

Homogenization of Hamilton-Jacobi-Bellman equations with respect to time-space shifts in a stationary ergodic medium

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Abstract

We consider a family $\{u_\varepsilon(t, x, \omega)\}$, $\varepsilon > 0$, of solutions of $\partial u_\varepsilon / \partial t + \frac{\varepsilon}{2} \Delta u_\varepsilon + H(t/\varepsilon, x/\varepsilon, \nabla u_\varepsilon, \omega) = 0$, $u_\varepsilon(T, x, \omega) = U(x)$, where the time-space dependence of the Hamiltonian $H(t, x, p, \omega)$ is realized through the shifts in a stationary ergodic random medium. For Hamiltonians, which are convex in p and satisfy certain growth and regularity conditions, we show the almost sure locally uniform in time and space convergence of $u_\varepsilon(t, x, \omega)$ as $\varepsilon \rightarrow 0$ to the solution $u(t, x)$ of a deterministic “effective” equation $\partial u / \partial t + \bar{H}(\nabla u) = 0$, $u(T, x) = U(x)$. The averaged Hamiltonian \bar{H} is given by a minimax formula.

1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\omega \in \Omega$. We are interested in the behavior of a family $\{u_\varepsilon(t, x, \omega)\}$, $\varepsilon > 0$, as $\varepsilon \rightarrow 0$, where $u_\varepsilon(t, x, \omega)$ is a solution of the following terminal value problem

$$\frac{\partial u_\varepsilon}{\partial t} + \frac{\varepsilon}{2} \Delta u_\varepsilon + H\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \nabla u_\varepsilon, \omega\right) = 0, \quad (t, x) \in [0, T) \times \mathbb{R}^d; \quad (1) \quad \boxed{\text{fvp}}$$

$$u_\varepsilon(T, x, \omega) = U(x). \quad (2) \quad \boxed{\text{fc}}$$

The Hamiltonian $H(t, x, p, \omega)$ is assumed to be convex in p . The dependence of H on (t, x) is realized through the shifts in a stationary ergodic random medium (see the next section for a detailed description). The terminal condition U is a uniformly continuous non-random function. Under some additional assumptions on H we obtain a homogenization result, i.e. we show that with probability one $u_\varepsilon(t, x, \omega)$ converges locally uniformly in t and x to a non-random limit $u(t, x)$. Function $u(t, x)$ is the unique solution of the Hamilton-Jacobi equation

$$\frac{\partial u}{\partial t} + \overline{H}(\nabla u) = 0, \quad (t, x) \in [0, T) \times \mathbb{R}^d, \quad (3) \quad \boxed{\text{hje}}$$

with the same terminal value $U(x)$. We also provide a minimax representation for the effective Hamiltonian \overline{H} .

Homogenization problems for this type of equations with or without a “viscous” term were extensively studied in the periodic, quasi- and almost- periodic settings. The standard approach is based on the construction of so called correctors. Roughly speaking, the corrector is the first non-trivial error term in a formal asymptotic expansion of u_ε around u . It is obtained as a solution of an auxiliary “cell” problem. The word “cell” refers to the fact that in the periodic case the problem is posed on a torus. The existence of correctors and the uniform convergence of u_ε to a limit were established for a wide range of first and second order partial differential equations. This method is very robust and does not require the Hamiltonian to be convex. The most essential assumption on H is the coercivity in p :

$$H(t, x, p, \omega) \rightarrow \infty \quad \text{as } |p| \rightarrow \infty$$

uniformly in t, x and ω .

The extension of homogenization results to the stationary ergodic setting presented a number of difficulties due to the lack of compactness. It was shown (see [9] and [10], Section 8) that correctors or even approximate correctors need not exist in general. In the case when H is convex in p and independent of t , the homogenization was proved by using variational methods in combination with some version of the ergodic theorem (see [10], [7]). A further step was taken in [2], which establishes homogenization for fully non-linear uniformly elliptic equations, whose solutions do not have a representation formula.

It turns out that the time-space averaging (versus just the space averaging) requires an additional control on u_ε (see, for example, [5] for periodic Hamilton-Jacobi equations, [8] for linear equations in stationary ergodic random media).

In this paper we present a method, which allows to obtain time-space homogenization results for Hamilton-Jacobi-Bellman equations in a stationary ergodic setting. It is essential for our approach that H is convex in p . To gain the necessary “additional control” we formulate and prove an “averaged” ergodic theorem for potentials (see Section 6).

In the case when

$$H(t, x, p, \omega) = \frac{1}{2} \sum_{i=1}^d p_i^2 - \sum_{i=1}^d b_i(t, x, \omega) p_i - W(t, x, \omega)$$

the homogenization problem for (\mathbb{I}) ^{fvp} is closely related to (see $\mathbb{[13]}$ ^{Sz}, $\mathbb{[10]}$ ^{LS2}, and $\mathbb{[7]}$ ^{KRV}) the quenched large deviations principle for a Brownian motion with a random drift $b = (b_1, b_2, \dots, b_d)$ in a random potential W . Both the drift and the potential are assumed to be stationary ergodic processes (see Remark $\mathbb{[2.1]}$ ^{abuse}).

The paper is organized as follows. In Section 2 we set up the notation, state the assumptions, formulate the main result, and discuss the idea of the proof. In Section 3 we prove the lower bound on u_ε . The construction of approximate super-solutions of (\mathbb{I}) ^{fvp} needed for the upper bound is carried out in Section 4. The proof of the upper bound on u_ε is given in Section 5. One of the essential ingredients of the proof is the ‘‘averaged’’ ergodic theorem for potentials. Since $\mathbb{[6]}$ ^{et} this theorem might be of separate interest, it is discussed in detail in Section 6.

2 Main result and preliminary discussion

For $x, y \in \mathbb{R}^k$ we denote by $\langle x, y \rangle$ the standard scalar product of x and y and by $|y|$ the Euclidean norm of y . If A is a set in \mathbb{R}^k then $|A|$ denotes the Lebesgue measure of A . $B(x, r)$ is a ball of radius r centered at x . If $f \in L^\alpha$ then $\|f\|_\alpha$ stands for its norm. \mathbb{E} is the expectation with respect to \mathbb{P} . $B(t)$, $t \geq 0$, is the path of the canonical Brownian motion. Borel σ -algebra on \mathbb{R}^{d+1} is denoted by \mathcal{B} .

Let $\{\tau_{(t,x)} : (t, x) \in \mathbb{R}^{d+1}\}$ be a group of measure preserving transformations acting ergodically on $(\Omega, \mathcal{F}, \mathbb{P})$. Assume that the map $(t, x, \omega) \mapsto \tau_{(t,x)}\omega$ from $\mathbb{R}^{d+1} \times \Omega$ to Ω is $\mathcal{B} \times \mathcal{F}$ measurable.

Let $H(p, \omega) : \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}$ be convex in p and satisfy the following assumptions:

(H1) For all $(p, \omega) \in \mathbb{R}^d \times \Omega$

$$c_1(|p|^\alpha - 1) \leq H(p, \omega) \leq c_2(|p|^\beta + 1)$$

for some positive constants c_1, c_2 and $1 < \alpha \leq \beta < \infty$.

(H2) $H(p, \tau_{(t,x)}\omega)$ is uniformly continuous in (t, x) uniformly in ω and locally uniformly in p , i. e. for every $l > 0$

$$\lim_{\delta \rightarrow 0} \sup_{|(t,x)| \leq \delta} \sup_{|p| \leq l} \sup_{\omega \in \Omega} |H(p, \tau_{(t,x)}\omega) - H(p, \omega)| = 0.$$

(H3) There exists a positive function $\nu(\delta) \rightarrow 0$ as $\delta \rightarrow 0$ and a constant $C > 0$ such that for $|(t, x)| \leq \delta$ and $\omega \in \Omega$

$$H(p, \tau_{(t,x)}\omega) \geq (1 + \nu(\delta))H((1 + \nu(\delta))^{-1}p, \omega) - C\nu(\delta).$$

These assumptions can be equivalently stated in terms of the Lagrangian

$$L(q, \omega) = \sup_{p \in \mathbb{R}^d} (\langle p, q \rangle - H(p, \omega)). \quad (4) \quad \boxed{\text{lagr}}$$

(L1) For all $(p, \omega) \in \mathbb{R}^d \times \Omega$

$$c_3(|q|^{\beta'} - 1) \leq L(q, \omega) \leq c_4(|q|^{\alpha'} + 1)$$

for some positive of constants c_3, c_4 and $\alpha' = \alpha/(\alpha - 1), \beta' = \beta/(\beta - 1)$.

(L2) $L(q, \tau_{(t,x)}\omega)$ is uniformly continuous in x uniformly in ω and locally uniformly in q .

(L3) There exists a positive function $\nu(\delta) \rightarrow 0$ as $\delta \rightarrow 0$ and a constant $C > 0$ such that for $|(t, x)| \leq \delta$ and $\omega \in \Omega$

$$L(q, \tau_{(t,x)}\omega) \leq (1 + \nu(\delta))L(q, \omega) + C\nu(\delta).$$

We assume that the terminal data $U(x)$ is uniformly continuous on \mathbb{R}^d . This implies that for every $\delta > 0$ there is a constant K_δ such that for all $x, y, \in \mathbb{R}^d$

$$|U(x) - U(y)| \leq K_\delta|x - y| + \delta. \quad (5) \quad \boxed{\text{uc}}$$

abuse

Remark 2.1. Functions $H(t, x, p, \omega)$, $b(t, x, \omega)$, and $W(t, x, \omega)$, which appeared in Introduction, should be understood as $H(p, \tau_{(t,x)}\omega)$, $b(\tau_{(t,x)}\omega)$, and $W(\tau_{(t,x)}\omega)$ respectively.

The effective Hamiltonian \bar{H} . The translation group $\{\tau_{(t,x)} : (t, x) \in \mathbb{R}^{d+1}\}$ acting on $L^2(\Omega, \mathcal{F}, \mathbb{P})$ will have infinitesimal generators $D_t, \nabla_i, i = 1, 2, \dots, d$, in the coordinate directions (see Section 7.1 and pp. 231-232 of [6]). The space gradient $\nabla = (\nabla_1, \nabla_2, \dots, \nabla_d)$, the divergence $\text{div} = \nabla_1 + \nabla_2 + \dots + \nabla_d$, and the Laplace operator $\Delta = \sum_{i=1}^d \nabla_i^2$ are defined in the usual way.

Let us denote by \mathbf{B} the space of measurable essentially bounded maps from Ω to \mathbb{R}^d and by \mathbf{D} the space of bounded probability densities $\Phi : \Omega \rightarrow \mathbb{R}$ relative to \mathbb{P} , which are bounded away from 0 and have essentially bounded time-space gradients. Define

$$\mathcal{E} = \{(b, \Phi) \in \mathbf{B} \times \mathbf{D} : D_t\Phi + \nabla \cdot (b\Phi) = \frac{1}{2}\Delta\Phi\}. \quad (6) \quad \boxed{\text{E}}$$

We shall always assume that the equation in $\boxed{\text{E}}$ is satisfied in the weak sense: with probability one for every $G \in C_0^\infty(\mathbb{R}^{d+1})$

$$\iint \Phi(\tau_{(t,x)}\omega) \partial_t G(t, x) + \langle (b\Phi - \frac{1}{2}\nabla\Phi)(\tau_{(t,x)}\omega), \nabla G(t, x) \rangle dt dx = 0. \quad (7) \quad \boxed{\text{ws}}$$

Define a convex function \bar{H} on \mathbb{R}^d by

$$\bar{H}(\theta) = \sup_{(b, \Phi) \in \mathcal{E}} \mathbb{E}[\langle \theta, b(\omega) \rangle - L(b(\omega), \omega)\Phi(\omega)]. \quad (8) \quad \boxed{\text{eh}}$$

The main result of this paper is the following theorem.

Theorem 2.1. Assume that $H(p, \omega)$ satisfies (H1)-(H3) and that the terminal condition $U(x)$ is uniformly continuous on \mathbb{R}^d . Let $u(t, x)$ be the unique solution of (3), (2) with \bar{H} given by (8). Then with probability one for every $l > 0$

$$\lim_{\varepsilon \rightarrow 0} \sup_{0 \leq t \leq T} \sup_{|x| \leq l} |u_\varepsilon(t, x, \omega) - u(t, x)| = 0.$$

Remark 2.2. To identify the effective Hamiltonian it is enough to consider the set of linear initial data $\{\langle p, x \rangle, p \in \mathbb{R}^d\}$. For each $p \in \mathbb{R}^d$ the terminal value problem for the effective equation (3) has an obvious solution $u(t, x) = \langle p, x \rangle - (T - t)\bar{H}(p)$. In particular, if we set $(t, x) = (0, 0)$ and $T = 1$ then we get $\bar{H}(p) = u(0, 0)$. Therefore if the homogenization result holds then

$$\bar{H}(p) = \lim_{\varepsilon \rightarrow 0} u_\varepsilon^p(0, 0, \omega) \text{ a.s.,}$$

where u_ε^p solves (1), (2) with $U(x) = \langle p, x \rangle$.

Shifts, rescaling, and variational formulae. Our assumptions on the Hamiltonian allow us to use a variational representation of $u_\varepsilon(t, x, \omega)$, which could be considered as the starting point of our analysis.

Denote by \mathcal{C} the space of essentially bounded controls $c = c(s, x)$. Consider the diffusion on \mathbb{R}^d

$$dx(s) = c(s, x(s))ds + dB(s), \quad x(t) = x, \quad s \geq t.$$

Let $Q_{t,x}^c$ be the measure on $C([t, \infty); \mathbb{R}^d)$ associated to this diffusion. For each $c \in \mathcal{C}$ and $\omega \in \Omega$ we set

$$v_c(t, x, \omega) = E^{Q_{t,x}^c} \left(U(x(T)) - \int_t^T L(c(s, x(s)), \tau_{(s, x(s))} \omega) ds \right).$$

Then

$$v(t, x, \omega) = \sup_{c \in \mathcal{C}} v_c(t, x, \omega)$$

is the solution of

$$\frac{\partial v}{\partial t} + \frac{1}{2} \Delta v + H(\nabla v, \tau_{(t,x)} \omega) = 0 \quad (9) \quad \boxed{\text{unsc}}$$

with $v(T, x) = U(x)$ (see, for example, [4]).

The uniqueness of solutions of the terminal value problem for (9) gives the following simple relation between $v(t, x, \cdot)$ and $v(0, 0, \cdot)$. If we fix an arbitrary $(t, x) \in [0, T) \times \mathbb{R}^d$ and set $U^x(y) = U(x + y)$ then the solution of (9) with the terminal data $v^{t,x}(T - t, y) = U^x(y)$ and $\omega' = \tau_{(t,x)} \omega$ is given by

$$v^{t,x}(s, y, \omega') = v^{t,x}(s, y, \tau_{(t,x)} \omega) = v(t + s, x + y, \omega), \quad (s, y) \in [0, T - t] \times \mathbb{R}^d.$$

In particular,

$$v(t, x, \omega) = v^{t,x}(0, 0, \tau_{(t,x)} \omega). \quad (10) \quad \boxed{\text{sh}}$$

An easy calculation shows that $u_\varepsilon(t, x) = \varepsilon v_\varepsilon(t/\varepsilon, x/\varepsilon)$, where v_ε solves the unscaled equation (9) with the terminal condition

$$v_\varepsilon(T/\varepsilon, x, \omega) = U(\varepsilon x)/\varepsilon.$$

This leads to the following variational formula for the solution to the terminal value problem (1), (2):

$$u^\varepsilon(t, x, \omega) = \sup_{c \in \mathcal{C}} E^{Q_{t/\varepsilon, x/\varepsilon}^{c_\varepsilon}} \left(U(\varepsilon x(T/\varepsilon)) - \varepsilon \int_{t/\varepsilon}^{T/\varepsilon} L(c(s, x(s)), \tau_{(s, x(s))} \omega) ds \right). \quad (11) \quad \boxed{\text{vf}}$$

We can also rescale the diffusion by setting $y_\varepsilon(s) = \varepsilon x(s/\varepsilon)$. Then $y_\varepsilon(s)$ solves

$$dy_\varepsilon(s) = c_\varepsilon(s, y_\varepsilon(s)) ds + \sqrt{\varepsilon} dB(s), \quad t \leq s \leq T, \quad y_\varepsilon(t) = x, \quad (12) \quad \boxed{\text{y}}$$

where $c_\varepsilon(s, y) = c(s/\varepsilon, y/\varepsilon)$. Writing $Q_{t, x}^{c_\varepsilon, \varepsilon}$ for the measure associated with $y_\varepsilon(s)$ we get

$$u^\varepsilon(t, x, \omega) = \sup_{c_\varepsilon \in \mathcal{C}} E^{Q_{t, x}^{c_\varepsilon, \varepsilon}} \left(U(y_\varepsilon(T)) - \int_t^T L(c_\varepsilon(s, y_\varepsilon(s)), \tau_{(s/\varepsilon, y_\varepsilon(s)/\varepsilon)} \omega) ds \right). \quad (13) \quad \boxed{\text{resc_vf}}$$

It is clear that we can drop the subscript in c_ε when using the above formula. Our goal is to show the almost sure locally uniform in t and x convergence of $u_\varepsilon(t, x, \omega)$ to $u(t, x)$, which is the solution of (3), (2). Since \bar{H} is convex, we can use Hopf-Lax-Oleinik representation

$$u(t, x) = \sup_{y \in \mathbb{R}^d} \left(U(y) - (T - t) \bar{L} \left(\frac{y - x}{T - t} \right) \right). \quad (14) \quad \boxed{\text{eff_vf}}$$

Ergodic theorem and a lower bound. Let $b \in \mathbf{B} \equiv L^\infty(\Omega; \mathbb{R}^d)$. Consider a Brownian motion $x(t)$ on \mathbb{R}^d starting from 0 at time 0 with a random drift $b(\tau_{(t, x)} \omega)$ and denote the corresponding measure on $C([0, \infty); \mathbb{R}^d)$ by $Q_{0, 0}^{b, \omega}$. Then this diffusion can be “lifted” to Ω as follows. Pick a starting point ω so that $b(\tau_{(t, x)} \omega)$ is defined for all $(t, x) \in \mathbb{R}^{d+1}$. Such points ω form a set of full measure (see [6], p. 232). Define $\omega(s) = \tau_{(s, x(s))} \omega$, $s \geq 0$. The measure $P^{b, \omega}$ induced on paths in Ω corresponds to a Markov process on Ω with the generator

$$\mathcal{A}_b = D_t + \frac{1}{2} \Delta + \langle b(\omega), \nabla \rangle.$$

If we can find a positive density Φ on Ω such that $\Phi d\mathbb{P}$ is an invariant ergodic probability measure for \mathcal{A}_b then by the ergodic theorem for every $F \in L^1(\Omega, \Phi d\mathbb{P})$

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t F(\omega(s)) ds = \int_\Omega F(\omega) \Phi(\omega) d\mathbb{P}$$

$P^{b, \omega}$ -a.s. or in $L^1(P^{b, \omega})$ for \mathbb{P} -a.e. ω .

Finding an invariant density for a given drift is a hard problem, since we work in Ω . It is clear though that \mathcal{E} is not empty, since it obviously contains pairs $(b, 1)$, where b is a constant. Moreover, it is easy to show that if $(b, \Phi) \in \mathcal{E}$ then $\Phi d\mathbb{P}$ is an *ergodic* invariant measure for \mathcal{A}_b , and for each pair in \mathcal{E} we can use the ergodic theorem.

If we view ω in the formula $\stackrel{\text{vf}}{(\text{II})}$ for u_ε as a parameter we may allow the controls c to be dependent on ω as well. Let us consider only stationary controls, i.e. $c(t, x, \omega) = b(\tau_{(t,x)}\omega)$, where b is such that $(b, \Phi) \in \mathcal{E}$ for some Φ . Setting $(t, x) = 0$ we obtain that $P^{b,\omega}$ -a.s. and in $L^1(P^{b,\omega})$

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \varepsilon x(T/\varepsilon) &= \lim_{\varepsilon \rightarrow 0} \varepsilon \int_0^{T/\varepsilon} b(\omega(s)) ds = T \int_{\Omega} b(\omega) \Phi(\omega) d\mathbb{P} \equiv Tm(b, \Phi); \\ \lim_{\varepsilon \rightarrow 0} \varepsilon \int_0^{T/\varepsilon} L(b(\omega(s)), \omega(s)) ds &= T \int_{\Omega} L(b(\omega), \omega) \Phi d\mathbb{P} \equiv Th(b, \Phi) \end{aligned} \quad (15) \quad \boxed{\text{mh}}$$

for a.e. ω with respect to \mathbb{P} . Therefore, changing from $Q_{0,0}^{b,\omega}$ to $P^{b,\omega}$ and using $\stackrel{\text{uc}}{(\text{5})}$ we get that for \mathbb{P} -a.e. ω

$$\liminf_{\varepsilon \rightarrow 0} u_\varepsilon(0, 0, \omega) \geq \sup_{b:(b,\Phi) \in \mathcal{E}} (U(Tm(b, \Phi)) - Th(b, \Phi)).$$

Rearranging the right-hand side we arrive at the inequality

$$\liminf_{\varepsilon \rightarrow 0} u_\varepsilon(0, 0, \omega) \geq \sup_{y \in \mathbb{R}^d} \left(U(y) - T\bar{L}\left(\frac{y}{T}\right) \right),$$

where

$$\bar{L}(q) = \inf_{\substack{b:(b,\Phi) \in \mathcal{E} \\ \mathbb{E}(b\Phi) = q}} h(b, \Phi) = \inf_{\substack{b:(b,\Phi) \in \mathcal{E} \\ \mathbb{E}(b\Phi) = q}} \mathbb{E}[L(b(\omega), \omega)\Phi(\omega)]. \quad (16) \quad \boxed{\text{e1}}$$

Observe that \bar{L} is convex. This is a simple consequence of $\stackrel{\text{e1}}{(\text{II6})}$ and the fact that if $(b_i, \Phi_i) \in \mathcal{E}$, $i = 1, 2$, then for every $\lambda \in [0, 1]$

$$\left(\frac{\lambda b_1 \Phi_1 + (1 - \lambda) b_2 \Phi_2}{\lambda \Phi_1 + (1 - \lambda) \Phi_2}, \lambda \Phi_1 + (1 - \lambda) \Phi_2 \right) \in \mathcal{E}.$$

This together with $\stackrel{\text{eh}}{(\text{8})}$ imply that \bar{L} is the convex conjugate of \bar{H} and

$$\liminf_{\varepsilon \rightarrow 0} u_\varepsilon(0, 0, \omega) \geq u(0, 0),$$

where u is the solution of $\stackrel{\text{hje}}{(\text{3})}$ with the terminal data U .

Notice $\stackrel{\text{sh}}{(\text{II0})}$ that this establishes an almost sure lower bound only at $(0, 0)$. The relation $\stackrel{\text{sh}}{(\text{II0})}$ and the translation invariance of \mathbb{P} imply that the lower bound holds for arbitrary (t, x) but in probability. More work needs to be done to obtain an almost sure locally uniform lower bound. The proof is similar to the one given in Section 4 of $\stackrel{\text{KRV}}{[7]}$ and is presented in the next section.

A few words about an upper bound. An upper bound is essentially obtained by comparison with a family of super-solutions of $\stackrel{\text{fvp}}{(\text{I})}$. The starting point

of the construction is the formula $\overset{\text{eh}}{(8)}$ for the effective Hamiltonian. The main idea is the same as in $\overset{\text{KRV}}{[7]}$. There are some difficulties in the construction due to the lack of control on the time derivatives. This problem was not present in the case of the time independent Hamiltonian.

3 Lower bound

We start with an auxiliary lemma, which is an immediate consequence of $\overset{\text{mh}}{(\text{I5})}$ and Egoroff's theorem.

$\boxed{\text{aux}}$ **Lemma 3.1.** *Let $(b, \Phi) \in \mathcal{E}$ and $m(b, \Phi)$, $h(b, \Phi)$ be as in $\overset{\text{mh}}{(\text{I5})}$. For every $\eta > 0$ there is a set N_η such that $\mathbb{P}(N_\eta) \geq 1 - \eta$ and*

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \sup_{\omega \in N_\eta} E^{Q_{0,0}^{b,\omega}} \left(\left| \varepsilon \int_0^{T/\varepsilon} b(\tau_{(s,x(s))}\omega) ds - Tm(b, \varphi) \right| \right) &= 0; \\ \lim_{\varepsilon \rightarrow 0} \sup_{\omega \in N_\eta} E^{Q_{0,0}^{b,\omega}} \left(\left| \varepsilon \int_0^{T/\varepsilon} L(b(\tau_{(s,x(s))}\omega), \tau_{(s,x(s))}\omega) ds - Th(b, \varphi) \right| \right) &= 0. \end{aligned}$$

Combining Lemma $\overset{\text{aux}}{3.1}$ with the variational formula $\overset{\text{vf}}{(\text{I1})}$ and the inequality $\overset{\text{uc}}{(\text{5})}$ we obtain the following statement (see Lemma 4.3 of $\overset{\text{KRV}}{[7]}$).

$\boxed{\text{1b}}$ **Lemma 3.2.** *Under the conditions of Lemma $\overset{\text{aux}}{3.1}$*

$$\lim_{\varepsilon \rightarrow 0} \inf_{\omega \in N_\eta} [u_\varepsilon(0, 0, \omega) - U(Tm(b, \Phi)) + Th(b, \Phi)] \geq 0. \quad (17) \quad \boxed{\text{1b0}}$$

The next lemma strengthens the estimate $\overset{\text{1b0}}{(\text{I7})}$.

$\boxed{\text{1bu}}$ **Lemma 3.3.** *Under the conditions of Lemma $\overset{\text{aux}}{3.1}$*

$$\lim_{r \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \inf_{\omega \in N_\eta} \inf_{\substack{|y| \leq r \\ 0 \leq t \leq r}} [u_\varepsilon(t, y, \omega) - U(Tm(b, \Phi)) + Th(b, \Phi)] \geq 0. \quad (18) \quad \boxed{\text{1b1}}$$

Proof. Let $\omega \in N_\eta$ and $x(t)$ be distributed according to $Q_{0,0}^{b,\omega}$. By Lemma $\overset{\text{aux}}{3.1}$ and Lemma $\overset{\text{1b}}{3.2}$ it is enough to construct a process $y(\cdot)$ such that

$$dy(s) = c(s, y(s), \omega) ds + dB(s), \quad t/\varepsilon \leq s \leq T/\varepsilon, \quad y(t/\varepsilon) = y/\varepsilon, \quad c(\cdot, \cdot, \omega) \in \mathcal{C},$$

and

$$\begin{aligned} \lim_{r \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \inf_{\omega \in N_\eta} \inf_{\substack{|y| \leq r \\ 0 \leq t \leq r}} \varepsilon \left(E^{Q_{t/\varepsilon, y/\varepsilon}^{c,\omega}} \left[y(T/\varepsilon) - \int_{t/\varepsilon}^{T/\varepsilon} L(c(s, y(s)), \tau_{(s,y(s))}\omega) ds \right] \right. \\ \left. - E^{Q_{0,0}^{b,\omega}} \left[x(T/\varepsilon) - \int_0^{T/\varepsilon} L(b(\tau_{(s,x(s))}\omega), \tau_{(s,x(s))}\omega) ds \right] \right) \geq 0. \end{aligned} \quad (19) \quad \boxed{\text{1b2}}$$

Fix an $n > 0$. Let $r < T/2$, $t \in [0, r]$, and $|y| \leq r$. We shall couple $y(\cdot)$ to $x(\cdot)$ as follows. Let A be the set of all continuous paths $x(\cdot)$ such that

$$\varepsilon|x(t/\varepsilon) - y/\varepsilon| \leq nr.$$

Suppose that $x(\cdot) \in A$ and set $t_1 = t + \varepsilon|x(t/\varepsilon) - y/\varepsilon|/n$. Then $t_1 \leq 2r < T$. Define

$$y(s) = \begin{cases} y/\varepsilon + x(s) - x(t/\varepsilon) + n(s - t/\varepsilon) \frac{(x(t/\varepsilon) - y/\varepsilon)}{|x(t/\varepsilon) - y/\varepsilon|}, & t/\varepsilon \leq s < t_1/\varepsilon; \\ x(s), & s \geq t_1. \end{cases}$$

Then for all $s \geq t/\varepsilon$ we have that $\varepsilon|x(s) - y(s)| \leq \varepsilon|x(t/\varepsilon) - y/\varepsilon| \leq nr$. Estimating the drift for $y(\cdot)$ we get $\|c\|_\infty \leq \|b\|_\infty + n$. Therefore,

$$\begin{aligned} & \varepsilon \left| \int_{t/\varepsilon}^{T/\varepsilon} L(c(s, y(s)), \tau_{(s, y(s))} \omega) ds - \int_0^{T/\varepsilon} L(b(\tau_{(s, x(s))} \omega), \tau_{(s, x(s))} \omega) ds \right| \leq \\ & \varepsilon \left| \int_{t/\varepsilon}^{t_1/\varepsilon} L(c(s, y(s)), \tau_{(s, y(s))} \omega) ds - \int_{t/\varepsilon}^{t_1/\varepsilon} L(b(\tau_{(s, x(s))} \omega), \tau_{(s, x(s))} \omega) ds \right| + \\ & \varepsilon \left| \int_0^{t/\varepsilon} L(b(\tau_{(s, x(s))} \omega), \tau_{(s, x(s))} \omega) ds \right| \leq 3r \sup_{|q| \leq \|b\|_\infty + n} |L(q, \omega)|. \end{aligned}$$

The last supremum is bounded uniformly in ω due to (L1).

Now assume that $x(\cdot) \notin A$. Define $y(s) = y/\varepsilon + B(s) - B(t/\varepsilon)$. Observe that for $|y| \leq r$, $0 \leq t \leq r$, and \mathbb{P} -a.e. ω

$$Q_{0,0}^{b,\omega}(A^c) \leq \frac{\varepsilon E^{Q_{0,0}^{b,\omega}} |x(t/\varepsilon) - y/\varepsilon|}{nr} \leq \frac{(\|b\|_\infty + 1)r + \sqrt{\varepsilon dr}}{nr} \leq \frac{\|b\|_\infty + 1}{n} + \frac{1}{n} \sqrt{\frac{\varepsilon d}{r}}.$$

Since on A^c the drift c is zero, each integrand in the time integrals of (I9) is bounded in absolute value by $\sup_{|q| \leq \|b\|_\infty} |L(q, \omega)|$. Moreover,

$$\varepsilon|y(T/\varepsilon) - x(T/\varepsilon)| \leq r + \|b\|_\infty T + \varepsilon|B(t/\varepsilon)|.$$

Therefore, if we restrict the expectation in the left-hand side of (I9) to A^c then the resulting quantity before we take the limits is bounded in absolute value by

$$(2T \sup_{|q| \leq \|b\|_\infty} |L(q, \omega)| + T\|b\|_\infty + r) Q_{0,0}^{b,\omega}(A^c) + \varepsilon \sup_{0 \leq t \leq r} E^{Q_{0,0}^{b,\omega}} |B(t/\varepsilon)|.$$

Taking the limit as $\varepsilon \rightarrow 0$ and then $r \rightarrow 0$ we get the bound

$$(2 \sup_{|q| \leq \|b\|_\infty} |L(q, \omega)| + \|b\|_\infty)(\|b\|_\infty + 1) \frac{T}{n}.$$

Letting $n \rightarrow \infty$ completes the proof. \square

Combining the relation $\stackrel{\text{sh}}{(\text{I}0)}$, Lemma $\stackrel{\text{lb}}{\text{3.3}}$, and the ergodic theorem we obtain the almost sure locally uniform lower bound.

Theorem 3.1. *Assume that $H(p, \omega)$ satisfies (H1) and (H2) and that the terminal condition $U(x)$ is uniformly continuous on \mathbb{R}^d . Let $u(t, x)$ be the unique solution of $\stackrel{\text{le}}{(\text{3})}$, $\stackrel{\text{fc}}{(\text{2})}$ with \bar{H} given by $\stackrel{\text{eh}}{(\text{8})}$. Then with probability one for every $l > 0$*

$$\liminf_{\varepsilon \rightarrow 0} \inf_{0 \leq t \leq T} \inf_{|x| \leq l} [u_\varepsilon(t, x, \omega) - u(t, x)] \geq 0.$$

Proof. We omit the proof, since it is essentially the same as the one of Theorem 2.1 in $\stackrel{\text{KV}}{[7]}$. \square

4 Construction of approximate super-solutions

Let $H(p, \omega)$ satisfy (H1)-(H3). Fix an arbitrary $\theta \in \mathbb{R}^d$. In this section we construct a family of functions $V_\delta(t, x, \omega)$, $\delta \in (0, \delta_0)$, with the mean zero stationary time-space gradients such that for a.e. ω

$$\partial_t V_\delta + \frac{1}{2} \Delta V_\delta + H(\theta + \nabla V_\delta, \tau_{(t,x)} \omega) \leq \bar{H}(\theta) + \nu(\delta), \quad (t, x) \in \mathbb{R}^{d+1}, \quad (20) \quad \boxed{\text{ss}}$$

where $\nu(\delta) > 0$, $\nu(\delta) \rightarrow 0$ as $\delta \rightarrow 0$. An upper bound on u^ε will be established in the next section using a probabilistic comparison argument and the ergodic theorem from Section 6.

Notation. Let $\mathbf{B}_r = \{b \in \mathbf{B} : \|b\|_\infty \leq r\}$ and

$$H_r(p, \omega) = \sup_{|q| \leq r} (\langle p, q \rangle - L(q, \omega)).$$

For each $k > 1$ set

$$\begin{aligned} \mathbf{E}_k &= \{\Phi : k^{-1} \leq \Phi(\omega) \leq k, \int \Phi(\omega) d\mathbb{P} = 1\}; \\ \mathbf{D}_k &= \{\Phi : k^{-1} \leq \Phi(\omega) \leq k, |\nabla \Phi| \leq k^2, \int \Phi(\omega) d\mathbb{P} = 1\} \subset \mathbf{E}_k. \end{aligned}$$

Then $\mathcal{E} = \cup_{k>0} \cup_{r>0} \mathcal{E}_{r,k}$, where

$$\mathcal{E}_{r,k} = \{(b, \Phi) \in \mathbf{B}_r \times \mathbf{D}_k : D_t \Phi + \nabla \cdot (b\Phi) = \frac{1}{2} \Delta \Phi\}.$$

If φ is a mollifier on \mathbb{R}^d and $F \in L^1(\Omega, \mathbb{P})$ we can define

$$F^\varphi(\omega) = \int F(\tau_{(0,x)} \omega) \varphi(x) dx. \quad (21) \quad \boxed{\text{mol}}$$

Then $F^\varphi \in L^1(\Omega, \mathbb{P})$ and is smooth under space shifts. Notice that if $\Phi \in \mathbf{E}_k$ then $\Phi^\varphi(\omega) \in \mathbf{D}_k$ provided k is large enough so that $\|\nabla \varphi\|_\infty \leq k$.

We start with the formula $\stackrel{\text{sh}}{(\text{8})}$ for the effective Hamiltonian and two applications of the minimax theorem.

2mm

Lemma 4.1. *There is a sequence of functions $\{F_n(\omega)\}$, $n \geq 1$, $F_n \in W^{1,\infty}(\Omega)$, $F_n(\tau_{(\cdot,\cdot)}\omega) \in C^\infty(\mathbb{R}^{d+1})$ a.s., such that*

$$\sup_n \sup_{\Phi \in \mathbf{D}_n} \left[\int [D_t F_n + \frac{1}{2} \Delta F_n + H_n(\theta + \nabla F_n, \omega)] \Phi d\mathbb{P} \right] \leq \bar{H}(\theta) + \frac{1}{n}. \quad (22) \quad \text{start}$$

Proof. From $\text{\textcircled{8}}$ we have

$$\bar{H}(\theta) \geq \sup_{(b,\Phi) \in \mathcal{E}_{r,k}} \mathbb{E}[(\langle \theta, b(\omega) \rangle - L(b(\omega), \omega)) \Phi(\omega)].$$

The definition of $\mathcal{E}_{r,k}$ contains a constraint. Let \mathcal{Y} be a space of nice test functions (for example, \mathcal{Y} could be a class of functions $F \in W^{1,\infty}(\Omega)$ convoluted with a mollifier from $C_0(\mathbb{R}^{d+1})$ to ensure the smoothness under the shifts). Using the fact that for $\Phi \in \mathbf{D}_k$

$$\begin{aligned} \inf_{F \in \mathcal{Y}} \mathbb{E} \left([D_t F + \langle b, \nabla F \rangle + \frac{1}{2} \Delta F] \Phi \right) &= \inf_{F \in \mathcal{Y}} \mathbb{E} \left([D_t F + \langle b, \nabla F \rangle] \Phi - \frac{1}{2} \langle \nabla F, \nabla \Phi \rangle \right) \\ &= \begin{cases} 0, & \text{if } D_t \Phi + \nabla \cdot (b\Phi) = \frac{1}{2} \Delta \Phi \text{ in the weak sense;} \\ -\infty, & \text{otherwise,} \end{cases} \end{aligned}$$

we can remove the constraint and get the inequality

$$\begin{aligned} \bar{H}(\theta) &\geq \sup_{\Phi \in \mathbf{D}_k} \sup_{b \in \mathbf{B}_r} \inf_{F \in \mathcal{Y}} \mathbb{E}[(D_t F + \langle b, \theta + \nabla F \rangle - L(b, \omega) + \frac{1}{2} \Delta F) \Phi] \\ &= \sup_{\Phi \in \mathbf{D}_k} \sup_{b \in \mathbf{B}_r} \inf_{F \in \mathcal{Y}} \mathbb{E}[(D_t F + \langle b, \theta + \nabla F \rangle - L(b, \omega)) \Phi - \frac{1}{2} \langle \nabla F, \nabla \Phi \rangle]. \end{aligned}$$

We would like to apply Sion's minimax theorem $\text{\textcircled{12}}$ and interchange the infimum and the supremum twice.

At first, let us look at the $\inf_{F \in \mathcal{Y}}$ and the $\sup_{b \in \mathbf{B}_r}$ for each fixed $\Phi \in \mathbf{D}_k$. The functional

$$\mathbb{E}[(D_t F + \langle b, \theta + \nabla F \rangle - L(b, \omega)) \Phi - \frac{1}{2} \langle \nabla F, \nabla \Phi \rangle]$$

is concave in b and is linear and continuous in F . The set $\mathbf{B}_r \subset \mathbf{B}$ is compact in the $*$ -weak topology. We only need to check that the functional is upper semi-continuous in b in the $*$ -weak topology (the same topology, in which \mathbf{B}_r is compact). Thus, we need to show that if $\mathbb{E}(b_n f) \rightarrow \mathbb{E}(b f)$ as $n \rightarrow \infty$ for each $f \in L^1(\Omega)$, where $b_n, b \in \mathbf{B}_r$, and

$$\mathbb{E}[(\langle \theta + \nabla F, b_n(\omega) \rangle - L(b_n(\omega), \omega) + \frac{1}{2} \Delta F + D_t F) \Phi(\omega)] \geq c$$

for some c , then the same inequality holds for the limiting function b . Clearly, it is enough to show that

$$\mathbb{E}[L(b_n(\omega), \omega) \Phi(\omega)] \leq c, \quad n = 1, 2, \dots, \Rightarrow \mathbb{E}[L(b(\omega), \omega) \Phi(\omega)] \leq c.$$

Since $b \in \mathbf{B}_r$ and $H(\cdot, \omega)$ is super-linear,

$$L(b(\omega), \omega) = \sup_{p \in \mathbb{R}^d} (\langle p, b(\omega) \rangle - H(p, \omega)) = \langle p(\omega), b(\omega) \rangle - H(p(\omega), \omega),$$

where $\|p\|_{L^\infty(\Omega)} \leq R(r) < \infty$. Therefore,

$$\begin{aligned}\mathbb{E}[L(b(\omega), \omega)\Phi(\omega)] &= \mathbb{E}[(\langle p(\omega), b(\omega) \rangle - H(p(\omega), \omega))\Phi(\omega)] \\ &= \lim_{n \rightarrow \infty} \mathbb{E}[(\langle p(\omega), b_n(\omega) \rangle - H(p(\omega), \omega))\Phi(\omega)] \\ &\leq \limsup_{n \rightarrow \infty} \mathbb{E}[L(b_n(\omega), \omega)\Phi(\omega)] \leq c,\end{aligned}$$

and we arrive at

$$\bar{H}(\theta) \geq \sup_{\Phi \in \mathcal{D}_k} \inf_{F \in \mathcal{Y}} \sup_{b \in \mathbf{B}_r} \mathbb{E}[(D_t F + \langle b, \theta + \nabla F \rangle - L(b, \omega))\Phi - \frac{1}{2}\langle \nabla F, \nabla \Phi \rangle].$$

For fixed Φ and F the supremum over $b \in \mathbf{B}_r$ can be taken inside the expectation. Recalling the definition of $H_r(p, \omega)$ we get

$$\bar{H}(\theta) \geq \sup_{\Phi \in \mathcal{D}_k} \inf_{F \in \mathcal{Y}} \mathbb{E}[(D_t F + H_r(\theta + \nabla F, \omega))\Phi - \frac{1}{2}\langle \nabla F, \nabla \Phi \rangle].$$

Now we would like to use the minimax theorem one more time. The functional

$$\mathbb{E}[(D_t F + H_r(\theta + \nabla F, \omega))\Phi - \frac{1}{2}\langle \nabla F, \nabla \Phi \rangle]$$

is convex in F and is linear in Φ . We need to work with the topology in which \mathbf{D}_k is compact (which corresponds to the convergence of Φ_n and $\nabla \Phi_n$ on each $f \in L^1(\Omega, \mathbb{P})$). The continuity of the functional in Φ for each F is then obvious (as long as $F \in \mathcal{Y}$ are required to have bounded derivatives up to at least the first order). We only need to show the lower semi-continuity in F but we even have the continuity in F , since we can work with the strong topology in \mathcal{Y} , and $H_r(p, \omega)$ is continuous in p .

Therefore, for each k and r

$$\bar{H}(\theta) \geq \inf_{F \in \mathcal{Y}} \sup_{\Phi \in \mathbf{D}_k} \mathbb{E}[(D_t F + \frac{1}{2}\Delta F + H_r(\theta + \nabla F, \omega))\Phi].$$

This immediately implies the statement of the lemma. \square

Let us briefly describe the idea of the next step. We would like to take some sort of weak limit of F_n and produce an F such that for almost all ω with respect to \mathbb{P} ,

$$D_t F(\omega) + \frac{1}{2}\Delta F(\omega) + H(\theta + \nabla F(\omega), \omega) \leq \bar{H}(\theta).$$

Unfortunately F may not exist but $\nabla F = g$ will exist in $L^\alpha(\Omega, \mathbb{P})$. Moreover, if we mollify F_n in x (see (21)) and consider $f_n^\varphi = D_t(F_n^\varphi) = (D_t F_n)^\varphi$ then we shall be able to produce a limiting object f_φ such that \mathbb{P} -a.s.

$$f_\varphi(\omega) + \frac{1}{2}\nabla \cdot g^\varphi(\omega) + \int H(\theta + g(\tau_{(0,x)}\omega), \tau_{(0,x)}\omega)\varphi(x) dx \leq \bar{H}(\theta), \quad (23) \quad \boxed{\text{targ}}$$

where

$$\nabla \cdot g^\varphi = \nabla \cdot \int g(\tau_{(0,x)}\omega)\varphi(x) dx = - \int \langle g(\tau_{(0,x)}\omega), (\nabla \varphi)(x) \rangle dx.$$

We shall end up with $g \in L^\alpha(\Omega, \mathbb{P})$ and a collection $f_\varphi \in L^1(\Omega, \mathbb{P})$, not necessarily smooth with respect to the shifts, which satisfies for almost all ω and each φ the compatibility condition

$$\int f_\varphi(\tau_{(t,x)}\omega) [\nabla \cdot G(t, x)] dt dx = \int \langle g^\varphi(\tau_{(t,x)}\omega), G_t(t, x) \rangle dt dx \quad (24) \quad \boxed{\text{compat}}$$

for $G \in C_0^\infty(\mathbb{R} \times \mathbb{R}^d; \mathbb{R}^d)$. We shall lose control of $\|f_\varphi\|_1$ as $\varphi \rightarrow \delta_0$ in $\mathcal{D}'(\mathbb{R}^d)$.

The details of the above construction are presented in the next theorem.

$\boxed{\text{comp0}}$

Theorem 4.1. *Let H satisfy (H1) and (H2). For every $\theta \in \mathbb{R}^d$ there is $g : \Omega \rightarrow \mathbb{R}^d$, $g \in L^\alpha(\Omega, \mathbb{P})$, and for every $r > 0$ there is a non-negative mollifier $\varphi \in C_0^\infty(\mathbb{R}^d)$, $\varphi(x) = 0$ outside of $B(0, r)$, and a function $f_\varphi \in L^1(\Omega, \mathbb{P})$ such that*

(a) $\mathbb{E}[f_\varphi] = 0$, $\mathbb{E}[g] = 0$;

(b) $(D_t, \nabla) \times (f_\varphi, g^\varphi) = 0$ in the sense of $\boxed{\text{compat}}$ (24);

(c) the inequality $\boxed{\text{targ}}$ (23) holds \mathbb{P} -a.s.;

(d) there is a constant $K_\varphi > 0$, which depends only on φ, c_1, α , and $\overline{H}(\theta)$, such that

$$f_\varphi(\omega) + \frac{c_1}{2} |g^\varphi(\omega)|^\alpha \leq K_\varphi \quad \mathbb{P}\text{-a.s.}$$

Proof. Let $f_n = D_t F_n$ and $g_n = \theta + \nabla F_n$, where $\{F_n\}$, $n \geq 1$, is constructed in Lemma 4.1.

Step 1. The sequence $\{g_n\}$ is uniformly integrable (see Theorem 5.2 of $\boxed{\text{KRV}}$ (7)). Take a weakly convergent subsequence and call it again $\{g_n\}$. We have $g_n \rightarrow g$ as $n \rightarrow \infty$ weakly in $L^1(\Omega, \mathbb{P})$ and $\mathbb{E}[H_n(g_n, \omega)] \leq \overline{H}(\theta)$. We shall show that $\mathbb{E}[H(g, \omega)] \leq \overline{H}(\theta)$ and $g \in L^\alpha(\Omega, \mathbb{P})$, where α is the same as in (H1).

Clearly, $H_k \leq H_n$ for $k \leq n$. Therefore,

$$\mathbb{E}[H_k(g_n, \omega)] \leq \overline{H}(\theta).$$

Let $q_k = q_k(\omega)$, $\|q_k\| \leq k$, be such that

$$H_k(g(\omega), \omega) = \sup_{\|q\| \leq k} [q \cdot g(\omega) - L(q, \omega)] = q_k(\omega) \cdot g(\omega) - L(q_k(\omega), \omega).$$

Then

$$\mathbb{E}[q_k \cdot g_n - L(q_k, \omega)] \leq \overline{H}(\theta).$$

Let $n \rightarrow \infty$ and use the weak convergence to get

$$\mathbb{E}[q_k \cdot g - L(q_k, \omega)] = \mathbb{E}[H_k(g, \omega)] \leq \overline{H}(\theta).$$

Finally we let $k \rightarrow \infty$. Using the lower bound on H we conclude that $g \in L^\alpha(\Omega, \mathbb{P})$.

Step 2. The only control we have on $\{f_n\}$, $n \geq 1$, is the following inequality (see Lemma 4.1)

$$\sup_{\Phi \in \mathbf{D}_n} \mathbb{E} \left[f_n + \frac{1}{2} \nabla \cdot g_n + H_n(g_n, \omega) \Phi \right] \leq \bar{H}(\theta).$$

This control is very weak. We mollify f_n and obtain some estimates on f_n^φ , $n \geq 1$.

Take a mollifier $\varphi \in C_0^\infty(\mathbb{R}^d)$, $\varphi(x) \geq 0$, such that

$$C_\varphi = \left(\int \left| \frac{\nabla \varphi(x)}{2\varphi(x)} \right|^{\alpha'} \varphi(x) dx \right)^{1/\alpha'} < \infty. \quad (25) \quad \boxed{\text{phi}}$$

Let $\Phi \in \mathbf{E}_k$. Recall that $\Phi^\varphi \in \mathbf{D}_k$ for all $k \geq c_\varphi = \|\nabla \varphi\|_\infty$. Define

$$A_{n,N} = \{\omega : |g_n(\omega)| \leq N\}, \quad g_{n,N}(\omega) = g_n(\omega) 1_{A_{n,N}}(\omega).$$

Since $\{g_n\}$ is uniformly integrable, there is an $N = N(k)$ such that for all n

$$\mathbb{E}[|g_n - g_{n,N}|] \leq \frac{1}{k}. \quad (26) \quad \boxed{\text{ui1}}$$

We have

$$\begin{aligned} \mathbb{E}[f_n^\varphi \Phi] &= \mathbb{E}[[f_n \Phi^\varphi] \leq \bar{H}(\theta) - \mathbb{E} \left[\left[\frac{1}{2} \nabla \cdot g_n + H_n(g_n, \omega) \right] \Phi^\varphi \right] \\ &= \bar{H}(\theta) + \mathbb{E} \left[\frac{1}{2} \nabla \Phi^\varphi \cdot g_n - H_n(g_n, \omega) \Phi^\varphi \right] \end{aligned} \quad (27) \quad \boxed{\text{s2}}$$

Observe that for all n

$$\begin{aligned} \mathbb{E}[\nabla \Phi^\varphi \cdot g_n] &\leq \mathbb{E}[|\nabla \Phi^\varphi| |g_n - g_{n,N}|] + \mathbb{E}[\nabla \Phi^\varphi \cdot g_{n,N}] \leq c_\varphi + \mathbb{E}[\nabla \Phi^\varphi \cdot g_{n,N}] \\ &= c_\varphi + \mathbb{E} \left[\Phi(\omega) \int (-\langle \nabla \varphi(x), g_{n,N}(\tau_{(0,x)} \omega) \rangle) dx \right] \\ &\leq c_\varphi + 2C_\varphi \mathbb{E} \left[\Phi(\omega) \left(\int |g_{n,N}(\tau_{(0,x)} \omega)|^\alpha \varphi(x) dx \right)^{1/\alpha} \right]. \end{aligned}$$

Moreover, for all $n \geq n_0(N)$

$$\begin{aligned} \mathbb{E}[H_n(g_n(\omega), \omega) \Phi^\varphi(\omega)] &\geq c_1 \mathbb{E}[(|g_{n,N}(\omega)|^\alpha - 1) \Phi^\varphi(\omega)] \\ &= c_1 \mathbb{E} \left[\Phi(\omega) \int |g_{n,N}(\tau_{(0,x)} \omega)|^\alpha \varphi(x) dx \right] - c_1. \end{aligned}$$

Set

$$\Theta_{n,N}(\omega) = \left(\int |g_{n,N}(\tau_{(0,x)} \omega)|^\alpha \varphi(x) dx \right)^{1/\alpha}.$$

Then substituting the above inequalities in $(\frac{52}{27})$ we obtain that for every $\Phi \in \mathbf{E}_k$ and all $n \geq n_1(k)$

$$\begin{aligned}\mathbb{E}[f_n^\varphi \Phi] &\leq \overline{H}(\theta) + \frac{c_\varphi}{2} + C_\varphi \mathbb{E}[\Phi(\omega) (\Theta_{n,N}(\omega) - c_1 \Theta_{n,N}^\alpha(\omega))] + c_1 \\ &\leq \overline{H}(\theta) + \frac{c_\varphi}{2} + C_\varphi \sup_{q \geq 0} (q - c_1 q^\alpha) + c_1 \leq M_\varphi.\end{aligned}$$

If Φ is not normalized then

$$\mathbb{E}[f_n^\varphi \Phi] \leq M_\varphi \mathbb{E}[\Phi].$$

Step 3. Consider the function Φ_n defined by

$$\Phi_n(\omega) = \begin{cases} \frac{1}{2} & \text{if } f_n^\varphi(\omega) \leq 0; \\ 2 & \text{if } f_n^\varphi(\omega) > 0. \end{cases}$$

Since $\mathbb{E}[f_n^\varphi] = 0$, we have

$$\mathbb{E}[f_n^\varphi : f_n^\varphi > 0] = -\mathbb{E}[f_n^\varphi : f_n^\varphi \leq 0] = \frac{1}{2} \mathbb{E}[|f_n^\varphi|].$$

Also $\Phi_n/\mathbb{E}[\Phi_n] \in \mathbf{E}_4$, and for all $n \geq n_1(4)$

$$\begin{aligned}\mathbb{E}[f_n^\varphi \Phi_n] &= 2\mathbb{E}[f_n^\varphi : f_n^\varphi > 0] + \frac{1}{2} \mathbb{E}[f_n^\varphi : f_n^\varphi \leq 0] \\ &= \frac{3}{4} \mathbb{E}[|f_n^\varphi|] \leq M_\varphi \mathbb{E}[\Phi_n] \leq 2M_\varphi.\end{aligned}$$

giving us the bound

$$\sup_{n \geq n_1(4)} \mathbb{E}[|f_n^\varphi|] \leq \frac{8}{3} M_\varphi.$$

Step 4. Let us take

$$\Phi_{k,n}(\omega) = \begin{cases} \frac{1}{k} & \text{if } f_n^\varphi(\omega) \leq \ell; \\ k & \text{if } f_n^\varphi(\omega) > \ell. \end{cases}$$

Recalling that

$$\mathbb{E}[f_n^\varphi : f_n^\varphi > \ell] = -\mathbb{E}[f_n^\varphi : f_n^\varphi \leq \ell],$$

we obtain

$$\begin{aligned}\mathbb{E}[f_n^\varphi \Phi_{k,n}] &= k \mathbb{E}[f_n^\varphi : f_n^\varphi > \ell] + \frac{1}{k} \mathbb{E}[f_n^\varphi : f_n^\varphi \leq \ell] = (k - \frac{1}{k}) \mathbb{E}[f_n^\varphi : f_n^\varphi > \ell] \\ &\leq M_\varphi \left[k \mathbb{P}[f_n^\varphi > \ell] + \frac{1}{k} \mathbb{P}[f_n^\varphi \leq \ell] \right] \leq M_\varphi \left[\frac{k}{\ell} \mathbb{E}[|f_n^\varphi|] + \frac{1}{k} \right].\end{aligned}$$

This gives us the bound

$$\sup_{n \geq n_1(k)} \mathbb{E}[f_n^\varphi : f_n^\varphi > \ell] \leq \frac{k M_\varphi}{k^2 - 1} \left[\frac{8k M_\varphi}{3\ell} + \frac{1}{k} \right].$$

Letting $\ell \rightarrow \infty$ we get

$$\lim_{\ell \rightarrow \infty} \sup_{n \geq n_1(k)} \mathbb{E}[f_n^\varphi : f_n^\varphi > \ell] \leq \frac{M_\varphi}{k^2 - 1}.$$

Since k was arbitrary, this implies the uniform integrability of the positive parts of $\{f_n^\varphi\}$, $n \geq 1$.

Step 5. Now we have to deal with the negative parts of $\{f_n^\varphi\}$. The difficulty is that they may not be uniformly integrable, and we have to find a way to split $f_{n,-}^\varphi = -(f_n^\varphi \wedge 0)$ (possibly just along a subsequence) into uniformly integrable and “bad” parts, so that the “bad” part plays no role in the limit. This is done using the following lemmas.

11 **Lemma 4.2.** *Let $\{h_n\}$, $n \geq 1$, be a sequence of non-negative functions and*

$$\sup_n \mathbb{E}[h_n] \leq C.$$

Then there is a subsequence $\{n_j\}$, $j \geq 1$, such that $h_{n_j} = \widehat{h}_{n_j} + r(h_{n_j})$, where

(a) $\widehat{h}_{n_j} = h_{n_j} 1_{\{h_{n_j} \leq \ell_j\}}$, $\ell_j \rightarrow \infty$ as $j \rightarrow \infty$, and $\{\widehat{h}_{n_j}\}$, $j \geq 1$, is uniformly integrable;

(b) remainder terms $r(h_{n_j})$ converge to zero in probability as $j \rightarrow \infty$.

The same is true for any slower growing sequence $\ell'_j \leq \ell_j$, $\ell'_j \rightarrow \infty$ as $j \rightarrow \infty$.

12 **Lemma 4.3.** *Let $\{h_n\}$, $n \geq 1$, be a sequence of non-negative functions such that*

$$\sup_n \mathbb{E}[h_n] \leq C,$$

and $\psi \in C_0^\infty(\mathbb{R}^d)$ be a non-negative mollifier. Apply Lemma 4.2 to $\{h_n^\psi\}$. Then the convergence of the remainder terms $r(h_{n_j}^\psi)$ to zero is locally uniform in x , i. e. for every $R > 0$ and $\varepsilon > 0$

$$\mathbb{P}(\omega : \sup_{\|x\| \leq R} r(h_{n_j}^\psi)(\tau_{(0,x)}\omega) \geq \varepsilon) \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

The proofs of Lemma 4.2 and Lemma 4.3 are elementary and are provided in the Appendix.

Return now to f_n^φ . Let $\varphi = \varphi_1 * \varphi_2$, where φ_1 satisfies (25). By Lemma 4.2 we can write (considering every subsequence as a whole sequence)

$$f_n^{\varphi_1} = m_n - b_n,$$

where $m_n = f_{n,+}^{\varphi_1} - \widehat{f_{n,-}^{\varphi_1}}$ are uniformly integrable, $b_n = r(f_{n,-}^{\varphi_1})$, and $b_n \rightarrow 0$ in probability. Taking a convolution with φ_2 in this decomposition we get

$$f_n^\varphi = m_n^{\varphi_2} - b_n^{\varphi_2}.$$

It is not hard to show that $m_n^{\varphi^2}$, $n \geq 1$, are still uniformly integrable. Apply Lemma 4.3 to $b_n^{\varphi^2}$ and get

$$b_n^{\varphi^2} = \widehat{b}_n^{\varphi^2} + r(b_n^{\varphi^2}),$$

where $\widehat{b}_n^{\varphi^2}$ are uniformly integrable and $r(b_n^{\varphi^2}) \rightarrow 0$ in probability locally uniformly with respect to the spacial shifts. Putting the two decompositions together we obtain

$$f_n^\varphi = k_n - r_n,$$

where $k_n = m_n^{\varphi^2} - \widehat{b}_n^{\varphi^2}$ and $r_n = r(b_n^{\varphi^2})$. This decomposition is “stable” in the following sense: further mollification and decomposition of the remainder terms r_n will only contribute uniformly integrable terms with the zero L^1 limit.

Step 6. Let $g \in L^\alpha(\Omega; \mathbb{R}^d)$ be a weak limit point of $\{g_n - \theta\}$, $n \geq 1$. Fix a small $r > 0$. Take $\varphi = \varphi_1 * \varphi_2$, where φ_i , $i = 1, 2$, are mollifiers, $\varphi \equiv 0$ outside of $B(0, r)$, and φ_1 satisfies (25). Decompose f_n^φ as in Step 5 and let $f_\varphi \in L^1(\Omega, \mathbb{P})$ be a weak limit point of k_n . It is clear that f_φ and g satisfy (a), (b), and (c).

To show part (d) we use (c) and the lower bound on H to get

$$f_\varphi(\omega) + \frac{1}{2} \nabla \cdot g^\varphi(\omega) + c_1 \int |g(\tau_{(0,x)}\omega)|^\alpha \varphi(x) dx \leq \overline{H}(\theta) + c_1. \quad (28) \quad \square$$

Applying the inequality

$$|\langle p, q \rangle| \leq \varepsilon |p|^\alpha + C_\varepsilon |q|^{\alpha'}$$

with $p = g(\tau_{(0,x)}\omega)$, $q = \nabla \varphi_1 / (2\varphi_1)$ and $\varepsilon = c_1/2$ and recalling that φ_1 satisfies (25) we obtain

$$\begin{aligned} \frac{1}{2} \nabla \cdot g^{\varphi_1}(\omega) + c_1 \int |g(\tau_{(0,x)}\omega)|^\alpha \varphi_1(x) dx = \\ c_1 \int |g(\tau_{(0,x)}\omega)|^\alpha \varphi_1(x) dx - \frac{1}{2} \int \langle g(\tau_{(0,x)}\omega), \nabla \varphi_1(x) \rangle dx \geq \\ \frac{c_1}{2} \int |g(\tau_{(0,x)}\omega)|^\alpha \varphi_1(x) dx - C_\varepsilon C_{\varphi_1}^{\alpha'}. \end{aligned}$$

Convoluting every term of the above inequality with φ_2 , substituting in (28), and applying Hölder’s inequality we get (d). \square

comp_cor

Corollary 4.1. *Let H satisfy (H1)-(H3). For each $\theta \in \mathbb{R}^d$ there is a positive function $\nu(\delta) \rightarrow 0$ as $\delta \rightarrow 0$ and a family of functions $(f_\delta, g_\delta) : \Omega \rightarrow \mathbb{R} \times \mathbb{R}^d$, $f_\delta \in L^1(\Omega, \mathbb{P})$, $g_\delta \in L^\alpha(\Omega, \mathbb{P})$, $\delta \in (0, \delta_0)$, such that*

- (a) $\mathbb{E}(f_\delta, g_\delta) = (0, 0)$ and $(D_t, \nabla) \times (f_\delta, g_\delta) = 0$ in the sense of (24)^{compat};
- (b) $f_\delta(\tau_{(\cdot, \cdot)}\omega), g_\delta(\tau_{(\cdot, \cdot)}\omega) \in C^\infty(\mathbb{R}^{d+1}; \mathbb{R}^{d+1})$ for \mathbb{P} -a.e. ω ;

(c) there is a constant $K_\delta > 0$, which depends only on δ, c_1, α , and $\overline{H}(\theta)$, such that for almost every ω with respect to \mathbb{P}

$$f_\delta(\tau_{(t,x)}\omega) + \frac{c_1}{2}|g_\delta(\tau_{(t,x)}\omega)|^\alpha \leq K_\delta \quad \text{for all } (t,x) \in \mathbb{R}^{d+1};$$

(d) for almost every ω with respect to \mathbb{P}

$$f_\delta(\tau_{(t,x)}\omega) + \nabla \cdot g_\delta(\tau_{(t,x)}\omega) + H(\theta + g_\delta(\tau_{(t,x)}\omega), \tau_{(t,x)}\omega) \leq \overline{H}(\theta) + \nu(\delta),$$

on \mathbb{R}^{d+1} .

Proof. The assumption (H3) will allow us to replace

$$\int H(\theta + g(\tau_{(0,x)}\omega), \tau_{(0,x)}\omega)\varphi(x) dx$$

in [\(23\)](#) with $H(\theta + g^\varphi(\omega), \omega)$ modulo a small error. See Lemma 6.3 of [\[7\]](#) for details. To gain the regularity with respect to space-time shifts we can mollify one more time in both t and x and again bring the mollification inside H . \square

Now we are ready to construct approximate super-solutions V_δ . Set

$$V_\delta(t, x, \omega) = \int_{(0,0) \rightarrow (t,x)} f_\delta(\tau_{(t,x(t))}\omega) dt + \langle g_\delta(\tau_{(t,x(t))}\omega), dx(t) \rangle, \quad (29) \quad \square$$

where $x(t)$ is any smooth path such that $x(0) = 0$ and $x(t) = x$. Since $(D_t, \nabla) \times (f_\delta, g_\delta) = 0$, the integral does not depend on the path. Main properties of V_δ are summarized in the next lemma.

Vprop **Lemma 4.4.** *Let $V_\delta(t, x, \omega)$ be given by [\(29\)](#). Then*

(a) for almost every ω with respect to \mathbb{P} and all $t, s \in \mathbb{R}$, $x, y \in \mathbb{R}^d$

$$V_\delta(0, 0, \omega) = 0, \quad V_\delta(s, y, \tau_{(t,x)}\omega) = V_\delta(t + s, x + y, \omega) - V_\delta(t, x, \omega);$$

(b) $\partial_t V_\delta(t, x, \omega) = f_\delta(\tau_{(t,x)}\omega)$, $\nabla V_\delta(t, x, \omega) = g_\delta(\tau_{(t,x)}\omega)$, and $V_\delta(t, x, \omega)$ satisfies [\(20\)](#);

(c) for all $t, s \in \mathbb{R}$ and $x, y \in \mathbb{R}^d$

$$\|V_\delta(t, x, \cdot) - V_\delta(s, y, \cdot)\|_1 \leq C(\delta)(|t - s| + |x - y|). \quad (30) \quad \square$$

(d) There are positive constants $C_1 = C_1(d)$ and $C_2 = C_2(\delta, c_1, \alpha, \overline{H}(\theta))$, such that with probability one for all $(t, x) \in \mathbb{R}^{d+1}$ and $\eta > 0$

$$V_\delta(t, x, \omega) \leq C_1 \eta^{-(d+1)} \int_0^\eta \int_{\|y\| \leq s} V_\delta(t - s, x - y, \omega) dy ds + C_2 \eta;$$

$$V_\delta(t, x, \omega) \geq C_1 \eta^{-(d+1)} \int_0^\eta \int_{\|y\| \leq s} V_\delta(t + s, x + y, \omega) dy ds - C_2 \eta.$$

Proof. Parts (a)-(c) are immediate from (29) and Corollary 4.1. We only need to prove (d).

Let δ, t, x, ω be fixed and (s, y) be an arbitrary vector in $\mathbb{R} \times \mathbb{R}^d$. Consider

$$W(s, y) := V_\delta(t + s, x + sy, \omega).$$

Then by part (c) of Corollary 4.1

$$\partial_s W(s, y) = f_\delta + \langle g_\delta, y \rangle \leq K_\delta - \frac{c_1}{2} \|g_\delta\|^\alpha + \langle g_\delta, y \rangle \leq K_1 + K_2 \|y\|^{\alpha'}.$$

Integrating we get for $s \geq 0$

$$V_\delta(t + s, x + sy, \omega) - V_\delta(t, x, \omega) \leq [K_1 + K_2 \|y\|^{\alpha'}] s. \quad (31) \quad \boxed{\text{osb}}$$

In particular, for $s > 0$

$$V_\delta(t + s, x + y, \omega) - V_\delta(t, x, \omega) \leq \left[K_1 + K_2 \left(\frac{\|y\|}{s} \right)^{\alpha'} \right] s.$$

Notice that (31) leads to the following a.s. estimates for fixed (t, x) and $\eta > 0$

$$\begin{aligned} V_\delta(t, x, \omega) &\leq \min_{\substack{0 \leq s \leq \eta \\ \|y\| \leq 1}} V_\delta(t - s, x - sy, \omega) + (K_1 + K_2)\eta \\ &\leq C_1 \eta^{-(d+1)} \int_0^\eta \int_{\|y\| \leq s} V_{\delta, \varepsilon}(t - s, x - y, \omega) dy ds + C_2 \eta \end{aligned}$$

The other inequality is obtained in a similar way. \square

5 Upper bound

We start with two lemmas (see [7], Lemma 4.1 and Lemma 4.2 for proofs). Let $y_\varepsilon(s)$ be given by (12) and

$$\xi_\varepsilon(t) = \int_t^T L(c(s, y_\varepsilon(s)), \tau_{(s/\varepsilon, y_\varepsilon(s)/\varepsilon)} \omega) ds. \quad (32) \quad \boxed{\text{xi}}$$

unif **Lemma 5.1.** *Assume (L1) and (5). Then in the variational formula (13), the supremum over \mathcal{C} can be replaced with the supremum over the subset $\mathcal{C}^* \subset \mathcal{C}$ of controls that satisfy the following condition: for each $\delta > 0$ there is $C_\delta > 0$, which depends only on δ and the constants in (L1) and (5), such that*

$$\sup_{x, \omega} E^{Q_{t, x}^{c, \varepsilon}} |\xi_\varepsilon(t)| \leq C_\delta (T - t + \sqrt{\varepsilon(T - t)}) + 2\beta\delta. \quad (33) \quad \boxed{\text{c*}}$$

In particular, for all $c \in \mathcal{C}^*$

$$\sup_{x, \omega} E^{Q_{t, x}^{c, \varepsilon}} \left[\int_t^T |c(s, y_\varepsilon(s))|^{\beta'} ds \right] \leq C_\delta (T - t + \sqrt{\varepsilon(T - t)}) + 2\beta\delta. \quad (34) \quad \boxed{\text{c*power}}$$

icmatch

Lemma 5.2. Assume the bounds (L1) and $\overline{\text{uc}}$ (5). There is a function $c(t) \rightarrow 0$ as $t \rightarrow 0$, such that for all $0 < \varepsilon \leq 1$

$$\sup_{x, \omega} |u_\varepsilon(T - t, x, \omega) - U(x)| \leq c(t).$$

The following theorem is the main result of this section.

Theorem 5.1. Assume that $H(p, \omega)$ satisfies (H1)-(H3) and that the terminal condition $U(x)$ is uniformly continuous on \mathbb{R}^d . Let $u(t, x)$ be the unique solution of $\overline{\text{ic}}$ (3), $\overline{\text{ic}}$ (2) with \overline{H} given by $\overline{\text{eh}}$ (8). Then with probability one for every $\ell > 0$

$$\limsup_{\varepsilon \rightarrow 0} \inf_{0 \leq t \leq T} \inf_{|x| \leq \ell} [u_\varepsilon(t, x, \omega) - u(t, x)] \geq 0.$$

Proof. By Lemma $\overline{\text{unif}}$ 5.1 the supremum in $\overline{\text{resc_vf}}$ (13) can be restricted to controls $c \in \mathcal{C}^*$. The estimate $\overline{\text{c*power}}$ (34) implies that $y_\varepsilon(T)$ is uniformly integrable with respect to $\{Q_{t,x}^{\varepsilon,c} : |x| \leq \ell, 0 \leq t \leq T, 0 < \varepsilon \leq 1, c \in \mathcal{C}^*\}$. Combining this with $\overline{\text{ic*}}$ (33) we get

$$\sup_{|x| \leq \ell} \sup_{0 \leq t \leq T} \sup_{c \in \mathcal{C}^*} \sup_{0 < \varepsilon \leq 1} E^{Q_{t,x}^{\varepsilon,c}} (|y_\varepsilon(t)| + |\xi_\varepsilon(t)|) < \infty. \quad (35) \quad \overline{\text{ui}}$$

Step 1. Without loss of generality we can assume that $(T - t) \geq r$ (see Lemma $\overline{\text{icmatch}}$ 5.2). Then using the uniform integrability we get that for every $\delta > 0$ there are $M = M(\delta)$ and $A = A(\delta, r)$ such that

$$\begin{aligned} & u^\varepsilon(t, x, \omega) - u(t, x) \\ &= \sup_{c \in \mathcal{C}^*} E^{Q_{t,x}^{c,\varepsilon}} (f(y_\varepsilon(T)) - \xi_\varepsilon(t)) - \sup_{y \in \mathbb{R}^d} \left[f(y) - (T - t)\overline{L} \left(\frac{y - x}{T - t} \right) \right] \\ &\leq \sup_{c \in \mathcal{C}^*} E^{Q_{t,x}^{c,\varepsilon}} [(f(y_\varepsilon(T)) - \xi_\varepsilon(t)) 1_{|y_\varepsilon(T) - x| \leq M}] + \delta \\ &\quad - \sup_{y \in \mathbb{R}^d} \left[f(y) - (T - t)\overline{L} \left(\frac{y - x}{T - t} \right) \right] \\ &\leq \sup_{c \in \mathcal{C}^*} E^{Q_{t,x}^{c,\varepsilon}} \left[\left((T - t)\overline{L} \left(\frac{y_\varepsilon(T) - x}{T - t} \right) - \xi_\varepsilon(t) \right) 1_{|y_\varepsilon(T) - x| \leq M} \right] + \delta \\ &= \sup_{c \in \mathcal{C}^*} E^{Q_{t,x}^{c,\varepsilon}} \left[\sup_{|\theta| \leq A} (\langle \theta, (y_\varepsilon(T) - x) \rangle - (T - t)\overline{H}(\theta) - \xi_\varepsilon(t)) 1_{|y_\varepsilon(T) - x| \leq M} \right] + \delta \\ &= \sup_{c \in \mathcal{C}^*} \int_{\{|y-x| \leq M\} \times \mathbb{R}} \sup_{|\theta| \leq A} (\langle \theta, y - x \rangle - (T - t)\overline{H}(\theta) - \xi) \mu_\varepsilon(dy, d\xi) + \delta, \end{aligned}$$

where μ_ε is the distribution of $(y_\varepsilon(T), \xi_\varepsilon(t))$ on $\mathbb{R}^d \times \mathbb{R}$ under $Q_{t,x}^{c,\varepsilon}$. Since \overline{H} is continuous, the supremum over θ can be restricted to the set $\{|\theta| \leq A, \theta \in \mathbb{Q}\}$, where \mathbb{Q} is the countable set of points in \mathbb{R}^d with rational coordinates. Since $L(\cdot, \omega)$ is super-linear, the integration can be further restricted to the set $\{(y, \xi) : |y - x| \leq M, \xi \leq B\}$ for a sufficiently large B (the proof is similar to the one of Lemma 4.1 in $\overline{\text{KR}}$ [7]). Therefore, to obtain an almost sure locally uniform upper

bound it is enough to show that for each $\theta \in \mathbb{R}^d$, $\ell, \eta > 0$ and all large enough M and B

$$\lim_{\varepsilon \rightarrow 0} \sup_{c \in \mathcal{C}^*} \sup_{|x| \leq \ell} \sup_{0 \leq t \leq T} Q_{t,x}^{\varepsilon,c} \{ \langle \theta, y_\varepsilon(T) - x \rangle - (T-t)\overline{H}(\theta) - \xi_\varepsilon(t) \geq \eta; A_{t,x} \} = 0, \quad (36) \quad \boxed{\text{meas}}$$

ω -a.s.. Here $A_{t,x} = \{|y_\varepsilon(T) - x| \leq M, \xi_\varepsilon(t) \leq B\}$.

Step 2. Since all computations below are valid for all $\omega \in \Omega'$, $\mathbb{P}(\Omega') = 1$, we fix $\omega \in \Omega'$ and drop it from the notation. Define

$$V_{\delta,\varepsilon}(t, x) = \varepsilon V_\delta(t/\varepsilon, x/\varepsilon).$$

By part (b) of Lemma [4.4](#) ^{prop} $V_{\delta,\varepsilon}(t, x)$ satisfies

$$\partial_t V_{\delta,\varepsilon} + \frac{\varepsilon}{2} \Delta V_{\delta,\varepsilon} + H(\theta + \nabla V_{\delta,\varepsilon}, \tau_{(t/\varepsilon, x/\varepsilon)} \omega) \leq \overline{H}(\theta) + \nu(\delta) \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d.$$

From Itô's formula for $V_{\delta,\varepsilon}(t, x) + \langle \theta, x \rangle$ and the above inequality we get

$$\begin{aligned} \langle \theta, y_\varepsilon(T) - x \rangle - (T-t)\overline{H}(\theta) - \xi_\varepsilon(t) &\leq V_{\delta,\varepsilon}(t, x) - V_{\delta,\varepsilon}(T, y_\varepsilon(T)) \\ &+ \sqrt{\varepsilon} \int_t^T \langle \nabla V_{\delta,\varepsilon}(s, y_\varepsilon(s)) + \theta, dB(s) \rangle + (T-t)\nu(\delta). \end{aligned} \quad (37) \quad \boxed{\text{ito1}}$$

We also used the fact that $\langle c, p \rangle \leq H(p, \omega) + L(c, \omega)$. Therefore, it is enough to show that for each $\theta \in \mathbb{R}^d$, $\ell, \eta > 0$, all sufficiently large M and B , and small $\delta > 0$

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \sup_{c \in \mathcal{C}^*} \sup_{|x| \leq \ell} \sup_{0 \leq t \leq T} Q_{t,x}^{\varepsilon,c} \{ \sqrt{\varepsilon} \int_t^T \langle \nabla V_{\delta,\varepsilon}(s, y_\varepsilon(s)), dB(s) \rangle \\ + V_{\delta,\varepsilon}(t, x) - V_{\delta,\varepsilon}(T, y_\varepsilon(T)) \geq \eta; A_{t,x} \} = 0. \end{aligned} \quad (38) \quad \boxed{\text{need}}$$

Observe that again by Itô's formula

$$\begin{aligned} V_{\delta,\varepsilon}(T, y_\varepsilon(T)) - V_{\delta,\varepsilon}(t, x) - \sqrt{\varepsilon} \int_t^T \langle \nabla V_{\delta,\varepsilon}(s, y_\varepsilon(s)), dB(s) \rangle \\ = \int_t^T \partial_s V_{\delta,\varepsilon}(s, y_\varepsilon(s)) + \langle \nabla V_{\delta,\varepsilon}(s, y_\varepsilon(s)), c(s, y_\varepsilon(s)) \rangle + \frac{\varepsilon}{2} \Delta V_{\delta,\varepsilon}(s, y_\varepsilon(s)) ds. \end{aligned} \quad (39) \quad \boxed{\text{ito2}}$$

Setting

$$F_{\delta,\varepsilon}(s, y) = \partial_s V_{\delta,\varepsilon}(s, y) + \langle \nabla V_{\delta,\varepsilon}(s, y), c(s, y) \rangle + \frac{\varepsilon}{2} \Delta V_{\delta,\varepsilon}(s, y),$$

we conclude from [\(38\)](#) ^{need} and [\(39\)](#) ^{ito2} that the relation

$$\lim_{\varepsilon \rightarrow 0} \sup_{c \in \mathcal{C}^*} \sup_{|x| \leq \ell} \sup_{0 \leq t \leq T} E^{Q_{t,x}^{\varepsilon,c}} \left(\left[\int_t^T F_{\delta,\varepsilon}(s, y_\varepsilon(s)) ds \right]^2; A_{t,x} \right) = 0$$

will imply the desired upper bound. The proof follows Step 2 of the Proof of Theorem 2.2 assuming **H4** in [7]. To complete the argument we only need to show that for every $\delta, a > 0$

$$\lim_{\varepsilon \rightarrow 0} \sup_{\substack{0 \leq t \leq T \\ |x| \leq a}} |V_{\delta, \varepsilon}(t, x, \omega)| = 0 \quad \mathbb{P}\text{-a.s.} \quad (40) \quad \boxed{\text{s1}}$$

Choosing $\eta = \delta/\varepsilon$ and rescaling the integrals in part (d) of Lemma 4.4 we get

$$|V_{\delta, \varepsilon}(t, x, \omega)| \leq C_1 \delta^{-(d+1)} \int_{|s| \leq \delta} \int_{|y| \leq |s|} |V_{\delta, \varepsilon}(t+s, x+y, \omega)| dy ds + C_2 \delta.$$

Therefore,

$$\sup_{\substack{0 \leq t \leq T \\ |x| \leq a}} |V_{\delta, \varepsilon}(t, x, \omega)| \leq C_1 \delta^{-(d+1)} \int_{|t| \leq T+\delta} \int_{|x| \leq a+\delta} |V_{\delta, \varepsilon}(t, x, \omega)| dx dt + C_2 \delta,$$

and (40) is an immediate consequence of the following ergodic theorem. \square

erg0 **Theorem 5.2 (Ergodic theorem).** *Let $(f, g) : \Omega \rightarrow \mathbb{R} \times \mathbb{R}^d$ satisfy the following conditions:*

$$(f, g) \in L^1(\Omega, \mathbb{P}), \quad \mathbb{E}(f, g) = 0, \quad (D_t, \nabla) \times (f, g) = 0 \text{ in the sense of (24).}$$

Define the “potential” $F : \mathbb{R}^{d+1} \rightarrow L^1(\Omega, \mathbb{P})$ by

$$F(t, x, \omega) = \int_{(0,0) \rightarrow (t,x)} f(\tau_{(t,x(t))}\omega) dt + \langle g(\tau_{(t,x(t))}\omega), dx(t) \rangle,$$

where $x(t)$ is a smooth path in \mathbb{R}^d such that $x(0) = 0$ and $x(t) = x$. Then for every bounded set $D \subset \mathbb{R}^{d+1}$

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \int_D |F(t/\varepsilon, x/\varepsilon, \omega)| dt dx = 0 \quad \mathbb{P}\text{-a.s.}$$

This theorem and its proof are the content of the next section.

6 An ergodic theorem for “box averages” of potentials

et

We shall work in \mathbb{R}^d instead of \mathbb{R}^{d+1} with the understanding that the time variable t does not play any special role in Theorem 5.2 and can be thought of as one of the space variables. The main result of this section is Theorem 6.3.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which \mathbb{R}^d acts as a group of measure preserving transformations $\tau_x : \Omega \rightarrow \Omega$, $x \in \mathbb{R}^d$, and \mathbb{P} is ergodic under this action. Assume that the map $(x, \omega) \mapsto \tau_x \omega$ from $\mathbb{R}^d \times \Omega$ to Ω is $\mathcal{B}(\mathbb{R}^d) \times \mathcal{F}$ measurable.

When ω is fixed and we are interested in $f(\tau_x\omega)$ as a function of x we sometimes write $f(x, \omega)$ or simply $f(x)$ instead of $f(\tau_x\omega)$.

For $k \in \{1, 2, \dots, d\}$ consider the subgroup of shifts $\{\tau_{(y,0)}, y \in \mathbb{R}^k\}$ generated by the first k coordinates. Since \mathbb{P} need not be ergodic under the action of this subgroup for $k < d$, we denote by \mathcal{I}^k a possibly non-trivial σ -algebra of subsets of \mathbb{R}^k , which are invariant under $\{\tau_{(y,0)}, y \in \mathbb{R}^k\}$. For additional information about ergodic elements of ergodic actions see [11].

Definition 6.1. A function $f : \Omega \rightarrow \mathbb{R}^d$, $f \in L^1(\Omega)$, is said to be a formal gradient if for every $\varphi \in C_0^\infty(\mathbb{R}^d)$ and for all $i, j = 1, 2, \dots, d$

$$\int f_i(\tau_x\omega) \partial_j \varphi(x) - f_j(\tau_x\omega) \partial_i \varphi(x) dx = 0 \quad \omega\text{-a.s.}$$

11c **Lemma 6.1.** Let $f \in L^1(\Omega)$. Then

$$\lim_{\delta \rightarrow 0} \sup_{|x| \leq \delta} \mathbb{E} |f(\tau_x\omega) - f(\omega)| = 0.$$

Proof. For almost all ω the function $f(\tau_x\omega)$ is locally integrable in x . Indeed, if B_R is a ball of radius R centered at 0 then by Fubini's theorem

$$\int \int_{B_R} |f(\tau_x\omega)| dx d\mathbb{P} = |B_R| \mathbb{E} |f| < \infty, \quad \text{and} \quad \int_{B_R} |f(\tau_x\omega)| dx \text{ is a.s. finite.}$$

Every locally integrable function is continuous in the mean, that is for every $R > 0$

$$g(\delta, \omega) = \sup_{|x| \leq \delta} \int_{B_R} |f(\tau_{x+y}\omega) - f(\tau_y\omega)| dy \rightarrow 0 \quad \text{as } \delta \rightarrow 0 \text{ for a.e. } \omega.$$

Moreover, for all $\delta \in [0, R]$

$$g(\delta, \omega) \leq 2 \sup_{|x| \leq \delta} \int_{x+B_R} |f(\tau_y\omega)| dy \leq 2 \int_{B_{2R}} |f(\tau_y\omega)| dy.$$

From the stationarity and the Lebesgue dominated convergence theorem we get that

$$\begin{aligned} \lim_{\delta \rightarrow 0} \sup_{|x| \leq \delta} \mathbb{E} |f(\tau_x\omega) - f(\omega)| &= \lim_{\delta \rightarrow 0} \sup_{|x| \leq \delta} \frac{1}{|B_R|} \mathbb{E} \int_{B_R} |f(\tau_{x+y}\omega) - f(\tau_y\omega)| dx \\ &\leq \frac{1}{|B_R|} \lim_{\delta \rightarrow 0} \mathbb{E} \sup_{|x| \leq \delta} \int_{B_R} |f(\tau_{x+y}\omega) - f(\tau_y\omega)| dx = \frac{1}{|B_R|} \lim_{\delta \rightarrow 0} \mathbb{E} g(\delta, \omega) = 0. \end{aligned}$$

□

The next lemma is an obvious consequence of the definition of f_δ and Lemma 6.1.

112 **Lemma 6.2.** Let $f \in L^1(\Omega; \mathbb{R}^k)$ and

$$f_\delta(\omega) = \int f(\tau_x \omega) \rho_\delta(x) dx,$$

where

$$\rho_\delta(x) = \delta^{-d} \rho(x/\delta), \quad \int \rho(x) dx = 1,$$

and $\rho(x)$ is a smooth function on \mathbb{R}^d equal to zero outside of the unit ball centered at 0. Then $f_\delta(\tau_x \omega)$ is smooth in x for a.e. ω , $\mathbb{E} f_\delta = \mathbb{E} f$, and $\lim_{\delta \rightarrow 0} \mathbb{E} |f_\delta(\omega) - f(\omega)| = 0$. Also if f is a formal gradient, then so is f_δ .

To each formal gradient we can associate a “potential” $F : \mathbb{R}^d \rightarrow L^1(\Omega)$ given by the formula

$$F(x, \omega) = \int_{0 \rightarrow x} \langle f(\tau_{x(t)} \omega), dx(t) \rangle, \quad (41) \quad \boxed{\text{int}}$$

where $x(t)$ is a Lipschitz continuous path connecting 0 and x .

pp **Lemma 6.3.** Let f be a formal gradient and F be as in $\frac{\text{int}}{(41)}$. Then F does not depend on the path $x(t)$, $F(0, \omega) = 0$ ω -a.s.,

$$F(x, \omega) - F(y, \omega) = F(x - y, \tau_y \omega),$$

and $\|F(x, \cdot) - F(y, \cdot)\|_1 \leq \|f\|_1 |x - y|$.

Proof. Consider $f_\delta(\tau_x \omega)$. For almost every ω it is a smooth function of x , and $\partial_i (f_\delta)_j = \partial_j (f_\delta)_i$. Therefore for a.e. ω function $f_\delta(\cdot, \omega) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ has a potential

$$F_\delta(x, \omega) = \int_{0 \rightarrow x} \langle f_\delta(\tau_{x(t)} \omega), dx(t) \rangle,$$

which does not depend on the path. Fix an arbitrary Lipschitz continuous path $x(t)$, $x(0) = 0$ and $x(T) = x$ for some $T > 0$. Then along this path

$$\mathbb{E} |F_\delta(x, \omega) - F(x, \omega)| \leq \mathbb{E} |f_\delta(\omega) - f(\omega)| \int_0^T |x'(t)| dt \rightarrow 0$$

as $\delta \rightarrow 0$ by Lemma $\frac{112}{6.2}$. This implies that $F(x, \omega)$ does not depend on the path.

Next let $x_1(t) = ty$, $t \in [0, 1]$, and $x_2(t) = (t - 1)(x - y)$, $t \in [1, 2]$. Then $x(t) = x_1(t)$ on $t \in [0, 1]$ and $x(t) = y + x_2(t)$ on $t \in [1, 2]$ is a path from 0 to y and

$$\begin{aligned} F(x, \omega) - F(y, \omega) &= \int \langle f(\tau_{x_2(t)+y} \omega), dx_2(t) \rangle \\ &= \int \langle f(\tau_{x_2(t)}(\tau_y \omega)), dx_2(t) \rangle = F(x - y, \tau_y \omega). \end{aligned}$$

In particular, $F(0, \omega) = 0$. The last statement also follows easily if we take the absolute value and the expectation in the above formula and observe that $x'_2(t) = x - y$. \square

We shall refer to a set $D \in \mathbb{R}^k$ as an admissible set if Wiener's ergodic theorem holds for $D_\alpha = \{\alpha x : x \in D\}$, $\alpha > 0$, that is for every $f \in L^1(\Omega)$

$$\lim_{\alpha \rightarrow \infty} \frac{1}{|D_\alpha|} \int_{D_\alpha} f(\tau_{(y,0)}\omega) dy = c_k(\omega) \quad \omega\text{-a.s. and in } L^1(\Omega),$$

where $c_k(\omega) = \mathbb{E}(f|I^k)(\omega)$. See [1] and [3] for more general results and discussion about admissible sets. In particular, if D is an open parallelepiped containing zero with edges parallel to coordinate axes then D is admissible. Observe also that if $D_2 \subset D_1$, D_1 and D_2 are both admissible then $D_1 \setminus D_2$ is also admissible. In particular, we can use this theorem when set D is a parallelepiped, which may not be open and may not contain the origin.

av **Theorem 6.1.** *Let $f = (f_1, \dots, f_d) \in L^1(\Omega; \mathbb{R}^d)$ be a formal gradient with mean zero and F be its normalized integral. Fix $k \in \{1, 2, \dots, d\}$ and an admissible set $D \subset \mathbb{R}^k$. Then*

$$\lim_{\varepsilon \rightarrow 0} \int_D f_k(\tau_{(y/\varepsilon, 0)}\omega) dy = 0 \quad \omega\text{-a.s.}$$

In particular, for every $x \in \mathbb{R}^d$

$$\lim_{\varepsilon \rightarrow 0} \varepsilon F(x/\varepsilon, \omega) = 0 \quad \omega\text{-a.s.}$$

Proof. By Wiener's ergodic theorem for a.e. ω and in $L^1(\Omega)$

$$\frac{1}{|D|} \int_D f_k(\tau_{(y/\varepsilon, 0)}\omega) dy = \frac{1}{|D_{1/\varepsilon}|} \int_{D_{1/\varepsilon}} f_k(\tau_{(y, 0)}\omega) dy \rightarrow c_k(\omega) \quad \text{as } \varepsilon \rightarrow 0,$$

where $c_k(\omega)$ is invariant under $\tau_{(y, 0)}$, $y \in \mathbb{R}^k$. Similarly, for any $z \in \mathbb{R}^d$

$$\frac{1}{|D|} \int_D f_k(\tau_{(y/\varepsilon, 0)}(\tau_z\omega)) dy \rightarrow c_k(\tau_z\omega) \quad \text{as } \varepsilon \rightarrow 0.$$

We claim that for every choice of $j \in \{k+1, \dots, d\}$ and $h \in \mathbb{R}$ the function $c_k(\omega)$ is invariant under the translation by $z = he_j$, where e_j is the j -th coordinate vector. Combining this statement with the semigroup property of the shifts and the invariance in the first k coordinates we conclude that $c_k(\omega)$ is invariant under τ_x , $x \in \mathbb{R}^d$. Then the ergodicity of \mathbb{P} implies that $c_k(\omega)$ is a constant a.s.. The mean zero property gives $c_k(\omega) \equiv 0$ a.s..

To prove our claim it is enough to show that for $z = he_j$, $j \in \{k+1, \dots, d\}$,

$$\int |c_k(\omega) - c_k(\tau_z\omega)| d\mathbb{P} = 0.$$

To simplify the notation we assume that $k = d - 1$ and $z = he_d$. The general

case is done similarly. For every $L > 0$ we have

$$\begin{aligned} \int |c_k(\omega) - c_k(\tau_z \omega)| d\mathbb{P} &\leq \int |c_k(\omega) - L^{-k} \int_{[0,L]^k} f_k(\tau_{(y,0)} \omega) dy| d\mathbb{P} \\ &\quad + L^{-k} \int \left| \int_{[0,L]^k} f_k(\tau_{(y,0)} \omega) - f_k(\tau_{(y,h)} \omega) dy \right| d\mathbb{P} \\ &\quad + \int |c_k(\tau_z \omega) - L^{-k} \int_{[0,L]^k} f_k(\tau_{(y,h)} \omega) dy| d\mathbb{P}. \end{aligned}$$

The first and the third terms in the right-hand side go to zero as $L \rightarrow \infty$. We only need to deal with the second integral. Since ω is fixed we drop it from the notation and also set $y = (y', y_k)$, $y' \in \mathbb{R}^{k-1}$. Then by Fubini's theorem

$$\begin{aligned} \int_{[0,L]^k} f_k(y, 0) - f_k(y, h) dy \\ = \int_{[0,L]^{k-1}} \left(\int_0^L f_k(y', y_k, 0) - f_k(y', y_k, h) dy_k \right) dy'. \end{aligned}$$

But f is a formal gradient and

$$\begin{aligned} \int_0^L f_k(y', y_k, 0) - f_k(y', y_k, h) dy_k \\ = F(y', L, 0) - F(y', 0, 0) - F(y', L, h) + F(y', 0, h) \\ = \int_0^h f_d(y', 0, r) - f_d(y', L, r) dr. \end{aligned}$$

Therefore,

$$\begin{aligned} L^{-k} \int \left| \int_{[0,L]^k} f_k(y, 0) - f_k(y, h) dy \right| d\mathbb{P} \\ = L^{-k} \int \left| \int_{[0,L]^{k-1}} \int_0^h f_d(y', 0, r) - f_d(y', L, r) dr dy' \right| d\mathbb{P} \leq \frac{2h}{L} \mathbb{E}|f_d| \rightarrow 0 \end{aligned}$$

as $L \rightarrow \infty$.

The second statement is basically the same as the case $k = 1$. Recalling the definition of F and following the steps of the above proof we get for $x \in \mathbb{R}^d$

$$\varepsilon F(x/\varepsilon, \omega) = \langle x, \varepsilon \int_0^{1/\varepsilon} f(\tau_{tx} \omega) dt \rangle \rightarrow 0 \text{ as } \varepsilon \rightarrow 0 \text{ a.s. and in } L^1(\Omega).$$

□

pav **Corollary 6.1.** *Let $D = [0, 1]^d$. Under the assumptions of Theorem [6.1](#)*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \int_D F(x/\varepsilon, \omega) dx = 0 \quad \omega\text{-a.s.}$$

Proof. Dropping ω from the notation and using the normalization $F(0) = 0$ we write

$$\int_D F(x) dx = \sum_{k=1}^d \int_{D_k} (1 - x_k) f_k(x_1, \dots, x_k, 0, \dots, 0) dx_1 \dots dx_k,$$

where $D_k = [0, 1]^k$. Rescaling gives

$$\varepsilon \int_D F(x/\varepsilon) dx = \sum_{k=1}^d \int_{D_k} (1 - x_k) f_k(x_1/\varepsilon, \dots, x_k/\varepsilon, 0, \dots, 0) dx_1 \dots dx_k. \quad (42) \quad \boxed{\text{resc}}$$

Approximating x_k by piecewise constant functions $\sum_{j=1}^n (j/n) 1_{[(j-1)/n, j/n]}$ we get

$$\begin{aligned} & \left| \int_{D_k} (1 - x_k) f_k(x_1/\varepsilon, \dots, x_k/\varepsilon, 0, \dots, 0) dx_1 \dots dx_k \right| \\ & \leq \left| \sum_{j=1}^n \left(1 - \frac{j}{n}\right) \int_{D_{k-1} \times [(j-1)/n, j/n]} f_k(x_1/\varepsilon, \dots, x_k/\varepsilon, 0, \dots, 0) dx_1 \dots dx_k \right| \\ & \quad + \left| \sum_{j=1}^n \int_{D_{k-1} \times [(j-1)/n, j/n]} \left(\frac{j}{n} - x_k\right) f_k(x_1/\varepsilon, \dots, x_k/\varepsilon, 0, \dots, 0) dx_1 \dots dx_k \right|. \end{aligned}$$

Each of the integrals in the first sum of the right-hand side goes to zero as $\varepsilon \rightarrow 0$ for every fixed n by Theorem [6.1](#)^{av}. The second sum is bounded above by

$$\frac{1}{n} \int_{D_k} |f_k(x_1/\varepsilon, \dots, x_k/\varepsilon, 0, \dots, 0)| dx_1 \dots dx_k \rightarrow \frac{|D_k|}{n} \mathbb{E}(|f_k| | \mathcal{I}^k)(\omega)$$

as $\varepsilon \rightarrow 0$. Letting $n \rightarrow \infty$ we obtain the result. \square

rep **Remark 6.1.** Observe that if in [\(42\)](#)^{resc} we replace $(1 - x_k)$ by any continuous function on D_k we still get the convergence to zero with essentially the same proof.

ue **Theorem 6.2.** Let $f \in L^1(\Omega; \mathbb{R}^d)$ be a formal gradient, $\mathbb{E}f = 0$, and $D = [0, 1]^d$. Then

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon \int_D |F(x/\varepsilon, \omega)| dx \leq \frac{1}{2} \mathbb{E} \left(\sum_{i=1}^d |f_i| \right) \quad \omega\text{-a.e.}$$

Proof. We shall start with a lemma.

113 **Lemma 6.4.** Let $D = [0, 1]^d$, F be a measurable function on D whose weak gradient $\nabla F = f \in L^1(D; \mathbb{R}^d)$. Then

$$\int_{D \times D} |F(x) - F(y)| dx dy \leq \frac{1}{2} \int_D \sum_{i=1}^d |f_i(x)| dx.$$

Proof. We have

$$\begin{aligned}
& \int_{D \times D} |F(x) - F(y)| dx dy \\
& \leq \sum_{i=1}^d \int_{D \times D} |F(\dots, y_{i-1}, x_i, x_{i+1}, \dots) - F(\dots, y_{i-1}, y_i, x_{i+1}, \dots)| dx dy \\
& = \sum_{i=1}^d \int_{D \times D} \left| \int_{y_i}^{x_i} f_i(\dots, y_{i-1}, s, x_{i+1}, \dots) ds \right| dx dy \\
& \leq 2 \sum_{i=1}^d \int_D s(1-s) |f_i(\dots, y_{i-1}, s, x_{i+1}, \dots)| dy_1 \dots, dy_{i-1} ds dx_{i+1} \dots dx_d \\
& \leq \frac{1}{2} \int_D \sum_{i=1}^d |f_i(x)| dx.
\end{aligned}$$

□

By Lemma [6.4](#) for a.e. ω

$$\begin{aligned}
\int_D \varepsilon |F(x/\varepsilon, \omega)| dx & \leq \varepsilon \int_D \left| F(x/\varepsilon, \omega) - \int_D F(y/\varepsilon, \omega) dy \right| dx + \varepsilon \left| \int_D F(y/\varepsilon, \omega) dy \right| \\
& \leq \varepsilon \int_{D \times D} |F(x/\varepsilon, \omega) - F(y/\varepsilon, \omega)| dx dy + \varepsilon \left| \int_D F(y/\varepsilon, \omega) dy \right| \\
& \leq \frac{1}{2} \sum_{i=1}^d \int_D |f_i(x/\varepsilon, \omega)| dx + \varepsilon \left| \int_D F(y/\varepsilon, \omega) dy \right|
\end{aligned}$$

Applying Wiener's ergodic theorem and Corollary [6.1](#) we get the result. □

erg1

Theorem 6.3. *Let $f \in L^1(\Omega)$ be a formal gradient, F be its normalized integral, and $\mathbb{E}f = 0$. Then for any bounded set $D \in \mathcal{B}(\mathbb{R}^d)$*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \int_D |F(x/\varepsilon, \omega)| dx = 0 \quad \omega\text{-a.s.}$$

Proof. At first, observe that it is enough to consider the case when $D = [0, a]^d$, $a > 0$. Rescaling by $1/a$ we see that without loss of generality we may assume that $D = [0, 1]^d$.

Step 1. Notice that if $f = \nabla F$, where $F(\omega)$ is a bounded function, then $\mathbb{E}f = 0$ and, obviously,

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \sup_{|x| \leq a} |F(\tau_{x/\varepsilon} \omega)| = 0 \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \varepsilon \int_D |F(\tau_{x/\varepsilon} \omega)| dx = 0.$$

Step 2. We show that for every $\eta > 0$ there is a function $F_\delta \in L^1(\Omega)$ with the integrable gradient $f_\delta = \nabla F_\delta$ such that $\mathbb{E} \left(\sum_{i=1}^d |f_i - (f_\delta)_i| \right) \leq \eta$.

Observe that if we choose a mollifier $\varphi = -\nabla \cdot g$, where $g : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is smooth and has a compact support, then

$$F(\omega) = \int \langle f(\tau_x \omega), g(x) \rangle dx \in L^1(\Omega),$$

and

$$\begin{aligned} \nabla_i F(\omega) &= \int \sum_{j=1}^d \nabla_i f_j(\tau_x \omega) g_j(x) dx = - \int \sum_{j=1}^d f_j(\tau_x \omega) \partial_i g_j(x) dx \\ &= - \int \sum_{j=1}^d f_i(\tau_x \omega) \partial_j g_j(x) dx = \int f_i(\tau_x \omega) \varphi(x) dx. \end{aligned}$$

Let $\rho(|x|)$ be a non-negative radially symmetric mollifier supported on $B(0, 1)$, $\int \rho(|x|) dx = 1$. For $\delta \in (0, 1)$ choose

$$\varphi(x) = \varphi_\delta(x) = \rho_\delta(|x|) - \rho_{1/\delta}(|x|).$$

It is easy to check that $\varphi_\delta = -\nabla \cdot g_\delta$, where

$$g_\delta(x) = -\frac{x}{|x|^d} \int_0^{|x|} s^{d-1} (\rho_\delta(s) - \rho_{1/\delta}(s)) ds.$$

Obviously, $g(x)$ is smooth and $g(x) \equiv 0$ for all $|x| \geq 1/\delta$. Set

$$F_\delta(\omega) = \int \langle f(\tau_x \omega) g_\delta(x) \rangle dx, \quad f_\delta(\omega) = \nabla F_\delta(\omega) = \int f(\tau_x \omega) \varphi_\delta(x) dx. \quad (43) \quad \boxed{\text{Fd}}$$

By Lemma [6.2](#) for sufficiently small $\delta > 0$

$$\mathbb{E} \left(\sum_{i=1}^d \left| f_i(\omega) - \int f_i(\tau_x \omega) \rho_\delta(|x|) dx \right| \right) \leq \eta/2. \quad (44) \quad \boxed{\text{sm}}$$

Moreover, by Wiener's ergodic theorem (see also Remark [6.1](#)) and the mean zero property of f_i , $i = 1, 2, \dots, d$,

$$\int f_i(\tau_x \omega) \rho_{1/\delta}(|x|) dx = \int_{B(0,1)} f_i(\tau_{y/\delta} \omega) \rho(|y|) dy \rightarrow 0 \text{ as } \delta \rightarrow 0 \text{ in } L^1(\Omega).$$

Therefore, $\mathbb{E} \left(\sum_{i=1}^d |f_i - (f_\delta)_i| \right) \leq \eta$ for sufficiently small δ .

Step 3. Let F_δ be as in Step 2. We show that there is a sequence of functions $\{F_\delta^N\}$, $N = 1, 2, \dots$, such that $|F_\delta^N(\omega)| \leq C(N, \delta)$, $|\nabla F_\delta^N| \leq C(N, \delta)$, and

$$\mathbb{E} \left[\sum_{j=1}^d |\partial_j F_\delta - \partial_j F_\delta^N| \right] \rightarrow 0 \text{ as } N \rightarrow \infty.$$

Let F_δ be as in [\(43\)](#). Define $f^N(\omega)$ by

$$f_i^N(\omega) = f_i(\omega)1_{\{|f_i(\omega)| \leq N\}}, \quad i = 1, 2, \dots, d.$$

Then $|f^N(\omega)| \leq \sqrt{d}N$. Set

$$F_\delta^N(\omega) = \int \sum_{i=1}^d f_i^N(\tau_x \omega) g_{\delta,i}(x) dx.$$

Then

$$\partial_j F_\delta^N(\omega) = - \int \sum_{i=1}^d f_i^N(\tau_x \omega) \partial_j g_{\delta,i}(x) dx,$$

and both F_δ^N and ∇F_δ^N are bounded. Moreover,

$$\begin{aligned} \mathbb{E} \left[\sum_{j=1}^d |\partial_j F_\delta - \partial_j F_\delta^N| \right] &= \mathbb{E} \left[\sum_{j=1}^d \left| \int \sum_{i=1}^d \partial_j (f_i(\tau_x \omega) - f_i^N(\tau_x \omega)) g_{\delta,i}(x) dx \right| \right] \\ &= \mathbb{E} \left[\sum_{j=1}^d \left| \int \sum_{i=1}^d (f_i(\tau_x \omega) - f_i^N(\tau_x \omega)) \partial_j g_{\delta,i}(x) dx \right| \right] \\ &\leq \mathbb{E} \left[\sum_{j=1}^d \sum_{i=1}^d \int |f_i(\tau_x \omega) - f_i^N(\tau_x \omega)| |\partial_j g_{\delta,i}(x)| dx \right] \\ &\leq C(\delta) \sum_{i=1}^d \mathbb{E} |f_i(\omega) - f_i^N(\omega)| \rightarrow 0 \text{ as } N \rightarrow \infty. \end{aligned}$$

Step 4. Writing $F = (F - F_\delta) + (F_\delta - F_\delta^N) + F_\delta^N$, applying [Theorem 6.2](#) to $F - F_\delta$ and $F_\delta - F_\delta^N$, and using Steps 1, 2, and 3 we see that for every $\eta > 0$ and all sufficiently large N

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \int_D |F(x/\varepsilon, \omega)| dx \leq \eta + \lim_{\varepsilon \rightarrow 0} \varepsilon \int_D |F_\delta^N(x/\varepsilon, \omega)| dx = \eta.$$

This completes the proof of [Theorem 6.3](#). □

A Appendix

Proof of Lemma 4.2. For $\ell = 1, 2, \dots$ define $h_n^\ell = h_n 1_{\{h_n \leq \ell\}}$. Then $0 \leq h_n^\ell \leq \ell$, and we can choose a subsequence $\{n_j\}$ such that for each ℓ functions $h_{n_j}^\ell \in L^\infty(\Omega)$ converge weakly as $j \rightarrow \infty$ to some h^ℓ , $0 \leq h^\ell \leq \ell$. Notice that

$$\mathbb{E} h^\ell = \lim_{j \rightarrow \infty} \mathbb{E} h_{n_j}^\ell \leq \liminf_{j \rightarrow \infty} \mathbb{E} h_{n_j} \leq C.$$

Since the sequence $\{h^\ell\}$, $\ell = 1, 2, \dots$, is non-decreasing, we can define $h(\omega) = \lim_{\ell \rightarrow \infty} h^\ell(\omega)$ a.s.. Then $\lim_{\ell \rightarrow \infty} \mathbb{E} h^\ell = \mathbb{E} h$.

Set

$$a_j^\ell = \mathbb{E} h_{n_j}^\ell, \quad a^\ell = \mathbb{E} h^\ell, \quad \text{and} \quad a = \mathbb{E} h.$$

Then

$$a_j^\ell \rightarrow a^\ell \text{ as } j \rightarrow \infty \quad \text{and} \quad a^\ell \nearrow a \text{ as } \ell \rightarrow \infty,$$

and we can choose a subsequence $\{\ell_j\}$, $\ell_j \rightarrow \infty$ such that $a_j^{\ell_j} \rightarrow a$ as $j \rightarrow \infty$. It is clear that for any slower growing sequence $\ell'_j \leq \ell_j$, $\ell'_j \rightarrow \infty$,

$$a_j^{\ell'_j} \rightarrow a \text{ as } j \rightarrow \infty.$$

We claim that $\widehat{h}_{n_j} = h_{n_j}^{\ell_j}$ and $r(h_{n_j}) = h_{n_j} - h_{n_j}^{\ell_j}$ satisfy conditions (a) and (b). Indeed, for each ℓ and all j such that $\ell_j \geq \ell$

$$\begin{aligned} \int_{\widehat{h}_{n_j} > \ell} \widehat{h}_{n_j} d\mathbb{P} &= \int_{h_{n_j}^{\ell_j} > \ell} h_{n_j}^{\ell_j} d\mathbb{P} \\ &= a_j^{\ell_j} - \int_{h_{n_j}^{\ell_j} \leq \ell} h_{n_j}^{\ell_j} d\mathbb{P} = a_j^{\ell_j} - \int h_{n_j}^\ell d\mathbb{P} = a_j^{\ell_j} - a_j^\ell. \end{aligned}$$

Thus,

$$\lim_{\ell \rightarrow \infty} \lim_{j \rightarrow \infty} \int_{\widehat{h}_{n_j} > \ell} \widehat{h}_{n_j} d\mathbb{P} = 0.$$

Also for every $\varepsilon \in (0, 1)$

$$\mathbb{P}(r(h_{n_j}) > \varepsilon) = \mathbb{P}(h_{n_j} - h_{n_j}^{\ell_j} > \varepsilon) \leq \mathbb{P}(h_{n_j} \geq \ell_j) \leq \frac{\mathbb{E} h_{n_j}}{\ell_j} \leq \frac{C}{\ell_j} \rightarrow 0 \text{ as } j \rightarrow \infty.$$

This completes the proof of Lemma [4.2](#). \square

The proof of Lemma [4.3](#) is based on the following observation.

13 **Lemma A.1.** *Let $h \geq 0$ be a measurable function and $\psi \geq 0$ be a nice mollifier on \mathbb{R}^d supported on the cube of side a around the origin. Then there is a $c > 0$, which depends only on ψ , such that for any $l > 0$ and ω*

$$h^\psi(\omega) > l \Rightarrow |\{x \in \mathbb{R}^d : |x| \leq a\sqrt{d}, h^\psi(\tau_{(0,x)}\omega) \geq cl\}| \geq c.$$

Proof of Lemma [4.3](#) assuming Lemma [A.1](#). For notational convenience we assume that $\{n_j\}$ is the whole sequence: $n_j = j$. We have

$$h_j^\psi = (\widehat{h^\psi})_j + r(h_j^\psi),$$

where $(\widehat{h^\psi})_j = h_j^\psi 1_{\{h_j^\psi \leq l_j\}}$ are uniformly integrable, $l_j \rightarrow \infty$, and $r(h_j^\psi) \rightarrow 0$ in probability. We need to show that the last convergence is locally uniform in x .

Fix $R > 0$ and $\varepsilon > 0$ and suppose that $r(h_j^\psi)(\tau_{(0,x)}\omega) \geq \varepsilon$ for some x in the ball of radius R around the origin. This, in fact, means that $h_j^\psi(\tau_{(0,x)}\omega) > l_j$.

From Lemma [A.1](#)^{[13](#)} it follows that $h_j^\psi(\tau_{0,y}\omega) \geq c\ell_j$ for y on a set of measure at least c in the ball $|y-x| \leq a\sqrt{d}$, where a, c depend only on ψ . Hence, for a ball B of radius $R + a\sqrt{d}$ around the origin

$$\int_B h_j^\psi(\tau_{(0,x)}\omega) dx \geq c^2\ell_j.$$

Summarizing the above, we have

$$\begin{aligned} \mathbb{P}(\omega : \sup_{|x| \leq R} r(h_j^\psi)(\tau_{(0,x)}\omega) \geq \varepsilon) &\leq \mathbb{P}\left(\omega : \int_B h_j^\psi(\tau_{(0,x)}\omega) dx \geq c^2\ell_j\right) \\ &\leq \frac{1}{c^2\ell_j} \mathbb{E} \int_B h_j^\psi(\tau_{(0,x)}\omega) dx \leq \frac{|B|}{c^2\ell_j} \mathbb{E} h_j^\psi \leq \frac{C|B|}{c^2\ell_j} \rightarrow 0 \text{ as } j \rightarrow \infty. \end{aligned}$$

□

Proof of Lemma [A.1](#)^{[13](#)}. We have $h^\psi(\omega) = \int h(\tau_{(0,x)}\omega)\psi(x)dx$, where ψ is a nice non-negative mollifier, bounded by A and supported on the cube of side a around the origin. We consider the cube K_b of size $b = \frac{a}{N}$ such that the cube of side a can be partitioned into a finite number N^d of cubes of side b . We have

$$\psi(x) \leq A \sum_{j=1}^{N^d} 1_{K_b^j}(x)$$

with $\{K_b^j\}$ being suitable translates of K_b . If $h^\psi(\omega) \geq \ell$, then for some j

$$\int h(\tau_{(0,y)}\omega) 1_{K_b^j}(y) dy \geq \frac{\ell}{AN^d}.$$

Since ψ is nice and $\int \psi(x) dx = 1$, it is easy to find c and N such that for any j ,

$$|\{x \in \mathbb{R}^d : |x| \leq a\sqrt{d}, c1_{K_b^j+x}(\cdot) \leq \psi(\cdot)\}| \geq c$$

We now have a lower bound

$$h^\psi(\tau_{(0,x)}\omega) = \int h(\tau_{(0,y)}\omega)\psi(y-x) dy \geq \frac{c\ell}{AN^d}.$$

provided $\psi(y-x) \geq c1_{K_b^j}(y)$, or $\psi(y) \geq c1_{K_b^j+x}(y)$

We have shown that there is a positive constant (call it again c) depending only on ψ such that, for any $\ell > 0$, if $h^\psi(\omega) \geq \ell$, then

$$|\{x \in \mathbb{R}^d : |x| \leq a\sqrt{d}, h^\psi(\tau_{(0,x)}\omega) \geq c\ell\}| \geq c.$$

□

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