Easton support. As in the proof of [1, Theorem] and the proof of Theorem 3 in the general case, since $\mathbb{P}^\kappa$ is κ-directed closed and each $\kappa \in \mathcal{K}$ has its supercompactness indestructible under κ-directed closed forcing, the cardinals less than or equal to κ in $V^{\mathbb{P}^\kappa}$ which are supercompact, strong, or measurable limits of strong cardinals are precisely the same as those in V. In addition, as in the proof of [1, Theorem], the cardinals $\delta < \kappa$ in $V^{\mathbb{P}^\kappa}$ and V which are either strong cardinals or measurable limits of strong cardinals are precisely the same as those cardinals which are either α strong for every $\alpha \in (\delta, \kappa)$ (in either $V^{\mathbb{P}^\kappa}$ or V) or are measurable limits of cardinals γ which are α strong for every $\alpha \in (\gamma, \kappa)$ (in either $V^{\mathbb{P}^\kappa}$ or V). This then allows us to conclude as in the proof of [1, Theorem] that $\mathbb{P}_\kappa$ as defined in $V^{\mathbb{P}^\kappa}$ is the same as $\mathbb{P}_\kappa$ as defined in V. Consequently, we will now show that $V^{\mathbb{P}^\kappa \times \mathbb{P}_\kappa} \models$ "κ is a limit of non-strong tall cardinals". This will be enough to complete the proof of Lemma 2.8. This is since $|\mathbb{P}_{<\kappa}| < \kappa$, so by the results of [16] and [13], if $V^{\mathbb{P}^\kappa \times \mathbb{P}_\kappa} \models$ "κ is a limit of non-strong tall cardinals", then $V^{\mathbb{P}^\kappa \times \mathbb{P}_\kappa \times \mathbb{P}_{<\kappa}} = V^{\mathbb{P}} \models$ "κ is a limit of non-strong tall cardinals".

To do this, work in $V^{\mathbb{P}^\kappa} = V_1$. Let δ_α be as in (3) of the definition of $\mathbb{P}_\kappa$. In analogy to the proofs of Lemmas 2.1 and 2.2, by the definition of $\mathbb{P}_\kappa$, we may write $\mathbb{P}_\kappa = \mathbb{R}^* * \dot{\mathbb{R}}^{**} * \dot{\mathbb{R}}$. Here, $|\mathbb{R}^*| < \delta_\alpha$, $\mathbb{R}^*$ is nontrivial, $\dot{\mathbb{R}}^{**} = \dot{\mathrm{A}}\mathrm{dd}(\delta', 1)$ for some measurable cardinal $\delta' > \delta_\alpha$, and $\dot{\mathbb{R}}$ is a term for the rest of $\mathbb{P}_\kappa$. Since $\Vdash_{\mathbb{R}^* * \dot{\mathbb{R}}^{**}}$ "$\dot{\mathbb{R}}$ is σ-strategically closed for σ the least inaccessible cardinal above δ'", as in the proof of Lemma 2.1, $V_1^{\mathbb{R}^* * \dot{\mathbb{R}}^{**} * \dot{\mathbb{R}}} = V_1^{\mathbb{P}_\kappa} \vDash$ "δ_α is $\delta' + 2$ strong" iff $V_1^{\mathbb{R}^* * \dot{\mathbb{R}}^{**}} \vDash$ "δ_α is $\delta' + 2$ strong". The remainder of the argument given in Lemma 2.1 then shows that $V_1^{\mathbb{R}^* * \dot{\mathbb{R}}^{**}} \vDash$ "δ_α is not $\delta' + 2$ strong". The argument given in the second paragraph of the proof of Lemma 2.2 then shows that $V_1^{\mathbb{R}^* * \dot{\mathbb{R}}^{**} * \dot{\mathbb{R}}} = V^{\mathbb{P}^\kappa \times \mathbb{P}_\kappa} \vDash$ "δ is a tall cardinal". Putting the previous two sentences together, we now have that for δ_α as in (3) of the definition of $\mathbb{P}_\kappa$, $V^{\mathbb{P}^\kappa \times \mathbb{P}_\kappa} \vDash$ "δ_α is a non-strong tall cardinal". Since there are unboundedly many such δ_α below κ, this completes the proof of Lemma 2.8.

□

Lemma 2.8 completes the proof of Theorem 5.

□

3 Concluding Remarks

In conclusion to this paper, we ask whether it is possible to prove versions of Theorems 3 – 5 in which the only tall cardinals are either strong or strongly compact. This seems to be an extremely challenging question to answer, both since every tall cardinal κ is automatically indestructible under (κ, ∞)-distributive forcing, and since by [4, Theorem 1], it is consistent to assume in a model containing a supercompact cardinal that the statement "κ is measurable iff κ is tall" is true. Thus, not only does forcing to add a non-reflecting stationary set of ordinals of small cofinality above a tall cardinal not destroy tallness as it does strong compactness, but the set of tall cardinals which might need to be eliminated in order to answer this question could be quite large. For instance, suppose we wish to force over a model with a supercompact cardinal λ and construct a model in which the least strongly compact cardinal is also the least strong cardinal and the only tall cardinals are either strong or strongly compact. If the statement "κ is measurable iff κ is tall" is true, then iteratively adding non-reflecting stationary sets of ordinals below λ to any cardinal δ which is either a strong cardinal or a non-strong tall cardinal will in fact destroy all measurable cardinals below λ. While this will, under the appropriate circumstances, preserve the strong compactness of λ, it will not preserve the strongness of λ, since after the forcing has been done, there will no longer be any measurable cardinals below λ. Finding a way around this problem seems to be quite difficult.

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