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Markov Processes and Related Fields

The Behaviour of the Specific Entropy in the Hydrodynamic Scaling Limit for Ginzburg – Landau Model

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Abstract. The paper studies the behaviour of the specific entropy for Ginzburg – Landau type models under the hydrodynamical scaling of time and space. It is shown that if the initial configurations possess a macroscopic profile, then for any fixed positive macroscopic time the specific microscopic entropy converges to the macroscopic entropy. The latter is defined in terms of the solution of the corresponding hydrodynamical equation, which is a non-linear diffusion equation. The above result is equivalent to the following statement: under the hydrodynamical scaling, for any positive macroscopic time the specific microscopic entropy relative to local Gibbs measures converges to zero.

KEYWORDS: entropy, Ginzburg – Landau model, local Gibbs measures, hydrodynamical scaling limit, hydrodynamical equation

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1. The description of the model

Let Λ_N be the one-dimensional periodic integer lattice $\mathbf{Z}/N\mathbf{Z}$, $N \in \mathbf{N}$. The variable $x_i \in \mathbf{R}$, $i = 1, \ldots, N$, attached to a lattice site, represents a charge at this site. The collection of all charges is a vector $x^{(N)}$ in $\mathbf{R}^N$, which we call the configuration. The nearest neighbour charges interact and the configuration $x^{(N)}$ changes in time. Applying the diffusive scaling of space and time, i.e. shrinking the spacing between charges by N and speeding up the time by N^2, we obtain a system of charges located at points i/N, $i = 1, \ldots, N$, of the circle $S = \mathbf{R}/\mathbf{Z}$. The evolution of $x^{(N)}(t)$ is described by the following system of stochastic differential equations:

$$
\begin{aligned}
dx_i(t) &= \frac{N^2}{2}(\varphi'(x_{i+1}(t)) - 2\varphi'(x_i(t)) + \varphi'(x_{i-1}(t)))\,dt \\
&\quad + N(d\beta_i(t) - d\beta_{i+1}(t)), \quad i \in \Lambda_N,
\end{aligned} \tag{1.1}
$$

where β_i, $i = 1, \ldots, N$, are independent Brownian motions and $\varphi \in C^2(\mathbf{R})$ satisfies the hypotheses $(\mathrm{H}_1) - (\mathrm{H}_3)$ listed below.

(H_1) $\displaystyle\int_{\mathrm{R}} \exp(-\varphi(x))\,dx = 1.$

(H_2) $\displaystyle\int_{\mathrm{R}} \exp(\alpha|\varphi'(x)| - \varphi(x))\,dx < \infty$ for any $\alpha > 0$.

(H_3) There exist positive constants $C_0,\ C_1,\ C_2,\ R$ and $p \in [1/2, 1]$ such that

$$\varphi''(x) \geqslant C_0 \qquad \text{for all } |x| \geqslant R; \tag{1.2}$$

$$\varphi''(x) \leqslant C_1\varphi(x) + C_2 \qquad \text{for all } x \in \mathbf{R}; \tag{1.3}$$

$$|\varphi'(x)| \leqslant C_1(|\varphi(x)|^p + 1) \qquad \text{for all } x \in \mathbf{R}. \tag{1.4}$$

For a simple example one can take $\varphi(x) = (x^2 - \log 2\pi)/2$.

The convexity assumption (1.2) implies the convergence of the integral

(H_3') $\displaystyle M(\lambda) \overset{\text{def}}{=} \int_{\mathrm{R}} \exp(\lambda x - \varphi(x))\,dx < \infty$ for any $\lambda \in \mathbf{R}$.

The infinitesimal generator of the diffusion process $x^{(N)}(t)$ is given by

$$
\begin{aligned}
\mathcal{L}_N &= \frac{N^2}{2}\sum_{i\in\Lambda_N}\Big(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\Big)^2 \\
&\quad - \frac{N^2}{2}\sum_{i\in\Lambda_N}(\varphi'(x_i) - \varphi'(x_{i+1}))\Big(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\Big).
\end{aligned} \tag{1.5}
$$

The generator $\mathcal{L}_N$ is formally symmetric with respect to the product measure

$$d\mu_N(x) = \exp\Big(-\sum_{i\in\Lambda_N}\varphi(x_i)\Big)\,dx_1\cdots dx_N \tag{1.6}$$

on $\mathbf{R}^N$ which is an invariant but non-ergodic measure for our diffusion. Non-ergodicity follows from the fact that $\mathcal{L}_N$ is degenerate in the direction of vector $(1, 1, \ldots, 1)$.

For $n \in \mathbf{N}$ and $y \in \mathbf{R}$ let

$$\mathbf{X}_{n,y} = \Big\{ x \in \mathbf{R}^k : \sum_{i=1}^{n} x_i = ny \Big\}. \tag{1.7}$$

Denote by $\mathcal{L}_{N,y}$ the restriction of $\mathcal{L}_N$ to the hyperplane $\mathbf{X}_{N,y}$. Then the operator $\mathcal{L}_{N,y}$ is strictly elliptic. The conditional measure

$$\mu_{N,y}(\cdot) = \mu_N\Big(\cdot \;\Big|\; \sum_{i\in\Lambda_N} x_i^{(N)} = Ny\Big), \tag{1.8}$$

supported on $\mathbf{X}_{N,y}$, is invariant and ergodic.

The Dirichlet form $D_{N,y}$ corresponding to $\mathcal{L}_{N,y}$ is defined by the relation

$$D_{N,y}(f) \stackrel{\text{def}}{=} - \int\limits_{\mathrm{X}_{N,y}} f \mathcal{L}_{N,y} f \, d\mu_{N,y} = \frac{N^2}{2} \int\limits_{\mathrm{X}_{N,y}} \sum_{i\in\Lambda_N} \Big(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_{i+1}} \Big)^2 d\mu_{N,y}. \tag{1.9}$$

It is non-degenerate for all non-constant $f \in W^{1,2}(\mathbf{X}_{N,y}, \mu_{N,y})$.

Assume that the process $x^{(N)}(\cdot)$ starts from a deterministic configuration $\eta^{(N)} = (\eta_1^{(N)}, \ldots, \eta_N^{(N)})$. Denote by $\mathsf{P}^{(N)}$ the measure on the space of continuous paths corresponding to the process $x^{(N)}(\cdot)$. Let ν_N^t be the distribution of charges in $\mathbf{R}^N$ at time $t > 0$. Since the total charge $\sum_{i\in\Lambda_N} x_i^{(N)}$ is preserved in time, the measure ν_N^t is supported on the hyperplane $\mathbf{X}_{N,y_N}$, where

$$y_N = \frac{1}{N} \sum_{i\in\Lambda_N} \eta_i^{(N)}.$$

Restricting ν_N^t to $\mathbf{X}_{N,y_N}$ we obtain a measure ν_{N,y_N}^t, which is absolutely continuous with respect to μ_{N,y_N}. Let

$$f_{N,y_N}^t \stackrel{\text{def}}{=} \frac{d\nu_{N,y_N}^t}{d\mu_{N,y_N}}.$$

The density f_{N,y_N}^t (up to a normalization constant) is the restriction to $\mathbf{X}_{N,y_N}$ of the solution to the following Cauchy problem:

$$\begin{aligned} \frac{\partial f}{\partial t} &= \mathcal{L}_N f \quad \text{on} \quad (0, +\infty) \times \mathbf{R}^N, \\ f\big|_{t=0} &= \delta_{\eta^{(N)}}. \end{aligned}$$

It is a consequence of ellipticity of $\mathcal{L}_{N,y_N}$ that f_{N,y_N}^t is a smooth and strictly positive function on $\mathbf{X}_{N,y_N}$ for all $t > 0$.

For any $t > 0$ the microscopic entropy $H(\nu^t_{N,y_N} \,|\, \mu_{N,y_N})$ relative to the invariant measure is given by

$$\begin{aligned} H(\nu^t_{N,y_N} \,|\, \mu_{N,y_N}) &= \int_{X_{N,y_N}} \log\Big(\frac{d\nu^t_{N,y_N}}{d\mu_{N,y_N}}\Big) d\nu^t_{N,y_N} \\ &= \int_{X_{N,y_N}} f^t_{N,y_N} \log f^t_{N,y_N} d\mu_{N,y_N}. \end{aligned}$$

Clearly, for our initial data $H(\nu^0_{N,y_N} \,|\, \mu_{N,y_N}) = +\infty$.

The relative entropy $H(\nu^t_{N,y_N} \,|\, \mu_{N,y_N})$ is decreasing in time: for $t > 0$

$$\begin{aligned} \frac{dH(\nu^t_{N,y_N} \,|\, \mu_{N,y_N})}{dt} &= -\frac{N^2}{2} \int_{X_{N,y_N}} \frac{1}{f^t_{N,y_N}} \sum_{i \in \Lambda_N} \Big(\frac{\partial f^t_{N,y_N}}{\partial x_i} - \frac{\partial f^t_{N,y_N}}{\partial x_{i+1}}\Big)^2 d\mu_{N,y_N} \\ &= -4D_{N,y_N}\Big(\sqrt{f^t_{N,y_N}}\,\Big) < 0. \end{aligned} \tag{1.10}$$

It was shown in [4] that if the average $N^{-1} \sum_{i \in \Lambda_N} \varphi(\eta_i^{(N)})$ is bounded uniformly in N and φ satisfies (H_1), (H_2), (H_3'), and the inequalities

$$-C \leqslant \varphi''(x) \leqslant C_1\varphi(x) + C_2, \qquad x \in \mathbf{R}, \tag{1.11}$$

hold for some positive constants C, C_1 and C_2, then

$$H(\nu^t_{N,y_N} \,|\, \mu_{N,y_N}) \leqslant C(t)N \qquad \text{for any} \;\; t > 0. \tag{1.12}$$

Moreover, for any sequence $\{t_N\}$ such that $t_N \to 0$ as $N \to \infty$ and $N^2 t_N$ is bounded away from 0, the corresponding constants $C(t_N)$ in (1.12) are still uniformly bounded in N.

We study the behaviour of the specific entropy $N^{-1}H(\nu^t_{N,y_N} \,|\, \mu_{N,y_N})$ as $N \to \infty$. We prove that it converges to the macroscopic entropy as $N \to \infty$ (Theorem 2.2). We also discuss the equivalent result about the behaviour of the specific entropy relative to local Gibbs measures (Theorem 2.3). The necessary definitions and the precise formulations of these results are given in Section 2. We obtain a lower bound on $N^{-1}H(\nu^t_{N,y_N} \,|\, \mu_{N,y_N})$ in Section 3. The proof of the upper bound is given in Section 4 modulo a number of technical lemmas, which we prove in Sections 5 and 6. We show the equivalence of Theorem 2.2 and Theorem 2.3 in Section 7. Appendices A and B contain limit theorems for densities, which we use throughout the paper, and a lemma from convex analysis, which is needed in Section 6.

2. The existence of the scaling limit and main results

At first we give a definition of an asymptotic macroscopic profile.

Definition 2.1. For each $N \in \mathbf{N}$ let ν_N be a probability measure on $\mathbf{R}^N$. We shall say that the set $\{\nu_N\}$ possesses an asymptotic macroscopic profile $m \in L^1(S)$ as $N \to \infty$ and denote this by $\nu_N \sim m$ if for every function $J \in C(S)$ and each $\delta > 0$

$$\lim_{N\to\infty} \nu_N \Big\{ \eta \in \mathbf{R}_N \,:\, \Big| \frac{1}{N} \sum_{i\in\Lambda_N} J\Big(\frac{i}{N}\Big)\eta_i - \int_S J(\theta) m(\theta)\, d\theta \Big| > \delta \Big\} = 0. \tag{2.1}$$

Our main assumption on the initial data is the existence of an asymptotic macroscopic profile, i.e.

$$\nu_N^0 = \delta_{\eta^{(N)}} \sim m_0 \tag{2.2}$$

for some $m_0 \in L^1(S)$. In particular,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i\in\Lambda_N} \eta_i^{(N)} = \int_0^1 m_0(\theta)\, d\theta \stackrel{\text{def}}{=} \bar{m}. \tag{2.3}$$

Then it is known (see Theorem 2.1 below) that for any $t > 0$

$$\nu_{N,y_N}^t \sim m(t, \cdot),$$

where $m(t, \cdot)$ is the solution of a non-linear diffusion equation with the initial condition m_0. We shall formulate this statement more precisely after introducing some notation.

Let M be defined as in (H_3') above. It is easy to check that

$$\rho = \log M(\lambda) \tag{2.4}$$

is a convex analytic function. Its convex conjugate

$$h(y) = \sup_{z\in\mathrm{R}} (yz - \log M(z)) \tag{2.5}$$

is also analytic and

$$h'^{-1}(\lambda) = \frac{M'(\lambda)}{M(\lambda)} = \rho'(\lambda). \tag{2.6}$$

The following theorem was proven in [4].

Theorem 2.1. *Assume that φ satisfies (H_1), (H_2), (H_3') and (1.11). Let the sequence $\eta^{(N)}$ of deterministic configurations satisfy (2.2) for some $m_0 \in L^1(S)$ and the average $N^{-1}\sum_{i\in\Lambda_N} \varphi(\eta_i^{(N)})$ be bounded uniformly in N. Let m be a weak solution of the equation*

$$\frac{\partial m}{\partial t} = \frac{1}{2}\frac{\partial^2 h'(m)}{\partial \theta^2}, \quad (t,\theta) \in (0,\infty) \times S, \tag{2.7}$$

with the initial condition m_0. Then $\nu_{N,y_N}^t \sim m(t, \cdot)$ locally uniformly in $t > 0$.

Remark 2.1. The above theorem is the extension of the results obtained earlier in [3] to the case of deterministic initial data. The main ingredient of the proof of Theorem 2.1 is the entropy estimate (1.12).

The main result of this paper is the following theorem.

Theorem 2.2. *Let the sequence* $\eta^{(N)}$, $N = 1, 2, \dots,$ *of deterministic configurations possess an asymptotic profile* $m_0 \in L^1(S)$ *as* $N \to \infty$ *in the sense of* (2.2). *Assume that* φ *satisfies* $(\mathrm{H}_1) - (\mathrm{H}_3)$ *and that there are constants* $C > 0$ *and* $\delta_0 > 0$ *such that*

$$\sum_{i \in \Lambda_N} \varphi^{2p+\delta_0}\big(\eta_i^{(N)}\big) \leqslant CN, \tag{2.8}$$

where p *is given in* (H_3). *Denote the average* $N^{-1} \sum_{i \in \Lambda_N} \eta_i^{(N)}$ *by* y_N. *Then for any* $t > 0$ *the specific microscopic entropy converges to the macroscopic entropy as* $N \to \infty$, *namely*

$$\lim_{N \to \infty} \frac{1}{N} H(\nu^t_{N,y_N} \,|\, \mu_{N,y_N}) = \int_S h(m(t,\theta))\, d\theta - h\Big(\int_S m(t,\theta)\, d\theta\Big),$$

where $m(t,\theta)$ *is the solution of* (2.7) *with the initial condition* m_0.

Remark 2.2. Notice that $\int_S m(t,\theta)\, d\theta$ does not depend on t and is equal to $\bar{m}$ (see (2.3)).

To every solution m of the hydrodynamical equation (2.7) we can associate a family of local Gibbs measures $\gamma_N^{m_t}$ on $\mathbf{R}^N$. Let $\lambda(t,\theta) = h'(m(t,\theta))$. Measures $\gamma_N^{m_t}$ are defined by their densities $g_N^{m_t}$ with respect to the invariant measure μ_N:

$$g_N^{m_t}(x) = \frac{d\gamma_N^{m_t}}{d\mu_N} = \frac{1}{Z_N^{m_t}} \exp\Big(\sum_{i \in \Lambda_N} \lambda\Big(t, \frac{i}{N}\Big) x_i\Big)$$

where $Z_N^{m_t}$ is the normalization constant, $Z_N^{m_t} = \prod_{i \in \Lambda_N} M(\lambda(t, i/N))$. Measures $\gamma_N^{m_t}$ depend only on the solution $m(t,\theta)$ and form a convenient set of reference measures. The conditional measure

$$\gamma_N^{m_t}\Big(\cdot \,\Big|\, \sum_{i \in \Lambda_N} x_i^{(N)} = Ny\Big)$$

and its density relative to $\mu_{N,y}$ will be denoted by $\gamma_{N,y}^{m_t}(\cdot)$ and $g_{N,y}^{m_t}$ respectively.

It is not difficult to show that for any $t \geqslant 0$

$$\gamma_N^{m_t} \sim m(t,\cdot).$$

If we start the process from the local equilibrium $\gamma_N^{m_0}$, then the measure ψ_N^t, which describes the distribution of this process at time t, will diverge from $\gamma_N^{m_t}$. But the specific entropy $N^{-1}H(\psi_N^t \,|\, \gamma_N^{m_t})$ approaches zero as $N \to \infty$. This result was proven in [7] as a part of the derivation of the scaling limit for this case.

The next theorem shows that even if the process starts very far away from the local equilibrium, the specific entropy relative to local Gibbs measures converges to zero as $N \to \infty$ for any $t > 0$. Thus, as far as the specific entropy is concerned, the process with the deterministic initial data in the hydrodynamical scaling limit immediately becomes indistinguishable from the one which started from the local equilibrium.

Theorem 2.3. *Under the assumptions of Theorem 2.2, for any fixed positive macroscopic time t the specific relative entropy*

$$\frac{1}{N} H(\nu_{N,y_N}^t \,|\, \gamma_{N,y_N}^{m_t}) \stackrel{\text{def}}{=} \frac{1}{N} \int\limits_{\mathrm{X}_{N,y_N}} f_{N,y_N}^t \log \frac{f_{N,y_N}^t}{g_{N,y_N}^t}\, d\mu_{N,y_N}$$

approaches zero as $N \to \infty$.

We show (see Lemma 7.1) that

$$\lim_{N\to\infty} \frac{1}{N} \int\limits_{\mathrm{X}_{N,y_N}} f_{N,y_N}^t \log g_{N,y_N}^t \, d\mu_{N,y_N} = \int\limits_0^1 h(m(t,\theta))\, d\theta - h(\bar{m}).$$

This implies the equivalence of Theorem 2.2 and Theorem 2.3.

We prove Theorem 2.2 by establishing the lower and upper bounds on the specific macroscopic entropy as $N \to \infty$.

3. Lower bound

We show that

$$\liminf_{N\to\infty} \frac{1}{N} H(\nu_{N,y_N}^t \,|\, \mu_{N,y_N}) \geqslant \int\limits_0^1 h(m(t,\theta))\, d\theta - h(\bar{m}). \tag{3.1}$$

Lemma 3.1. *Let φ satisfy* (H_1), (H_3'). *Assume that a sequence of real numbers $\{y_N\}$ and probability measures $\{\nu_N\}$ satisfy the following conditions:*

(a) *for each N measure ν_N is supported on $\mathbf{X}_{N,y_N}$* (see (1.7))*;*

(b) *$H(\nu_N \,|\, \mu_{N,y_N}) \leqslant CN$, where μ_{N,y_N} is defined by* (1.8)*;*

(c) $\nu_N \sim m$ *for some function* m *such that* $h \circ m \in L^1(S)$.

Then

$$\liminf_{N\to\infty} \frac{1}{N} H(\nu_N \mid \mu_{N,y_N}) \geqslant \int_S h(m(\theta))\, d\theta - h\Big(\int_S m(\theta)\, d\theta\Big).$$

Proof. The entropy inequality implies that for every $J \in C(S)$

$$\begin{aligned} \frac{1}{N} H(\nu_N \mid \mu_{N,y_N}) \quad \geqslant \quad & \frac{1}{N} \mathsf{E}_{\nu_N} \sum_{i\in\Lambda_N} J\Big(\frac{i}{N}\Big) x_i \\ & - \frac{1}{N} \log \mathsf{E}_{\mu_{N,y_N}} \exp\Big(\sum_{i\in\Lambda_N} J\Big(\frac{i}{N}\Big) x_i\Big). \end{aligned} \tag{3.2}$$

Condition (c) means that

$$\frac{1}{N} \sum_{i\in\Lambda_N} J\Big(\frac{i}{N}\Big) x_i \longrightarrow \int_S J(\theta) m(\theta)\, d\theta$$

as $N \to \infty$ in probability relative to ν_N. Next we prove the uniform integrability of $N^{-1} \sum_{i\in\Lambda_N} |x_i|$ under ν_N, which allows us to conclude that

$$\lim_{N\to\infty} \frac{1}{N} \mathsf{E}_{\nu_N} \sum_{i\in\Lambda_N} J\Big(\frac{i}{N}\Big) x_i = \int_S J(\theta) m(\theta)\, d\theta.$$

Lemma 3.2. *Let* φ *satisfy* (H_1) *and* (H_3'). *Define*

$$A_{N,R} = \Big\{x \in \mathbf{X}_{N,y_N} \,:\, N^{-1} \sum_{i\in\Lambda_N} |x_i| > R\Big\}.$$

Then under assumptions (a) *and* (b) *of Lemma* 3.1

$$\lim_{R\to\infty} \int_{A_{N,R}} \frac{1}{N} \sum_{i\in\Lambda_N} |x_i|\, d\nu_N = 0$$

uniformly in N.

Proof. It can be shown (see, for example, [6]) that (H_3') implies the existence of a function $\omega : [0,\infty) \to [1,\infty)$, which is symmetric, convex and satisfies

$$\lim_{|x|\to\infty} \frac{|x|}{\omega(|x|)} = 0 \quad \text{and} \quad \int_{\mathbb{R}} \exp(\omega(|x|) - \varphi(x))\, dx < \infty. \tag{3.3}$$

For any $\varepsilon > 0$ and all sufficiently large R we obtain

$$
\begin{aligned}
&\int_{A_{N,R}} N^{-1} \sum_{i\in\Lambda_N} |x_i|\, d\nu_N \\
&\qquad \leqslant \varepsilon \int_{A_{N,R}} \omega\Big(N^{-1} \sum_{i\in\Lambda_N} |x_i|\Big)\, d\nu_N \quad \leqslant \varepsilon \int_{A_{N,R}} N^{-1} \sum_{i\in\Lambda_N} \omega(|x_i|)\, d\nu_N \\
&\qquad \leqslant \varepsilon N^{-1} \log \int_{\mathrm{X}_{N,y_N}} \exp\Big(\sum_{i\in\Lambda_N} \omega(|x_i|)\Big)\, d\mu_{N,y_N} + \varepsilon N^{-1} H(\nu_N \mid \mu_{N,y_N}) \\
&\qquad \leqslant C\varepsilon
\end{aligned}
$$

uniformly in N. This follows from (3.3), Theorem A.1 and condition (b) of Lemma 3.1. Hence

$$\lim_{R\to\infty} \int_{A_{N,R}} N^{-1} \sum_{i\in\Lambda_N} |x_i|\, d\nu_N = 0$$

uniformly in N. □

To compute the limit of the second term in the right-hand side of (3.2) we notice at first that by assumption (c) we have

$$\lim_{N\to\infty} y_N = \int_S m(\theta)\, d\theta.$$

It follows from Theorems A.1 and A.2 that

$$
\begin{aligned}
\lim_{N\to\infty} \frac{1}{N} \log \mathsf{E}_{\mu_{N,y_N}} \exp\Big(\sum_{i\in\Lambda_N} J\Big(\frac{i}{N}\Big)x_i\Big) \quad &\leqslant \int_S \log M\big(J(\theta)\big)\, d\theta \\
&\quad + h\Big(\int_S m(\theta)\, d\theta\Big), \qquad (3.4)
\end{aligned}
$$

and, therefore,

$$
\begin{aligned}
&\liminf_{N\to\infty} \frac{1}{N} H(\nu_N \mid \mu_{N,y_N}) \\
&\qquad \geqslant \sup_{J\in C(S)} \Big(\int_S J(\theta)m(\theta) - \log M\big(J(\theta)\big)\, d\theta\Big) - h\Big(\int_S m(\theta)\, d\theta\Big) \\
&\qquad = \int_S h(m(\theta))\, d\theta - h\Big(\int_S m(\theta)\, d\theta\Big),
\end{aligned}
$$

as claimed. □

The lower bound (3.1) follows immediately from (1.12), Theorem 2.1 and Lemma 3.1.

4. Upper bound

We have to show that

$$\limsup_{N\to\infty} \frac{1}{N} H(\nu^t_{N,y_N} \mid \mu_{N,y_N}) \leqslant \int_S h(m(t,\theta))\, d\theta - h(\bar{m}). \tag{4.1}$$

The proof depends on a number of lemmas. The precise statements of these lemmas are given in appropriate steps below but the proofs are presented in the next two sections.

Step 1. We start with a martingale decomposition.

Lemma 4.1 (Martingale Decomposition). *Let $(\Omega, \mathcal{G}_0, \mathsf{P})$ be a probability space and $\mathcal{G}_0 \supset \mathcal{G}_1 \supset \cdots \supset \mathcal{G}_k$ be a decreasing sequence of σ-algebras. For a non-negative random variable X_0 such that $\mathsf{E}\, X_0 |\log X_0| < \infty$, define $X_i = \mathsf{E}(X_0 \mid \mathcal{G}_i)$, $i = 0, 1, \dots, k$. Then*

$$\mathsf{E}\, X_0 \log X_0 = \sum_{i=1}^{k} \mathsf{E}\,\mathsf{E}\left(X_{i-1} \log \frac{X_{i-1}}{X_i} \;\Big|\; \mathcal{G}_i\right) + \mathsf{E}\, X_k \log X_k.$$

The proof is straightforward, and we omit it.

We apply the above lemma in the following context. Consider f^t_{N,y_N} as a random variable on the probability space $(\mathbf{X}_{N,y_N}, \mathcal{G}_0, \mu_{N,y_N})$, where $\mathcal{G}_0$ is the Borel σ-algebra on $\mathbf{X}_{N,y_N}$. Fix a large number $l \in \mathbf{N}$. Divide the circle S into $k_N = [N/(2l+1)]$ equal parts, which we call "boxes". Each box contains $(2l+1)$ or $(2l+2)$ sites. Let $B_i \subset \Lambda_N$ be the set of indexes corresponding to the ith box and

$$\bar{x}_i = \frac{1}{|B_i|} \sum_{j\in B_i} x_j, \qquad i = 1, \dots, k_N,$$

where $|B_i|$ is the number of elements in B_i. For $i = 1, \dots, k_N$ define

$$\mathcal{G}_i = \Big\{\sigma\text{-subalgebra of } \mathcal{G}_0 \text{ generated by the averages } \bar{x}_1, \bar{x}_2, \dots, \bar{x}_i \text{ and by } x_j,\ j \in \bigcup_{n=i+1}^{k_N} B_n\Big\}.$$

Let $f^t_i = \mathsf{E}_{\mu_{N,y_N}}(f^t_{N,y_N} \mid \mathcal{G}_i)$. The dependence of $\mathcal{G}_i$ and f^t_i on N will not be reflected in the notation. By Lemma 4.1,

$$\begin{aligned} \frac{1}{N} H(\nu^t_{N,y_N} \mid \mu_{N,y_N}) &= \frac{1}{N} \mathsf{E}_{\mu_{N,y_N}} f^t_{N,y_N} \log f^t_{N,y_N} \\ &= \frac{1}{N} \sum_{i=1}^{k_N} \mathsf{E}_{\mu_{N,y_N}} \mathsf{E}_{\mu_{N,y_N}} \left(f^t_{i-1} \log \frac{f^t_{i-1}}{f^t_i} \;\Big|\; \mathcal{G}_i\right) \\ &\quad + \frac{1}{N} \mathsf{E}_{\mu_{N,y_N}} f^t_{k_N} \log f^t_{k_N}. \end{aligned}$$

Since the entropy decreases in time, we can average in t and obtain that for any $t_0 \in (0, t)$

$$\begin{aligned}
&\frac{1}{N} H(\nu^t_{N,y_N} \,|\, \mu_{N,y_N}) \\
&\qquad \leqslant \frac{1}{N(t-t_0)} \int_{t_0}^{t} H(\nu^s_{N,y_N} \,|\, \mu_{N,y_N})\, ds \\
&\qquad = \frac{1}{N(t-t_0)} \int_{t_0}^{t} \sum_{i=1}^{k_N} \mathsf{E}_{\mu_{N,y_N}} \mathsf{E}_{\mu_{N,y_N}} \Big(f^s_{i-1} \log \frac{f^s_{i-1}}{f^s_i} \,\Big|\, \mathcal{G}_i \Big)\, ds \\
&\qquad\quad + \frac{1}{N(t-t_0)} \int_{t_0}^{t} \mathsf{E}_{\mu_{N,y_N}} f^s_{k_N} \log f^s_{k_N}\, ds. \qquad (4.2)
\end{aligned}$$

Step 2. We show that the first term in the right-hand side of (4.2) can be made arbitrarily small as $N \to \infty$.

At first we apply a version of the logarithmic Sobolev inequality (see Theorem 4.1 below) to each term of the sum in the right-hand side of (4.2).

Let $n \in \mathbf{N}$, $y \in \mathbf{R}$ and $\mathbf{X}_{n,y}$ be defined by (1.7). Denote the Lebesgue measure on $\mathbf{X}_{n,y}$ by $\sigma_{n,y}$.

Theorem 4.1. *Let $\varphi \in C^2(\mathbf{R})$ satisfy* (H$_1$) *and the convexity condition* (1.2). *Define the probability measure $\mu_{n,y}$ on the hyperplane $\mathbf{X}_{n,y}$ by*

$$d\mu_{n,y} = \frac{1}{Z_{n,y}} \exp\Big(-\sum_{i=1}^{l} \varphi(x_i)\Big)\, d\sigma_{n,y},$$

where $Z_{n,y}$ is the normalization constant. Then for any smooth $f : \mathbf{R}^n \to \mathbf{R}_+$ such that $\mathsf{E}_{\mu_{n,y}} f = 1$ we have the inequality

$$\mathsf{E}_{\mu_{n,y}} f \log f \leqslant C(n, C_0, R) \int_{\mathrm{X}_{l,y}} \sum_{i=1}^{n-1} \Big(\frac{\partial \sqrt{f}}{\partial x_i} - \frac{\partial \sqrt{f}}{\partial x_{i+1}} \Big)^2 d\mu_{n,y}, \qquad (4.3)$$

where R is defined in (1.2).

Remark 4.1. It is known (see, for example, [1] and [2]) that if $\varphi''(x) \geqslant C_0 > 0$ for all $x \in \mathbf{R}$, then $C(n, C_0, 0) = 4n^2/C_0$. Since we need merely the existence of some constant $C(n, C_0, R)$, we can relax the assumption on φ and require the convexity of φ only for large $|x|$. (See pp. 201–202 of [5] on how to obtain one logarithmic Sobolev inequality directly from another.)

Let $\mu_i \stackrel{\text{def}}{=} \mu_{N,y_N}(\,\cdot \mid \mathcal{G}_i)$ and $\tilde{\mu}_i$ be the marginal of μ_i on the ith box. Then $f_i = \mathsf{E}_{\mu_i} f_{N,y_N} = \mathsf{E}_{\tilde{\mu}_i} f_{i-1}$. It is easy to see that $\tilde{\mu}_i$ is exactly in the form described in Theorem 4.1 ($n = |B_i|$, $y = \bar{x}_i$). Let

$$\partial_{j-1,j} = \Big(\frac{\partial}{\partial x_{j-1}} - \frac{\partial}{\partial x_j}\Big).$$

Setting $\tilde{f}_i^s = f_{i-1}^s / f_i^s$ and applying (4.3) with $f = \tilde{f}_i^s$ we obtain

$$\begin{aligned} \mathsf{E}_{\tilde{\mu}_i} \tilde{f}_i^s \log \tilde{f}_i^s &\leqslant C(l, C_0, R) \int\limits_{X_{l,\bar{x}_i}} \sum_{j-1,j\in B_i} \Big(\partial_{j-1,j} \sqrt{\tilde{f}_i^s}\,\Big)^2 \, d\tilde{\mu}_i \\ &= \frac{C(l,C_0,R)}{4 f_i^s} \int\limits_{X_{l,\bar{x}_i}} \sum_{j-1,j\in B_i} \frac{\Big(\partial_{j-1,j} f_{i-1}^s\Big)^2}{f_{i-1}^s} \, d\tilde{\mu}_i. \qquad (4.4) \end{aligned}$$

Moreover,

$$\partial_{j-1,j} f_{i-1}^s = \partial_{j-1,j} \, \mathsf{E}_{\mu_{i-1}} f_{N,y_N}^s = \mathsf{E}_{\mu_{i-1}} \partial_{j-1,j} f_{N,y_N}^s, \quad j-1, j \in B_i. \qquad (4.5)$$

The term where $\partial_{j-1,j}$ is applied to μ_{i-1} vanishes because μ_{i-1} depends only on the sum of x_j, $j \in \cup_{n=i}^{k_N} B_n$, and $\partial_{j-1,j}$ applied to this sum is equal to zero. Using (4.5) and Hölder's inequality we find that

$$\begin{aligned} \int\limits_{X_{l,\bar{x}_i}} \sum_{j-1,j\in B_i} \frac{(\partial_{j-1,j} f_{i-1}^s)^2}{f_{i-1}^s} \, d\tilde{\mu}_i &= \int\limits_{X_{l,\bar{x}_i}} \sum_{j-1,j\in B_i} \frac{(\mathsf{E}_{\mu_{i-1}} \partial_{j-1,j} f_{N,y_N}^s)^2}{f_{i-1}^s} \, d\tilde{\mu}_i \\ &\leqslant \int\limits_{X_{l,\bar{x}_i}} 4 \sum_{j-1,j\in B_i} \mathsf{E}_{\mu_{i-1}} \big(\partial_{j-1,j} \sqrt{f_{N,y_N}^s}\,\big)^2 \, d\tilde{\mu}_i \\ &= 4 \, \mathsf{E}_{\mu_i} \sum_{j-1,j\in B_i} \big(\partial_{j-1,j} \sqrt{f_{N,y_N}^s}\,\big)^2. \qquad (4.6) \end{aligned}$$

From (4.6), (4.4) and (4.2) we obtain

$$\begin{aligned} \frac{1}{N} H(\nu_{N,y_N}^t \,|\, \mu_{N,y_N}) &\leqslant \frac{C(l,C_0,R)}{N^3 (t-t_0)} \int_{t_0}^{t} D_{N,y_N}\big(\sqrt{f_{N,y_N}^s}\,\big) \, ds \\ &\quad + \frac{1}{N(t-t_0)} \int_{t_0}^{t} \mathsf{E}_{\mu_{N,y_N}} f_k^s \log f_k^s \, ds, \end{aligned}$$

where the Dirichlet form $D_{N,y_N}(\cdot)$ is given by (1.9).

The inequality (1.10) and the entropy bound (1.12) imply the following estimate on time averages of the Dirichlet form:

$$\int_{t_0}^{t} D_{N,y_N}\big(\sqrt{f^s_{N,y_N}}\,\big)\,ds = H(\nu^{t_0}_{N,y_N}\,|\,\mu_{N,y_N}) - H(\nu^{t}_{N,y_N}\,|\,\mu_{N,y_N}) \leqslant C(t_0)N.$$

This allows us to conclude that the first term in the right-hand side of (4.2) converges to zero as $N \to \infty$.

Step 3. We show that the last term in (4.2) admits the desired upper bound. Denote by $\bar{\nu}^s_{k_N}$ ($\bar{\mu}_{k_N}$) the joint distribution of averages $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{k_N}$ under ν^t_{N,y_N} (μ_{N,y_N} respectively). We have $d\bar{\nu}^s_{k_N} = f^s_{k_N}\, d\bar{\mu}_{k_N}$ and

$$\mathsf{E}_{\mu_{N,y_N}} f^s_{k_N} \log f^s_{k_N} = \mathsf{E}_{\nu^s_{N,y_N}} \log \frac{d\bar{\nu}^s_{k_N}}{d\sigma_{k_N}} - \mathsf{E}_{\nu^s_{N,y_N}} \log \frac{d\bar{\mu}_{k_N}}{d\sigma_{k_N,y_N}}, \tag{4.7}$$

where σ_{k_N,y_N} is the Lebesgue measure on $\mathbf{X}_{k_N,y_N}$.

In Section 5 we prove

Lemma 4.2. *Under the assumptions of Theorem 2.2, for any $t > t_0 > 0$*

$$\limsup_{l\to\infty} \limsup_{N\to\infty} \frac{1}{N(t-t_0)} \int_{t_0}^{t} \mathsf{E}_{\nu^s_{N,y_N}} \log \frac{d\bar{\nu}^s_{k_N}}{d\sigma_{k_N,y_N}}\, ds \leqslant 0.$$

To finish the proof of Theorem 2.2, we only have to deal with the last term in (4.7).

For $n \in \mathbf{N}$ define

$$h_n(y) \stackrel{\text{def}}{=} -\frac{1}{n} \log \int_{\mathrm{X}_{n,y}} \exp\Big(-\sum_{j=1}^{n} \varphi(x_j)\Big)\, d\sigma_{n,y}.$$

Then

$$\begin{aligned} \frac{1}{N} \log \frac{d\bar{\mu}_{k_N}}{d\sigma_{k_N,y_N}} &= -\frac{1}{N} \sum_{i=1}^{k_N} |B_i|\, h_{|B_i|}(\bar{x}_i) \\ &\quad - \frac{1}{N} \log \int_{\mathrm{X}_{N,y_N}} \exp\Big(-\sum_{i\in\Lambda_N} \varphi(x_i)\Big)\, d\sigma_{N,y_N}. \end{aligned}$$

By Theorem A.1 we know that

$$\lim_{n\to\infty} h_n(y) = h(y) \qquad \text{locally uniformly in } y,$$

and, since $\lim_{N\to\infty} y_N = \bar{m}$, we have that

$$\lim_{N\to\infty} h_N(y_N) = h(\bar{m}).$$

Therefore, we arrive at the estimate

$$\limsup_{N\to\infty} \frac{1}{N} H(\nu^t_{N,y_N} \mid \mu_{N,y_N}) \leqslant \limsup_{N\to\infty} \frac{1}{N(t-t_0)} \int_{t_0}^{t} \mathsf{E}_{\nu^s_{N,y_N}} \sum_{i=1}^{k_N} |B_i| h_{|B_i|}(\bar{x}_i)\, ds - h(\bar{m}) \quad (4.8)$$

for any $l \in \mathbf{N}$ and $t_0 \in (0, t)$.

Step 4. For $j \in \Lambda_N$ define the shift operator T_j on $\mathbf{R}^N$ and also on functions $f : \mathbf{R}^N \to \mathbf{R}^N$ by

$$T_j x_i = x_{i+j}, \qquad i \in \Lambda_N,$$

and

$$T_j f(x) = f(T_j x).$$

We also set for any $n \in \mathbf{N}$

$$\bar{x}_{i,n} = \frac{1}{2n+1} \sum_{|j-i| \leqslant n} x_j. \quad (4.9)$$

Notice that the left-hand side of (4.8) does not depend on the way we divide the circle S into boxes. This observation implies that

$$\limsup_{N\to\infty} \frac{1}{N} H(\nu^t_{N,y_N} \mid \mu_{N,y_N}) \leqslant \limsup_{l\to\infty} \limsup_{N\to\infty} \frac{1}{N(t-t_0)} \times \int_{t_0}^{t} \mathsf{E}_{\nu^s_{N,y_N}} \frac{1}{2l+1} \sum_{j=0}^{2l} \sum_{i=1}^{k_N} |B_i| h_{|B_i|}(T_j \bar{x}_i)\, ds - h(\bar{m}). \quad (4.10)$$

Remark 4.2. If N is divisible by $(2l+1)$ then, clearly, $|B_i| = 2l+1$ and

$$\frac{1}{N} \mathsf{E}_{\nu^s_{N,y_N}} \frac{1}{2l+1} \sum_{j=0}^{2l} \sum_{i=1}^{k_N} |B_i| h_{|B_i|}(T_j \bar{x}_i)\, ds = \frac{1}{N} \mathsf{E}_{\nu^s_{N,y_N}} \sum_{i\in\Lambda_N} h_{2l+1}(\bar{x}_{i,l}).$$

Otherwise there is a difference between the left-hand side and the right-hand side in the above expression, which can not be a priori neglected even in the limit (as $N \to \infty$ and then $l \to \infty$), since functions h_n are neither bounded nor uniformly continuous.

Next we obtain an estimate on h_n, which is uniform in n.

Lemma 4.3. *Let $\varphi \in C^2(\mathbf{R})$ satisfy* (H_1), (1.2) *and* (1.4) *with $p = 1$. Then there is a constant $M > 0$ such that*

$$|h_n(x)| \leqslant M(|\varphi(x)| + 1), \qquad x \in \mathbf{R},$$

uniformly in n.

The proof of this lemma is given in Section 6.

The next lemma combined with Lemma 4.3 provides us with the uniform integrability, which we need when passing from the convergence in probability (relative to ν^s_{N,y_N}) to the convergence in $L^1(\mathbf{X}_{N,y_N}, d\nu^s_{N,y_N})$.

Lemma 4.4. *Under the assumptions of Theorem* 2.2 *there is $q > 1$ and a constant C such that*

$$\mathsf{E}_{\nu^s_{N,y_N}} \Big(\frac{1}{N} \sum_{i \in \Lambda_N} |\varphi(x_i)| \Big)^q \leqslant C$$

uniformly in N and locally uniformly in $s > 0$.

We postpone the proof of this lemma until Section 6.

We use Lemma 4.3 and Lemma 4.4 to justify the following replacements. At first we replace the expression under the integral sign in (4.10) with

$$\mathsf{E}_{\nu^s_{N,y_N}} \sum_{i \in \Lambda_N} h(\bar{x}_{i,l}), \tag{4.11}$$

where $\bar{x}_{i,l}$ is defined by (4.9) with $n = l$. Then we substitute the averages over small macroscopic blocks $\bar{x}_{i,\varepsilon N}$ for the averages over large microscopic blocks $\bar{x}_{i,l}$ obtaining

$$\begin{aligned} \limsup_{N\to\infty} \frac{1}{N} H(\nu^t_{N,y_N} \,|\, \mu_{N,y_N}) \;\leqslant\; & \lim_{\varepsilon\to 0} \limsup_{l\to\infty} \limsup_{N\to\infty} \frac{1}{(t-t_0)} \\ & \times \int_{t_0}^{t} \frac{1}{N} \mathsf{E}_{\nu^s_{N,y_N}} h(\bar{x}_{i,\varepsilon N})\, ds - h(\bar{m}). \end{aligned} \tag{4.12}$$

This substitution is based on the following two-block estimate.

Theorem 4.2 (two-block estimate). *Under the assumptions of Theorem* 2.1, *for any* $t_2 > t_1 > 0$

$$\lim_{\varepsilon\to 0}\limsup_{l\to\infty}\limsup_{N\to\infty}\int_{t_1}^{t_2}\mathsf{E}_{\nu^s_{N,y_N}}\frac{1}{N}\sum_{i\in\Lambda_N}|\bar{x}_{i,l}-\bar{x}_{i,\varepsilon N}|\,ds=0.$$

This theorem is an immediate consequence of the entropy estimate (1.12) and Theorem 4.7 of [3].

The existence of the scaling limit implies that for any $u\in S$, $\varepsilon>0$ and $\delta>0$

$$\nu^s_{N,y_N}\Big\{x\in\mathbf{X}_{N,y_N}\ :\ \Big|\bar{x}_{[uN],\varepsilon N}-\frac{1}{2\varepsilon}\int_{u-\varepsilon}^{u+\varepsilon}m(s,\theta)\,d\theta\Big|>\delta\Big\}\longrightarrow 0$$

as $N\to\infty$. Thus we can replace the expression under the integral sign in (4.12) with

$$\frac{1}{N}\sum_{i\in\Lambda_N}h\Big(\frac{1}{2\varepsilon}\int_{i/N-\varepsilon}^{i/N+\varepsilon}m(s,\theta)\,d\theta\Big). \tag{4.13}$$

But (4.13) is the integral sum for

$$\int_S h\Big(\frac{1}{2\varepsilon}\int_{u-\varepsilon}^{u+\varepsilon}m(s,\theta)\,d\theta\Big)\,du.$$

Then, since h and m are continuous functions, we can pass to the limit as $\varepsilon\to 0$. Finally we let $t_0\to t$ and obtain the upper bound (4.1).

5. Proof of Lemma 4.2

For this lemma we need, in fact, only assumptions (H_1), (H'_3), the inequality (1.3) and the entropy estimate (1.12).

For each N and $s\in[t_0,t]$ we introduce a new process $x^{(N)}_{s_N}(\cdot)$, which satisfies (1.1) up to time $s_N<s$. At time s_N we "switch off" the drift, and then the process $x^{(N)}_{s_N}$ evolves as a Gaussian diffusion process with constant covariance. For all quantities associated with the new process we keep the same notation as for $x^{(N)}$ but add a subscript s_N. In particular, the corresponding measure on the space of continuous paths will be denoted by $\mathsf{P}^{(N)}_{s_N}$.

We have

$$\frac{1}{(t-t_0)N}\int_{t_0}^{t} \mathsf{E}_{\nu^s_{N,y_N}} \log \frac{d\bar{\nu}^s_{k_N}}{d\sigma_{k_N,y_N}}\,ds$$

$$= \frac{1}{(t-t_0)N}\int_{t_0}^{t} \mathsf{E}_{\nu^s_{N,y_N}} \log \frac{d\bar{\nu}^s_{k_N}}{d\nu^s_{s_N,k_N}}\,ds$$

$$+ \frac{1}{(t-t_0)N}\int_{t_0}^{t} \mathsf{E}_{\nu^s_{N,y_N}} \log \frac{d\nu^s_{s_N,k_N}}{d\sigma_{k_N,y_N}}\,ds. \qquad (5.1)$$

The idea of the proof is to choose the "switch off" time s_N in such a way that both terms in the right-hand side of (5.1) can be bounded above by quantities, which vanish when we pass to the limit as $N \to \infty$ and then as $l \to \infty$. It turns out that the optimal choice for s_N is given by $s_N = s - k_N/N^3$, where $k_N = [N/(2l+1)]$.

We rewrite the system (1.1) using matrix notation,

$$dx(t) = N^2 A^{(N)} b^{(N)}(x(t))\,dt + NG^{(N)}\,d\beta^{(N)},$$

where $\beta^{(N)}$ is a standard N-dimensional Brownian motion, $A^{(N)} = G^{(N)}G^{(N)*}$ and

$$G^{(N)} = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -1 \\ -1 & 0 & 0 & \dots & 1 \end{pmatrix}, \qquad b^{(N)}(x) = -\frac{1}{2}\begin{pmatrix} \varphi'(x_1) \\ \varphi'(x_2) \\ \dots \\ \dots \\ \varphi'(x_N) \end{pmatrix}. \qquad (5.2)$$

We now give a formal description of $x^{(N)}_{s_N}$. The process $x^{(N)}_{s_N}$ is defined by the system

$$dx(t) = N^2 A^{(N)} b^{(N)}_{s_N}(t, x(t))\,dt + NG^{(N)}\,d\beta^{(N)}(t)$$

where

$$b^{(N)}_{s_N}(t,\cdot) = \begin{cases} b^{(N)}(\cdot) & \text{if } t \leqslant s_N, \\ 0 & \text{if } t > s_N, \end{cases} \qquad (5.3)$$

and $G^{(N)}$, $b^{(N)}(x)$ are given by (5.2). Assume that $x^{(N)}_{s_N}(0) = \eta^{(N)}$.

To estimate the first term in the right-hand side of (5.1), we notice that by the convexity of function $x \log x$ and Jensen's inequality

$$\mathsf{E}_{\nu^s_{N,y_N}} \log \frac{d\bar{\nu}^s_{k_N}}{d\nu^s_{s_N,k_N}} \leqslant \mathsf{E}_{\mathsf{P}^{(N)}} \log \frac{d\mathsf{P}^{(N)}}{d\mathsf{P}^{(N)}_{s_N}}\bigg|_{\mathcal{F}_s},$$

where $\mathcal{F}_s$ is the σ-algebra up to time s.

Proposition 5.1.

$$\frac{1}{N}\,\mathsf{E}_{\mathsf{P}^{(N)}}\log\frac{d\mathsf{P}^{(N)}}{d\mathsf{P}^{(N)}_{s_N}}\bigg|_{\mathcal{F}_s}=\frac{N}{8}\,\mathsf{E}_{\mathsf{P}^{(N)}}\int_{s_N}^{s}(\varphi'(x_i(\tau))-\varphi'(x_{i+1}(\tau)))^2\,d\tau. \tag{5.4}$$

Proof. By Girsanov's formula

$$\log\frac{d\mathsf{P}^{(N)}_{s_N}}{d\mathsf{P}^{(N)}}\bigg|_{\mathcal{F}_s} = -\int_0^s (b^{(N)}-b^{(N)}_{s_N})\,d\bar{x}-\frac{N^2}{2}\int_0^s\langle b^{(N)} - b^{(N)}_{s_N}, A^{(N)}(b^{(N)}-b^{(N)}_{s_N})\rangle\,d\tau$$

where $\bar{x}=x-N^2\int_0^s A^{(N)}b^{(N)}\,d\tau$ is a $\mathsf{P}^{(N)}$-martingale and $\langle\cdot,\cdot\rangle$ denotes a scalar product in $\mathbf{R}^N$. Since b_{s_N} coincides with b up to time s_N and vanishes for all $\tau>s_N$, we obtain

$$\log\frac{d\mathsf{P}^{(N)}}{d\mathsf{P}^{(N)}_{s_N}}\bigg|_{\mathcal{F}_s}=\int_{s_N}^{s} b^{(N)}\,d\bar{x}+\frac{N^2}{2}\int_{s_N}^{s}\langle b^{(N)},A^{(N)}b^{(N)}\rangle\,d\tau.$$

Taking expectation with respect to $\mathsf{P}^{(N)}$ in the above equality and using (5.2) we obtain (5.4). □

Proposition 5.2. *Let* $s_N=s-\varepsilon_N>t_0/2$ *and let* φ *satisfy* (1.3). *Assume also that the entropy estimate* (1.12) *holds. Then*

$$\frac{1}{(t-t_0)}\,\mathsf{E}_{\mathsf{P}^{(N)}}\int_{t_0}^{t}\int_{s_N}^{s}(\varphi'(x_i(\tau))-\varphi'(x_{i+1}(\tau)))^2\,d\tau\,ds\leqslant NC(t,t_0)\varepsilon_N.$$

Proof. Step 1. Applying Itô's formula to $\sum_{i\in\Lambda_N}\varphi(x_i(s))$ and using (1.1) we find that for any $t_2>t_1>0$

$$\begin{aligned}\frac{1}{2}\,\mathsf{E}_{\mathsf{P}^{(N)}}\int_{t_1}^{t_2}\sum_{i\in\Lambda_N}(\varphi'(x_i(s))-\varphi'(x_{i+1}(s)))^2\,ds &= \frac{1}{N^2}\,\mathsf{E}_{\mathsf{P}^{(N)}}\Big(\sum_{i\in\Lambda_N}\varphi(x_i(t_1))-\sum_{i\in\Lambda_N}\varphi(x_i(t_2))\Big)\\ &\quad+\mathsf{E}_{\mathsf{P}^{(N)}}\int_{t_1}^{t_2}\sum_{i\in\Lambda_N}\varphi''(x_i(s))\,ds.\end{aligned} \tag{5.5}$$

Since we assumed that $\varphi''(x) \leqslant C_1\varphi(x) + C_2$, the last term in the above equality is bounded by

$$C_1\, \mathsf{E}_{\mathsf{P}^{(N)}} \int_{t_1}^{t_2} \sum_{i\in\Lambda_N} \varphi(x_i(s))\, ds + C_2 N(t_2 - t_1).$$

Step 2. We show that for any $t_0 > 0$ there exists a constant $C = C(t_0)$ such that for all N and all $t > t_0$

$$\frac{1}{N}\, \mathsf{E}_{\nu^t_{N,y_N}} \sum_{i\in\Lambda_N} \varphi(x_i) \leqslant C(t_0). \tag{5.6}$$

By the entropy inequality and (1.12), for any $\delta \in (0,1)$

$$\frac{1}{N}\, \mathsf{E}_{\nu^t_{N,y_N}} \sum_{i\in\Lambda_N} \varphi(x_i) \leqslant \frac{1}{N\delta} \log \mathsf{E}_{\mu_{N,y_N}} \exp\Big(\delta \sum_{i\in\Lambda_N} \varphi(x_i)\Big) + \frac{1}{\delta}\, C(t_0). \tag{5.7}$$

By Theorem A.1 the first term in the right-hand side of (5.7) converges to a limit as $N \to \infty$. The inequality (5.6) follows.

Step 3. Combining the results of Steps 1 and 2 we conclude that

$$\begin{aligned} \frac{1}{(t-t_0)}\, \mathsf{E}_{\mathsf{P}^{(N)}} \int_{t_0}^{t}\int_{s_N}^{s} (\varphi'(x_i(\tau)) - \varphi'(x_{i+1}(\tau)))^2\, d\tau\, ds \\ \leqslant \frac{2}{N^2(t-t_0)}\, \mathsf{E}_{\mathsf{P}^{(N)}} \int_{t_0}^{t} \Big(\sum_{i\in\Lambda_N} \varphi(x_i(s_N)) - \sum_{i\in\Lambda_N} \varphi(x_i(s))\Big)\, ds \\ + NC(t_0)\varepsilon_N. \end{aligned} \tag{5.8}$$

Notice that

$$\begin{aligned} \mathsf{E}_{\mathsf{P}^{(N)}} \int_{t_0}^{t} \Big(\sum_{i\in\Lambda_N} \varphi(x_i(s_N)) - \sum_{i\in\Lambda_N} \varphi(x_i(s))\Big)\, ds \\ = \mathsf{E}_{\mathsf{P}^{(N)}} \int_{t_0-\varepsilon_N}^{t_0} \sum_{i\in\Lambda_N} \varphi(x_i(s))\, ds - \mathsf{E}_{\mathsf{P}^{(N)}} \int_{t-\varepsilon_N}^{t} \sum_{i\in\Lambda_N} \varphi(x_i(s))\, ds \\ \leqslant NC(t_0)\varepsilon_N, \end{aligned} \tag{5.9}$$

where the last inequality follows from Step 2. Relations (5.8) and (5.9) imply the statement of Proposition 5.2. □

Propositions 5.1 and 5.2 allow us to conclude that for $s_N = s - \varepsilon_N > t_0/2$

$$\frac{1}{N(t-t_0)} \int_{t_0}^{t} \mathsf{E}_{\nu^s_{N,y_N}} \log \frac{d\bar{\nu}^s_{k_N}}{d\nu^s_{s_N,k_N}} \leqslant N^2 C(t,t_0)\varepsilon_N.$$

In particular, for $\varepsilon_N = k_N/N^3$, where $k_N = [N/(2l+1)]$, the right-hand side of the above inequality converges to zero when we let $N \to \infty$ and then $l \to \infty$.

We turn now to the last term in (5.1). After time s_N the box averages $(\bar{x}_{s_N,1}, \bar{x}_{s_N,2}, \dots, \bar{x}_{s_N,k_N})$ undergo a Gaussian diffusion in $\mathbf{R}^{k_N}$ with parameters $(0, k_N^2 A^{(k_N)})$, where $A^{(k_N)} : \mathbf{R}^{k_N} \to \mathbf{R}^{k_N}$ is given by

$$A^{(k_N)} = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & -1 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ -1 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}.$$

Let $\tilde{A}^{(k_N)}$ be the restriction of $A^{(k_N)}$ to $\mathbf{X}_{k_N,y_N}$. Then $\tilde{A}^{(k_N)}$ is non-degenerate and

$$\begin{aligned} &\frac{1}{N} \log \frac{d\nu^s_{s_N,k_N}}{d\sigma_{k_N,y_N}}(\,\cdot\,) \\ &\quad = \frac{1}{N} \log \Big[\big((2\pi k_N^2 \varepsilon_N)^{k_N-1} \det \tilde{A}^{(k_N)} \big)^{-1/2} \\ &\qquad \times \int_{\mathrm{X}_{k_N,y_N}} \exp\big(-(x-\cdot)^T \tilde{A}^{(k_N)^{-1}} (x-\cdot)/(2\varepsilon_N) \big)\, d\nu^{s_N}_{s_N,k_N} \Big] \\ &\quad \leqslant -\frac{k_N-1}{2N} \log\big(2\pi k_N^2 \varepsilon_N\big) - \frac{1}{2N} \log\det \tilde{A}^{(k_N)}. \end{aligned} \tag{5.10}$$

The $\det \tilde{A}^{(k_N)}$ is equal to the product of non-zero eigenvalues of $A^{(k_N)}$.

Proposition 5.3. *For $k = 3, 4, \dots$ we have*

$$\begin{aligned} \det \tilde{A}^{(k)} &= 2^{k-1} \prod_{j=1}^{k-1} \Big(1 - \cos \frac{2\pi j}{k}\Big); \\ \lim_{k\to\infty} \frac{1}{k} \log(\det \tilde{A}^{(k)}) &= 2\log 2 + 4 \int_0^{1/2} \log\sin(\pi x)\, dx. \end{aligned}$$

Proof. Denote $2\pi j/k$ by $\alpha_j^{(k)}$. It is easy to check that vectors

$$e_j^{(k)} = (e^{i\alpha_j^{(k)}}, \dots, e^{ik\alpha_j^{(k)}}), \quad j = 0, \dots, k-1,$$

are independent eigenvectors for $A^{(k)}$ with eigenvalues $2\,(1-\cos\alpha_j^{(k)})$ respectively. Taking the product of non-zero eigenvalues ($j=1,\dots,k-1$) we obtain the first part of the proposition.

To prove the second statement, notice that

$$\begin{aligned}\frac{1}{k}\log\prod_{j=1}^{k-1}\Big(1-\cos\frac{2\pi j}{k}\Big) &= \frac{k-1}{k}\log 2+\sum_{i=1}^{k-1}\frac{1}{k}\log\sin^2\frac{\pi j}{k}\\ &= \frac{k-1}{k}\log 2+4\sum_{j=1}^{[k/2]}\frac{1}{k}\log\sin\frac{\pi j}{k}. \qquad (5.11)\end{aligned}$$

The last sum is the integral sum for the convergent integral

$$\int_0^{1/2}\log\sin(\pi x)\,dx.$$

Passing to the limit as $k\to\infty$ concludes the proof. □

Let $s-s_N=k_N/N^3$. Since $k_N=[N/(2l+1)]$, we obtain from (5.10) and Proposition 5.3 that

$$\lim_{l\to\infty}\lim_{N\to\infty}\Big(-\frac{k_N-1}{2N}\log(2\pi k_N^2\varepsilon_N)-\frac{1}{2N}\log\det\tilde{A}^{(k_N)}\Big)=0.$$

The proof of Lemma 4.2 is complete.

6. Proofs of Lemma 4.3 and Lemma 4.4

Proof of Lemma 4.3. It is clear that for any $x'<x''$

$$\begin{aligned}\int_{x'}^{x''}\exp(-h_n(x))\,dx &= \int_{nx'\leqslant\sum_{j=1}^n x_j\leqslant nx''}\exp\Big(-\sum_{j=1}^n\varphi(x_j)\Big)\,d^nx\\ &\geqslant \int_{\substack{x'\leqslant x_j\leqslant x''\\ j=1,\dots,n}}\exp\Big(-\sum_{j=1}^n\varphi(x_j)\Big)\,d^nx\\ &= \Big(\int_{x'}^{x''}\exp(-\varphi(x))\,dx\Big)^n.\end{aligned}$$

If h_n and φ are monotone increasing for $x>x_0>0$, then

$$(x''-x')\exp(-nh_n(x'))\geqslant(x''-x')^n\exp(-n\varphi(x''))\quad\text{for } x',x''\in[x_0,\infty),$$

i.e. $h_n(x') \leqslant -(1-1/n)\log(x''-x') + \varphi(x'')$, which implies

$$h_n(x) \leqslant \varphi(x+1) \qquad \text{for } x > x_0. \tag{6.1}$$

Similarly, if h_n and φ are monotone decreasing for $x < -x_0$, then

$$h_n(x) \leqslant \varphi(x-1) \qquad \text{for } x < -x_0. \tag{6.2}$$

Since φ is convex for large $|x|$ and $\int \exp(-\varphi(x))\,dx < \infty$, we know that it is monotone increasing for large positive x and is monotone decreasing for large negative x. Moreover,

$$\varphi'(x) \geqslant C_0 x - C \qquad \text{for } x > x_0$$

and

$$\varphi'(x) \leqslant C_0 x + C \qquad \text{for } x < -x_0$$

for some $x_0 > 0$. These inequalities hold also for h_n'. Indeed, if $x > x_0$, then

$$h_n'(x) = \frac{\displaystyle\int_{\mathrm{X}_{n,x}} \frac{1}{n}\sum_{j=1}^n \varphi'(x_j)\exp\Big(-\sum_{j=1}^n \varphi(x_j)\Big)d\sigma_{n,x}}{\displaystyle\int_{\mathrm{X}_{n,x}} \exp\Big(-\sum_{j=1}^n \varphi(x_j)\Big)d\sigma_{n,x}} \geqslant C_0 x - C.$$

Similarly, $h_n'(x) \leqslant C_0 x + C$ for $x < -x_0$. This implies monotonicity of h_n for large $|x|$. Using the condition on the derivative of φ we find that

$$|\varphi(x \pm 1)| \leqslant C(|\varphi(x)| + 1) \tag{6.3}$$

for some constant $C > 0$. The statement of the lemma now follows from (6.1)-(6.3) by noticing that h_n are bounded below uniformly in n. The last observation is a consequence of monotonicity of h_n for large $|x|$ and Theorem A.1. □

Proof of Lemma 4.4. Let $q = \min\{2, 1+\delta_0/2p\}$. Without loss of generality we can assume that $\varphi \geqslant 0$. Let

$$H_N^{(q)}(s) = \int_{\mathrm{X}_{N,y_N}} f_{N,y_N}^s \left|\log f_{N,y_N}^s\right|^q d\mu_{N,y_N}.$$

By Lemma B.1 we have

$$\begin{aligned} \mathsf{E}_{\nu_{N,y_N}^s}\Big(\frac{1}{N}\sum_{i\in\Lambda_N}\varphi(x_i)\Big)^q \quad &\leqslant \quad \frac{C_q}{N^q} + \frac{4}{N^q}\log^q\Big(1 + \mathsf{E}_{\mu_{N,y_N}}\exp\Big(\sum_{i\in\Lambda_N}\varphi(x_i)\Big)\Big) \\ &\quad + \frac{2}{N^q}H_N^{(q)}(s). \end{aligned} \tag{6.4}$$

The second term in the right-hand side of (6.4) is clearly bounded since

$$\frac{1}{N}\log \mathsf{E}_{\mu_{N,y_N}} \exp\Big(\sum_{i\in\Lambda_N}\varphi(x_i)\Big)$$

converges to a limit as $N\to\infty$. Thus we have reduced the problem to proving that for our choice of q

$$H_N^{(q)}(s)\leqslant C_q N^q \tag{6.5}$$

locally uniformly in $s>0$.

The proof of (6.5) is similar to the proof of (1.12) given in [4]. Even though the function $x|\log x|^q$ is convex only for $x\geqslant \exp(1-q)$, it will not affect our considerations. There exists a smooth convex function $K(x)$ such that

$$x|\log x|^q\leqslant K(x)\leqslant B+x|\log x|^q,\quad x\geqslant 0, \tag{6.6}$$

for some $B>O$. We have

$$\begin{aligned}\frac{d}{ds}\int\limits_{X_{N,y_N}} K(f^s_{N,y_N})\,d\mu_{N,y_N} &= \int\limits_{X_{N,y_N}} K'(f^s_{N,y_N})\mathcal{L}_N f^s_{N,y_N}\,d\mu_{N,y_N}\\ &= -\int\limits_{X_{N,y_N}} K''(f^s_{N,y_N})\sum_{i\in\Lambda_N}\Big(\frac{\partial f^s_{N,y_N}}{\partial x_i}-\frac{\partial f^s_{N,y_N}}{\partial x_{i+1}}\Big)^2 d\mu_{N,y_N}\leqslant 0\end{aligned}$$

and, therefore, if (6.5) holds for some $s_0>0$ with $C_q=C_q(s_0)$, then it holds for all $s\geqslant s_0$ with $C_q\leqslant C_q(s_0)+B$. In the next step we argue that if we let s_0 depend on N in such a way that $s_0(N)\to 0$ as $N\to\infty$ and $s_0(N)N^2$ is bounded below by some $\tau>0$, then the sequence $H_N^{(q)}(s_0(N))/N^q$ remains bounded by a constant $C=C(\tau)$. This will finish the proof of (6.5).

Step 2. Let $s_N=\tau N^{-2}$, $\tau>0$, and define a non-speeded process $z^{(N)}(\tau)=x^{(N)}(s_N)$. Then $z^{(N)}(\tau)$ satisfies the system

$$dz_i(\tau)=\frac{1}{2}(\varphi'(z_{i-1})-2\varphi(z_i)+\varphi(z_{i+1}))d\tau+d\beta_i(\tau)-d\beta_{i+1}(\tau), \tag{6.7}$$

with initial conditions $z_i(0)=\eta_i^{(N)},\ i\in\Lambda_N$.

By abuse of notation we write f^τ_{N,y_N}, ν^τ_{N,y_N}, $H_N^{(q)}(\tau)$ instead of $f^{s_N}_{N,y_N}$, $\nu^{s_N}_{N,y_N}$, $H_N^{(q)}(s_N)$.

We need to show that

$$H_N^{(q)}(\tau) = \int\limits_{\mathrm{X}_{N,y_N}} f^\tau_{N,y_N} |\log f^\tau_{N,y_N}|^q \, d\mu_{N,y_N} \leqslant C(\tau) N^q$$

uniformly in N. This is done by "comparison" of $z^{(N)}$ with a Gaussian process $w^{(N)}$ which solves

$$dw_i^{(N)}(\tau) = d\beta_i^{(N)}(\tau) - d\beta_{i+1}^{(N)}(\tau), \qquad i \in \Lambda_N,$$

with the same initial data.

Denote by r^τ_{N,y_N} the density of the finite dimensional distribution of $w^{(N)}(\tau)$ with respect to μ_{N,y_N}. Then

$$\int\limits_{\mathrm{X}_{N,y_N}} f^\tau_{N,y_N} |\log f^\tau_{N,y_N}|^q \, d\mu_{N,y_N}$$
$$\leqslant \int\limits_{\{f^\tau_{N,y_N} \geqslant 1\}} f^\tau_{N,y_N} \left| \log \left(\frac{f^\tau_{N,y_N}}{r^\tau_{N,y_N}} \cdot r^\tau_{N,y_N} \right) \right|^q d\mu_{N,y_N} + \max_{0 \leqslant x \leqslant 1} x |\log x|^q.$$

It is evident that for any $0 < x \leqslant y,\ xy \geqslant 1$,

$$0 \leqslant \log(xy) \leqslant 2 \log y.$$

We have

$$\int\limits_{\{f^\tau_{N,y_N} \geqslant 1\}} f^\tau_{N,y_N} \left| \log \left(\frac{f^\tau_{N,y_N}}{r^\tau_{N,y_N}} \cdot r^\tau_{N,y_N} \right) \right|^q d\mu_{N,y_N}$$
$$\leqslant 2^q \int\limits_{\{f^\tau_{N,y_N} \geqslant 1\}} f^\tau_{N,y_N} \Big(\max \Big\{ \log \frac{f^\tau_{N,y_N}}{r^\tau_{N,y_N}}, \log r^\tau_{N,y_N} \Big\} \Big)^q d\mu_{N,y_N}$$
$$\leqslant 2^q \Big(\int\limits_{\mathrm{X}_{N,y_N}} f^\tau_{N,y_N} \left| \log \frac{f^\tau_{N,y_N}}{r^\tau_{N,y_N}} \right|^q d\mu_{N,y_N} + \int\limits_{\{r^\tau_{N,y_N} \geqslant 1\}} f^\tau_{N,y_N} \log^q r^\tau_{N,y_N} \, d\mu_{N,y_N} \Big).$$

The first term of the sum above will be estimated by Girsanov's formula. At this point our assumption (2.8) on the initial data comes into play. The fact that r^τ_{N,y_N} is known explicitly will allow us to obtain a bound on the second term. The following two propositions complete the proof of Lemma 4.4.

Proposition 6.1. *Assume that* $\{\eta^{(N)}\}$ *satisfies* (2.8) *where* p *is defined in* (H_3). *Then*

(i) $\mathsf{E}_{\nu^\tau_{N,y_N}} \sum_{i\in\Lambda_N} |\varphi(z_i^{(N)})|^{2pq} \leqslant C(T)N$ *for all* $\tau \in [0,T]$, $T<\infty$;

(ii) $\int_{\mathrm{X}_{N,y_N}} f^\tau_{N,y_N} \left|\log \frac{f^\tau_{N,y_N}}{r^\tau_{N,y_N}}\right|^q d\mu_{N,y_N} \leqslant C(\tau)N^q$ *for any* $\tau>0$.

Proposition 6.2. *For any* $\tau>0$ *there exists a constant* $C(\tau)$ *such that*

$$\int_{r^\tau_{N,y_N}\geqslant 1} f^\tau_{N,y_N} \log^q r^\tau_{N,y_N}\, d\mu_{N,y_N} \leqslant C(\tau)N^q.$$

Proof of Proposition 6.1. To simplify the notation we drop the superscript in $z^{(N)}$.

(i) First we prove a slightly more general statement. Let $\psi(x)$ be any smooth convex function which satisfies the inequality

$$\psi''(x) \leqslant C_1\psi(x) + C_2. \tag{6.8}$$

By Itô's formula

$$d\psi(z_i) = \psi''(z_i)\,d\tau + \psi'(z_i)\,dz_i.$$

Taking summation over i, integrating from 0 to τ, applying (6.7) and computing the expectation we obtain

$$\begin{aligned}
&\mathsf{E}_{\mathsf{P}_N} \sum_{i\in\Lambda_N} \psi(z_i(\tau))\\
&= \sum_{i\in\Lambda_N} \psi(\eta_i^{(N)}) + \int_0^\tau \mathsf{E}_{\mathsf{P}_N} \psi''(z_i(s))\,ds\\
&\quad - \frac{1}{2}\int_0^\tau \mathsf{E}_{\mathsf{P}_N} \sum_{i\in\Lambda_N} (\psi'(z_{i+1}(s)) - \psi'(z_i(s)))(\varphi'(z_{i+1}(s)) - \varphi'(z_i(s)))\,ds\\
&\leqslant \sum_{i\in\Lambda_N} \psi(\eta_i^{(N)}) + C_2N\tau + C_1\int_0^\tau \mathsf{E}_{\mathsf{P}_N} \sum_{i\in\Lambda_N} \psi(z_i(s))\,ds.
\end{aligned}$$

In the last inequality we used convexity of φ and ψ and condition (6.8). An application of the Gronwall inequality yields

$$\mathsf{E}_{\nu^\tau_{N,y_N}} \sum_{i\in\Lambda_N} \psi(z_i) \leqslant \Big(\sum_{i\in\Lambda_N} \psi(\eta_i^{(N)}) + N(1-\exp(-C\tau))\Big)\exp(C\tau). \tag{6.9}$$

In our case we can take $\psi(x)$ essentially to be equal to $|\varphi(x)|^{2pq}$ modifying the latter on a finite interval as necessary. Then (6.9) implies that for $\tau \in [0,T]$

$$\mathsf{E}_{\nu^{\tau}_{N,y_N}} \sum_{i\in\Lambda_N} |\varphi(z_i)|^{2pq} \leqslant C(T)N.$$

(ii) We show that

$$\int\limits_{\mathrm{X}_{N,y_N}} K\left(\frac{f^{\tau}_{N,y_N}}{r^{\tau}_{N,y_N}}\right) r^{\tau}_{N,y_N}\, d\mu_{N,y_N} \leqslant C(\tau)N^q$$

where K satisfies (6.6). This immediately implies (ii). By convexity of K,

$$\begin{aligned}\int\limits_{\mathrm{X}_{N,y_N}} K\left(\frac{f^{\tau}_{N,y_N}}{r^{\tau}_{N,y_N}}\right) r^{\tau}_{N,y_N}\, d\mu_{N,y_N} &\leqslant \mathsf{E}_{Q_N} K\left(\frac{d\mathsf{P}_N}{dQ_N}\right)\Big|_{\mathcal{F}_\tau} \\ &\leqslant B + \mathsf{E}_{\mathsf{P}_N} \left|\log \frac{d\mathsf{P}_N}{dQ_N}\Big|_{\mathcal{F}_\tau}\right|^q,\end{aligned}$$

where Q_N is the measure on continuous paths associated with $w^{(N)}$. In the same way as in the proof of Proposition 5.1 we obtain

$$\begin{aligned}&\mathsf{E}_{\mathsf{P}_N} \left|\log \frac{d\mathsf{P}_N}{dQ_N}\Big|_{\mathcal{F}_\tau}\right|^q \\ &= \mathsf{E}_{\mathsf{P}_N} \left|\int\limits_0^\tau b^{(N)}(z(s))\, d\bar{z}(s) + \frac{1}{2}\int\limits_0^\tau \langle b^{(N)}(z), A^{(N)}b^{(N)}(z)\rangle(s)\, ds\right|^q \\ &\leqslant 2^{q-1}\left|\mathsf{E}_{\mathsf{P}_N} \int\limits_0^\tau b^{(N)}(z(s))\, d\bar{z}(s)\right|^q \\ &\quad + C(\tau)\, \mathsf{E}_{\mathsf{P}_N} \int\limits_0^\tau \langle b^{(N)}(z), A^{(N)}b^{(N)}(z)\rangle^q(s)\, ds\end{aligned}$$

with

$$\bar{z}(s) = z(s) - \int\limits_0^\tau A^{(N)}b^{(N)}(z(s))\, ds.$$

From the inequality $a^q \leqslant C(q)(a^2+1)$, where $q \in [1,2]$, $a > 0$, and Itô's isometry we find that

$$
\begin{aligned}
\mathsf{E}_{\mathsf{P}_N} \left| \log \frac{d\mathsf{P}_N}{dQ_N}_{\mathcal{F}_\tau} \right|^q &\leqslant \tilde{C}(q)\Big(\mathsf{E}_{\mathsf{P}_N} \int_0^\tau \langle b^{(N)}(z), A^{(N)} b^{(N)}(z)\rangle(s)\, ds + 1\Big) \\
&\quad + C(\tau)\, \mathsf{E}_{\mathsf{P}_N} \int_0^\tau \langle b^{(N)}(z(s)), A^{(N)} b^{(N)}(z(s))\rangle^q \, ds \\
&< \tilde{C}(q,\tau)\Big(\mathsf{E}_{\mathsf{P}_N} \int_0^\tau \langle b^{(N)}(z), A^{(N)} b^{(N)}(z)\rangle^q(s)\, ds + 1\Big).
\end{aligned}
$$

Notice that by (5.2) and (1.4)

$$
\begin{aligned}
\langle b^{(N)}(z), A(N) b^{(N)}(z)\rangle &= \frac{1}{2} \sum_{i\in\Lambda_N} (\varphi'(z_{i+1}) - \varphi'(z_i))^2 \\
&\leqslant 4C_1^2 \Big(\sum_{i\in\Lambda_N} |\varphi(z_i)|^{2p} + N\Big).
\end{aligned}
$$

Therefore by the Hölder inequality and part (i)

$$
\begin{aligned}
&\mathsf{E}_{\mathsf{P}_N} \int_0^\tau \langle b^{(N)}(z(s)), A(N) b^{(N)}(z(s))\rangle^q \, ds \\
&\qquad \leqslant C\Big(N^{q-1}\, \mathsf{E}_{\mathsf{P}_N} \int_0^\tau \sum_{i\in\Lambda_N} |\varphi(z_i(s))|^{2pq}\, ds + N^q \Big) \leqslant C(\tau) N^q
\end{aligned}
$$

This completes the proof of Proposition 6.1. □

Proof of Proposition 6.2. Let $\tilde{A}^{(N)}$ be the restriction of $A^{(N)}$ to $\mathbf{X}_{N,y_N}$. Then by the definition of r^τ_{N,y_N} for any z such that $r^\tau_{N,y_N}(z) \geqslant 1$ we obtain

$$
\begin{aligned}
0 &\leqslant \frac{1}{N} \log r^\tau_{N,y_N}(z) \\
&= -\frac{1}{N} \log \frac{d\mu_{N,y_N}}{d\sigma_{N,y_N}}(z) + \frac{1}{N} \log \Big[\big((2\pi\tau)^{N-1} \det \tilde{A}^{(N)} \big)^{-1/2} \\
&\quad \times \exp\Big(-\frac{1}{2\tau} \langle z - \eta^{(N)}, (\tilde{A}^{(N)})^{-1} (z - \eta^{(N)})\rangle \Big) \Big] \\
&\leqslant -\frac{N-1}{2N} \log(2\pi\tau) - \frac{1}{2N} \log(\det \tilde{A}^{(N)}) + \frac{1}{N} \sum_{i\in\Lambda_N} \varphi(z_i) \\
&\quad + \frac{1}{N} \log \int_{\mathrm{X}_{N,y_N}} \exp\Big(-\sum_{i\in\Lambda_N} \varphi(x_i)\Big)\, d\sigma_{N,y_N}.
\end{aligned}
$$

The first term in the right-hand side is evidently bounded. The second and the fourth terms have finite limits as $N \to \infty$ by Proposition 5.3 (with $k = N$) and Theorem A.1 respectively. The third term is the only one which depends on z. We conclude that

$$\begin{aligned}
&\frac{1}{N^q} \int\limits_{r^\tau_{N,y_N} \geqslant 1} f^\tau_{N,y_N} \log^q r^\tau_{N,y_N}\, d\mu_{N,y_N} \\
&\qquad \leqslant \; C(\tau) + \frac{2^{q-1}}{N^q} \int\limits_{\mathrm{X}_{N,y_N}} \Big| \sum_{i\in\Lambda_N} \varphi(z_i)\Big|^q d\nu^\tau_{N,y_N} \\
&\qquad \leqslant \; C(\tau) + \frac{2^{q-1}}{N^q} \int\limits_{\mathrm{X}_{N,y_N}} N^{q-1/2p}\Big(\sum_{i\in\Lambda_N} |\varphi(z_i)|^{2pq}\Big)^{1/2p} d\nu^\tau_{N,y_N} \\
&\qquad = \; C(\tau) + 2^{q-1}\Big(\mathsf{E}_{\nu^\tau_{N,y_N}} \frac{1}{N} \sum_{i\in\Lambda_N} |\varphi(z_i)|^{2pq}\Big)^{1/2p} \leqslant \tilde{C}(\tau)
\end{aligned}$$

by part (i) of Proposition 6.1. □

Lemma 4.4 is proved □

7. Proof of Theorem 2.3

Lemma 7.1. *Under conditions of Theorem* 2.2

$$\lim_{N\to\infty} \frac{1}{N} \int\limits_{\mathrm{X}_{N,y_N}} f^t_{N,y_N} \log g^t_{N,y_N}\, d\mu_{N,y_N} = \int\limits_0^1 h(m(t,\theta))\, d\theta - h(\bar{m})$$

where $\bar{m}$ *is defined by* (2.3).

Proof. Let $m(t,\theta)$ be the solution of (2.7) with initial condition m_0 and $\lambda(t,\theta) = h'(m(t,\theta))$. Denote $\lambda(i/N,\theta)$ by λ^i_N. Recall that $y_N = \sum_{i\in\Lambda_N} \eta^{(N)}_i/N$. The density g^t_{N,y_N} of a local Gibbs measure $\gamma^{m_t}_N$ with respect to the invariant measure μ_{N,y_N} is easily computed:

$$g^t_{N,y_N}(x) = \frac{\exp\Big(\sum_{i\in\Lambda_N} \lambda^i_N x_i\Big) \int\limits_{\mathrm{X}_{N,y_N}} \exp\Big(-\sum_{i\in\Lambda_N} \varphi(x_i)\Big)\, d\sigma_{N,y_N}}{\int\limits_{\mathrm{X}_{N,y_N}} \exp\Big(\sum_{i\in\Lambda_N} \lambda^i_N x_i - \sum_{i\in\Lambda_N} \varphi(x_i)\Big)\, d\sigma_{N,y_N}}, \quad x \in \mathbf{X}_{N,y_N}.$$

Hence

$$\begin{aligned}
\frac{1}{N}\int\limits_{X_{N,y_N}} \log g^t_{N,y_N}\, d\nu^t_{N,y_N}
&= \frac{1}{N}\int\limits_{X_{N,y_N}} \sum_{i\in\Lambda_N} \lambda^i_N x_i\, d\nu^t_{N,y_N} \\
&\quad - \frac{1}{N}\log \int\limits_{X_{N,y_N}} \exp\Big(\sum_{i\in\Lambda_N} \lambda^i_N x_i - \sum_{i\in\Lambda_N}\varphi(x_i)\Big)\, d\sigma_{N,y_N} \\
&\quad + \frac{1}{N}\log \int\limits_{X_{N,y_N}} \exp\Big(-\sum_{i\in\Lambda_N}\varphi(x_i)\Big) d\sigma_{N,y_N}.
\end{aligned}$$

By Theorem A.1 and Theorem A.2 we obtain

$$\lim_{N\to\infty}\frac{1}{N}\log \int\limits_{X_{N,y_N}} \exp\Big(-\sum_{i\in\Lambda_N}\varphi(x_i)\Big)\, d\sigma_{N,y_N} = -h(\bar m);$$

$$\begin{aligned}
\lim_{N\to\infty}\frac{1}{N}\log \int\limits_{X_{N,y_N}} \exp\Big(\sum_{i\in\Lambda_N} \lambda^i_N x_i - \sum_{i\in\Lambda_N}\varphi(x_i)\Big)\, d\sigma_{N,y_N} \\
= -\int_0^1 \log M(\lambda(t,\theta))\, d\theta.
\end{aligned}$$

The last equality follows from (2.6), (A.2) and the definition of λ. By the existence of the scaling limit (2.1) and Lemma 3.2 we have that

$$\lim_{N\to\infty} \mathsf{E}_{\nu^t_{N,y_N}} \left|\frac{1}{N}\sum_{i\in\Lambda_N}\lambda^i_N x_i - \int_0^1 \lambda(t,\theta) m(t,\theta)\, d\theta\right| = 0. \tag{7.1}$$

This completes the proof of Lemma 7.1:

$$\begin{aligned}
\lim_{N\to\infty}\frac{1}{N}\int\limits_{X_{N,y_N}} \log g^t_{N,y_N}\, d\nu^t_{N,y_N} &= \int_0^1 \big(\lambda m - \log M(\lambda)\big)(t,\theta)\, d\theta - h(\bar m) \\
&= \int_0^1 h(m(t,\theta))\, d\theta - h(\bar m).
\end{aligned}$$

□

A. Limit theorems for densities

Let φ satisfy (H$_1$), (H$_2$) and (H$_3'$) and functions M, ρ, h be defined by (H$_3'$), (2.4), (2.5) respectively.

The following theorem was proven in [3]:

Theorem A.1. *Let φ_N be the density of $\sum_{i\in\Lambda_N} X_i/N$, where $X_1,\dots,X_N$ are independent identically distributed random variables with the common density $\exp(-\varphi(x))$. Then*

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N}\log\varphi_N(x) &= -h(x),\\ \lim_{N\to\infty}\frac{1}{N}\frac{\varphi_N'(x)}{\varphi_N(x)} &= -h'(x)\end{aligned}$$

uniformly on compact x-intervals.

We prove a generalization of this theorem to the series of independent random variables $X_N^1,\ X_N^2,\dots,\ X_N^N$, which are not identically distributed but whose distributions vary slightly with i. More precisely, fix a function $\lambda\in C([0,1])$. Denote $\lambda(i/N)$ by λ_N^i, the sup-norm of λ by $\|\lambda\|$, and the set $\{\lambda\in C([0.1]) : \|\lambda\|\leqslant R\}$ by B_R^λ. Assume that X_N^i has the density

$$a(x,\lambda_N^i)=\frac{1}{M(\lambda_N^i)}\exp(\lambda_N^i x-\varphi(x)) \tag{A.1}$$

with respect to the Lebesgue measure, $i=1,2,\dots,N$ and $N\in\mathbf{N}$.

Theorem A.2. *Let $X_N^1,\ X_N^2,\dots,\ X_N^N$ be defined as above and a_N^λ be the density of $\sum_{i\in\Lambda_N} X_N^i/N$. Then*

$$\lim_{N\to\infty}\frac{1}{N}\log a_N^\lambda(x)=-I^\lambda(x)$$

uniformly on any set of the form $\{|x|\leqslant R_1\}\times B_{R_2}^\lambda$, $R_1,R_2<\infty$. The rate function I^λ is given by the relation

$$I^\lambda(x)=\sup_z\Big\{zx-\int_0^1\log\frac{M(\lambda(\theta)+z)}{M(\lambda(\theta))}d\theta\Big\}. \tag{A.2}$$

The proof of this theorem is based on the local limit theorem, which we now formulate. Define for any $s\in\mathbf{R}$

$$b(y,s)=\frac{1}{M(s)}\exp\big(s(y+\rho'(s))-\varphi(y+\rho'(s))\big). \tag{A.3}$$

Then it is easy to check that

$$\begin{aligned}\int b(y,s)\,dy &= 1,\\ \int yb(y,s)\,dy &= 0,\\ \int y^2 b(y,s)\,dy &= \frac{M''(s)}{M(s)} - \Big(\frac{M'(s)}{M(s)}\Big)^2 = \rho''(s).\end{aligned}$$

Theorem A.3. *Let $Y_N^1, \ldots, Y_N^N$ be independent random variables and the density of Y_N^i with respect to the Lebesgue measure is given by* (A.3), *$i = 1, 2, \ldots, N$. Let b_N^λ be the density of $\sum_{i\in\Lambda_N} Y_N^i/\sqrt{N}$. Then for any non-negative integer k the functions b_N^λ belong to $C^k(\mathbf{R})$ for all $N \geqslant N_0(\|\lambda\|, k)$ and*

$$\lim_{N\to\infty} \frac{d^k}{dy^k} b_N^\lambda(y) = \frac{1}{\sqrt{2\pi}\sigma(\lambda)} \frac{d^k}{dy^k} \exp\Big(-\frac{y^2}{2\sigma^2(\lambda)}\Big)$$

uniformly on $\mathbf{R} \times B_R^\lambda$, $R < \infty$. Here $\sigma^2(\lambda) = \int_0^1 \rho''(\lambda(\theta))d\theta$.

The proof of Theorem A.3 is similar to the proof of Lemma 3.3 of [3] and is omitted.

Proof of Theorem A.2. From (A.1) and (A.3) we see that X_N^i has the same distribution as $Y_N^i + \rho'(\lambda_N^i)$. This implies the equivalence in law

$$\frac{1}{N}\sum_{i\in\Lambda_N} X_N^i \sim \frac{1}{\sqrt{N}}\Big(\frac{1}{\sqrt{N}}\sum_{i\in\Lambda_N} Y_N^i\Big) + \frac{1}{N}\sum_{i\in\Lambda_N}\rho'(\lambda_N^i),$$

from which we obtain that

$$a_N^\lambda(x) = \sqrt{N} b_N^\lambda\Big(\sqrt{N}\Big(x - \frac{1}{N}\sum_{i\in\Lambda_N}\rho'(\lambda_N^i)\Big)\Big).$$

By Theorem A.3 we have

$$\lim_{N\to\infty}\frac{1}{N}\log a_N^\lambda\Big(\frac{1}{N}\sum_{i\in\Lambda_N}\rho'(\lambda_N^i)\Big) = \lim_{N\to\infty}\frac{1}{N}\big(\log\sqrt{N} + \log b_N^\lambda(0)\big) = 0. \quad \text{(A.4)}$$

Let $z_0 = z_0(y)$ be the point where the supremum in (A.2) is attained and $\tilde\lambda(\theta) = \lambda(\theta) + z_0$. Introduce a new set of random variables $\tilde X_N^i$, $i = 1, \ldots, N$, with densities

$$\tilde a(x, \tilde\lambda_N^i) = \frac{1}{M(\tilde\lambda_N^i)}\exp(\tilde\lambda_N^i x - \varphi(x)).$$

Then by (A.4) the density $a_N^{\tilde{\lambda}}$ of the average $\sum_{i\in\Lambda_N} \tilde{X}_N^i/N$ satisfies

$$\lim_{N\to\infty} \frac{1}{N} \log a_N^{\tilde{\lambda}}\Big(\frac{1}{N}\sum_{i\in\Lambda_N} \rho'(\tilde{\lambda}_N^i)\Big) = 0. \tag{A.5}$$

Since $a(x,\lambda_N^i) = \tilde{a}(x,\tilde{\lambda}_N^i)\exp(-z_0 x)M(\tilde{\lambda}_N^i)/M(\lambda_N^i)$, using the properties of convolution we can compute that

$$a_N^{\lambda}(x) = a_N^{\tilde{\lambda}}(x)\exp(-z_0 xN)\prod_{i=1}^N (M(\tilde{\lambda}_N^i)/M(\lambda_N^i)).$$

From the last equation we obtain

$$\begin{aligned}\frac{1}{N}\log a_N^{\lambda}\Big(\frac{1}{N}\sum_{i\in\Lambda_N}\rho'(\tilde{\lambda}_N^i)\Big) &= \frac{1}{N}\log a_N^{\tilde{\lambda}}\Big(\frac{1}{N}\sum_{i\in\Lambda_N}\rho'(\tilde{\lambda}_N^i)\Big) - \frac{z_0}{N}\sum_{i\in\Lambda_N}\rho'(\tilde{\lambda}_N^i)\\ &\quad + \frac{1}{N}\sum_{i\in\Lambda_N}\log\frac{M(\tilde{\lambda}_N^i)}{M(\lambda_N^i)}.\end{aligned} \tag{A.6}$$

By our choice of $z_0(x)$ we find

$$\frac{1}{N}\sum_{i\in\Lambda_N}\rho'(\tilde{\lambda}_N^i) = \frac{1}{N}\sum_{i\in\Lambda_N}\frac{M'(\lambda_N^i+z_0)}{M(\lambda_N^i+z_0)} \longrightarrow \int_0^1 \frac{M'(\lambda(\theta)+z_0)}{M(\lambda(\theta)+z_0)}\,d\theta = x \tag{A.7}$$

as $N\to\infty$. The statement of the theorem follows now from (A.5), (A.6), (A.7) and Lemma A.1.

Lemma A.1. *Let* $I_N^{\lambda}(y) = -N^{-1}\log a_N^{\lambda}(y)$ *where* a_N^{λ} *defined as in Theorem* A.2*. Then for any* $y_1, y_2 \in \{|y| \leqslant R\}$

$$|I_N^{\lambda}(y_1) - I_N^{\lambda}(y_2)| \leqslant C(R,\|\lambda\|)|y_1 - y_2|.$$

Proof. We have

$$I_N^{\lambda}(y) = -\frac{1}{N}\log\int_{\mathrm{X}_{N,y}}\prod_{i=1}^N \frac{\exp\big(\lambda_N^i x_i - \varphi(x_i)\big)}{M(\lambda_N^i)}\,d\sigma_{N,y}. \tag{A.8}$$

Computing the derivative we find

$$\frac{d}{dy}I_N^{\lambda}(y) = -\frac{1}{N}\sum_{i\in\Lambda_N}\lambda_N^i + \frac{1}{N}\int_{\mathrm{X}_{N,y}}\sum_{i\in\Lambda_N}\varphi'(x_i)\,d\mu_{N,y}^{\lambda}(x) \tag{A.9}$$

where the measure $d\mu^{\lambda}_{N,y}(x)$ is given by

$$\frac{d\mu^{\lambda}_{N,y}(x)}{d\sigma_{N,y}} = \frac{\exp\Big(\sum_{i\in\Lambda_N}(\lambda^i_N x_i - \varphi(x_i))\Big)}{\int\limits_{X_{N,y}} \exp\Big(\sum_{i\in\Lambda_N}(\lambda^i_N x_i - \varphi(x_i))\Big)\, d\sigma_{N,y}}.$$

The first term of (A.9) is bounded by $\|\lambda\|$. To estimate the second term, we apply Jensen's inequality: for any $\delta > 0$

$$\begin{aligned}
&\int\limits_{X_{N,y}} \sum_{i\in\Lambda_N} |\varphi'(x_i)| d\mu^{\lambda}_{N,y}(x) \\
&\qquad \leqslant \frac{1}{\delta N}\log \int\limits_{X_{N,y}} \exp\Big(\delta \sum_{i\in\Lambda_N}|\varphi'(x_i)|\Big)\, d\mu^{\lambda}_{N,y}(x) \\
&\qquad \leqslant \frac{1}{\delta N}\log \frac{\int\limits_{X_{N,y}} \exp\Big(\sum_{i\in\Lambda_N}(\delta|\varphi'(x_i)| + \|\lambda\|\,|x_i| - \varphi(x_i))\Big)\, d\sigma_{N,y}}{\int\limits_{X_{N,y}} \exp\Big(-\sum_{i\in\Lambda_N}(\|\lambda\|\,|x_i| + \varphi(x_i))\Big)\, d\sigma_{N,y}}.
\end{aligned}$$

Let

$$\begin{aligned}
\varphi_{\delta,\lambda}(x) &= -\delta|\varphi'(x)| - \|\lambda\|\,|x| + \varphi(x) + \log C_{\delta,\lambda}, \\
\varphi_{\lambda}(x) &= \|\lambda\|\,|x| + \varphi(x) + \log C_{\lambda},
\end{aligned}$$

where the constants $C_{\delta,\lambda}$ and C_λ are chosen to turn $\exp(-\varphi_{\delta,\lambda})$ and $\exp(-\varphi_\lambda)$ into probability densities. Now we apply Theorem A.1 twice with φ replaced by $\varphi_{\delta,\lambda}$ and φ_λ to see that

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{\delta N}\log \int\limits_{X_{N,y}} \exp\Big(\delta \sum_{i\in\Lambda_N}|\varphi'(x_i)|\Big)\, d\mu^{\lambda}_{N,y}(x) \\
&\qquad = \frac{1}{\delta}\log\frac{C_{\delta,\lambda}}{C_\lambda} + \frac{1}{\delta}(h_\lambda(y) - h_{\delta,\lambda}(y))
\end{aligned}$$

uniformly on compact y-intervals. Rate functions $h_{\delta,\lambda}$ and h_λ are defined as in (2.5) with a suitable choice of index for φ. They are bounded on bounded y-intervals. This concludes the proof. □

Theorem A.2 is proved. □

B. One lemma from convex analysis

Lemma B.1. *Let μ be a probability measure on $\mathbf{R}^n$ and f and g be non-negative μ-measurable functions. Assume also that $\int g\,d\mu = 1$. Then for any $q \in [1,2]$*

$$\int f^q g\,d\mu \leqslant C_q + 2^{2q-2}\log^q\Big(\int e^f\,d\mu + 1\Big) + 2^{q-1}\int |\log g|^q g\,d\mu,$$

where C_q is a positive constant.

Proof. The statement of the lemma is a consequence of the convexity of the function

$$F(u) = \exp(u^\alpha) - u^\alpha \qquad \text{for } u \geqslant 0,\ \alpha = 1/q \in [1/2, 1]. \tag{B.1}$$

We write

$$\begin{aligned}\int f^q g\,d\mu &= \int\limits_{f\geqslant \log g} f^q g\,d\mu + \int\limits_{f<\log g} f^q g\,d\mu \\ &\leqslant \int\limits_{f\geqslant \log g} f^q g\,d\mu + \int\limits_{f<\log g} |\log g|^q g\,d\mu. \end{aligned} \tag{B.2}$$

Then we estimate the first term in the right-hand side of the above inequality.

$$\int\limits_{f\geqslant \log g} f^q g\,d\mu \leqslant 2^{q-1}\int (f-\log g)_+^q g\,d\mu + 2^{q-1}\int\limits_{f\geqslant \log g} |\log g|^q g\,d\mu. \tag{B.3}$$

By Jensen's inequality for function (B.1) we have

$$\begin{aligned}&\exp\Big(\int (f-\log g)_+^q g\,d\mu\Big)^{1/q} - \Big(\int (f-\log g)_+^q g\,d\mu\Big)^{1/q} \\ &\qquad\leqslant \int (\exp{(f-\log g)_+} - (f-\log g)_+)g\,d\mu. \end{aligned} \tag{B.4}$$

Since $u \leqslant (q/e)e^{u/q}$ for any $q > 0$ and $u \geqslant 0$, we obtain from (B.4)

$$\begin{aligned}&\exp\Big(\int (f-\log g)_+^q g\,d\mu\Big)^{1/q} \\ &\quad\leqslant \frac{q}{e}\Big(\int \exp{(f-\log g)_+} g\,d\mu\Big)^{1/q} + \int \exp{(f-\log g)_+} g\,d\mu \\ &\quad= \frac{q}{e}\Big(\int\limits_{f\geqslant\log g} e^f\,d\mu + \int\limits_{f<\log g} g\,d\mu\Big)^{1/q} + \int\limits_{f<\log g} e^f\,d\mu + \int\limits_{f<\log g} g\,d\mu \\ &\quad\leqslant \frac{q}{e}\Big(\int e^f\,d\mu + 1\Big)^{1/q} + \int e^f\,d\mu + 1. \end{aligned} \tag{B.5}$$

Applying the function $\log^q(\cdot)$ to both sides of (B.5) and taking into account the inequality $cu + u^q \leqslant (c+1)u^q$ for $u \geqslant 1$, we conclude that

$$\int (f - \log g)_+^q g \, d\mu \leqslant \log^q \Big(\Big(\frac{q}{e} + 1\Big)\Big(\int e^f \, d\mu + 1\Big)\Big). \tag{B.6}$$

Relations (B.2), (B.3) and (B.6) imply the statement of the lemma. □

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References

[1] D. Bakry and M. Emery (1985) Diffusions hypercontractives. Séminaire de prob., *Lect. Notes Math.* **1123**, Springer, Berlin-New York, 177–206.

[2] J.-D. Dueschel and D. Stroock (1989) *Large Deviations.* Pure and Applied Mathematics **137**, Academic Press, Boston.

[3] M.Z. Guo, G.C. Papanicolau and S.R.S. Varadhan (1988) Nonlinear diffusion limit for a system with nearest neighbor interactions. *Commun. Math. Phys.* **118**, 31–59.

[4] Sh. Lu (1995) Hydrodynamic scaling limits with deterministic initial configurations. *Ann. Prob.* **23** (4), 1831–1852.

[5] D.W. Stroock (1993) Logarithmic Sobolev inequalities for Gibbs states. Dirichlet forms (Varenna, 1992), Springer, Berlin, 194–228.

[6] S.R.S. Varadhan (1993) Entropy methods in hydrodynamic scaling. In: *Nonequilibrium Problems in Many-Particle Systems* (*Montecatini,* 1992), Lect. Notes Math. **1551**, Springer, Berlin, 112–145.

[7] H.-T. Yau (1991) Relative entropy and hydrodynamics of a Ginzburg–Landau model. *Lett. Math. Phys.* **22**, 63–80.