The Wholeness Axioms and the Class of Supercompact Cardinals *†

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Abstract

We show that certain relatively consistent structural properties of the class of supercompact cardinals are also relatively consistent with the Wholeness Axioms.

1 Introduction and Preliminaries

The Wholeness Axiom WA, introduced by Paul Corazza in [6] and [10], is intended as a weakening of Kunen's inconsistency result of [16] concerning the nonexistence of a nontrivial elementary embedding from the universe to itself. In addition, in [12], Hamkins gave a stratification of WA into countably many axioms $\mathrm{WA}_0, \mathrm{WA}_1, \ldots, \mathrm{WA}_\infty$, with WA_∞ the original Wholeness Axiom WA. More specifically, work in the language $\{\in, \mathbf{j}\}$ extending the usual language of set theory $\{\in\}$, where $\mathbf{j}$ is a unary function symbol representing the elementary embedding. For $n \in \{0, 1, 2, 3, \ldots, \infty\}$, WA_n is defined as:

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1. (Elementarity) All instances of $\varphi(x) \leftrightarrow \varphi(\mathbf{j}(x))$ for φ a formula in the language $\{\in\}$.

2. (Separation) All instances of the Separation Axiom for Σ_n formulae in the full language $\{\in, \mathbf{j}\}$.

3. (Nontriviality) The axiom $\exists x[\mathbf{j}(x) \neq x]$.

The purpose of this paper is to augment the studies of the Wholeness Axioms found in [6], [10], [8], [12], [9], [7], and [4] and show that some relatively consistent structural properties of the class of supercompact cardinals are also relatively consistent with the Wholeness Axioms. Specifically, we prove the following.

Theorem 1 *If the Wholeness Axiom* WA_0 *is consistent, then it is consistent with the following:*

1. *The classes of supercompact and strongly compact cardinals coincide precisely, except at measurable limit points.*

2. *Each supercompact cardinal* κ *is Laver indestructible [17] under* κ*-directed closed forcing.*

3. *Each non-supercompact strongly compact cardinal* κ *has its strong compactness indestructible under* κ*-directed closed forcing not changing* $\wp(\kappa)$*.*

Theorem 2 *If the Wholeness Axiom* WA_0 *is consistent, then it is consistent with* GCH *and level by level equivalence between strong compactness and supercompactness.*

Theorem 3 *If the existence of an* I_3 *cardinal is consistent, then the (full) Wholeness Axiom* WA *is consistent with the following:*

1. *The classes of supercompact and strongly compact cardinals coincide precisely, except at measurable limit points.*

2. *Each supercompact cardinal* κ *is Laver indestructible under* κ*-directed closed forcing.*

3. *Each non-supercompact strongly compact cardinal* κ *has its strong compactness indestructible under* κ*-directed closed forcing not changing* $\wp(\kappa)$*.*

Theorem 4 *If the existence of an* I_3 *cardinal is consistent, then the (full) Wholeness Axiom* WA *is consistent with* GCH *and level by level equivalence between strong compactness and supercompactness.*

A few remarks concerning the above theorems are now in order. By the work of [10] and [9], ZFC + WA_0 implies the existence of a proper class of supercompact limits of supercompact cardinals (and much more). Further, by the work of Menas [18], if $\alpha < \kappa$ and κ is the α^{th} measurable limit of either supercompact or (non-supercompact) strongly compact cardinals, then κ is strongly compact but is not supercompact. Thus, Theorems 1 – 4 are meaningful. In addition, the indestructibility for non-supercompact strongly compact cardinals found in Theorems 1 and 3 was first discussed in [1]. Also, property (1) of Theorems 1 and 3 was first introduced and established by Kimchi and Magidor in [15], and the property given in Theorems 2 and 4 ("level by level equivalence between strong compactness and supercompactness", i.e., for $\kappa < \lambda$ regular cardinals, κ is λ strongly compact iff κ is λ supercompact, except possibly if κ is a measurable limit of cardinals δ which are λ supercompact) was first discussed and shown to be relatively consistent by the author and Shelah in [5]. Consequently, Theorems 1 – 4 demonstrate that certain interesting structural properties that the class of supercompact cardinals may possess are also relatively consistent with the Wholeness Axioms.

Before beginning the proofs of Theorems 1 – 4, we very briefly mention some preliminary material concerning notation and terminology. For $\alpha < \beta$ ordinals, (α, β) is as in usual interval notation. When forcing, $q \geq p$ means that *q is stronger than p*. If $G \subseteq \mathbb{P}$ is V-generic, we will abuse notation somewhat and use both $V[G]$ and $V^{\mathbb{P}}$ to denote the generic extension by $\mathbb{P}$. We will also occasionally abuse notation by writing x when we really mean $\check{x}$. For κ a regular cardinal and α an arbitrary ordinal, $\mathrm{Add}(\kappa, \alpha)$ is the standard partial ordering for adding α many Cohen subsets of κ.

The partial ordering $\mathbb{P}$ is *κ-directed closed* if every directed subset of $\mathbb{P}$ of size less than κ has an upper bound. $\mathbb{P}$ is *κ-strategically closed* if in the two person game in which the players construct an increasing sequence $\langle p_\alpha \mid \alpha \leq \kappa \rangle$, where player I plays odd stages and player II plays even stages (choosing the trivial condition at stage 0), player II has a strategy which ensures the game can always be continued. $\mathbb{P}$ is *$<\kappa$-strategically closed* if $\mathbb{P}$ is δ-strategically closed for every cardinal $\delta < \kappa$. If $\mathbb{P} = \langle \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle \mid \alpha < \kappa \rangle$ is an Easton support iteration of length κ and $0 \le \gamma < \delta < \kappa$, we will abuse notation by writing $\mathbb{P}_{\gamma,\delta}$ for both the portion of the iteration *strictly between γ and δ* (i.e., we will use this notation in the proofs of Theorems 1, 3, and 4) and the portion of the iteration *between γ and δ but including δ* (particularly in the proof of Theorem 2, where both usages will occur). It will be clear from the context exactly which of the two of these is meant.

As in [11], if $\mathcal{A}$ is a collection of partial orderings, then the *lottery sum* is the partial ordering $\oplus\mathcal{A} = \{\langle \mathbb{P}, p \rangle \mid \mathbb{P} \in \mathcal{A} \text{ and } p \in \mathbb{P}\} \cup \{0\}$, ordered with 0 below everything and $\langle \mathbb{P}, p \rangle \le \langle \mathbb{P}', p' \rangle$ iff $\mathbb{P} = \mathbb{P}'$ and $p \le p'$. Intuitively, if G is V-generic over $\oplus\mathcal{A}$, then G first selects an element of $\mathcal{A}$ (or as Hamkins says in [11], "holds a lottery among the posets in $\mathcal{A}$") and then forces with it. The terminology "lottery sum" is due to Hamkins, although the concept of the lottery sum of partial orderings has been around for quite some time and has been referred to at different junctures via the expressions "disjoint sum of partial orderings," "side-by-side forcing," and "choosing which partial ordering to force with generically."

Finally, we mention that we are assuming a reasonable familiarity with standard concepts in large cardinals and forcing, as found, e.g., in [13] or [14]. We do note explicitly that we will say *κ is $<\lambda$ supercompact* if κ is δ supercompact for every $\delta < \lambda$. In addition, *an* I_3 *cardinal* κ is a cardinal such that there exists an elementary embedding $j : V_\lambda \to V_\lambda$ having critical point κ with λ the supremum of *the critical sequence associated with* j, i.e., $\lambda = \bigcup_{i<\omega} \kappa_i$, where $\kappa = \kappa_0 = \mathrm{cp}(j)$ and $\kappa_{i+1} = j(\kappa_i)$. Additional information on I_3 cardinals may be found in [14].

2 The Proofs of Theorems 1 – 4

We turn now to the proofs of our theorems. We will provide full details for the proof of Theorem 1 and indicate how the proofs of our remaining results follow from earlier work, which is found both in this paper and elsewhere. In particular, the proofs of Theorems 3 and 4 will really only be proof sketches. We begin with the proof of Theorem 1.

Proof: Assume that WA_0 is consistent. As Hamkins remarks in [12], this means that there is a model $\langle V, \in j\rangle$, where $\langle V, \in\rangle$ is a model of ZFC and $j : V \to V$ is a nontrivial amenable elementary embedding. We take the structure $\langle V, \in j\rangle$ as our ground model. We also let $\langle \kappa_i \mid i < \omega\rangle$ be the critical sequence associated with j. As shown in [12],

$$V_{\kappa_0} \prec V_{\kappa_1} \prec \cdots \prec V$$

is an elementary chain of models, with $V = \bigcup_{i<\omega} V_{\kappa_i}$. In particular, elementarity implies that $V_{\kappa_0} \models$ "There is a proper class of supercompact limits of supercompact cardinals".

Note that by the work of [10] and [9], for each $i < \omega$, $V \models$ "κ_i is a supercompact limit of supercompact cardinals". Therefore, since $V_{\kappa_j} \prec V$ for any $j < \omega$, for each $i < \omega$ and all $j > i$, $V_{\kappa_j} \models$ "κ_i is a supercompact limit of supercompact cardinals". In particular, for any $i < \omega$, there is a cardinal $\kappa_i^* > \kappa_i$ such that in both $V_{\kappa_{i+1}}$ and V, κ_i^* is the least supercompact cardinal greater than κ_i.

We now follow the plan of attack found in the proof of the Main Theorem of [12] by first defining a class partial ordering $\mathbb{P}_{\kappa_0}$ in V_{κ_0} such that after forcing with $\mathbb{P}_{\kappa_0}$ over V_{κ_0}, the resulting model satisfies properties (1) – (3) of Theorem 1. We give the definition used in the proof of [2, Theorem 2], as opposed to the original one employed in the proof of the Theorem of [1]. Specifically, working in V_{κ_0}, let $\mathcal{K}$ be the class of supercompact cardinals. Let $\mathcal{D} = \langle \delta_\alpha \mid \alpha \in (\mathrm{Ord})^{V_{\kappa_0}}\rangle$ enumerate in increasing order all regular limits of strong cardinals. For any ordinal δ, define σ_δ as the successor of the smallest regular cardinal greater than or equal to the supremum of the supercompact cardinals below δ, or ω if there are no supercompact cardinals below δ.[1] The partial ordering $\mathbb{P}_{\kappa_0}$ with which we force is the proper class Easton support iteration $\langle\langle \mathbb{P}_\delta, \dot{\mathbb{Q}}_\delta\rangle \mid \delta \in (\mathrm{Ord})^{V_{\kappa_0}}\rangle$ which begins by adding a Cohen subset of ω and then does trivial forcing except when $\delta \in \mathcal{D}$. At such a stage, $\dot{\mathbb{Q}}_\delta$ has the form $\dot{\mathbb{Q}}_{\delta,1} * \dot{\mathbb{Q}}_{\delta,2}$, where $\dot{\mathbb{Q}}_{\delta,1}$ is a term for the lottery sum of all δ-directed closed partial orderings having rank below the least V_{κ_0}-strong cardinal δ' above δ, and $\dot{\mathbb{Q}}_{\delta,2}$ is a term for the standard partial ordering (see [5]) which adds a non-reflecting stationary set of ordinals of cofinality

[1] In [2], σ_δ is defined as the smallest regular cardinal greater than or equal to the supremum of the supercompact cardinals below δ, or ω if there are no supercompact cardinals below δ. This difference in definition, however, is inessential, and does not affect any of the proofs given in [2].

$\sigma_{\delta'}$ to δ'.[2]

We continue by following the proof of the Main Theorem of [12], taking the liberty to quote verbatim when appropriate. We also refer readers to [12] for any missing or unexplained details. We begin by letting $G_{\kappa_0} \subseteq \mathbb{P}_{\kappa_0}$ be V-generic. Next, we consider the partial ordering $j(\mathbb{P}_{\kappa_0}) = \mathbb{P}_{\kappa_1}$. Observe that if $\delta < \kappa$ and $\varphi(x)$ is the formula in the language of set theory which says either "x is a strong cardinal" or "x is a supercompact cardinal", then by the fact that $V_{\kappa_0} \prec V_{\kappa_1}$, for $\delta < \kappa_0$, $V_{\kappa_0} \vDash \varphi(\delta)$ iff $V_{\kappa_1} \vDash \varphi(\delta)$. This means we may write $j(\mathbb{P}_{\kappa_0})$ as $\mathbb{P}_{\kappa_0} * \dot{\mathbb{P}}_{\kappa_0,\kappa_1}$. Further, since $V_{\kappa_1} \vDash$ "κ_0 is supercompact", by forcing above a condition opting for trivial forcing in the lottery sum held at stage κ_0, we may assume that $V[G_{\kappa_0}] \vDash$ "$\mathbb{P}_{\kappa_0,\kappa_1}$ is κ_0^+-directed closed".

Force now to obtain a $V[G_{\kappa_0}]$-generic object $G_{\kappa_0,\kappa_1} \subseteq \mathbb{P}_{\kappa_0,\kappa_1}$, which also provides a V-generic object $G_{\kappa_1} = G_{\kappa_0} * G_{\kappa_0,\kappa_1} \subseteq \mathbb{P}_{\kappa_1}$. Since $j''G_{\kappa_0} = G_{\kappa_0} \subseteq G_{\kappa_1}$, as usual, j lifts (in $V[G_{\kappa_1}]$) to $j : V[G_{\kappa_0}] \to V[G_{\kappa_1}]$ with $j(G_{\kappa_0}) = G_{\kappa_1}$. In addition, as in [12], $V_{\kappa_0}[G_{\kappa_0}] \prec V_{\kappa_1}[G_{\kappa_1}]$. Next, consider the partial ordering $j(\mathbb{P}_{\kappa_1}) = \mathbb{P}_{\kappa_0} * \dot{\mathbb{P}}_{\kappa_0,\kappa_2} = \mathbb{P}_{\kappa_1} * \dot{\mathbb{P}}_{\kappa_1,\kappa_2}$. As before, by forcing above a condition opting for trivial forcing in the lottery sum held at stage κ_1, since $V_{\kappa_2} \vDash$ "κ_1 is supercompact", we may assume that $V[G_{\kappa_1}] \vDash$ "$\mathbb{P}_{\kappa_1,\kappa_2}$ is κ_1^+-directed closed". Therefore, since $j''G_{\kappa_0,\kappa_1}$ is a directed subset of $\mathbb{P}_{\kappa_1,\kappa_2}$ in $V[G_{\kappa_1}]$ having size κ_1, by the directed closure of $\mathbb{P}_{\kappa_1,\kappa_2}$ in $V[G_{\kappa_1}]$, there is a master condition q_1 for $j''G_{\kappa_0,\kappa_1}$. Let $G_{\kappa_1,\kappa_2} \subseteq \mathbb{P}_{\kappa_1,\kappa_2}$ be a $V[G_{\kappa_1}]$-generic object containing q_1. We now have a V-generic object $G_{\kappa_2} = G_{\kappa_1} * G_{\kappa_1,\kappa_2} \subseteq \mathbb{P}_{\kappa_2}$ such that $j''G_{\kappa_1} \subseteq G_{\kappa_2}$. By continuing inductively in this manner for ω many steps, we obtain V-generic objects $G_{\kappa_n} \subseteq \mathbb{P}_{\kappa_n}$ for every $n \in \omega - \{0\}$ and master conditions q_n for $j''G_{\kappa_{n-1},\kappa_n}$ such that $q_n \in G_{\kappa_n,\kappa_{n+1}}$ and $j''G_{\kappa_n} \subseteq G_{\kappa_{n+1}}$. This means that working in $V[G_{\kappa_{n+1}}]$, it is always possible to lift j to $j : V[G_{\kappa_n}] \to V[G_{\kappa_{n+1}}]$ and have that $j(G_{\kappa_n}) = G_{\kappa_{n+1}}$ and $V_{\kappa_n}[G_{\kappa_n}] \prec V_{\kappa_{n+1}}[G_{\kappa_{n+1}}]$. This produces the elementary chain of models of length ω

$$V_{\kappa_0}[G_{\kappa_0}] \prec V_{\kappa_1}[G_{\kappa_1}] \prec \cdots \prec V_{\kappa_n}[G_{\kappa_n}] \prec \cdots.$$

Let $\overline{V} = \bigcup_{n \in \omega} V_{\kappa_n}[G_{\kappa_n}]$. Since $\overline{V}$ is the union of an elementary chain of models, the theory of $\overline{V}$

[2] We refer readers to [5] for the exact definition and properties of the standard partial ordering for adding a non-reflecting stationary set of ordinals of cofinality δ to the regular cardinal λ. We note only that this partial ordering is both δ-directed closed and $<\lambda$-strategically closed.

is the theory of each $V_{\kappa_n}[G_{\kappa_n}]$. In particular, the theory of $\overline{V}$ is the same as the theory of $V_{\kappa_0}[G_{\kappa_0}]$, so $\overline{V}$ is a model of ZFC satisfying properties (1) – (3) of Theorem 1. Also, because for every $n \in \omega$ we have already lifted j to $j : V[G_{\kappa_n}] \to V[G_{\kappa_{n+1}}]$, we have defined a map $j : \overline{V} \to \overline{V}$. The argument that j is elementary, and consequently, that $\langle \overline{V}, \in j\rangle$ is a model of WA_0, is now the same as in [12]. Specifically, if $\overline{V} \vDash \varphi(x)$, then because we have an elementary chain of models, $V_{\kappa_n}[G_{\kappa_n}] \vDash \varphi(x)$ for sufficiently large n. An application of j now yields that $V_{\kappa_{n+1}}[G_{\kappa_{n+1}}] \vDash \varphi(j(x))$ for such n, so once again, the fact that we have an elementary chain of models yields that $\overline{V} \vDash \varphi(j(x))$. In addition, since $j \restriction V[G_{\kappa_n}]$ was defined in $V[G_{\kappa_{n+1}}]$, it follows that $j \restriction V_{\kappa_n}[G_{\kappa_n}] \in \overline{V}$. Hence, $j : \overline{V} \to \overline{V}$ is amenable, so $\langle \overline{V}, \in, j\rangle$ is a model of WA_0. This completes the proof of Theorem 1.

□

Turning now to the proof of Theorem 2, let $\langle V, \in, j\rangle$ be a model for WA_0. As in [12], we may assume that $V \vDash$ GCH as well. In addition, as in the proof of Theorem 1, let $\langle \kappa_i \mid i < \omega\rangle$ be the critical sequence generated by j.

As in the proof of Theorem 1, we first define a class partial ordering $\mathbb{P}_{\kappa_0}$ such that after forcing over V_{κ_0} with $\mathbb{P}_{\kappa_0}$, the resulting model satisfies GCH and level by level equivalence between strong compactness and supercompactness. We start by defining the partial orderings $\mathbb{P}^0_{\delta,\lambda}$, $\mathbb{P}^1_{\delta,\lambda}[S]$, and $\mathbb{P}^2_{\delta,\lambda}[S]$ of [5, Section 1] in an arbitrary ground model $\overline{V} \vDash$ ZFC. So that readers are not overly burdened, we abbreviate our definitions and descriptions somewhat. Full details may be found by consulting [5].

Fix $\delta < \lambda$, $\lambda \geq \aleph_1$ regular cardinals in our ground model $\overline{V}$. The first notion of forcing $\mathbb{P}^0_{\delta,\lambda}$ is once again the standard partial ordering for adding a non-reflecting stationary set of ordinals S of cofinality δ to λ^+. Next, work in $V_1 = \overline{V}^{\mathbb{P}^0_{\delta,\lambda}}$, letting $\dot{S}$ be a term always forced to denote S. $\mathbb{P}^2_{\delta,\lambda}[S]$ is the usual partial ordering for introducing a club set C which is disjoint to S (and therefore makes S non-stationary).

We fix now in V_1 a $\clubsuit(S)$ sequence $X = \langle x_\alpha \mid \alpha \in S\rangle$, the existence of which is given by [5, Lemma 1]. We are ready to define in V_1 the partial ordering $\mathbb{P}^1_{\delta,\lambda}[S]$. First, since each element of S has cofinality δ, the proof of [5, Lemma 1] shows each $x \in X$ can be assumed to be such that order type$(x) = \delta$. Then, $\mathbb{P}^1_{\delta,\lambda}[S]$ is defined as the set of all 4-tuples $\langle w, \alpha, \bar{r}, Z\rangle$ satisfying the following properties.

1. $w \in [\lambda^+]^{<\lambda}$.
2. $\alpha < \lambda$.
3. $\bar{r} = \langle r_i \mid i \in w\rangle$ is a sequence of functions from α to $\{0, 1\}$, i.e., a sequence of subsets of α.
4. $Z \subseteq \{x_\beta \mid \beta \in S\}$ is a set such that if $z \in Z$, then for some $y \in [w]^\delta$, $y \subseteq z$ and $z - y$ is bounded in the β such that $z = x_\beta$.

The ordering on $\mathbb{P}^1_{\delta,\lambda}[S]$ is given by $\langle w^1, \alpha^1, \bar{r}^1, Z^1\rangle \leq \langle w^2, \alpha^2, \bar{r}^2, Z^2\rangle$ iff the following hold.

1. $w^1 \subseteq w^2$.
2. $\alpha^1 \leq \alpha^2$.
3. If $i \in w^1$, then $r^1_i \subseteq r^2_i$.
4. $Z^1 \subseteq Z^2$.
5. If $z \in Z^1 \cap [w^1]^\delta$ and $\alpha^1 \leq \alpha < \alpha^2$, then $|\{i \in z \mid r^2_i(\alpha) = 0\}| = |\{i \in z \mid r^2_i(\alpha) = 1\}| = \delta$.

By [5, Lemma 4], $\mathbb{P}^0_{\delta,\lambda} * (\mathbb{P}^1_{\delta,\lambda}[\dot{S}] \times \mathbb{P}^2_{\delta,\lambda}[\dot{S}])$ is equivalent to $\mathrm{Add}(\lambda^+, 1) * \dot{\mathrm{Add}}(\lambda, \lambda^+)$ and so is λ-directed closed. In particular, $\mathbb{P}^0_{\delta,\lambda} * \mathbb{P}^2_{\delta,\lambda}[\dot{S}]$ is equivalent to $\mathrm{Add}(\lambda^+, 1)$, and after forcing with $\mathbb{P}^0_{\delta,\lambda} * \mathbb{P}^2_{\delta,\lambda}[\dot{S}]$, $\mathbb{P}^1_{\delta,\lambda}[\dot{S}]$ is equivalent to $\mathrm{Add}(\lambda, \lambda^+)$. By the remark on [5, middle of page 108] and the fact that $\mathbb{P}^0_{\delta,\lambda}$ is δ-directed closed, $\mathbb{P}^0_{\delta,\lambda} * \mathbb{P}^1_{\delta,\lambda}[\dot{S}]$ is δ-directed closed. By [5, Lemma 5], $\mathbb{P}^0_{\delta,\lambda} * \mathbb{P}^1_{\delta,\lambda}[\dot{S}]$ preserves GCH, cardinals and cofinalities, is λ^{++}-c.c., and is $<\lambda$-strategically closed.

We are now in a position to define $\mathbb{P}_{\kappa_0}$. We quote verbatim from [5, page 121] when appropriate. Work in V_{κ_0}. Let $\langle \delta_i \mid i \in (\mathrm{Ord})^{V_{\kappa_0}}\rangle$ enumerate the inaccessible cardinals, and let $\lambda_i > \delta_i$ be the least regular cardinal such that $V_{\kappa_0} \vDash$ "δ_i isn't λ_i supercompact" if such a λ_i exists. If no such λ_i exists, i.e., if δ_i is supercompact, then let $\lambda_i = \Omega$, where we think of Ω as some giant "ordinal" larger than any $\alpha \in (\mathrm{Ord})^{V_{\kappa_0}}$. If possible, choose $\theta_i < \delta_i$ as the least regular cardinal such that

$\theta_i < \delta_k < \delta_i$ implies $\lambda_k < \delta_i$ (whenever $k < i$). Note that θ_i is undefined for δ_i iff δ_i is a limit of cardinals which are $<\delta_i$ supercompact because for $k < i$, if δ_k is $<\delta_i$ supercompact, then $\lambda_k \geq \delta_i$.

We define now a class Easton support iteration $\mathbb{P}_{\kappa_0} = \langle\langle\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha\rangle \mid \alpha \in (\mathrm{Ord})^{V_{\kappa_0}}\rangle$ as follows.

1. $\mathbb{P}_0$ is trivial.

2. Assuming $\mathbb{P}_\alpha$ has been defined, $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$, where $\dot{\mathbb{Q}}_\alpha$ is a term for the trivial partial ordering unless α is regular and for some inaccessible $\delta = \delta_i < \alpha$ with θ_i defined, either δ_i is α supercompact or $\alpha = \lambda_i$. Under these circumstances, $\dot{\mathbb{Q}}_\alpha$ is a term for

$$(\prod_{\{i<\alpha \ \mid \ \delta_i \text{ is } \alpha \text{ supercompact}\}} (\mathbb{P}^0_{\theta_i,\alpha} * \mathbb{P}^2_{\theta_i,\alpha}[\dot{S}_{\theta_i,\alpha}]) * \prod_{\{i<\alpha \ \mid \ \delta_i \text{ is } \alpha \text{ supercompact}\}} \mathbb{P}^1_{\theta_i,\alpha}[\dot{S}_{\theta_i,\alpha}])$$

$$\times$$

$$(\prod_{\{i<\alpha \ \mid \ \alpha=\lambda_i\}} \mathbb{P}^0_{\theta_i,\alpha} * \prod_{\{i<\alpha \ \mid \ \alpha=\lambda_i\}} \mathbb{P}^1_{\theta_i,\alpha}[\dot{S}_{\theta_i,\alpha}])$$

$= (\dot{\mathbb{P}}^0_\alpha * \dot{\mathbb{P}}^1_\alpha) \times (\dot{\mathbb{P}}^2_\alpha * \dot{\mathbb{P}}^3_\alpha)$, with the proviso that elements of $\dot{\mathbb{P}}^0_\alpha$ and $\dot{\mathbb{P}}^2_\alpha$ will have full support, and elements of $\dot{\mathbb{P}}^1_\alpha$ and $\dot{\mathbb{P}}^3_\alpha$ will have support less than α.

Observe that the definition of $\mathbb{P}_{\kappa_0}$ easily implies that $\mathbb{P}_{\kappa_0}$ is an initial segment of $j(\mathbb{P}_{\kappa_0}) = \mathbb{P}_{\kappa_1}$. We consider now the partial ordering $\mathbb{P}_{\kappa_1}$. It is a folklore fact that if δ is $<\gamma$ supercompact and γ is measurable, then δ is γ supercompact.[3] Thus, for any δ_i with $i < \kappa_0$, since κ_0 is a measurable cardinal in both V and V_{κ_1}, it is impossible to have $\lambda_i = \kappa_0$. By the definition of $\mathbb{P}_{\kappa_0}$, this means that $\dot{\mathbb{Q}}_{\kappa_0}$ has the form $\dot{\mathbb{P}}^0_{\kappa_0} * \dot{\mathbb{P}}^1_{\kappa_0}$, so by our remarks on directed closure in the paragraph immediately following the definition of the ordering on $\mathbb{P}^1_{\delta,\lambda}[S]$, $\Vdash_{\mathbb{P}_{\kappa_0}}$ "$\dot{\mathbb{P}}^0_{\kappa_0} * \dot{\mathbb{P}}^1_{\kappa_0}$ is equivalent to a κ_0-directed closed partial ordering having size κ_0^+". Further, by the definition of θ_i and the fact that $V \vDash$ "κ_0 is a supercompact limit of supercompact cardinals" (which as we have already observed means that $V_{\kappa_1} \vDash$ "κ_0 is a supercompact limit of supercompact cardinals"), if $\delta_i > \kappa_0$ is an inaccessible cardinal for which θ_i is defined, then $\theta_i \geq \kappa_0$.[4] In addition, by the definition of θ_i and the fact that $V_{\kappa_1} \vDash$ "κ^*_0 is supercompact", if $\delta_i > \kappa^*_0$ is an inaccessible cardinal for which θ_i is defined, then

[3]If $\langle\mu_\alpha \mid \alpha < \gamma\rangle$ is a sequence of normal measures over $P_\delta(\alpha)$ and ν is a normal measure over γ, then $\mu = \{x \subseteq P_\delta(\gamma) \mid \{\alpha < \gamma \mid x \cap P_\delta(\alpha) \in \mu_\alpha\} \in \nu\}$ is easily verified as being a normal measure over $P_\delta(\gamma)$.

[4]Note that by the fact κ_0 is a limit of cardinals which are $<\kappa_0$ supercompact, θ_{κ_0} is undefined.

$\theta_i \geq \kappa_0^*$. Therefore, by our remarks on directed closure in the paragraph immediately following the definition of the ordering on $\mathbb{P}^1_{\delta,\lambda}[S]$ and the definition of $\mathbb{P}_{\kappa_0}$, we may write $\mathbb{P}_{\kappa_1} = \mathbb{P}_{\kappa_0} * \dot{\mathbb{P}}_{\kappa_0,\kappa_0^*} * \dot{\mathbb{P}}_{\kappa_0^*,\kappa_1}$, where $\Vdash_{\mathbb{P}_{\kappa_0}}$ "$\dot{\mathbb{P}}_{\kappa_0,\kappa_0^*}$ is equivalent to a κ_0-directed closed partial ordering and has size $\gamma < \kappa_1$" and $\Vdash_{\mathbb{P}_{\kappa_0} * \dot{\mathbb{P}}_{\kappa_0,\kappa_0^*}}$ "$\dot{\mathbb{P}}_{\kappa_0^*,\kappa_1}$ is equivalent to a κ_0^*-directed closed partial ordering and has size κ_1". In particular, since $\kappa_0^* > \kappa_0^+$, $\Vdash_{\mathbb{P}_{\kappa_0} * \dot{\mathbb{P}}_{\kappa_0,\kappa_0^*}}$ "$\dot{\mathbb{P}}_{\kappa_0^*,\kappa_1}$ is equivalent to a κ_0^+-directed closed partial ordering and has size κ_1".

Again as in the proof of Theorem 1, force to obtain a V-generic object $G_{\kappa_1} \subseteq \mathbb{P}_{\kappa_1}$. However, this time, factor G_{κ_1} as $G_{\kappa_0} * G_{\kappa_0,\kappa_0^*} * G_{\kappa_0^*,\kappa_1}$, where G_{κ_0} is V-generic over $\mathbb{P}_{\kappa_0}$, G_{κ_0,κ_0^*} is $V[G_{\kappa_0}]$-generic over $\mathbb{P}_{\kappa_0,\kappa_0^*}$, and $G_{\kappa_0^*,\kappa_1}$ is $V[G_{\kappa_0}][G_{\kappa_0,\kappa_0^*}]$-generic over $\mathbb{P}_{\kappa_0^*,\kappa_1}$. In exactly the same manner as in the proof of Theorem 1, since $j''G_{\kappa_0} = G_{\kappa_0} \subseteq G_{\kappa_1}$, j lifts in $V[G_{\kappa_1}]$ to $j : V[G_{\kappa_0}] \to V[G_{\kappa_1}]$, with $j(G_{\kappa_0}) = G_{\kappa_1}$ and $V_{\kappa_0}[G_{\kappa_0}] \prec V_{\kappa_1}[G_{\kappa_1}]$. Now, in analogy to the proof of Theorem 1, consider the partial ordering $j(\mathbb{P}_{\kappa_1}) = \mathbb{P}_{\kappa_0} * \dot{\mathbb{P}}_{\kappa_0,\kappa_0^*} * \dot{\mathbb{P}}_{\kappa_0^*,\kappa_1} * \dot{\mathbb{P}}_{\kappa_1,\kappa_1^*} * \dot{\mathbb{P}}_{\kappa_1^*,\kappa_2} = \mathbb{P}_{\kappa_1} * \dot{\mathbb{P}}_{\kappa_1,\kappa_1^*} * \dot{\mathbb{P}}_{\kappa_1^*,\kappa_2}$. In $V[G_{\kappa_1}]$, $j''G_{\kappa_0,\kappa_0^*}$ is a directed subset of $\mathbb{P}_{\kappa_1,\kappa_1^*}$ having size γ. Since $V[G_{\kappa_1}] \vDash$ "$\mathbb{P}_{\kappa_1,\kappa_1^*}$ is equivalent to a κ_1-directed closed partial ordering" and $\kappa_1 > \gamma$, we can find a master condition $q_{0,0}$ for $j''G_{\kappa_0,\kappa_0^*}$. Let G_{κ_1,κ_1^*} be a $V[G_{\kappa_1}]$-generic object over $\mathbb{P}_{\kappa_1,\kappa_1^*}$ containing $q_{0,0}$. This means that working in $V[G_{\kappa_1}][G_{\kappa_1,\kappa_1^*}]$, we may now lift j to $j : V[G_{\kappa_0}][G_{\kappa_0,\kappa_0^*}] \to V[G_{\kappa_1}][G_{\kappa_1,\kappa_1^*}]$. In $V[G_{\kappa_1}][G_{\kappa_1,\kappa_1^*}]$, $j''G_{\kappa_0^*,\kappa_1}$ is a directed subset of $\mathbb{P}_{\kappa_1^*,\kappa_2}$ having size κ_1. Since $V[G_{\kappa_1}][G_{\kappa_1,\kappa_1^*}] \vDash$ "$\mathbb{P}_{\kappa_1^*,\kappa_2}$ is κ_1^+-directed closed", we can find a master condition $q_{0,1}$ for $j''G_{\kappa_0^*,\kappa_1}$. Let $G_{\kappa_1^*,\kappa_2}$ be a $V[G_{\kappa_1}][G_{\kappa_1,\kappa_1^*}]$-generic object over $\mathbb{P}_{\kappa_1^*,\kappa_2}$ containing $q_{0,1}$. Working in $V[G_{\kappa_1}][G_{\kappa_1,\kappa_1^*}][G_{\kappa_1^*,\kappa_2}] = V[G_{\kappa_2}]$, lift j to $j : V[G_{\kappa_0}][G_{\kappa_0,\kappa_0^*}][G_{\kappa_0^*,\kappa_1}] \to V[G_{\kappa_1}][G_{\kappa_1,\kappa_1^*}][G_{\kappa_1^*,\kappa_2}]$, i.e., working in $V[G_{\kappa_2}]$, j lifts to $j : V[G_{\kappa_1}] \to V[G_{\kappa_2}]$. We may now continue inductively for ω many steps, building an ω sequence of generic objects and master conditions, and thereby complete the proof of Theorem 2 as in the proof of Theorem 1.

$\square$

To prove Theorems 3 and 4, let $j : V_\lambda \to V_\lambda$ be an elementary embedding in our ground model V which has critical point κ and witnesses that j is an I_3 embedding. As before, let $\langle \kappa_i \mid i < \omega \rangle$ be the critical sequence associated with j, with $\lambda = \bigcup_{i<\omega} \kappa_i$. Suppose $\mathbb{P}_\lambda \in V$ and $\mathbb{P}_\lambda \subseteq V_\lambda$, with $\dot{G}_\lambda$ a $\mathbb{P}_\lambda$-name for a V-generic filter over $\mathbb{P}_\lambda$ and $\dot{G}_{\kappa_n}$ a $\mathbb{P}_{\kappa_n}$-name for $\dot{G}_\lambda \restriction \kappa_n$. Suppose further that

$q = \langle \dot{q}_i \mid i < \lambda \rangle \in \mathbb{P}_\lambda$ is a condition obtained inductively satisfying the following three conditions.

1. $q \restriction \kappa_1$ is trivial.

2. For each $n \geq 1$, $\dot{q}_{\kappa_n} \in \dot{\mathbb{Q}}_{\kappa_n}$ is a name for a master condition for $\{j(p_{\kappa_n}) \mid p \in \dot{G}_{\kappa_n}\}$.

3. For each $n \geq 1$, $\dot{q} \restriction (\kappa_n, \kappa_{n+1}) \in \dot{\mathbb{P}}_{\kappa_n, \kappa_{n+1}}$ is a name for a master condition for $\{j(p) \restriction (\kappa_n, \kappa_{n+1}) \mid p \in \dot{G}_{\kappa_n}\}$.

The proof of [7, Theorem 1.2] shows that if G_λ is a V-generic filter containing q, then j lifts to $j : (V_\lambda)^{V[G_\lambda]} \to (V_\lambda)^{V[G_\lambda]}$ witnessing that j is an I_3 embedding with critical point κ. However, the inductive construction of the master conditions mentioned in the proofs of Theorems 1 and 2 (specifically, the sequence $q = \langle q_n \mid n < \omega \rangle$ of master conditions built at the n^{th} stage of the induction in the proof of Theorem 1 or the sequence $q = \langle \langle q_{n,0}, q_{n,1} \rangle \mid n < \omega \rangle$ of master conditions built at the n^{th} stage of the induction in the proof of Theorem 2) provides us with a way of constructing the requisite q required in the proofs of Theorems 3 and 4, where for $\mathbb{P}_{\kappa_n}$ as defined in either Theorem 1 or Theorem 2, $\mathbb{P}_\lambda$ is taken as the inverse limit of $\langle \mathbb{P}_{\kappa_n} \mid n < \omega \rangle$. Therefore, by the work of [10], $\langle (V_\lambda)^{V[G_\lambda]}, \in, j \rangle$ is a model for WA. By construction, depending upon how $\mathbb{P}_\lambda$ is defined, $(V_\lambda)^{V[G_\lambda]}$ is a model for the conclusions of either Theorem 3 or Theorem 4. This completes our sketch of the proofs of Theorems 3 and 4.

$\square$

3 Concluding Remarks

In conclusion to this paper, we make several remarks. First, as Corazza has shown in [10], if κ is the critical point of an elementary embedding witnessing WA, then κ is super-n-huge for every $n \in \omega$. In addition, as Hamkins has observed in [12], the same fact follows if κ is the critical point of an elementary embedding witnessing WA_0. Thus, the proofs of Theorems 2 and 4 establish the consistency (relative to very strong hypotheses) of GCH and level by level equivalence between strong compactness and supercompactness with the existence of super-n-huge cardinals for every

$n \in \omega$. We conjecture that the consistency of these hypotheses with the existence of a super-n-huge cardinal for a specific $n \in \omega$ can be established relative to the existence of that kind of super-n-huge cardinal alone, and that the consistency of these hypotheses relative to any particular form of huge cardinal can be established relative to exactly that form of huge cardinal.

In addition, we note that our methods of proof are amenable to the establishment of the relative consistency of other properties known to be consistent with the class of supercompact cardinals with WA_0 and WA. For example, in [3], relative to ZFC and the existence of a class $\mathcal{K}$ of supercompact cardinals, the consistency of the theory T = "ZFC + $\mathcal{K}$ is the class of supercompact cardinals + The classes of supercompact and strongly compact cardinals coincide precisely, except at measurable limit points + Every measurable cardinal κ is κ^+ supercompact" was shown. The partial ordering $\mathbb{P}$ used to establish this theorem is an Easton support iteration which can be fit into the rubric of the partial orderings discussed in this paper. Consequently, by forcing over a model witnessing the conclusions of either Theorem 2 or Theorem 4, it is possible to establish the relative consistency of T with WA_0 and WA in an analogous manner to what was just done. For further details on the definition of $\mathbb{P}$, which is somewhat complicated, we refer readers to [3].

Finally, for the same reasons as in [12] and [7], it is unknown for $1 \le n \le \infty$ whether the consistency of just WA_n implies the consistency of WA_n with the conclusions of Theorems 1 and 2. We finish by asking if this is indeed the case.

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