Homogenization of stochastic Hamilton-Jacobi equations: brief review of methods and applications

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Abstract

Homogenization problems for stochastic partial differential equations recently received a lot of attention. This paper gives a short review of two homogenization methods for Hamilton-Jacobi type equations in stationary ergodic random media. It also discusses the connection between the homogenization for a viscous Hamilton-Jacobi equation with a quadratic in momenta Hamiltonian and large deviations of diffusions in a random potential.

1 Introduction

We consider a family of functions $u_\varepsilon(t,x,\omega)$, $\varepsilon>0$, such that each u_ε is the viscosity solution of the Cauchy problem for the following Hamilton-Jacobi type equation

$$\frac{\partial u_\varepsilon}{\partial t}=\varepsilon\sigma^2\Delta_x u_\varepsilon+H\left(\frac{x}{\varepsilon},\nabla_x u_\varepsilon,\omega\right),\quad (t,x)\in[0,\infty)\times\mathbb{R}^d, \tag{1}$$

with the initial condition $u_\varepsilon(0,x,\omega)=f(x)$, where σ is a constant, f is a uniformly continuous non-random function on $\mathbb{R}^d$, and ω is an element of probability space Ω. When $\sigma>0$ we refer to (1) as a viscous Hamilton-Jacobi equation. We assume that $H(x,p,\omega)$ is a stationary ergodic random process in x. A more detailed description is given in Section 3.

We are interested in the behavior of $u_\varepsilon(t,x,\omega)$ as $\varepsilon\to 0$. When ε is small we can consider x and x/ε as two spatial variables that are on widely separated scales. A small change in x results in a huge fluctuation of x/ε and, as a consequence, of the Hamiltonian $H(\cdot,p,\omega)$. Due to numerous cancellations the averaging occurs, and u_ε converges uniformly on compact subsets of $[0,\infty)\times\mathbb{R}^d$ to the deterministic function $u(t,x)$.

This scenario is known to work ([22], [10], [16]) in the case when $H(x,p)$ is periodic in x. Moreover, $u(t,x)$ is shown to be the viscosity solution of the *effective equation*

$$\frac{\partial u}{\partial t} = \overline{H}(\nabla_x u), \quad (t,x)\in[0,\infty)\times\mathbb{R}^d, \tag{2}$$

with the same initial condition f as in (1). The effective Hamiltonian $\overline{H}(p)$ depends only on p and is non-random.

It was shown in [27] ($\sigma = 0$) and [24], [20] ($\sigma > 0$) that $u_\varepsilon(t,x,\omega)\to u(t,x)$ for a.e. ω uniformly on compact sets of $[0,\infty)\times\mathbb{R}^d$. Paper [25] ($\sigma = 0$) establishes the locally uniform convergence of u_ε to u in $L^1(\Omega)$. Moreover, [25] contains the central limit theorem for this convergence in several special cases. Main assumptions on the Hamiltonian are the convexity and superlinearity in p as well as some regularity in p and x. The precise conditions on the Hamiltonian and the initial data vary from paper to paper, and we shall not state them here.

The paper is organized as follows. In the next section we discuss (1) with periodic Hamitonians. Section 3 is devoted to basic definitions and examples for the stochastic case. Then we restrict our attention to convex Hamiltonians and review known variational representations for solutions of (1) in Section 4. Sections 5 and 6 present two homogenization methods for (1). We finish the paper with an application of homogenization results to large deviations for diffusions in random media.

2 Periodic Hamilton-Jacobi equations

Since the stationary ergodic case is a natural generalization of the periodic case, we start with a short discussion of the standard approach for periodic Hamiltonians and point out some obstacles, which do not allow to extend this method from periodic to random Hamiltonians. More details can be found in [22], [10], [16], [24], [23], [4]. For a broader view of periodic and random homogenization the reader is referred to monographs [2], [17], and [3].

Assume that the Hamiltonian $H(x,p)$ is 1-periodic in each x_i, $i=1,2,\dots,d$. Consider the formal expansion

$$u_\varepsilon(t,x) = u(t,x) + \varepsilon u_1(t,x,x/\varepsilon) + O(\varepsilon^2) \tag{3}$$

and substitute it into (1). Equate the terms of order zero with respect to ε and denote x/ε by y. We get the equation

$$\frac{\partial u}{\partial t}(t,x) = \sigma^2\Delta_y u_1(t,x,y) + H(y,\nabla_x u(t,x) + \nabla_y u_1(t,x,y)).$$

Fixing (t,x), treating y as an independent variable, and setting

$$P = \nabla_x u(t,x), \quad \lambda = \frac{\partial u}{\partial t}(t,x), \quad \text{and} \quad U(y) = u_1(t,x,y),$$

we arrive at the following problem: for each $P \in \mathbb{R}^d$ find a constant $\lambda \in \mathbb{R}$ and a function $U(y)$, which is 1-periodic in each y_i, $i = 1, 2, \ldots, d$, and satisfies the equation

$$\lambda = \sigma^2 \Delta U(y) + H(y, P + \nabla U(y)), \quad y \in \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d. \tag{4}$$

Since (4) is considered on the torus, i.e. periodic cell, this problem is often referred to as a *cell problem.* Function $U(y)$ is called a *corrector.*

For the rest of this section we assume that $\sigma = 0$. At first, we state a version of the classical result about the existence of solutions to the cell problem ([22]; [10], Lemma 2.1). Case $\sigma > 0$ is considered in [13], Theorem 5.

Theorem 2.1. *Let $H(x,p)$ be 1-periodic in each x_i, $i = 1, 2, \ldots, d$, Lipschitz continuous on $[0,1]^d \times \{p \in \mathbb{R}^d : |p| \leqslant R\}$ for each $R > 0$, and*

$$\lim_{|p|\to\infty} H(x,p) = +\infty \quad \textit{uniformly in } x.$$

Then for each $P \in \mathbb{R}^d$ there exists a unique constant $\lambda = \lambda(P)$ and a periodic Lipschitz continuous viscosity solution U of the cell problem (4).

Let us remark that the computation of the effective Hamiltonian, in general, is a very difficult task. If $H(x,p)$ is convex in p then there are variational formulas for $\overline{H}$ ([5], [6]; [11] and references therein). In particular, the following minimax formula holds ([7], [13]):

$$\overline{H}(P) = \min_{U \in C^2(\mathbb{T}^d)} \max_{y \in \mathbb{T}^d} (H(y, P + \nabla U(y)) + \sigma^2 \Delta U(y)). \tag{5}$$

Paper [14] gives numerical computations of $\overline{H}$ for several cases in dimension 2.

If for each P there is a unique $\lambda(P)$ and a corrector $U(y)$, then we immediately get homogenization for affine initial data $f(x) = a + P \cdot x$ with the effective Hamiltonian $\overline{H}(p) = \lambda(p)$.

Indeed, let $f(x) = a + P \cdot x$. Observe that $u(t,x) = a + P \cdot x + t\lambda(P)$ is a viscosity solution of (2) with $\overline{H}(P) = \lambda(P)$. Consider

$$\hat{u}_\varepsilon(t,x) = u(t,x) + \varepsilon U(x/\varepsilon).$$

It is easy to check that $\hat{u}_\varepsilon(t,x)$ solves (1) with the initial data

$$\hat{f}(x) = a + P \cdot x + \varepsilon U(x/\varepsilon), \text{ and}$$

$$\sup_{(t,x)\in[0,\infty)\times\mathbb{R}^d} |\hat{u}_\varepsilon(t,x) - u(t,x)| = \varepsilon \sup_{y \in \mathbb{T}^d} |U(y)|.$$

The contraction property of Hamilton-Jacobi semigroup implies that

$$\sup_{(t,x)\in[0,\infty)\times\mathbb{R}^d} |\hat{u}_\varepsilon(t,x) - u_\varepsilon(t,x)| \leqslant \sup_{x\in\mathbb{R}^d} |f(x) - \hat{f}(x)| = \varepsilon \sup_{y\in\mathbb{T}^d} |U(y)|.$$

Therefore,

$$\sup_{(t,x)\in[0,\infty)\times\mathbb{R}^d} |u_\varepsilon(t,x) - u(t,x)| \leqslant 2\varepsilon \sup_{y\in\mathbb{T}^d} |U(y)| \leqslant C\varepsilon.$$

This establishes the homogenization for affine initial data and identifies the effective Hamiltonian $\overline{H}(P)$ as the unique constant $\lambda(P)$ in the cell problem.

The homogenization for general initial data can be obtained either using properties of Hamilton-Jacobi semigroup ([22], [16], $\sigma = 0$) or the perturbed test function method ([10], $\sigma \geqslant 0$).

If $H(x, p)$ is not periodic then we face several problems. The most important problem is the lack of compactness, as we need to solve (4) not on the torus but on $\mathbb{R}^d$. Moreover, λ may not be unique, and the corrector U (when it exists) need not be bounded. Notice that our discussion of homogenization for affine initial data suggests that we need $\varepsilon U(y/\varepsilon) \to 0$ as $\varepsilon \to 0$. In a non-periodic case, it is natural to impose the following growth condition at infinity

$$\lim_{|y| \to \infty} |y|^{-1} U(y) = 0. \tag{6}$$

It is easy to see that this condition is necessary for the uniqueness of λ ([27], [23]): the cell problem for $H(p, x) = |p|^\alpha$, $\alpha \geqslant 1$, admits infinitely many solutions

$$\lambda(P) = |P + Q|^\alpha, \quad U(y) = Q \cdot y + a, \quad Q \in \mathbb{R}^d,$$

while, clearly, the one which corresponds to $\overline{H}(p) = |p|^\alpha$, is $\lambda(P) = |P|^\alpha$ with $U(y) \equiv a$.

On the other hand, the existence for each P of a Lipschitz continuous corrector with a sublinear growth at infinity guarantees the uniqueness of λ ([23], Proposition 1.2, [25], Theorem 4.1) and implies the homogenization on $\mathbb{R}^d$ for all uniformly continuous initial data and very general Hamiltonians $H(x, p)$ ([25], Theorem 4.1).

For more information about of correctors for (1) in the stochastic setting see [23] and [24], Section 8.

3 Definitions and examples

Random environment. Stationary ergodic random medium is modelled as a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, on which $\mathbb{R}^d$ acts as a group of measure preserving transformations $\tau_x : \Omega \to \Omega$, $x \in \mathbb{R}^d$, so that $\mathbb{P}(A) = \mathbb{P}(\tau_x A)$ for all $x \in \mathbb{R}^d$ and $A \in \mathcal{F}$. Probability measure $\mathbb{P}$ is assumed to be ergodic under translations, i.e. the probability of every set $A \in \mathcal{F}$ that is invariant under all τ_x, $x \in \mathbb{R}^d$, is equal to zero or one. We assume also that the map $(x, \omega) \mapsto \tau_x \omega$ is jointly measurable on $\mathbb{R}^d \times \Omega$.

Example 1. (Periodic environment.) Periodic case can be considered as a particular example of a stationary random case with $\Omega = \mathbb{T}^d$, $\mathcal{F}$ is the Borel σ-algebra on $\mathbb{T}^d$, $\mathbb{P}$ the Lebesgue measure, and $\tau_x \omega = \omega + x \,(\mathrm{mod}\ 1)$, $x \in \mathbb{R}^d$.

Quasi-periodic and almost periodic cases can also be embedded in the stationary ergodic setting.

Example 2. (Poissonian environment (see, for example, [28]).) Ω is the set of locally finite simple point measures on $\mathbb{R}^d$, i.e. $\omega(\{x\}) \in \{0, 1\}$ for all $x \in \mathbb{R}^d$,

$\omega = \sum_i \delta_{x_i},\ x_i \in \mathbb{R}^d,\ \omega(K) < \infty$ for any compact $K \subset \mathbb{R}^d$. For $x \in \mathbb{R}^d$ and $\omega \in \Omega$ we define $\tau_x\omega = \sum_i \delta_{x_i - x}$.

$\mathcal{F}$ is generated by applications: $\omega \to \omega(A) \in \mathbb{N} \cup \{+\infty\},\ A \in \mathcal{B}(\mathbb{R}^d)$, where $\mathcal{B}(\mathbb{R}^d)$ is Borel σ-algebra on $\mathbb{R}^d$.

$\mathbb{P}$ is Poisson law with intensity ν: for every $A \in \mathcal{B}(\mathbb{R}^d)$ the random variable $\omega(A)$ has Poisson distribution with parameter $\nu|A|$, where $|A|$ is the d-dimensional volume of A. Moreover, for each $m \in \mathbb{N}$ and disjoint sets $A_i,\ i = 1, 2, \dots, m,\ |A_i| < \infty$, the random variables $\omega(A_i),\ i = 1, 2, \dots, m$ are independent.

Example 3. (Random chessboard.) We shall consider $\mathbb{R}^2$ but the model has an obvious generalization to $\mathbb{R}^d$.

Put two independent one dimensional Poissonian environments (see Example 2) on coordinate axes x and y respectively and independently toss a fair coin. Realization of a pair of such environments gives collections of points $\{x_i\}$ and $\{y_i\}$ on coordinate axes x and y. Lines $x = x_i$ and $y = y_i$ produce a grid in $\mathbb{R}^2$ with a random cell size. A coin toss determines the color of the cell containing the origin. The color of other cells is determined by the chessboard pattern. This procedure gives a realization of a random chessboard environment. For a more formal description see, for example, [8].

Diffusion processes on $\mathbb{R}^d$ with a unique invariant distribution (such as the Ornstein-Uhlenbeck process) can be used to construct other examples of stationary ergodic environments.

Hamiltonian. Define $H(p, x, \omega) = \tilde{H}(p, \tau_x\omega)$, where $\tilde{H}(p, \omega)$ is a continuous function of $p \in \mathbb{R}^d$ for each ω and

$$\lim_{|p|\to\infty} \tilde{H}(p, \omega) = +\infty \tag{7}$$

uniformly in ω. We also assume that $\tilde{H}$ behaves regularly under the shifts, i.e. for every $R > 0$

$$\lim_{\delta\to 0} \sup_{\omega\in\Omega} \sup_{|x|\leqslant\delta} \sup_{|p|\leqslant R} |\tilde{H}(p, \tau_x\omega) - \tilde{H}(p, \omega)| = 0. \tag{8}$$

For homogenization results we shall need further assumptions on H, which vary depending on the method and can be found in references given in an appropriate section.

Example 4. One of the most important examples is the case when $\tilde{H}(p, \omega)$ is quadratic in p, in particular,

$$\tilde{H}(p, \omega) = \frac{1}{2}|p|^2 + b(\omega) \cdot p - V(\omega), \tag{9}$$

where b and V are bounded and continuous under the shifts. Homogenization problem for this type of Hamiltonians is closely related to quenched large deviations for diffusions in random media ($\sigma > 0$), which we discuss in the last section.

4 Convex case and variational formulas

From now on we assume that $\tilde{H}(p,\omega)$ is convex in p. Moreover, we suppose that there are constants $1 < \alpha \leqslant \beta$ and $c_1, c_2 > 0$ such that for all $p \in \mathbb{R}^d$ and $\omega \in \Omega$

$$c_1(|p|^\alpha - 1) \leqslant \tilde{H}(p,\omega) \leqslant c_2(|p|^\beta + 1). \tag{10}$$

These growth assumptions can be relaxed ([24]) but here we shall focus our attention on Hamiltonians, which satisfy (10).

Let $\tilde{L}(q,\omega)$ be the Legendre transform of $\tilde{H}(p,\omega)$,

$$\tilde{L}(q,\omega) = \sup_{p\in\mathbb{R}^d}(p\cdot q - \tilde{H}(p,\omega)), \quad L(x,q,\omega) = \tilde{L}(q,\tau_x\omega). \tag{11}$$

Then $\tilde{L}(q,\omega)$ is also locally uniformly continuous under the shifts (see (8)) and satisfies the convexity and growth assumptions similar to (10) with exponents $1 < \beta' \leqslant \alpha'$, $\alpha' = (\alpha-1)^{-1}\alpha$, $\beta' = (\beta-1)^{-1}\beta$, namely, for some positive c_3 and c_4, all $q \in \mathbb{R}^d$ and $\omega \in \Omega$

$$c_3(|q|^{\beta'} - 1) \leqslant \tilde{L}(q,\omega) \leqslant c_4(|q|^{\alpha'} + 1). \tag{12}$$

The convexity assumption is crucial for both methods that we discuss here as it allows one to use the variational representation of solutions of (1). Let us stress that in the periodic case the convexity of Hamiltonian plays no role. But to the best of my knowledge all currently available homogenization techniques for the stochastic equation (1) depend on the convexity assumption as they rely on variational formulas for the solutions. One should mention that a recent work [4] establishes stochastic homogenization of fully non-linear uniformly elliptic equations, whose solutions do not have representation formulas.

Rescaling and shifts. Let $v(t,x,\omega)$ be a (viscosity) solution of

$$\frac{\partial v}{\partial t} = \sigma^2\Delta_x v + H(x,\nabla_x v,\omega), \quad (t,x) \in [0,\infty)\times\mathbb{R}^d, \tag{13}$$

with the initial condition $g(x)$, and for a fixed $y \in \mathbb{R}^d$ let $v^y(t,x,\omega)$ be a (viscosity) solution of (13) with the initial condition $g(x+y)$. The uniqueness of (viscosity) solutions implies the following simple relation

$$v(t,x+y,\omega) = v^y(t,x,\tau_y\omega).$$

In particular,

$$v(t,y,\omega) = v^y(t,0,\tau_y\omega). \tag{14}$$

Homogenization problem for (1) can be restated as a scaling limit problem, namely,

$$u_\varepsilon(t,x,\omega) = \varepsilon v_\varepsilon(\varepsilon^{-1}t,\varepsilon^{-1}x,\omega),$$

where $v_\varepsilon(t,x,\omega)$ solves the ε-independent equation (13) with the initial condition $v_\varepsilon(0,x,\omega) = \varepsilon^{-1}f(\varepsilon x)$. When $f(x) = p\cdot x$ functions v_ε do not depend on ε. We are interested in the behavior of v_ε under the hyperbolic scaling of the time and space.

Property (14) leads to the equation

$$u_\varepsilon(t,x,\omega) = u_\varepsilon^y(t,0,\tau_{\varepsilon^{-1}y}\omega), \tag{15}$$

where u_ε^y solves (1) with the initial condition $f(x+y)$. If we were interested, say, only in the convergence in probability or in $L^1(\mathbb{P})$, we could use (15) to reduce the problem to the study of the behavior of $u_\varepsilon(t,0,\cdot)$.

Variational formula, case $\sigma = 0$. Denote by $\Pi_{s,x}^{t,y}$ a set of all Lipschitz paths $x(\cdot)$ on $[s,t]$ such that $x(s) = x$ and $x(t) = y$, and set

$$v(t,x,\omega) = \sup_{x(\cdot)\in\Pi_{0,x}} \left(g(x(t)) - \int_0^t L(x(s),x'(s),\omega)\,ds \right) \tag{16}$$

$$= \sup_{y\in\mathbb{R}^d} \left(g(y) - \inf_{x(\cdot)\in\Pi_{s,x}^{t,y}} \int_0^t L(x(s),x'(s),\omega)\,ds \right), \tag{17}$$

where $\Pi_{0,x}$ is the collection of Lipschitz paths with only one fixed end $x(0) = x$. Under certain assumptions on L (see [12]) function $v(t,x,\omega)$ is the viscosity solution of (13), $\sigma = 0$. Define

$$I(s,x,t,y,\omega) = \inf_{x(\cdot)\in\Pi_{s,x}^{t,y}} \int_s^t L(x(s),x'(s),\omega)\,ds. \tag{18}$$

$I(s,x,t,y,\omega)$ plays a special role, since, on the one hand, it completely defines the solution by relation (17), and, on the other hand, it has a subadditive property, which allows to pass to the limit after the rescaling. Moreover, for each fixed $(s,y)\in[0,\infty)\times\mathbb{R}^d$ it is a viscosity solution (see Remark 4.1) of (13) on $(s,\infty)\times\mathbb{R}^d$, and

$$I(s,x,s,y,\omega) = \begin{cases} 0, & \text{if } x = y \\ +\infty, & \text{if } x \neq y. \end{cases}$$

After the rescaling we get

$$u_\varepsilon(t,x,\omega) = \sup_{y\in\mathbb{R}^d} (f(y) - I_\varepsilon(0,x,t,y,\omega)), \tag{19}$$

where

$$I_\varepsilon(s,x,t,y,\omega) = \inf_{y(\cdot)\in\Pi_{s,x}^{t,y}} \int_s^t L\left(\frac{y(r)}{\varepsilon}, y'(r), \omega\right)\,dr. \tag{20}$$

Let us also remark that when the Hamiltonian depends only on p, as in the effective equation, we get the classical Hopf-Lax-Oleinik formula for solutions of (2)

$$u(t,x) = \sup_{y\in\mathbb{R}^d} \left(f(y) - t\overline{L}\left(\frac{y-x}{t}\right)\right). \tag{21}$$

Variational formula, case $\sigma > 0$. Let $\mathcal{C}$ be a set of all measurable maps $c : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ such that $\sup_{(s,x)\in[0,t]\times\mathbb{R}^d} |c(s,x)| < \infty$ for each $t > 0$. Consider the diffusion on $\mathbb{R}^d$

$$x(t) = x + \int_0^t c(s, x(s))\, ds + \sqrt{2}\sigma B(t),$$

where $B(\cdot)$ is the standard d-dimensional Brownian motion. Denote by Q_x^c the corresponding measure on the path space $C([0,\infty))$. Then the function

$$v(t, x, \omega) = \sup_{c\in\mathcal{C}} E^{Q_x^c} \left(g(x(t)) - \int_0^t L(x(s), c(s, x(s)), \omega)\, ds \right), \tag{22}$$

where $L(x, q, \omega)$ is given by (11), is the solution of (13) with the initial condition $g(x)$. By rescaling we get a representation formula for the solution of (1):

$$u_\varepsilon(t, x, \omega) = \sup_{c\in\mathcal{C}} E^{Q_{x/\varepsilon}^c} \left(f(\varepsilon x(t/\varepsilon)) - \varepsilon \int_0^{t/\varepsilon} L(x(s), c(s, x(s)), \omega)\, ds \right). \tag{23}$$

We can rescale the diffusion and write

$$u_\varepsilon(t, x, \omega) = \sup_{c\in\mathcal{C}} E^{Q_x^{\varepsilon,c}} \left(f(y(t)) - \int_0^t L\left(\frac{y(s)}{\varepsilon}, c\left(\frac{s}{\varepsilon}, \frac{y(s)}{\varepsilon}\right), \omega\right) ds \right), \tag{24}$$

where $Q_x^{\varepsilon,c}$ corresponds to the process $y(\cdot) = \varepsilon x(\cdot/\varepsilon)$:

$$y(t) = x + \int_0^t c\left(\frac{s}{\varepsilon}, \frac{y(s)}{\varepsilon}\right) ds + \sqrt{2\varepsilon}\sigma B(t).$$

Remark 4.1. Variational formulas stated above give viscosity solutions of the corresponding equations under some additional assumptions on H or L ([15], [12]). We do not state them here. Instead, we consider variational formulas (19) and (23) as the starting point of our analysis and concentrate on the study of the behavior of the quantities given by these formulas.

Effective Hamiltonian. There are several ways to characterize the effective Hamiltonian $\overline{H}$, but, as we mentioned before, even for periodic Hamiltonians the explicit or numeric calculation of the effective Hamiltonian is a challenging problem.

We start with a simple observation: function $u^P(t, x) = P\cdot x + t\overline{H}(P)$ satisfies

$$u_t = \overline{H}(\nabla_x u), \quad u(0, x) = P \cdot x.$$

If we assume that the homogenization result holds then we get the following limiting expression for $\overline{H}$: for every $P \in \mathbb{R}^d$

$$\overline{H}(P) = \lim_{\varepsilon\to 0} u_\varepsilon(1, 0, \omega) \quad \text{a.s.} \tag{25}$$

where $u_\varepsilon(t,x,\omega)$ is the solution of (1) with the initial condition $f(x) = P \cdot x$. Therefore, to completely characterize the effective Hamiltonian it is enough to deal just with linear initial data.

Let

$$\mathcal{B}_0 = \{\tilde{F} : \Omega \to \mathbb{R}^d \,|\, \tilde{F} \in L^\infty(\mathbb{P}),\ \mathbb{E}\tilde{F} = 0\}. \tag{26}$$

Set $F(x,\omega) = \tilde{F}(\tau_x\omega)$, then $F(\cdot,\omega) \in L^\infty_{\text{loc}}(\mathbb{R}^d)$ a.s.. Consider

$$\begin{aligned}\lambda(P,\tilde{F},\omega) &= \sup_{x\in\mathbb{R}^d} \big(H(x, P + F(x,\omega),\omega) + \sigma^2 \operatorname{div} F(x,\omega)\big)\\ &\overset{\text{def}}{=} \sup_{\varphi\in C_0^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d} H(x, P + F(x,\omega),\omega)\,\varphi(x) - \sigma^2 F(x,\omega)\cdot\nabla_x\varphi(x)\,dx.\end{aligned}$$

Clearly, λ is invariant under the shifts: $\lambda(P,\tilde{F},\tau_x\omega) = \lambda(P,\tilde{F},\omega)$ a.s.. Since $\mathbb{P}$ is ergodic, $\lambda(P,\tilde{F},\cdot)$ is constant a.s., and we can drop the dependence on ω.

Define

$$\overline{H}(P) = \inf_{\tilde{F}\in\mathcal{B}_0} \lambda(P,\tilde{F}) \text{ a.s..} \tag{27}$$

To emphasize the connection with the minimax formula for the periodic case (5) we can rewrite (27) in the following equivalent way ([24])

$$\begin{aligned}\overline{H}(P) &= \inf \sup_{x\in\mathbb{R}^d} \big(H(x, P + \nabla_x U(x,\omega),\omega) + \sigma^2\,\Delta_x U(x,\omega)\big)\\ &= \inf \operatorname*{ess\,sup}_{\omega\in\Omega} \Big(\tilde{H}(P + \tilde{F}(\omega),\omega) + \sigma^2 \operatorname{div}\tilde{F}(\omega)\Big) \quad \text{a.s.,}\end{aligned} \tag{28}$$

where the infimum is taken over all Lipschitz continuous functions $U : \mathbb{R}^d \times \Omega \to \mathbb{R}$, whose gradient $(\nabla_x U)(x,\omega) = \tilde{F}(\tau_x\omega)$ is stationary under the shifts and has mean zero. The supremum in the first line is understood in the viscosity sense.

See Section 6 for yet another formula the effective Hamiltonian (case $\sigma > 0$).

5 Method 1: Subadditive Ergodic Theorem

The main idea of this approach is to find a family of objects (for example, special solutions or sets), which, on the one hand, characterize or generate solutions u_ε, and on the other hand, have a subadditive property. Then some version of the subadditive ergodic theorem implies the existence of the limit for that family, and the generating property allows to extend the result to the whole collection u_ε.

This method is well-known and its applications are so numerous that we shall give only one very interesting reference [19], which summarized the developments of the subadditive ergodic theory and its first applications to various mathematical problems before 1973. For improvements of Kingman's theorem see, for example, [9], [21], and [1].

We shall describe this method for (1) with $\sigma = 0$ (see [27] and [25]). The case $\sigma > 0$ is more subtle, and the reader is referred to [24] (in particular, Section 6). Paper [24] establishes the homogenization result for even more general viscous

Hamilton-Jacobi equations than (1). The subadditive ergodic theorem is also used in [4] to obtain a stochastic homogenization for fully nonlinear uniformly elliptic and parabolic partial differential equations.

Without loss of generality we can assume that $\tilde{L}(q,\omega)$ is non-negative. Indeed, (12) implies that it is bounded below, and we can always add a constant to make it non-negative. In the end we shall be able to simply subtract this constant and return to $\tilde{L}$.

For each $[a,b) \subset \mathbb{R}$, $b > a$, and $q \in \mathbb{R}^d$ define a random set function

$$\mu_q([a,b),\omega) = I(a,0,b,bq,\omega),$$

where $I(s,x,t,y,\omega)$ was defined in (18). For every integer z set

$$\tau_z \mu_q([a,b),\omega) = \mu_q(z + [a,b),\omega).$$

Then making the change of variable in the integral and using the stationarity of L we get

$$\begin{aligned} \mu_q(z + [a,b),\omega) &= I(a+z,0,b+z,(b+z)q,\omega) \\ &= I(a,0,b,bq,\tau_{zq}\omega) = \mu_q([a,b),\tau_{zq}\omega). \end{aligned}$$

Obviously, for each interval $[a,b)$ the function $\mu_q([a,b),\omega)$ is $\mathcal{F}$-measurable. Moreover, μ_q has the following properties:

(1) (Subadditivity) $\mu_q([a,c),\omega) \leqslant \mu_q([a,b),\omega) + \mu_q([b,c),\omega)$ for all $a < b < c$ and $\omega \in \Omega$.

(2) (Regularity) $0 \leqslant \mu_q([a,b),\omega) \leqslant C(b-a)$, where $C = C(q,c_3,c_4,\alpha')$.

(3) (Stationarity) For each $[a,b)$, $a < b$, random variables $\tau_z \mu_q([a,b),\omega)$ and $\mu_q([a,b),\omega)$ have the same distribution for all $z \in \mathbb{Z}$.

A version of the subadditive ergodic theorem (see Theorem 2.7 in [1] and Proposition 1 in [8]) then implies that there exists an $\mathcal{F}$-measurable function $\mu(q,\omega)$ and a set of full measure $\Omega' \subset \Omega$ such that for every $\omega \in \Omega'$ and every interval $[a,b)$, $a < b$,

$$\lim_{t\to\infty} |t(b-a)| \mu_q(t[a,b),\omega) = \mu(q,\omega).$$

Using the regularity of μ_q and the ergodicity of $\mathbb{P}$ it is easy to show that $\mu(q,\omega)$ is non-random.

Next notice that $I_\varepsilon(0,0,t,y,\omega) = \varepsilon \mu_{y/t}([0,t/\varepsilon),\omega)$. Therefore, for each $\omega \in \Omega'$ we have the limit

$$\lim_{\varepsilon\to 0} I_\varepsilon(0,0,t,y,\omega) = t\mu(y/t).$$

The above conclusion can also be obtained using the classical Kingman's subadditive ergodic theorem [18] ([25], Lemma 3.3).

Let $u_\varepsilon(t,x,\omega)$ be the unique viscosity solution of (1) with $\sigma = 0$ and $f(x) = P \cdot x$, $P \in \mathbb{R}^d$. Then

$$u_\varepsilon(t,0,\omega) = \sup_{y\in\mathbb{R}^d} (P \cdot y - I_\varepsilon(0,0,t,y,\omega)).$$

From (12) and the definition of I_ε it is not difficult to see ([25], Lemma 3.4) that for all $t \in [0,T]$ the supremum over y can be replaced with the supremum over a compact set $\{y \in \mathbb{R}^d : |y| \leqslant R(T)\}$, where $R(T)$ is independent of ε. Then the equicontinuity of $I_\varepsilon(0,0,\cdot,\cdot,\omega)$ on compact subsets of $(0,\infty)\times\mathbb{R}^d$ ([25], Lemma 3.1) permits us to take the limit inside the supremum and conclude that for each $t > 0$ and $\omega \in \Omega'$

$$\lim_{\varepsilon\to 0} u_\varepsilon(t,0,\omega) = \sup_{y\in\mathbb{R}^d} (P\cdot y - t\mu(y/t)) = u^P(t,0) = t\overline{H}(P).$$

This implies that $\overline{H}(P)$ is the convex conjugate of μ. Moreover, we also get the convergence in probability of $u_\varepsilon(t,x,\omega)$ to $u^P(t,x)$ for $(t,x) \in (0,\infty)\times\mathbb{R}^d$.

Properties of Hamilton-Jacobi semigroup and standard uniform bounds on u_ε, $\nabla_x u_\varepsilon$ and $\partial u_\varepsilon/\partial t$ allow to strengthen the above result to the almost sure locally uniform convergence on $[0,\infty)\times\mathbb{R}^d$ ([27], Theorem 1).

The argument based on the subadditive ergodic theorem does not immediately give the minimax representation of $\overline{H}$ (28). This representation was obtained in [23].

6 Method 2: Ergodic and Minimax Theorems

This method is also based on the variational formula for solutions. A lower bound on the family $u_\varepsilon(t,x,\omega)$, $\varepsilon \in (0,1]$, is obtained by an application of the standard ergodic theorem. A matching upper bound is derived using Sion's minimax theorem combined with compactness arguments and the maximum principle.

Let $u_\varepsilon(t,x,\omega)$ be a solution of the "viscous" Hamilton-Jacobi equation (1). We start with the variational formula (22) with $x = 0$. Since the supremum over $\mathcal{C}$ is taken for each ω, we can let c depend on ω. At first, we consider a very special class of controls. This will lead to a lower bound on $u_\varepsilon(t,0,\omega)$, $\varepsilon \in (0,1]$.

We choose c to be independent of t and stationary in x so that $c(x,\omega) = b(\tau_x\omega)$, $x \in \mathbb{R}^d$, for some $b \in L^\infty(\mathbb{P})$. Such choice of control allows us to "lift" the diffusion $Q_0^{c,\omega}$ to Ω. To each starting point ω and a continuous path $x(t) \in \mathbb{R}^d$ we can associate a path $\omega(t) = \tau_{x(t)}\omega \in \Omega$. The induced measure $P^{b,\omega}$ defines a Markov process on Ω with the generator

$$\mathcal{A}_b = \sigma^2\Delta + b(\omega)\cdot\nabla,$$

where $\nabla = (D_1, D_2, \ldots, D_d)$, $\Delta = \sum_{i=1}^d D_i^2$, and D_i, $i = 1,2,\ldots,d$, are infinitesimal generators of the translation group $\{\tau_x : x \in \mathbb{R}^d\}$ acting on $L^2(\Omega,\mathcal{F},\mathbb{P})$:

$$D_i f(\omega) = \lim_{h\to 0} \frac{f(\tau_{he_i}\omega) - f(\omega)}{h}, \quad i = 1,2,\ldots,d.$$

If we could find a strictly positive density $\Phi(\omega)$ such that $\Phi d\mathbb{P}$ is an invariant ergodic probability measure for $\mathcal{A}_b$, then for every $F \in L^1(\Phi d\mathbb{P})$ we would have by the ergodic theorem

$$\lim_{\varepsilon\to 0} \varepsilon \int_0^{t/\varepsilon} F(\tau_{x(s)}\omega)\, ds = \lim_{\varepsilon\to 0} \varepsilon \int_0^{t/\varepsilon} F(\omega(s))\, ds = t\mathbb{E}(F\Phi)$$

$P^{b,\omega}$-a.s. or in $L^1(P^{b,\omega})$ for $\mathbb{P}$-a.e. ω. Unfortunately, we can not solve the problem of finding the invariant measures for the diffusion $P^{b,\omega}$ on Ω. Thus, we have to further restrict the set of allowed controls and hope that we still will be able to prove the matching upper bound.

Let us denote by $\mathbf{B}$ the space of measurable essentially bounded maps from Ω to $\mathbb{R}^d$ and by $\mathbf{D}$ the space of bounded probability densities $\Phi : \Omega \to \mathbb{R}$ relative to $\mathbb{P}$, which are bounded away from 0 and have essentially bounded gradients. Define

$$\mathcal{E} = \{(b, \Phi) \in \mathbf{B} \times \mathbf{D} : \nabla \cdot (b\Phi) = \sigma^2 \Delta\Phi\}. \tag{29}$$

We assume that the equation in (29) is satisfied in the weak sense: with probability one for every $G \in C_0^\infty(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \left(b\Phi - \sigma^2\nabla\Phi\right)(\tau_x\omega) \cdot \nabla_x G(x)\, dx = 0. \tag{30}$$

In short, we consider all pairs (b, Φ) such that $\Phi\, d\mathbb{P}$ is an invariant ergodic measure for the diffusion $P^{b,\omega}$ and $\Phi\, d\mathbb{P}$ is equivalent to $\mathbb{P}$. Return now to the variational formula (23) with $x = 0$. For each $c(t, x, \omega) = b(\tau_x\omega)$, $(b, \Phi) \in \mathcal{E}$, we obtain by the ergodic theorem that $P^{b,\omega}$-a.s. and in $L^1(P^{b,\omega})$

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \varepsilon x(t/\varepsilon) = \lim_{\varepsilon\to 0} \varepsilon \int_0^{t/\varepsilon} b(\omega(s))\, ds = t\int_\Omega b(\omega)\Phi(\omega)\, d\mathbb{P} \stackrel{\text{def}}{=} tm(b, \Phi);\\
&\lim_{\varepsilon\to 0} \varepsilon \int_0^{t/\varepsilon} L(b(\omega(s)), \omega(s))\, ds = t\int_\Omega L(b(\omega), \omega)\Phi\, d\mathbb{P} \stackrel{\text{def}}{=} th(b, \Phi)
\end{aligned} \tag{31}$$

for a.e. ω with respect to $\mathbb{P}$. Therefore, changing from $Q_{0,0}^{b,\omega}$ to $P^{b,\omega}$ and using the uniform continuity of f we get that for $\mathbb{P}$-a.e. ω

$$\liminf_{\varepsilon\to 0} u_\varepsilon(t, 0, \omega) \geqslant \sup_{b:(b,\Phi)\in\mathcal{E}} (f(tm(b, \Phi)) - th(b, \Phi)).$$

Rearranging the right-hand side we arrive at the inequality

$$\liminf_{\varepsilon\to 0} u_\varepsilon(t, 0, \omega) \geqslant \sup_{y\in\mathbb{R}^d} \left(f(y) - t\overline{L}\left(\frac{y}{t}\right)\right), \tag{32}$$

where

$$\overline{L}(q) = \inf_{\substack{b:(b,\Phi)\in\mathcal{E}\\ \mathbb{E}(b\Phi)=q}} h(b, \Phi) = \inf_{\substack{b:(b,\Phi)\in\mathcal{E}\\ \mathbb{E}(b\Phi)=q}} \mathbb{E}[\tilde{L}(b(\omega), \omega)\Phi(\omega)]. \tag{33}$$

Observe that $\overline{L}$ is convex. This is a simple consequence of (33) and the fact that if $(b_i, \Phi_i) \in \mathcal{E}$, $i = 1, 2$, then for every $\lambda \in [0, 1]$

$$\left(\frac{\lambda b_1\Phi_1 + (1-\lambda)b_2\Phi_2}{\lambda\Phi_1 + (1-\lambda)\Phi_2}, \lambda\Phi_1 + (1-\lambda)\Phi_2\right) \in \mathcal{E}.$$

Therefore, by Hopf-Lax-Oleinik formula (21)

$$\liminf_{\varepsilon\to 0} u_\varepsilon(t,0,\omega) \geqslant u(t,0),$$

where u is the solution of (2) with the initial condition f and $\overline{H}$ is the convex conjugate of $\overline{L}$. This gives us yet another candidate for the effective Hamiltonian:

$$\overline{H}(p) = \sup_{q\in\mathbb{R}^d}(p\cdot q - \overline{L}(q)) = \sup_{(b,\Phi)\in\mathcal{E}} \mathbb{E}[(p\cdot b(\omega) - \tilde{L}(b(\omega),\omega))\Phi(\omega)]. \tag{34}$$

Notice that (32) establishes an almost sure lower bound only at $x = 0$. The relation (14) and the translation invariance of $\mathbb{P}$ imply that the lower bound holds for arbitrary x but in probability. More work needs to be done to obtain an almost sure locally uniform lower bound. The detailed proof can be found in Section 4 of [20].

Next we turn to an upper bound for u_ε. This is a more difficult task, and we only sketch main ideas.

We have already proved the lower bound (32) for $u_\varepsilon(t,0,\omega)$ with general initial data. Below we indicate how to get the matching upper bound with the effective Hamiltonian (34) in the case when f is linear and relate (34) to the earlier definitions (27) and (28).

Observe that the supremum in (34) is taken over $(b,\Phi)\in\mathcal{E}$, which contains a constraint (29). To remove this constraint we introduce a "Lagrange multiplier" U with the stationary mean zero gradient $\nabla U = \tilde{F}$ and formally obtain the following chain of equalities

$$\begin{aligned}
\overline{H}(P) &= \sup_{(b,\Phi)\in\mathcal{E}} \mathbb{E}[[P\cdot b(\omega) - \tilde{L}(b(\omega),\omega)]\Phi(\omega)]\\
&= \sup_{\Phi}\sup_{b}\inf_{U}\ \mathbb{E}[[P\cdot b(\omega) - \tilde{L}(b(\omega),\omega) + \mathcal{A}_b U]\Phi(\omega)]\\
&= \sup_{\Phi}\inf_{U}\sup_{b}\ \mathbb{E}[[P\cdot b(\omega) - \tilde{L}(b(\omega),\omega) + \mathcal{A}_b U]\Phi(\omega)]\\
&= \sup_{\Phi}\inf_{U}\sup_{b}\ \mathbb{E}[[(P+(\nabla U)(\omega))\cdot b(\omega) - \tilde{L}(b(\omega),\omega) + \sigma^2\mathrm{div}(\nabla U)]\Phi(\omega)]\\
&= \sup_{\Phi}\inf_{U}\ \mathbb{E}[[\tilde{H}(P+(\nabla U)(\omega),\omega) + \sigma^2\mathrm{div}(\nabla U)(\omega)]\Phi(\omega)]\\
&= \inf_{U}\sup_{\Phi}\ \mathbb{E}[[\tilde{H}(P+\tilde{F}(\omega),\omega) + \sigma^2\mathrm{div}\tilde{F}(\omega)]\Phi(\omega)]\\
&= \inf_{U}\operatorname*{ess\,sup}_{\omega}\,[\tilde{H}(P+\tilde{F}(\omega),\omega) + \sigma^2\mathrm{div}\tilde{F}(\omega)].
\end{aligned}$$

The last expression gives us (28). We have used the fact that

$$\inf_{U}\ \mathbb{E}[\varphi\mathcal{A}_b U] = -\infty$$

unless $\varphi\, d\mathbb{P}$ is an invariant measure for $\mathcal{A}_b$, in which case it is 0. This formal computation suggests that for every $\delta > 0$, there exists a U_δ with the stationary mean zero gradient such that

$$H(x, P+\nabla_x U(x,\omega),\omega) + \sigma^2\Delta_x U(x,\omega) \leqslant \overline{H}(P) + \delta \quad \text{in } \mathcal{D}'(\mathbb{R}^d) \text{ a.s. in } \omega.$$

U_δ is a weak object and some work needs to be done before we can use it as a test function and obtain the upper bound by comparison.

The interchanges of the infimum and the supremum also need justification and should be performed more carefully. This justification is based on a compactification argument and Sion's minimax theorem [26]. The details are given in Section 5 of [20].

Assume that we can construct a sufficiently regular $U_\delta(x, \omega)$ with the mean zero stationary gradient. We can normalize U_δ so that $U_\delta(0, \omega) = 0$ a.s.. Then we are able to obtain the matching upper bound on $u_\varepsilon(t, 0, \omega)$ with linear initial data $f(x) = P \cdot x$ essentially by comparing $u_\varepsilon(t, x, \omega)$ with

$$\hat{u}_\varepsilon(t, x, \omega) = P \cdot x + t\overline{H}(P) + \varepsilon U_\delta(x/\varepsilon, \omega),$$

which is a super-solution of (1) with the initial condition $\hat{f}_\varepsilon(x, \omega) = P \cdot x + \varepsilon U_\delta(x/\varepsilon, \omega)$, and satisfies

$$\lim_{\varepsilon \to 0} \hat{u}_\varepsilon(t, 0, \omega) = t\overline{H}(p) = u(t, 0).$$

The lack of regularity of $(\nabla U_\delta)(\omega)$ may not allow a straightforward application of the ergodic theorem to show that the error term is small, i.e. with probability one for all $\ell > 0$

$$\lim_{\varepsilon \to 0} \sup_{|x| \leqslant \ell} \varepsilon |U_\delta(x/\varepsilon, \omega)| = 0.$$

For the rigorous proof of this comparison result and the extension to general uniformly continuous initial data under some additional assumptions on H the reader is referred to Sections 6 and 7 of [20].

7 Applications

We discuss the connection between the homogenization result for (1) with H given by (9) and large deviations for diffusions in an absorbing random media. Paper [24] states a general version of the large deviations result. Applications related to combustion and the propagation of fronts arising as asymptotic limits of reaction-diffusion equations in a random environment are given in [27] and [24].

Large deviations for diffusions in an absorbing random environment. Let $b : \Omega \to \mathbb{R}^d$ and $V : \Omega \to [0, \infty)$ be bounded measurable functions. Fix an ω and consider the diffusion

$$x(t) = x + \int_0^t b(\tau_{x(s)}\omega)\, ds + B(t),$$

where $B(\cdot)$ is the standard d-dimensional Brownian motion. An element of the path space for this diffusion is denoted by w and the corresponding probability measure on the path space by $Q_x^{b,\omega}(dw)$. The diffusion starts from x at time 0 and has the generator

$$\mathcal{A}_{b,\omega} = \frac{1}{2}\Delta_x + b(\tau_x \omega) \cdot \nabla_x.$$

Introduce the absorption into our random medium by "switching on" the potential V and consider quenched path measures

$$\tilde{Q}_{t,\omega}(dw) = \frac{1}{S_{t,\omega}} \exp\left(-\int_0^t V(\tau_{x(s)}\omega)\,ds\right) Q_0^{b,\omega}(dw),$$

where the normalization factor $S_{t,\omega}$ is given by

$$S_{t,\omega} = E_{Q_0^{b,\omega}} \exp\left(-\int_0^t V(\tau_{x(s)}\omega)\,ds\right). \tag{35}$$

$S_{t,\omega}$ is the probability that the particle diffusing from the origin has not been absorbed up to time t.

Remark 7.1. A detailed study of these measures for Brownian motion ($b \equiv 0$) in a Poissonian environment and potential

$$V(\tau_x\omega) = \sum_i W(x - x_i),\ \ x \in \mathbb{R}^d,\ \ \omega = \sum_i \delta_{x_i}, \tag{36}$$

where $W : \mathbb{R}^d \to [0,\infty)$, $W \not\equiv 0$, is bounded, measurable, and compactly supported, was done by A. S. Sznitman ([28]). Borrowing his informal definition of quenched path measures, one can say that "measures $\tilde{Q}_{t,\omega}$ describe the behavior of the trajectory of a particle diffusing from the origin in a typical realization of a partially absorbing random medium conditioned on the atypical event that it has not been absorbed up to a (long) time t" ([28], p.XI). Potential V in (36) is not bounded and, thus, does not fit into our framework. It would be natural to extend homogenization results to allow unbounded potentials, which have sufficiently many moments.

If we are interested in large deviations for $x(t)/t$ as $t \to \infty$ under the quenched path measures then the main object to study is the $\mathbb{P}$-a.e. limit of the logarithmic moment generating function:

$$\begin{aligned}\Lambda(\lambda) &= \lim_{t\to\infty} \frac{1}{t} \log E_{Q_{t,\omega}} \exp(\lambda \cdot x(t)) \qquad (37)\\ &= \lim_{t\to\infty} \frac{1}{t} \log E_{Q_0^{b,\omega}} \exp\left(\lambda \cdot x(t) - \int_0^t V(\tau_{x(s)}\omega)\,ds\right) - \lim_{t\to\infty} \frac{1}{t} \log S_{t,\omega}.\end{aligned}$$

The large deviations rate function is then obtained as the convex conjugate of $\Lambda(\lambda)$.

By the Feynman-Kac formula the quantity

$$v^\lambda(t,x,\omega) = E_{Q_x^{b,\omega}} \exp\left(\lambda \cdot x(t) - \int_0^t V(\tau_{x(s)}\omega)\,ds\right)$$

solves the following Cauchy problem:

$$\frac{\partial v^\lambda}{\partial t} = \frac{1}{2}\Delta_x v^\lambda + b(x,\omega)\cdot \nabla_x v^\lambda - V(\tau_x\omega)\,v^\lambda; \quad v^\lambda(0,x,\omega) = \exp(\lambda \cdot x).$$

Therefore,

$$\Lambda(\lambda) = \lim_{t\to\infty} \frac{1}{t} \log v^\lambda(t,0,\omega) - \lim_{t\to\infty} \frac{1}{t} \log v^0(t,0,\omega).$$

If we define

$$u_\varepsilon^\lambda(t,x,\omega) = \varepsilon \log v^\lambda \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \omega \right)$$

then the problem reduces to the study of the behavior of $u_\varepsilon^\lambda(1,0,\omega)$ as $\varepsilon \to 0$. It is easy to check that u_ε^λ solves

$$\frac{\partial u_\varepsilon^\lambda}{\partial t} = \frac{\varepsilon}{2} \Delta_x u_\varepsilon^\lambda + \frac{1}{2} |\nabla_x u_\varepsilon^\lambda|^2 + b(x/\varepsilon,\omega) \cdot \nabla_x u_\varepsilon^\lambda - V(\tau_{x/\varepsilon}\omega), \quad u_\varepsilon^\lambda(0,x,\omega) = \lambda \cdot x.$$

This is a stochastic homogenization problem for (1) with

$$H(p,x,\omega) = \frac{1}{2}|p|^2 + b(x,\omega) \cdot p - V(x,\omega).$$

Therefore,

$$\Lambda(\lambda) = \overline{H}(\lambda) - \overline{H}(0).$$

Remark 7.2. When $b \equiv 0$ and V is given by (36) the asymptotics of $S_{t,\omega}$ is known ([28], p. 196). In particular, $\mathbb{P}$-a.s.

$$\overline{H}(0) = \lim_{t\to\infty} \frac{1}{t} \log S_{t,\omega} = 0.$$

Moreover, available estimates on the large deviation rate function ([28], p. 248) imply that the effective Hamiltonian $\overline{H}(p)$ in this case has a "flat piece" near the origin, that is $\overline{H}(p) = 0$ for all $|p| \leqslant c$ for some $c > 0$.

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