

ON THE CAUCHY PROBLEM FOR THE GENERALIZED POROUS MEDIUM EQUATION

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ABSTRACT

We prove the existence of a weak solution of the Cauchy problem in classes of growing functions for the generalized porous medium equation $u_t = \Delta\varphi(u)$ under broad assumptions on φ . In particular, functions $\varphi(s) \sim s^m \ln^p s$, $m \geq 1$, $p \geq -1$, and $\varphi(s) \sim \exp(s^p)$, $p > 0$, (as $s \rightarrow +\infty$) are included. We give sufficient conditions on the growth of the initial data as $|x| \rightarrow \infty$, which, in general, can not be improved, as we illustrate by examples. A lower bound on the existence time is also obtained. Under the convexity assumption on φ we prove the uniqueness of a weak solution.

1. Introduction

In this note we study the Cauchy problem for the generalized porous medium equation

$$\mathcal{L}(u) \equiv u_t - \Delta\varphi(u) = 0, \quad (x, t) \in S_T = \mathbb{R}^N \times (0, T), \quad (1)$$

with initial data

$$u(x, 0) = u_0(x) \geq 0, \quad x \in \mathbb{R}^N \quad (2)$$

where $u_0 \in L_{loc}^\infty(\mathbb{R}^N)$ and φ satisfies the following conditions:

(A1) $\varphi \in C^1([0, +\infty)) \cap C^3((0, \infty))$, $\varphi^{(i)}(s) > 0$ for $s > 0$, $i = 0, 1$;

(A2) $\int_1^{+\infty} \frac{dr}{\varphi^{-1}(r)} = +\infty$.

Let T be the existence time, i.e. the largest $T > 0$ for which the problem (1.1), (1.2) has a weak solution in S_T . (The definition of a weak solution will be given in the next section.) We are interested in the following questions: under what conditions on the growth of u_0 at infinity the problem (1.1), (1.2) has a solution in some S_T and how T depends on u_0 and φ .

Since the early 1950's equations of type (1) were studied extensively (see, for example, review [12]). Today there is a relatively complete theory for the case when $\varphi(s) = s^m$, $m > 0$. For $m \geq 1$ the function $\varphi(s) = s^m$ satisfies (A1) and (A2) while for $m \in (0, 1)$ we have that

$$\int_1^{+\infty} \frac{dr}{\varphi^{-1}(r)} < +\infty. \quad (3)$$

In the latter case the problem (1), (2) has the unique weak solution in S_∞ for any $u_0 \in L_{loc}^1(\mathbb{R}^N)$ (see [10]). Paper [7] extends these results for more general φ with polynomial growth and [14] gives a proof of the global existence for an arbitrary monotone graph φ , $0 \in \varphi(0)$, which satisfies (3). If $\varphi(s) = s^m$, $m \geq 1$, then for the existence of a solution in some S_T , $T > 0$, it is necessary to impose certain conditions on the behavior of $u_0(x)$ as $|x| \rightarrow +\infty$ (see [1] for $m = 1$ and [2] for $m > 1$). Below we formulate a well known result for the porous medium equation ($m > 1$).

Theorem 1.1 ([2], [3]). *Let $\varphi(s) = s^m$, $m > 1$, $u_0 \in L^1_{loc}(\mathbb{R}^N)$, $u_0 \geq 0$ a.e.. The problem (1), (2) has a weak solution in S_T for some $T \in (0, +\infty]$ if and only if*

$$l(u_0) \equiv \lim_{\rho \rightarrow +\infty} \sup_{R \geq \rho} R^{-N-2/(m-1)} \int_{|x| \leq R} u_0(x) dx < +\infty.$$

Moreover $T \geq c(m, N)/l(u_0)^{m-1}$ for $l(u_0) \neq 0$ and $T = +\infty$ for $l(u_0) = 0$.

The uniqueness theorem without any growth assumptions about the solution for the porous medium equation was proven in [5]. Theorem 1.1 was generalized in [6] for monotone functions $\varphi(s)$, $\varphi(0) = 0$, which for all $s > 0$ can be estimated above and below by powers of s with exponents greater than 1. Papers [11] and [8] contain one-dimensional results for more general φ .

In this note we prove the existence of a weak solution in classes of growing functions under broad assumptions on φ (see (A1), (A2) and some technical conditions in Theorems 2.1 and 2.2). The sufficient conditions on the growth of u_0 as $|x| \rightarrow +\infty$ given in Theorems 2.1 and 2.2, in general, can not be improved, which is illustrated by examples in Section 5. The lower bound on the existence time T is also obtained. Under a convexity assumption on φ we prove the uniqueness of a weak solution.

Precise formulations of the results are given in Section 2. Sections 3 and 4 contain the proofs of existence and uniqueness theorems. Further, in Section 5, we give several examples and continue the discussion of the results. Two technical lemmas are presented in the Appendix.

2. Main results

Throughout the paper we use the following notation:

$$\Psi(s) = \int_{\varphi(1)}^{\varphi(s)} \frac{dr}{\varphi^{-1}(r)} \equiv \int_1^s \frac{\varphi'(\xi)}{\xi} d\xi, \quad s \geq 1; \quad (4)$$

$$\psi = \Psi^{-1}, \quad \Psi(\psi(r)) \equiv r, \quad r \geq 0;$$

$$\lambda = \lim_{\rho \rightarrow +\infty} \inf_{s \geq \rho} \frac{\varphi'(s)\varphi'''(s)}{(\varphi''(s))^2}; \quad (5)$$

$$\mu = \lim_{\rho \rightarrow +\infty} \sup_{s \geq \rho} \frac{s\varphi''(s)}{\varphi'(s)}; \quad (6)$$

$$B_\rho = \{x \in \mathbb{R}^N : |x| < \rho\}, \quad \rho > 0.$$

We denote positive constants by $c_i, C_i, i \in \mathbb{N}$.

Definition 2.1. A non-negative function $u(x, t)$ defined in $S_T, T \in (0, +\infty]$, is said to be a weak solution of the problem (1), (2) if

$$u \in C([0, T]; L_{loc}^1(\mathbb{R}^N)) \cap L_{loc}^\infty(S_T)$$

and satisfies the equation (1) in the sense of distributions and the initial condition (2) in $L_{loc}^1(\mathbb{R}^N)$.

We start with existence theorems.

Theorem 2.1. Let φ and the initial data u_0 be such that:

- a) $\varphi(s)$ satisfies (A1), $\varphi''(s) > 0$ for $s \geq s_0$, and $\lambda > -\infty$;
- b) $u_0 \in L_{loc}^\infty(\mathbb{R}^N), u_0 \geq 0$ a.e. and

$$\lim_{\rho \rightarrow +\infty} \operatorname{ess\,sup}_{|x| \geq \rho} \frac{\varphi'(u_0(x))}{|x|^2} = \gamma \in [0, +\infty).$$

Let $T_1 = (2\gamma(N + 2 - 2\lambda))^{-1}$ if $\gamma > 0$ and $T_1 = +\infty$ if $\gamma = 0$.

Then there exists a weak solution of the problem (1), (2) in S_{T_1} , which satisfies the following growth condition:

$$\varphi'(u(x, t))(1 + |x|^2)^{-1} \in L^\infty(\mathbb{R}^N \times [0, \tau]) \quad (7)$$

for any $\tau \in (0, T_1)$.

Remark. The positivity of the second derivative of $\varphi(s)$ for large values of s immediately implies that φ satisfies (A2).

Theorem 2.2. Let φ and the initial data u_0 be such that:

- a) $\varphi(s)$ satisfies (A1), (A2), and $\mu < +\infty$;
- b) $u_0 \in L_{loc}^\infty(\mathbb{R}^N)$, $u_0 \geq 0$ a.e. and

$$\lim_{\rho \rightarrow +\infty} \operatorname{ess\,sup}_{|x| \geq \rho} \frac{\Psi(u_0(x) + 1)}{|x|^2} = \sigma \in [0, +\infty).$$

Let $T_2 = (2\sigma(2 + \mu N))^{-1}$ if $\sigma > 0$ and $T_2 = +\infty$ if $\sigma = 0$.

Then for any $T < T_2$ there exists a weak solution of the problem (1), (2) in S_T , which satisfies the following growth condition:

$$\Psi(u(x, t) + 1)(1 + |x|^2)^{-1} \in L^\infty(\mathbb{R}^N \times [0, \tau]) \quad (8)$$

for any $\tau \in (0, T)$.

Remark. It is easy to show (see Lemma A.1) that if φ satisfies (A1) and (A2) then $\mu \geq 0$ and if, in addition, $\varphi''(s) > 0$ for large s then $\lambda \leq 1$. In particular, the expressions for T_1 and T_2 above always make sense. We would

like to mention that for the existence theorems the differentiability of φ at 0 is not needed: continuity is sufficient.

Consider the conditions $\lambda > -\infty$ and $\mu < +\infty$ in more detail. Assume that φ satisfies (A1).

1) Let $\varphi''(s) > 0$ for $s \geq s_0$. It can be shown that if $\lambda > -\infty$ then $\varphi'(s)$ grows not slower than some positive power of s as $s \rightarrow +\infty$. Notice also that there is absolutely no restriction on how fast φ can grow at infinity since the inequality $\lambda > -\infty$ certainly holds if $\varphi'''(s) \geq 0$ for large s .

2) If $\mu < +\infty$ then, clearly, $\varphi'(s)$ grows at infinity not faster than some positive power of s . The inequality $\mu < +\infty$ on its own puts no restrictions from below on the growth of $\varphi(s)$ as $s \rightarrow +\infty$. It holds, for example, if $\varphi''(s) \leq 0$ for large s . It is the condition (A2) that does not allow φ to grow too slowly.

In particular, if φ satisfies the conditions of both Theorem 2.1 and Theorem 2.2 then for all sufficiently large s

$$cs^{\nu_1} \leq \varphi'(s) \leq Cs^{\nu_2}, \quad (9)$$

where c , C and $\nu_1 \leq \nu_2$ are positive constants. The following example indicates that in many such cases both theorems give the same result.

Example 1. Function $\varphi(s) = s^m \ln^p(s + e)$ for $m \geq 1$, $p \geq -1$ satisfies the conditions of Theorem 2.2 with $\mu = m - 1$. It also satisfies the conditions of Theorem 2.1 for a slightly narrower range of parameters: $m > 1$, $p \geq -1$, in which case $\lambda = (m - 2)/(m - 1)$. We can compute that $\gamma = \sigma(m - 1)$. Therefore $T_1 = T_2 = (2\sigma(N(m - 1) + 2))^{-1}$.

In Section 5 we continue the discussion about the relationship between

Theorem 2.1 and Theorem 2.2 and also give examples for which the above theorems are sharp.

Let us now formulate uniqueness theorems.

Theorem 2.3. *Suppose that φ and u_0 satisfy all the conditions of Theorem 2.1. Let $u(x, t)$ and $v(x, t)$ be weak solutions of the problem (1), (2) in S_T , $T > 0$, which satisfy (7) for any $\tau \in (0, T)$. Then for any $t \in (0, T)$*

$$u(x, t) = v(x, t) \quad \text{a.e. in } \mathbb{R}^N.$$

The next theorem uses an additional assumption about the convexity of φ which is not needed for existence (see Theorem 2.2). Typical examples where Theorem 2.3 does not apply and Theorem 2.4 does are given by $\varphi(s) \sim s \ln^p s$, $p \geq 0$. Let us point out that when T_2 is finite, the uniqueness interval T'_2 obtained in the theorem below is smaller than T_2 . This should certainly be attributed to insufficiency of our technique for this case.

Theorem 2.4. *Suppose that φ and u_0 satisfy all the conditions of Theorem 2.2 and $\varphi''(s) \geq 0$ for $s \geq s_0$. Let $u(x, t)$ and $v(x, t)$ be weak solutions of the problem (1), (2) in S_T , $T > 0$ such that*

$$K = \lim_{\rho \rightarrow +\infty} \operatorname{ess\,sup}_{\substack{|x| \geq \rho \\ t \in [0, T]}} \frac{\Psi(\max\{u(x, t), v(x, t)\} + 1)}{|x|^2} < +\infty. \quad (10)$$

Define $T'_2 = (2K(2 + 3\mu))^{-1}$. Then for any $t \in (0, T'_2)$.

$$u(x, t) = v(x, t) \quad \text{a.e. in } \mathbb{R}^N.$$

3. Proofs of existence theorems

At first we prove a lemma, which reduces the existence problem to the construction of a suitable a priori bound for the solution.

Lemma 3.1. *Let $u_0 \in L_{loc}^\infty(\mathbb{R}^N)$, $u_0(x) \geq 0$ a.e. and there exists a function $g \in C_{x,t}^{1,2}(S_\tau) \cap C(S_\tau \cup \{t = 0\})$, $\tau > 0$ such that*

$$\mathcal{L}(g) \geq 0 \text{ in } S_\tau, \quad g(x, t) \geq \delta > 0, \quad g(x, 0) \geq u_0(x) \text{ a.e..}$$

Then there exists a weak solution of the problem (1), (2) in S_τ and

$$0 \leq u(x, t) \leq g(x, t) \text{ a.e. in } S_\tau.$$

Proof. Let $k_0 = [\delta^{-1}] + 1$ where $[y]$ stands for the integer part of y . Then $g > 1/k_0$. Applying Lemma A.2 with $b(x) = g(x, 0)$ and $w(x) = u_0(x)$ we obtain sequences $\{u_{0k}\}_{k=k_0+1}^{+\infty}$, $u_{0k} \in L^\infty(B_k)$, and $\{u_{0kj}\}_{j=k}^{+\infty}$, $u_{0kj} \in C(B_k)$, which satisfy the conditions 1), 2) of Lemma A.2. For every pair $(k, j) \in \mathbb{N}^2$, where $j > k > k_0$, consider the following problem:

$$\begin{aligned} \mathcal{L}(u) &= 0, & (x, t) &\in D_k = B_k \times (0, \tau), \\ u(x, t) &= g(x, t), & (x, t) &\in \Gamma_k = \partial B_k \times [0, \tau), \\ u(x, 0) &= u_{0kj}(x), & x &\in B_k. \end{aligned}$$

This problem has a unique classical solution $u_{kj} \in C_{x,t}^{2,1}(D_k) \cap C(\bar{D}_k \setminus \{t = \tau\})$ (see [13]). Since the sequence $\{u_{0kj}\}_{j=k}^{j=+\infty}$ satisfies (42), the comparison principle yields (see, for example, [4], p. 395–396)

$$\begin{aligned} \frac{1}{k} &\leq u_{kj}(x, t) \leq g(x, t), & (x, t) &\in \bar{D}_k \setminus \{t = \tau\}, \\ u_{(k+1)j}(x, t) &\leq u_{kj}(x, t) \leq u_{k(j+1)}(x, t), & (x, t) &\in \bar{D}_k \setminus \{t = \tau\}, \end{aligned}$$

where $j > k > k_0$. Passing to the limit as $j \rightarrow +\infty$ and then as $k \rightarrow +\infty$ we obtain a function $u \in L_{loc}^\infty(S_\tau)$ which satisfies the inequalities

$$0 \leq u(x, t) \leq g(x, t).$$

The proof that $u(x, t)$ is a weak solution of the problem (1), (2) can be carried out along the lines of [9], p. 41–42. \square

Using the above lemma we can now prove the existence theorems.

Proof of Theorem 2.1. We prove the existence of a solution u_δ in S_{T_δ} where $T_\delta = (2(\gamma + \delta)(N + 2 - 2\lambda + 2\delta))^{-1} < T_1$ for any $0 < \delta < 1/2$ and check that u_δ satisfies (7) for any $\tau \in (0, T_\delta)$. After we prove the uniqueness Theorem 2.3 (Section 4) we immediately obtain the statement of Theorem 2.1.

For an arbitrary $\delta \in (0, 1/2)$ we construct an upper barrier g_δ for which all the conditions of Lemma 3.1 hold in S_{T_δ} . By the assumption b) there is a constant $\eta = \eta(\delta) \geq \max\{\varphi'(s_0), r_1\}$ (see the line just above (23) for the definition of $r_1 = r_1(\delta)$) such that

$$\varphi'(u_0(x)) \leq (\gamma + \delta)|x|^2 + \eta \quad \text{a.e. in } \mathbb{R}^N$$

Let

$$\xi = \xi(\delta) = (2(N + 2 - 2\lambda + 2\delta))^{-1};$$

$$D = D(\delta) = \eta T_\delta^{2\xi N}.$$

Define

$$g_\delta(x, t) = \varphi'^{-1} \left(\frac{\xi|x|^2}{T_\delta - t} + \frac{D}{(T_\delta - t)^{2\xi N}} \right). \quad (11)$$

Denoting the argument of the function φ'^{-1} in (11) by $r = r(x, t)$ we find that

$$\begin{aligned} \mathcal{L}(g_\delta)(x, t) &= \left(\frac{\xi|x|^2}{(T_\delta - t)^2} + \frac{2\xi ND}{(T_\delta - t)^{2\xi N + 1}} \right) (\varphi'^{-1}(r))' \\ &\quad - \frac{2\xi N}{T_\delta - t} r(\varphi'^{-1}(r))' - \frac{4\xi^2|x|^2}{(T_\delta - t)^2} (r(\varphi'^{-1}(r))')' \\ &= \left\{ \frac{\xi|x|^2}{(T_\delta - t)^2} (1 - 2\xi(N + 2 - 2\lambda + 2\delta)) \right. \\ &\quad \left. - \frac{4\xi^2|x|^2}{(T_\delta - t)^2} \left(\frac{r(\varphi'^{-1}(r))''}{(\varphi'^{-1}(r))'} - (\delta - \lambda) \right) \right\} \times (\varphi'^{-1}(r))' \geq 0 \end{aligned}$$

by (23) and the inequality $r(x, t) \geq \eta \geq r_1$. Clearly $g_\delta(x, t) \geq \varphi'^{-1}(\eta) > 0$ and $g_\delta(x, 0) \geq u_0(x)$ a.e. in \mathbb{R}^N . Therefore by Lemma 3.1 there exists a weak solution $u_\delta(x, t)$ of the problem (1), (2) and moreover

$$u_\delta(x, t) \leq g_\delta(x, t) \quad \text{in } S_{T_\delta}.$$

From the last inequality and (11) it follows that $u_\delta(x, t)$ satisfies (7) for any $\tau \in (0, T_\delta)$. The proof of Theorem 2.1 is now complete. \square

Proof of Theorem 2.2. Define $\delta > 0$ from the equation $T = T(\delta) = (2(\sigma + \delta)(2 + (\mu + \delta)N))^{-1}$. We construct an upper barrier g_δ which satisfies all the conditions of Lemma 3.1.

Let $\eta = \eta(\delta) > 0$ be a number for which the inequality

$$\Psi(u_0(x) + 1) \leq (\sigma + \delta)|x|^2 + \eta, \quad x \in \mathbb{R}^N.$$

holds a.e. in \mathbb{R}^N (see b)). Let

$$\begin{aligned} \xi &= \xi(\delta) = (2(2 + (\mu + \delta)N))^{-1} \\ D(t) &= \frac{1}{\mu + \delta} \left(\frac{((\mu + \delta)\eta + C_9)T^{2\xi(\mu + \delta)N}}{(T - t)^{2\xi(\mu + \delta)N}} - C_9 \right), \end{aligned}$$

where C_9 is the same as in (34). Define

$$g_\delta(x, t) = \psi \left(\frac{\xi|x|^2}{T-t} + D(t) \right). \quad (12)$$

Denoting the argument of the function ψ in (12) by $r(x, t)$ we find that

$$\begin{aligned} \mathcal{L}(g_\delta)(x, t) &= \\ &= \left\{ \frac{(\xi - 4\xi^2)|x|^2}{(T-t)^2} + D'(t) - \frac{2\xi N}{T-t} \varphi'(\psi(r)) \Big|_{r=r(x,t)} \right\} \times \psi'(r) \Big|_{r=r(x,t)}. \end{aligned} \quad (13)$$

In the derivation of (13) we used the following identity:

$$\begin{aligned} (\varphi(\psi(r)))'' &= (\varphi'(\psi(r))\psi'(r))' \\ &= (\Psi'(\psi(r))\psi(r)/\Psi'(\psi(r)))' = \psi'(r), \quad r \geq 0. \end{aligned}$$

Taking into account the inequality (34) we can estimate $\mathcal{L}g_\delta(x, t)$ from below:

$$\begin{aligned} \mathcal{L}(g_\delta)(x, t) \geq & \left\{ \frac{(1 - 2\xi(2 + (\mu + \delta)N)\xi|x|^2)}{(T-t)^2} + D'(t) \right. \\ & \left. - \frac{2\xi N(\mu + \delta)}{T-t} D(t) + \frac{2\xi NC_9}{T-t} \right\} \psi'(r) \Big|_{r=r(x,t)} = 0. \end{aligned}$$

Clearly $g_\delta(x, t) \geq \psi(\eta) > 0$ and $g_\delta(x, 0) \geq u_0(x)$ a.e. in \mathbb{R}^N . Therefore by Lemma 3.1 there exists a weak solution $u_\delta(x, t)$ of the problem (1), (2) which satisfies

$$u_\delta(x, t) \leq g_\delta(x, t) \quad \text{in } S_T.$$

From the last inequality and (12) it follows that $u_\delta(x, t)$ satisfies (8) for any $\tau \in (0, T)$. This completes the proof of Theorem 2.2. \square

4. Proofs of uniqueness theorems

Proof of Theorem 2.3. This is a variation of the standard argument (see, for example, already cited works [11] and [3]).

Let $u(x, t)$, $v(x, t)$ be weak solutions of the problem (1), (2) defined in some S_T , $T \in (0, +\infty]$ which satisfy the growth condition (7) for all $\tau \in (0, T)$. We show that $u(x, \tau)$ and $v(x, \tau)$ coincide a.e. for any $\tau \in (0, T)$.

The Definition 2.1 implies that for any function $f \in C_0^\infty(\bar{S}_\tau)$, $0 < \tau < T$,

$$\begin{aligned} \int_{\mathbb{R}^N} f(x, \tau)(u(x, \tau) - v(x, \tau))dx \\ = \int_0^\tau \int_{\mathbb{R}^N} [(u - v)f_t + (\varphi(u) - \varphi(v))\Delta f]dxdt. \end{aligned} \quad (14)$$

Let

$$a(x, t) = \begin{cases} \frac{\varphi(u(x, t)) - \varphi(v(x, t))}{u(x, t) - v(x, t)}, & \text{if } u(x, t) \neq v(x, t). \\ \varphi'(u(x, t)), & \text{if } u(x, t) = v(x, t). \end{cases}$$

Then the equation (14) can be rewritten as follows:

$$\int_{\mathbb{R}^N} f(x, \tau)(u(x, \tau) - v(x, \tau))dx = \int_0^\tau \int_{\mathbb{R}^N} (u - v)(f_t + a\Delta f)dxdt. \quad (15)$$

Notice that by convexity of φ and (7) for all $t \in [0, \tau]$

$$0 \leq a(x, t) \leq C_\tau(|x|^2 + 1), \quad \text{a.e. in } x \in \mathbb{R}^N. \quad (16)$$

Fix an arbitrary $R_0 > 0$ and a function

$$\omega \in C_0^\infty(\mathbb{R}^N), \quad 0 \leq \omega \leq 1, \quad \omega = 0 \quad \text{for } |x| \geq R_0.$$

Let $\{a_n\}$ be any sequence of positive functions such that

$$\begin{aligned} a_n &\in C^\infty(\mathbb{R}^N \times [0, \tau]); \\ a_n(x, t) &\leq (C_\tau + 1)(|x|^2 + 1) \quad (x, t) \in \mathbb{R}^N \times [0, \tau]. \end{aligned} \quad (17)$$

For any $R > R_0 + 1$ there exists a unique classical solution $f_n(x, t)$ of the problem

$$Lf_n \equiv f_{nt} + a_n \Delta f_n = 0, \quad 0 < t < \tau, \quad |x| < R, \quad (18)$$

$$f_n(x, t)|_{|x|=R} = 0, \quad 0 \leq t \leq \tau \quad (19)$$

$$f_n(x, \tau) = \omega(x), \quad |x| < R; \quad (20)$$

Construct a sequence of cutoff functions $\theta_h \in C_0^\infty(\mathbb{R}^N)$, $0 < h < \frac{1}{2}$, with the following properties:

i) $0 \leq \theta_h \leq 1$, $\theta_h(x) = 1$ in B_{R-2h} and $\theta_h(x) = 0$ outside B_{R-h} ;

ii) $\|\nabla \theta_h\|_{L^\infty} \leq C_1 h^{-1}$; $\|\Delta \theta_h\|_{L^\infty} \leq C_1 h^{-2}$.

Choose

$$f(x, t) = \begin{cases} f_n(x, t)\theta_h(x), & |x| < R, \quad 0 \leq t \leq \tau \\ 0, & |x| \geq R, \quad 0 \leq t \leq \tau. \end{cases}$$

Then from (15) we find that

$$\begin{aligned} \int_{\mathbb{R}^N} \omega(x)(u(x, \tau) - v(x, \tau))dx &= \int_0^\tau \int_{\mathbb{R}^N} (u - v)(a - a_n)\theta_h \Delta f_n dx dt \\ &\quad + \int_0^\tau \int_{\mathbb{R}^N} (\varphi(u) - \varphi(v))(f_n \Delta \theta_h + 2\nabla f_n \nabla \theta_h) dx dt \end{aligned}$$

and therefore by (7)

$$\begin{aligned} \left| \int_{\mathbb{R}^N} \omega(x)(u(x, \tau) - v(x, \tau))dx \right| &\leq c(R, \tau) \int_0^\tau \int_{B_R} |a - a_n| |\Delta f_n| dx dt \\ &\quad + \int_0^\tau \int_{B_R} |\varphi(u) - \varphi(v)| \cdot |f_n \Delta \theta_h + 2\nabla f_n \nabla \theta_h| dx dt = c(R, \tau)I_n + J_{nh} \end{aligned}$$

Since R_0 and $\omega(x)$ are arbitrary, it is enough to show that the sum $c(R, \tau)I_n + J_{nh}$ can be made smaller than any given positive number under a suitable choice of sequence $\{a_n\}$ and parameters h and $R > R_0 + 1$.

Lemma 4.1. *For any $\varepsilon > 0$ there exists a sequence of positive functions $\{a_n\}$ such that the conditions (17) hold and $I_n < \varepsilon$ for all $n \in \mathbb{N}$.*

The proof of this lemma can be found, for example, in [3], p. 80.

We proceed with the second integral.

$$J_{nh} \leq C_1 \int_0^\tau \int_{B_{R-h} \setminus B_{R-2h}} |\varphi(u) - \varphi(v)| \left(\frac{|f_n|}{h^2} + \frac{|\nabla f_n|}{h} \right) dx dt.$$

Since f_n is a solution of (18) and $f_n = 0$ for $|x| = R$ then

$$\begin{aligned} \sup_{\substack{R-2h < |x| < R \\ 0 < t < \tau}} |f_n(x, t)| &\leq 2h \sup_{\substack{R-2h < |x| < R \\ 0 < t < \tau}} |\nabla f_n(x, t)|, \\ \lim_{h \rightarrow 0^+} \sup_{\substack{R-2h < |x| < R \\ 0 < t < \tau}} |\nabla f_n(x, t)| &= \sup_{\substack{|x|=R \\ 0 < t < \tau}} |\nabla f_n(x, t)| = \sup_{\substack{|x|=R \\ 0 < t < \tau}} \left| \frac{\partial f_n(x, t)}{\partial \nu} \right| \end{aligned}$$

and

$$\begin{aligned} J_n &\equiv \limsup_{h \rightarrow 0^+} J_{nh} \\ &\leq \tau C_2 R^{N-1} \sup_{\substack{|x|=R \\ 0 < t < \tau}} \left| \frac{\partial f_n(x, t)}{\partial \nu} \right| \lim_{h \rightarrow 0^+} \operatorname{ess\,sup}_{\substack{R-2h < |x| < R \\ 0 < t < \tau}} |\varphi(u) - \varphi(v)|. \end{aligned} \quad (21)$$

Using (7) and (16) we find that for any $t \in [0, \tau]$ and almost all $x \in \mathbb{R}^N$

$$\begin{aligned} |\varphi(u(x, t)) - \varphi(v(x, t))| &= a(x, t) |u(x, t) - v(x, t)| \\ &\leq C_\tau (1 + |x|^2) \varphi'^{-1}(C_\tau (1 + |x|^2)). \end{aligned}$$

It is easy to check that for all $s > s_0$ the following identity holds:

$$\frac{\varphi'(s) \varphi'''(s)}{(\varphi''(s))^2} = -r \frac{d^2 \varphi'^{-1}(r)}{dr^2} \bigg/ \frac{d\varphi'^{-1}(r)}{dr} \bigg|_{r=\varphi'(s)}. \quad (22)$$

Since $\lambda > -\infty$ (see (5) and assumption a)) then for any $\delta > 0$ there is a $r_1 > r_0 = \varphi(s_0)$ such that

$$r \frac{(\varphi'^{-1}(r))''}{(\varphi'^{-1}(r))'} \leq -\lambda + \delta \quad \text{for any } r \geq r_1. \quad (23)$$

Therefore

$$\varphi'^{-1}(r) \leq C_3(1 + r^{1-\lambda+\delta}), \quad r \geq r_1,$$

and we arrive at the inequality

$$|\varphi(u(x, t)) - \varphi(v(x, t))| \leq C_4(1 + |x|^2)^{2-\lambda+\delta} \quad (24)$$

where $C_4 = C_4(\tau, \lambda, \delta)$. Substituting (24) into (21) yields

$$J_n \leq C_5 R^{N+3-2\lambda+2\delta} \sup_{\substack{|x|=R \\ 0 < t < \tau}} \left| \frac{\partial f_n(x, t)}{\partial \nu} \right|. \quad (25)$$

To estimate the normal derivative of f_n in the right hand side of (25) we need an upper bound on f_n . We claim that

$$f_n(x, t) \leq q(x, t) \stackrel{\text{def}}{=} M e^{\alpha(\tau-t)} (1 + |x|^2)^{-\beta}, \quad 0 \leq t \leq \tau, \quad |x| \leq R. \quad (26)$$

where $\alpha \geq 4\beta(\beta + 1)(C_\tau + 1)$, β is an arbitrary positive number and M is large enough to ensure that

$$q(x, \tau) \geq \omega(x) = f_n(x, \tau), \quad |x| \leq R.$$

Since $\text{supp } \omega(x) \subset B_{R_0}$ the constant M does not depend on R . Computing Lq (see (18)) we find

$$Lq \leq (4\beta(\beta + 1)(C_\tau + 1) - \alpha)q \leq 0.$$

Moreover $q(x, t) \geq f_n(x, t) = 0$, $|x| = R$, $0 \leq t \leq \tau$ by (20). The inequality (26) follows by the maximum principle.

On the set $A_R = \{(x, t) \mid R - 1 \leq |x| \leq R, 0 < t < \tau\}$ let us construct a function $p(x, t)$ such that $p(x, t) \geq f_n(x, t)$ and $p(x, t) = 0$ for $|x| = R, 0 < t < \tau$. Then

$$\frac{\partial(p(x, t) - f_n(x, t))}{\partial \nu} \leq 0 \quad \text{for } |x| = R, 0 < t < \tau.$$

Since $\frac{\partial f_n(x, t)}{\partial \nu} \leq 0$ for $|x| = R, 0 < t < \tau$ we conclude that

$$\sup_{\substack{|x|=R \\ 0 < t < \tau}} \left| \frac{\partial f_n(x, t)}{\partial \nu} \right| \leq \sup_{\substack{|x|=R \\ 0 < t < \tau}} \left| \frac{\partial p(x, t)}{\partial \nu} \right|. \quad (27)$$

For $N \geq 3$ define $p(x, t) = \frac{d_1}{|x|^{N-2}} + d_2 = p(x)$ where d_1, d_2 satisfy the following linear system:

$$\frac{d_1}{(R-1)^{N-2}} + d_2 = \frac{Me^{\alpha\tau}}{(1+(R-1)^2)^\beta}; \quad (28)$$

$$\frac{d_1}{R^{N-2}} + d_2 = 0 \quad (29)$$

Using (26), (28) and (29) we obtain

$$\begin{aligned} \Delta p &= 0, & R-1 < |x| < R; \\ p(x) &\geq f_n(x, \tau) = 0, & R-1 < |x| < R; \\ p(x) &\geq f_n(x, t), & |x| = R-1, 0 < t < \tau; \\ p(x) &= f_n(x, t) = 0, & |x| = R, 0 < t < \tau \end{aligned}$$

The formula for $p(x, t)$ implies that

$$\left| \frac{\partial p}{\partial \nu} \right|_{|x|=R} = \frac{d_1(N-2)}{R^{N-1}} \leq C_6 R^{-2\beta}.$$

From (25), (27) and the above inequality we find that

$$J_n \leq C_7 R^{N+3-2\lambda+2\delta-2\beta}, \quad (30)$$

where $C_7 = C_7(\lambda, N, \omega, R_0, \delta, \tau)$. By setting $p(x) = d_1 \ln |x| + d_2$ for $N = 2$ and $p(x) = d_1 |x| + d_2$ for $N = 1$ we arrive at (30) in these cases as well.

Choose $\beta > -\lambda + \delta + (N + 3)/2$, sufficiently large $R > R_0 + 1$ and sufficiently small $h > 0$ to obtain $J_{nh} < \varepsilon$. The proof of Theorem 2.3 is complete. \square

Proof of Theorem 2.4. The proof is similar to the one of the Theorem 2.3. The main difficulty is to obtain a suitable upper bound on $f_n(x, t)$.

Fix an arbitrary $\delta > 0$. Let $K_{i\delta} = K + i\delta$, $i = 1, 2$ and define

$$\tau = \tau(\delta) = \left(2K_{2\delta}(2 + 3(\mu + 2\delta))\right)^{-1}.$$

It is enough to prove uniqueness for $t = \tau$. Proceed as in the proof of Theorem 2.3. Notice that for all $t \in [0, \tau]$

$$0 \leq a(x, t) \leq \varphi'(\psi(K_\delta |x|^2 + C_\delta)), \quad \text{a.e. in } x \in \mathbb{R}^N. \quad (31)$$

The lemma below is a modification of Lemma 4.1 and can be proven similarly.

Lemma 4.2. *For any $\varepsilon > 0$ there exists a sequence of positive functions $\{a_n\}$ such that*

$$\begin{aligned} a_n &\in C^\infty(\mathbb{R}^N \times [0, \tau]); \\ a_n(x, t) &\leq \varphi'(\psi(K_\delta |x|^2 + C_8)), \quad (x, t) \in \mathbb{R}^N \times [0, \tau], \end{aligned} \quad (32)$$

for some $C_8 = C_8(\delta)$ and $I_n < \varepsilon$ for all $n \in \mathbb{N}$.

In the same way as in the proof of Theorem 2.3 we arrive at the inequality (21). The condition $\mu < +\infty$ implies that for any $\delta > 0$ there exists $s_1 = s_1(\delta) > 1$ such that

$$\varphi''(s) \leq (\mu + \delta) \frac{\varphi'(s)}{s} \quad \text{for all } s \geq s_1. \quad (33)$$

Integrating the above inequality over $[s_1, s]$ and substituting $s = \psi(r)$ we obtain

$$\varphi'(\psi(r)) \leq (\mu + \delta)r + C_9, \quad r \geq 0, \quad C_9 = C_9(\delta). \quad (34)$$

Using (31), growth condition (10) and (34) we find that

$$\begin{aligned} |\varphi(u) - \varphi(v)| &= a(x, t)|u(x, t) - v(x, t)| \\ &\leq \varphi'(\psi(K_\delta|x|^2 + C_\delta))\psi(K_\delta|x|^2 + C_\delta) \\ &\leq ((\mu + \delta)K_\delta|x|^2 + C_{10})\psi(K_\delta|x|^2 + C_\delta), \end{aligned} \quad (35)$$

where C_{10} depends on δ .

We construct an upper bound on $f_n(x, t)$. Define

$$q(x, t) = M \int_{r(x, t)}^{+\infty} \zeta^{-\frac{N}{2}} \psi^{-\beta}(\zeta) d\zeta,$$

where $M > 0$, $\beta > 1 + \frac{3}{2}(\mu + 2\delta)$ and

$$r(x, t) = \frac{K_\delta|x|^2 + C_8}{1 - 4\beta K_\delta t}.$$

It is easy to check that

$$\int_1^{+\infty} \zeta^{-\frac{N}{2}} \psi^{-\beta}(\zeta) d\zeta < +\infty. \quad (36)$$

Indeed, by (33)

$$\varphi'(s) \leq C_{11}s^{\mu+\delta} \quad \text{for all } s \geq s_1$$

and therefore

$$\Psi(s) = \int_1^s \frac{\varphi'(r)}{r} dr \leq C_{12}s^{\mu+\delta} \quad s \geq s_1.$$

Taking the inverse we obtain

$$\psi(r) \geq C_{13}r^{\frac{1}{\mu+\delta}}, \quad r \geq \Psi(s_1),$$

which implies (36). Notice that $\tau < (4\beta K_\delta)^{-1}$ and therefore $q(x, t) > 0$ for all $(x, t) \in \mathbb{R}^N \times [0, \tau]$. Therefore we can find large enough M , independent of R and such that

$$q(x, \tau) \geq \omega(x) = f_n(x, t), \quad |x| \leq R.$$

By (19) we also have that

$$q(x, t) \geq f_n(x, t) = 0, \quad |x| = R, \quad 0 \leq t \leq \tau.$$

Since $r = r(x, t)$ satisfies the relations

$$r_t \geq \beta |\nabla r|^2 \quad \text{and} \quad \Delta r = \frac{N}{2} \frac{|\nabla r|^2}{r},$$

we compute that

$$\begin{aligned} Lq &= q_t + a_n \Delta q \\ &\leq \frac{M}{r^{\frac{N}{2}} \psi^\beta(r)} \left(\beta |\nabla r|^2 \frac{a_n \psi'(r)}{\psi(r)} - r_t + a_n \frac{N}{2} \frac{|\nabla r|^2}{r} - a_n \Delta r \right) \\ &\leq \frac{M}{r^{\frac{N}{2}} \psi^\beta(r)} \left(\beta |\nabla r|^2 \frac{\varphi'(\psi(K_\delta |x|^2 + C_8))}{\varphi'(\psi(r))} - r_t + a_n \frac{N}{2} \frac{|\nabla r|^2}{r} - a_n \Delta r \right) \leq 0 \end{aligned}$$

and conclude from the maximum principle that

$$f_n(x, t) \leq q(x, t) \quad \text{for all } t \in [0, \tau], \quad |x| \leq R.$$

In the same way as in the proof of Theorem 2.3 we derive the inequality

$$\left| \frac{\partial f_n}{\partial \nu} \right|_{|x|=R} \leq C_{14} q(x, t) \Big|_{|x|=R-1}.$$

Hence by (21), (35) and our choice of β

$$\begin{aligned}
J_n &\leq C_{15}R^{N-1}((\mu + \delta)K_\delta|x|^2 + C_{10})\psi(K_\delta|x|^2 + C_\delta)q(x, t) \Big|_{|x|=R-1} \\
&\leq C_{16}R^{N-1} \int_{r(x,0)}^{+\infty} \zeta^{1-\frac{N}{2}} \psi^{1-\beta}(\zeta) d\zeta \Big|_{|x|=R-1} \\
&\leq C_{16}R^{N-1} \int_{r(x,0)}^{+\infty} \zeta^{1-\frac{N}{2} + \frac{1-\beta}{\mu+\delta}} d\zeta \Big|_{|x|=R-1} \\
&\leq C_{17}R^{3+2\frac{1-\beta}{\mu+\delta}} \rightarrow 0 \quad \text{as } R \rightarrow \infty.
\end{aligned}$$

□

5. Examples and further discussion

The results presented in Theorems 2.1 and 2.2 are sharp for the following radial explicit solutions:

Example 2. For any $T > 0$ and $H \in \mathbb{R}$ the function

$$u(x, t) = (4\pi(T - t))^{-\frac{N}{2}} \exp\left(\frac{|x|^2}{4(T - t)} + H\right)$$

is a solution of the heat equation with the existence time T . Since $\mu = 0$ and $\sigma = (4T)^{-1}$, we obtain that $T_2 = T$.

Example 3. Let $\varphi(s) = s^m$, $m > 1$. Then φ satisfies the conditions of Theorem 2.1 and 2.2. The function

$$u(x, t) = \frac{1}{(T - t)^\kappa} \left(\frac{\kappa(m - 1)|x|^2}{2mN(T - t)^{\frac{2\kappa}{N}}} + H \right)_+^{\frac{1}{m-1}},$$

where $T > 0$, $H \in \mathbb{R}$, $\kappa = (m - 1 + 2/N)^{-1}$ and $(y)_+ = \max\{0, y\}$ is a weak solution of the problem (1), (2) in the maximum domain S_T . In this case $\sigma = \gamma/(m - 1) = (2T(N(m - 1) + 2))^{-1}$ and $T_1 = T_2 = T$.

Example 4. Let $\varphi(s) = \exp(ms)$, $m > 0$. Then φ satisfies the condition a) of Theorem 2.1 with $\lambda = 1$. The function

$$u(x, t) = \frac{1}{m} \ln \left(\frac{|x|^2 + H}{2mN(T-t)} \right), \quad T > 0, \quad H \geq 2mNT$$

is a solution of the problem (1), (2) in the maximum domain S_T . In this case $\gamma = (2NT)^{-1}$ and $T_1 = T$.

An interesting question is whether one really needs two theorems or, more precisely, two different descriptions of the growth conditions on the initial data to guarantee the existence of a solution in some S_T . Here is a heuristic explanation of why it seems natural to distinguish between the case when φ grows quickly and the case when φ grows slowly as $s \rightarrow +\infty$ (but still satisfies (A2)). For the sake of the argument let us assume that

$$\int_0^1 \frac{\varphi'(s)}{s} ds < +\infty$$

and define slightly modified versions of Ψ and ψ by

$$\begin{aligned} \tilde{\Psi}(s) &= \int_0^s \frac{\varphi'(\zeta)}{\zeta} d\zeta, \quad s \geq 0 \\ \tilde{\psi} &= \tilde{\Psi}^{-1}. \end{aligned}$$

Under the change of variables $w = \tilde{\Psi}(u)$ the equation (1) takes the form

$$w_t = \varphi'(\tilde{\psi}(w))\Delta w + |\nabla w|^2. \quad (37)$$

If φ grows slowly (say, $\varphi(s) \sim s \ln^p s$, $p \geq -1$) then the major contribution to the right hand side of (37) comes from the second term. It is easy to check that the function

$$z(x, t) = \frac{|x|^2}{4(\tau - t)} + H, \quad (x, t) \in S_\tau,$$

for any $H > 0$ and $\tau > 0$ solves the equation $z_t = |\nabla z|^2$, $(x, t) \in S_\tau$, and also satisfies the inequality $z_t \leq \varphi'(\tilde{\psi}(z))\Delta z + |\nabla z|^2$. Returning to the original notation and solving (1) with the initial condition

$$u_0(x) = \tilde{\psi}(z(x, 0)) = \tilde{\psi}\left(\frac{|x|^2}{4\tau} + H\right) \quad (38)$$

we expect the solution $u(x, t)$ (if it exists) to blow up not later than at $t = \tau$. On the other hand by Theorem 2.2 the solution of (1) with initial function (38) exists in S_T for any $T < \tau_\mu = \frac{2\tau}{2+\mu N}$. For $\varphi(s) \sim s \ln^p s$, $p \geq -1$, mentioned above (and for any φ such that $\mu = 0$) we have $\tau_\mu = \tau_0 = \tau$. The above observation suggests that the blow up of the solution in this case is due to the quadratic term $|\nabla w|^2$. If we allow φ to grow faster, then the first term in the right-hand side of (37) will start to interfere. Indeed, for φ from the “intermediate” range, i.e. satisfying (9), both terms are comparable when applied to $z(x, t)$. For fast growing φ (say, $\varphi(s) \sim \exp s$) the first term takes the lead and therefore requires us to reconsider our conditions on initial data, which is done in Theorem 2.1.

A. Auxiliary lemmas

Lemma A.1. *Let φ satisfy (A1) and (A2). Then $\mu \geq 0$. If, in addition, $\varphi''(s) > 0$ for all $s \geq s_0$ then $\lambda \leq 1$.*

Proof. Assume that $\mu < 0$. Then (6) implies that for some $\varepsilon \in (0, -\mu)$ and all $s \geq s_1 = s_1(\varepsilon) > 0$

$$\frac{s\varphi''(s)}{\varphi'(s)} < -\varepsilon.$$

Therefore

$$0 < \int_{s_1}^{+\infty} \frac{\varphi'(s)}{s} ds < - \int_{s_1}^{+\infty} \frac{\varphi''(s)}{\varepsilon} ds = - \frac{\varphi'(+\infty) - \varphi'(s_1)}{\varepsilon}. \quad (39)$$

From (39) and (A1) we find that $0 < \varphi'(+\infty) < \varphi'(s_1) < +\infty$ and consequently

$$\int_{s_1}^{+\infty} \frac{\varphi'(s)}{s} ds < +\infty,$$

which contradicts (A2).

Let $\varphi''(s) > 0$ for $s \geq s_0$. Assume that $\lambda > 1$. Then (see (5)) for some $\varepsilon > 0$ and all $s > s_2 = s_2(\varepsilon) \geq s_0$ we have

$$\frac{\varphi'(s)\varphi'''(s)}{(\varphi''(s))^2} > 1 + \varepsilon.$$

rewrite this inequality in the following way:

$$\frac{\varphi'''(s)}{\varphi''(s)} > (1 + \varepsilon) \frac{\varphi''(s)}{\varphi'(s)}, \quad s > s_2.$$

Integrating the above inequality twice over the interval $[s_2, s]$, $s > s_2$, we obtain

$$-\frac{1}{\varepsilon}(\varphi'(s))^{-\varepsilon} > c_1(s - s_2) - c_2 \quad \text{for all } s > s_2. \quad (40)$$

Since $\varphi'(s) > 0$ for $s > 0$ the previous line yields $\varphi'(s_2 + \frac{c_2}{c_1} - 0) = +\infty$. This conclusion contradicts the assumption (A1).

The following technical lemma was proven in [9].

Lemma A.2. *Let $w \in L_{loc}^\infty(\mathbb{R}^N)$, $b \in C(\mathbb{R}^N)$, $b \geq \frac{1}{k_0} > 0$, $k_0 \in \mathbb{N}$, and $0 \leq w \leq b$ a.e.. Then there exists a sequence of positive functions $w_k \in L^\infty(B_k)$, $k > k_0$, which possesses the following properties:*

1) $w_{k+1} \leq w_k$ a.e. in B_k , $w_k \rightarrow w$ as $k \rightarrow +\infty$ a.e. in B_n ($n \in \mathbb{N}$), and

$$\frac{1}{k} \leq w_k \leq b \quad \text{a.e. in } B_k; \quad (41)$$

2) for any $k > k_0$ there exists a sequence of positive functions $w_{kj} \in C(B_k)$, $j > k$, which satisfies the following conditions:

$$\frac{1}{k} \leq w_{kj} \leq b; \quad w_{(k+1)j} \leq w_{kj} \leq w_{k(j+1)} \quad \text{in } B_k; \quad (42)$$

$$w_{kj}(x) = b(x) \quad \text{on } |x| = k;$$

$$w_{kj} \rightarrow w_k \quad \text{as } j \rightarrow +\infty \quad \text{a.e. in } B_{k-1}.$$

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REFERENCES

- [1] D. G. Aronson, *Widder's inversion theorem and the initial distribution problems*, SIAM J. Math. Anal., 12, no. 4, 1981, 639–651.
- [2] D. G. Aronson and L. A. Caffarelli, *The initial trace of a solution of the porous medium equation*, Trans. Amer. Math. Soc., 280, no. 1, 1983, 351–366.
- [3] P. Bénilan, M. G. Crandall, and M. Pierre, *Solutions of the porous medium equation in \mathbf{R}^N under optimal conditions on initial values*, Indiana Univ. Math. J., 33, no. 1, 1984, 51–87.
- [4] L. Collatz, *Functional analysis and numerical mathematics*, Academic Press, New York, 1966. Translated from the German by Hansjörg Oser.

- [5] B. E. J. Dahlberg and C. E. Kenig, *Nonnegative solutions of the porous medium equation*, Comm. Partial Differential Equations, 9, no. 5, 1984, 409–437.
- [6] B. E. J. Dahlberg and C. E. Kenig, *Nonnegative solutions of generalized porous medium equations*, Rev. Mat. Iberoamericana, 2, no. 3, 1986, 267–305.
- [7] B. E. J. Dahlberg and C. E. Kenig, *Nonnegative solutions to fast diffusions*, Rev. Mat. Iberoamericana, 4, no. 1, 1988, 11–29.
- [8] A. L. Gladkov, *On the solvability of the Cauchy problem with an initial function that arbitrarily increases at infinity for some degenerate quasilinear parabolic equations*, Differential'nye Uravneniya, 27, no. 2, 1991, 243–250. Translation in Differential Equations 27, no. 2, 1991, 170–176.
- [9] A. L. Gladkov, *The Cauchy problem for some degenerate quasilinear parabolic equations with absorption*, Sibirsk. Mat. Zh., 34, no. 1, 1993, 47–64. Translation in Siberian Math. J. 34, no. 1, 1993, 37–54.
- [10] M. A. Herrero and M. Pierre, *The Cauchy problem for $u_t = \Delta u^m$ when $0 < m < 1$* , Trans. Amer. Math. Soc., 291, no. 1, 1985, 145–158.
- [11] A. S. Kalashnikov, *The Cauchy problem in a class of growing functions for equations of unsteady filtration type*, Vestnik Moskov. Univ. Ser. I. Mat. Mekh. (Russian), , no. 6, 1963, 17–27.
- [12] A. S. Kalashnikov, *Some problems of the qualitative theory of second-order nonlinear degenerate parabolic equations*, Uspekhi Mat. Nauk, 42, no. 2, 1987, 135–176.

- [13] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'tseva, *Linear and quasilinear equations of parabolic type*, American Mathematical Society, Providence, R.I., 1967. Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23.
- [14] B. Song, *The Cauchy problem for $u_t = \Delta\varphi(u)$ and the condition $1/\varphi^{-1} \in L^1$* , 1992. Preprint.