

# THRESHOLD PHENOMENA FOR THE ANISOTROPIC CONTACT PROCESS

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## Abstract

This paper reviews results on the isotropic contact process on an infinite homogeneous tree and selected results of [6] on the symmetric anisotropic process on the tree  $\mathcal{T}_{2d}$  of degree  $2d$  for  $d > 1$ , while it is also meant to serve as a friendly guide through some ideas of the approach taken in [6] that borrows a collection of tools from dynamical systems and carries further an analogy between the contact process and the random walk on  $\mathcal{T}_{2d}$ . One goal is to gain insight into the picture of explosion of the symmetric anisotropic contact process at the weak/strong survival threshold, which may be quantified by saying that, at this threshold, the critical exponent of the log of the expected size of the set of all infected ends of the tree does not exceed  $1/2$ , with equality  $1/2$  holding if and only if the process is isotropic. We end with a discussion of the contact process on less regular trees, on subperiodic trees, and with a number of open problems.

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# 1 Introduction

## 1.1 Stochastic Growth Model Near a Discontinuous Threshold

The contact process, introduced by HARRIS [4] on  $\mathbf{Z}^d$  and surveyed to some extent in LIGGETT [16] (see also [14, 17]), serves to model epidemics, tumor growth, competition, and computer viruses on networks, to name a few. It is an example for a stochastic growth process which, along with branching random walks and percolation processes in spaces with hyperbolic geometries, exhibits an *intermediate phase* not present in the corresponding processes living in spaces with Euclidean geometry. This is the *weak survival region*, in which the infection survives forever with positive probability but, with probability one, eventually vacates every compact subset of the ambient space. Regular trees provide a fruitful “laboratory” for direct analysis of the contact process and the branching random walk along with their phase transitions.

The symmetric contact process on an infinite homogeneous tree  $\mathcal{T}_{2d}$  of degree  $2d$ , for  $d > 1$ , experiences an “explosion” at the *phase transition* between weak and strong survival, which geometrically is reflected upon the set of infected ends of  $\mathcal{T}_{2d}$  increasing from a very thin fractal set to a large set of ends – to the entire set of ends in the isotropic case – at the time when we cross the phase transition from weak survival. This picture may be quantified in that, at this threshold, the critical exponent of the log of the expected size of the set of all infected ends of the tree does not exceed  $1/2$ , with equality  $1/2$  holding if and only if the symmetric process is isotropic. We remark that in this note we view the contact process on  $\mathcal{T} = \mathcal{T}_{2d}$  of even degree in the *anisotropic* setup and the contact process on  $\mathcal{T} = \mathcal{T}_{d+1}$  for every integer  $d \geq 1$  in the *isotropic* setting.

This paper, as did some recent lectures I have presented on the subject, attempts to survey part of the literature on the isotropic contact process on infinite, locally finite trees and a variety of results in HUETER [6] on the symmetric anisotropic contact process, which describe the above-mentioned phenomenon at the discontinuous threshold. It is meant to be a friendly guide through [6] and to overview a few ideas of the approach taken up in [6] – while omitting the rather long technical details thereof – that combines a couple of vigorous concepts in dynamical systems with probabilistic machinery and invokes connections between the contact process and the random walk. The literature on related stochastic growth models, percolation, or models in statistical mechanics will not be addressed here so as not to exceed space.

Section 1 will introduce definitions and background on the contact process, while Section 2 will highlight some facts on the isotropic process and will motivate ideas that will throw further light on the anisotropic process in Section 5. The results in Section 3 will describe the behaviour of the process near criticality, near the discontinuous threshold. Section 4 will indicate the necessary background on Gibbs states, taken on the route of analysis in Section 5 and will point the reader to another example, the self-affine sets in which Gibbs states naturally arise. Section 6 will explain the analogies between the contact process and the (branching) random walk, uncovering an equation common to both, the contact process

and random walk on  $\mathcal{T}_{2d}$ . Section 7 will initiate a discussion on how the geometry of a graph may influence the phases. Can the contact process feel the geometry of a graph ?

## 1.2 Anisotropic Contact Process

We shall restrict our attention to a symmetric anisotropic contact process on an infinite homogeneous tree  $\mathcal{T}_{2d} = \mathcal{T}$  of degree  $2d$ . Each vertex  $x$  of  $\mathcal{T}$  has exactly  $2d - 1$  children. The tree  $\mathcal{T}$  is *homogeneous* in that for any two vertices  $x$  and  $y$  there is an isometry that maps  $x$  to  $y$ . Associated with each of the  $2d$  emanating edges is a nonnegative infection rate  $\lambda_i$ . An *anisotropic contact process* on the tree  $\mathcal{T}_{2d}$  is a continuous time Markov process  $A_t$  on the set of finite subsets of (the vertex set of)  $\mathcal{T}$  that evolves as follows. Infected sites (members of  $A_t$ ) recover at rate 1 and upon recovery are removed from  $A_t$ . Healthy sites (members of  $A_t^c$ ) become infected at a rate that equals the sum of the infection rates attached to the edges leading to infected nearest neighbours and upon infection are added to  $A_t$ . Under the default probability measure  $\mathbf{P}$ , the initial state  $A_0$  is the singleton set  $\{1\}$ , where 1 is a distinguished element of  $\mathcal{T}$ , called the *root* or *root vertex* of the tree. The rates mentioned belong to two independent sets of independent Poisson variables, one set of which is attached to the vertices and the other set of which is attached to the edges of  $\mathcal{T}$ , as explained in Section 1.5. The contact process can be shown to enjoy the strong Markov property (Section 1.5). The symmetry assumption that we will further describe in Section 1.3 below guarantees that the same infection rate be attached to an edge when the “infection crosses the edge” forwards and backwards. Specifically, the symmetry assumption guarantees that each infection rate be used twice for the set of emanating edges of each vertex. Hence, in the notion explained in Section 1.3 below, the set of infection rates is  $\lambda_{a_1}, \dots, \lambda_{a_d}, \lambda_{a_1^{-1}}, \dots, \lambda_{a_d^{-1}}$ , and for each of the  $d$  letters  $j$ , we assume  $\lambda_j = \lambda_{j^{-1}}$ . If the infection rates are all equal, then the contact process is called *isotropic*. Note that, alternatively, to define an anisotropic contact process, one may vary the recovery rates instead of the infection rates.

## 1.3 The Tree and $d$ Generators

From the perspective of an anisotropic contact process, different vertices of the underlying homogeneous tree at distance  $n$  from the root vertex look different. Therefore, it is useful to describe and identify each vertex of the tree by a finite address, a finite word from a finite alphabet. Even though to label the  $2d$  edges the alphabet needs  $2d$  letters in general, in the symmetric setting at hand,  $d$  letters will suffice. Thus, let  $\mathcal{A}_+ = \{a_1, a_2, \dots, a_d\}$  be a set of  $d$  letters, let  $\mathcal{A}_- = \{a_1^{-1}, a_2^{-1}, \dots, a_d^{-1}\}$  be the set of formal inverses of the letters in  $\mathcal{A}_+$ , and set  $\mathcal{A} = \mathcal{A}_+ \cup \mathcal{A}_-$ . The free group  $\mathcal{G}$  with generators  $\mathcal{A}_+$  is the set of finite reduced words from the alphabet  $\mathcal{A}$  (a word is *reduced* if no letter  $a \in \mathcal{A}$  is adjacent to its inverse), where multiplication is concatenation followed by reduction and the group identity 1 is the empty word. There is a natural bijection between  $\mathcal{G}$  and the set of vertices of  $\mathcal{T}$ , in which  $g, h \in \mathcal{G}$  are mapped to adjacent vertices of  $\mathcal{T}$  if and only if  $gh^{-1} \in \mathcal{A}$ . In other

words, vertices are uniquely represented by finite reduced words from  $\mathcal{A}$ . In the subsequent discussion, we shall not be careful to distinguish between vertices of  $\mathcal{T}$  and the words (or group elements) representing them, and we shall refer to  $\mathcal{G}$  as the vertex set of  $\mathcal{T}$ . For any vertex  $z$ , denote by  $|z|$  the length of its representative word. Note that  $|z|$  as well is the distance from vertex  $z$  to vertex 1 in the graph  $\mathcal{T}$ . For every integer  $n \geq 0$ , let  $\mathcal{G}_n$  denote the set of all vertices  $x \in \mathcal{G}$  at distance  $n$  from the root vertex (i.e.  $|x| = n$ ).

In a canonical way, the bijection between  $\mathcal{T}$  and  $\mathcal{G}$  induces a bijection between the natural boundary of  $\mathcal{T}$  and the *geometric boundary*  $\Omega$ , the set of infinite reduced words from the alphabet  $\mathcal{A}$ . A *geodesic* in  $\mathcal{T}$  is a finite or infinite sequence of distinct vertices  $v_1, v_2, \dots$  such that for every  $i \geq 1$ , the vertices  $v_i$  and  $v_{i+1}$  are nearest neighbours. An *end* of  $\mathcal{T}$  is an equivalence class of infinite geodesics, two geodesics being equivalent if and only if the sets of vertices through which they pass differ in at most finitely many vertices. If  $\omega = x_1 x_2 \dots \in \Omega$  then  $\omega$  corresponds to the end of  $\mathcal{T}$  represented by the infinite geodesic that passes through the vertices  $1, x_1, x_1 x_2, \dots$  in succession. For each real number  $\alpha \in (0, 1)$ , there is a natural metric  $d_\alpha$  on  $\Omega$ , defined by

$$d_\alpha(\omega, \omega') = \alpha^{N(\omega, \omega')}, \quad (1.1)$$

where  $N(\omega, \omega')$  is the largest integer  $n$  such that the sequences  $\omega$  and  $\omega'$  agree in entries  $1, 2, \dots, n$ . For any choice of  $\alpha$ , the corresponding topology on  $\Omega$  is the topology of coordinatewise convergence. For any vertex  $z$  of  $\mathcal{T}$ , define  $\mathcal{T}(z)$  to be the set of vertices  $v$  such that the geodesic segment from 1 to  $v$  passes through  $z$ , equivalently, such that the unique word representing  $z$  is a prefix of the word representing  $v$ . Similarly, define  $\Omega(z)$  to be the set of infinite reduced words  $\omega = x_1 x_2 \dots$  such that, for some finite  $n$ , the word  $z$  is represented by the word  $x_1 x_2 \dots x_n$ . Observe that, for every integer  $n \geq 1$ , the set  $\{\Omega(z) : |z| = n\}$  is a finite open cover of the geometric boundary  $\Omega$ . Finally, define  $\Sigma$  to be the set of all doubly infinite reduced words  $\xi = (x_n)_{n=-\infty}^\infty$  from  $\mathcal{A}$ .

We like to point out that, if  $\lambda_{a_1}, \dots, \lambda_{a_d}, \lambda_{a_1^{-1}}, \dots, \lambda_{a_d^{-1}}$  denotes the set of infection rates, then the *symmetric* anisotropic contact process will mean that, for each of the  $d$  letters  $j$ , we assume  $\lambda_j = \lambda_{j^{-1}}$ .

## 1.4 Three Regimes

The symmetric contact process on a homogeneous tree of degree 3 or larger with infection rates  $\{\lambda_i\}_{i \in \mathcal{A}_+}$  distinguishes itself from the symmetric process on the integer lattice  $\mathbf{Z}^d$  in that there are two different survival regions. We start with some definitions. The contact process is said to become *extinct* (or die out) if, with probability one,  $A_t = \emptyset$  eventually. The contact process is said to *survive strongly* if with positive probability,  $|A_t| \rightarrow \infty$  and, for all  $x \in \mathcal{T}$ , for arbitrarily large values of  $t$ , we have  $x \in A_t$ . If the contact process does not become extinct, and thus, survives yet does not survive strongly, then it is said to *survive weakly*. Equivalently, strong survival is often called *local survival* and weak survival is often called *global survival* (because the process survives somewhere on the graph but

not locally). More formally, on  $\mathcal{T}_{2d}$  for  $d > 1$ , there is a partition of the parameter space  $[0, \infty)^d = \mathbf{R}_+^d \ni \lambda = (\{\lambda_a\}_{a \in \mathcal{A}_+})$  into three regions  $\mathcal{R}_1, \mathcal{R}_2$  and  $\mathcal{R}_3$  – called *extinction phase*, *weak survival phase* and *strong survival phase* – such that

- (a) if  $\lambda \in \mathcal{R}_1$ , then  $A_t = \emptyset$  eventually with probability 1,
- (b) if  $\lambda \in \mathcal{R}_2$ , then  $\mathbf{P}(|A_t| \rightarrow \infty) > 0$ , but  $\forall x \in \mathcal{T}$ ,  $\mathbf{P}(x \in A_t \text{ for arbitrarily large } t) = 0$ ,
- (c) if  $\lambda \in \mathcal{R}_3$ , then with positive probability,  $|A_t| \rightarrow \infty$  and, for all  $x \in \mathcal{T}$ , for arbitrarily large values of  $t$ , we have  $x \in A_t$ .

On  $\mathcal{R}_1$ , the contact process is called *subcritical*, on  $\mathcal{R}_2$ , *weakly supercritical*, and on  $\mathcal{R}_3$ , *strongly supercritical*. Note that the definitions do not depend on  $x$  since  $\mathcal{T}$  is a connected graph. On  $\mathbf{Z}^d$ , we observe that  $\mathcal{R}_2 = \emptyset$  and that the contact process dies out at the positive finite critical value (BEZUIDENHOUT AND GRIMMETT [1]). Observe that the contact process is not stable, that is, the dichotomy arises that either  $|A_t| \rightarrow \infty$  or  $|A_t| \rightarrow 0$  as  $n \rightarrow \infty$ , that is, with probability zero,  $|A_t| = k$  infinitely often for  $0 < k < \infty$ . This may be seen as follows. Fix an integer  $k \geq 1$ . Consider the sequence of times when  $|A_t|$  returns to  $k$ , defined by  $T_0 = -1$  and  $T_n = \inf\{t \geq T_{n-1} + 1 : |A_t| = k\}$  for every integer  $n \geq 1$ . If  $T_n = \infty$  for some  $n \geq 1$ , then set  $T_m = \infty$  for  $m > n$ . There is positive probability  $p_k$  that, if there are at most  $k$  infected vertices, the contact process becomes extinct in less than one unit of time. By the strong Markov property, we collect  $\mathbf{P}(T_n < \infty | T_{n-1} < \infty) \leq 1 - p_k$ . In light of

$$\begin{aligned} \mathbf{P}(T_n < \infty) &\leq (1 - p_k) \mathbf{P}(T_{n-1} < \infty) \\ &\leq (1 - p_k)^{n-1} \mathbf{P}(T_1 < \infty) \\ &\leq (1 - p_k)^{n-1} (1 - p_1) \end{aligned}$$

and the fact that  $\{T_n < \infty\}$  is a decreasing sequence, we arrive at

$$\lim_{n \rightarrow \infty} \mathbf{P}(T_n < \infty) = \mathbf{P}(\bigcap_{n \geq 1} \{T_n < \infty\}) = 0.$$

Since  $k > 0$  was arbitrary, we conclude that the contact process will stop returning to every  $k \geq 1$  with probability one. Furthermore on the event of survival, with probability zero,  $|A_t| = 0$ . Consequently, on the event of survival,  $|A_t| \rightarrow \infty$  as  $n \rightarrow \infty$ , with probability one.

The contact process has been a subject of intense study, initially only on the integer lattices, since it was introduced by HARRIS [4] about three decades ago. Motivated by observations in percolation, PEMANTLE [20] demonstrated that there are two distinct (non-degenerate) critical values  $\lambda_1 < \lambda_2$ , and thus, there is weak survival between those two values for the isotropic contact process on  $\mathcal{T}_{d+1}$  if  $d \geq 3$ . LIGGETT [15] established weak survival for the binary tree (the case  $d = 2$ ) while STACEY [24] gave an alternative proof for weak survival of the process on  $\mathcal{T}_{d+1}$  for every  $d \geq 2$ . HUETER [6] showed that the symmetric anisotropic contact process on  $\mathcal{T}_{2d}$  has weak survival when  $d \geq 2$  (see also Section 3). In

addition, the *critical* isotropic contact process on  $\mathcal{T}_{d+1}$  at the first critical value  $\lambda_1$  becomes extinct (see PEMANTLE [20] for  $d \geq 3$ , MORROW, SCHINAZI, AND ZHANG [19] for  $d \geq 2$ , and HUETER [6] for the symmetric anisotropic process on  $\mathcal{T}_{2d}$  for  $d \geq 2$ ) and at the second critical value  $\lambda_2$  survives weakly (a consequence of results of LALLEY AND SELLKE [13] in the isotropic case and HUETER [6] in the symmetric anisotropic case).

## 1.5 Percolation Structure and Strong Markov Property

**Percolation structure.** The contact process may be constructed via the usual *percolation structure* (HARRIS [5]) on  $\mathcal{T} \times (0, \infty)$ , that is, as a system of independent Poisson processes attached to vertices and ordered pairs of neighbouring vertices. For each vertex  $x \in \mathcal{T}$ , the Poisson process attached to  $x$  has rate 1 and determines the recovery times, specifically, at every occurrence time, site  $x$  recovers if it is infected. For each ordered pair  $(x, xi)$ ,  $i \in \mathcal{A}$ , of neighbouring vertices, the Poisson process attached to  $(x, xi)$  has rate  $\lambda_i$ , the occurrence times being precisely those times when an infection at  $x$  may jump to  $xi$ . Occurrences in these Poisson processes are marked on a system of directed rays  $\{x\} \times [0, \infty)$  connected to the vertices  $x$  of  $\mathcal{T}$ , so that (A) at each occurrence time  $t$  of the Poisson process attached to  $(x, xi)$  an *infection arrow* is drawn from  $(x, t)$  to  $(xi, t)$ , and (B) at each occurrence time  $t$  of the Poisson process attached to  $x$  a *recovery mark* \* is attached to  $(x, t)$ . There are no simultaneous occurrences of infection arrows and/or recovery marks in the percolation structure. At time  $t$ , the contact process now consists of all those vertices  $y$  for which there is a (directed) path through the percolation structure, the system of rays and arrows just described, that begins at the root vertex 1, ends at  $(y, t)$ , and does not pass through any recovery marks \*. An *infection trail* is a connected path in the percolation structure that does not pass through any recovery marks.

**Strong Markov Property.** The contact process exhibits the strong Markov property (see LALLEY AND SELLKE [13], Section 2.2). Let  $G, F_1, F_2, \dots, F_k$  be pairwise nonoverlapping parts of the tree  $\mathcal{T}$ , and let  $S_1, S_2, \dots, S_k$  be stopping times determined by the percolation structure over  $G$ . Then conditional on the percolation structure over  $G$ , the post- $S_i$  portions of the percolation structures over the sets  $F_i$  are independent, and for each  $i$ , the post- $S_i$  percolation structure over  $F_i$  has the same distribution as the entire percolation structure over  $F_i$ .

## 2 Some Facts about the Isotropic Contact Process on $\mathcal{T}_{d+1}$

We return to the isotropic case and let  $\lambda > 0$  denote the infection rate and  $0 < \lambda_1 < \lambda_2 < \infty$  the critical values of the contact process for  $d > 1$ . It turns out that the set of all vertices at distance  $n$  that are ever infected plays a significant role. First, the exponential rate of its size enables us to recognize the one among the 3 phases in which the process is living. Second, it tells us about the nature of phase transition. Specifically, we shall be interested in the size of jump of its cardinality at the second phase transition. Recall  $\mathcal{G}_n$  to be the

set of all vertices in  $\mathcal{T}_{d+1}$  at distance  $n$  from the root vertex. For any fixed vertex  $x \in \mathcal{G}_n$ , define

$$u_n = u_n(\lambda) = \mathbf{P}(x \in A_t \text{ for some } t > 0).$$

This probability is a function in  $n$  (but does not depend on the particular vertex  $x$  at distance  $n$ ). Since in order for the infection to reach distance  $n + m$ , it first needs to reach distance  $n$  and continue to travel for distance  $m$  and in view of a possibly created backbone of the infection, the monotonicity properties and the Markov property of the contact process, we obtain  $u_{n+m} \geq u_n u_m$  for all  $m, n \geq 1$ . A supermultiplicativity argument shows that

$$\lim_{n \rightarrow \infty} u_n(\lambda)^{1/n} = \beta = \beta(\lambda) \tag{2.1}$$

exists and that  $u_n(\lambda) \leq \beta(\lambda)^n$  for all  $n \geq 0$ . The function  $\beta$  is strictly increasing in  $\lambda$  on  $(0, \lambda_2)$  (LALLEY [12], Proposition 9) and is continuous on  $(0, \lambda_2)$  and left-continuous everywhere (SCHONMANN [23], Theorem 2). Let  $\mathbf{E}$  denote expectation under the probability measure  $\mathbf{P}$ . Let  $\mathcal{Y}_n$  denote the set of all vertices in  $\mathcal{G}_n$  that are ever infected. The expected value  $\mathbf{E}|\mathcal{Y}_n|$  is related to the function  $\beta$  via

$$\begin{aligned} \mathbf{E}|\mathcal{Y}_n| &= \mathbf{E} \sum_{x \in \mathcal{G}_n} 1_{\{x \text{ is ever infected}\}} \\ &= \sum_{x \in \mathcal{G}_n} \mathbf{P}(x \text{ is ever infected}) \\ &= \sum_{x \in \mathcal{G}_n} u_n(\lambda) \\ &= d^n u_n(\lambda). \end{aligned}$$

Hence, for every  $\varepsilon > 0$  and all sufficiently large  $n$ ,

$$(d\beta[1 - \varepsilon])^n \leq \mathbf{E}|\mathcal{Y}_n| \leq (d\beta)^n \tag{2.2}$$

and

$$\lim_{n \rightarrow \infty} (\mathbf{E}|\mathcal{Y}_n|)^{1/n} = d\beta(\lambda). \tag{2.3}$$

Write  $\theta_1(\lambda) = (\mathbf{E}|\mathcal{Y}_n|)^{1/n} = d\beta(\lambda)$ . We have seen in (2.2) that  $\mathbf{E}|\mathcal{Y}_n|$  decays or increases exponentially unless  $\theta = 1$ . More precisely, by embedding a Galton-Watson tree with mean offspring number  $\theta_1$  in the set of infected vertices of the contact process, by relying on the notion of “downward infection trails”, and by embedding the set of infected vertices of the contact process in a Galton-Watson tree with mean offspring number  $\theta_1$ , respectively (see [6], Sections 2 and 5 for constructions of those GW trees), it follows that  $\lambda > \lambda_1$  implies  $\theta_1 > 1$  and  $\lambda < \lambda_1$  implies  $\theta_1 \leq 1$ . Since  $\lambda_1 < \lambda_2$  and the function  $\beta$  is continuous at  $\lambda_1$ , it follows that  $\beta(\lambda_1) = 1/d$ ,  $\beta(\lambda) < 1/d$  for  $\lambda < \lambda_1$ , and  $\beta(\lambda) > 1/d$  for  $\lambda > \lambda_1$ . We shall see ((3.2) below) that  $\beta(\lambda_2) = 1/\sqrt{d}$  and  $\beta$  is discontinuous at  $\lambda_2$ . As a consequence, an enormous explosion is expected to happen for  $A_t$ , that is, the size of the jump of  $\mathbf{E}|\mathcal{Y}_n|$  at  $\lambda_2$  is such that  $\mathbf{E}|\mathcal{Y}_n|$  increases to its square and  $\theta_1(\lambda_2) = \sqrt{d}$  and  $\theta_1(\lambda_2+) = d$ . In other words,

$\mathbf{E}|\mathcal{Y}_n|$  grows like  $d^{n/2}$  when  $\lambda = \lambda_2$  and  $\mathbf{E}|\mathcal{Y}_n|$  grows like  $d^n$  when  $\lambda = \lambda_2 + \cdot$ . The quantity  $\theta_1$  will also indicate the size of the set of all infected ends of the tree, more precisely, its Hausdorff dimension (consult Theorem 3 below).

In the anisotropic setup, more care is required, though, to come up with functions analogous to those introduced in the isotropic framework (consult also [6]). We shall survey and summarize the method detailed in [6] without writing out the technical steps and details to keep the exposition gentle.

### 3 Anisotropic Process Crossing from Weak Survival

The anisotropic contact process has a richer structure than the isotropic one. The question arises on how the isotropic process is embedded in the anisotropic world, especially, how it is behaved near the second phase transition  $\overline{\mathcal{R}}_2 \cap \overline{\mathcal{R}}_3$  as compared to the non-isotropic process. What does the weak/strong survival phase transition look like when crossed from either side of the transition? Our starting point is the result that the symmetric anisotropic process with infection vector  $\lambda \in \mathbf{R}_+^d$  on  $\mathcal{T}_{2d}$ ,  $d > 1$ , has weak survival. (The asymmetric case is more complicated. But this avenue is treated elsewhere.)

**Theorem 1 (Hueter [6])** *Consider the symmetric anisotropic contact process on  $\mathcal{T}_{2d}$ . The weak survival region  $\mathcal{R}_2$  is nonempty unless  $d = 1$  and enjoys the following properties:*

(a) *Every line in the interior of the first quadrant in  $\mathbf{R}^d$  that passes through the origin has an intersection with  $\mathcal{R}_2$  that is a line segment,* (b) *the region  $\mathcal{R}_2$  has positive  $d$ -dimensional Lebesgue measure, and* (c)  *$\mathcal{R}_2$  is a connected region and is symmetric in the  $d$  parameters  $\lambda_i$ .*

**Theorem 2 (Hueter [6])** *The critical symmetric anisotropic contact process on  $\mathcal{T}_{2d}$  for  $d > 1$  at  $\overline{\mathcal{R}}_1 \cap \overline{\mathcal{R}}_2$  becomes extinct and at  $\overline{\mathcal{R}}_2 \cap \overline{\mathcal{R}}_3$  survives weakly.*

In the weak survival region  $\mathcal{R}_2$ , with probability one, the infection eventually vacates every finite subset of vertices of the tree and can only survive in the boundary of the tree. Hence, it has a well-defined limit set. Define the *limit set*  $\Lambda$  of the contact process on  $\mathcal{T}$  to be the (random) set of  $\omega = x_1 x_2 \dots \in \Omega$  such that each vertex  $x_1 x_2 \dots x_k$  of  $\omega$  is infected at some time. If the contact process is supercritical, then on the event of survival,  $\Lambda$  is nonempty and compact (relative to any of the metrics  $d_\alpha$  defined in (1.1)).

**Theorem 3 (Hueter [6])** *For  $\lambda \in \mathcal{R}_2$ , almost surely on the event of survival, the Hausdorff dimension  $\delta(\lambda)$  of  $\Lambda$  (relative to the metric  $d_\alpha$ ) is given by*

$$\delta(\lambda) = -\frac{\log \theta_1(\lambda)}{\log \alpha}, \quad (3.1)$$

where  $\theta_1(\lambda)$  is the top eigenvalue of a Perron-Frobenius transition matrix of infection probabilities of the contact process (as explained later in Section 5). Furthermore, if  $\delta_H(\Omega)$  denotes the Hausdorff dimension of  $\Omega$ ,

$$\delta(\lambda) \leq \frac{1}{2} \delta_H(\Omega), \quad (3.2)$$

with equality holding if and only if the underlying symmetric contact process is isotropic and  $\lambda \in \overline{\mathcal{R}}_2 \cap \overline{\mathcal{R}}_3$  (at the transition between weak and strong survival).

The equality in (3.2) confirms a conjecture that was first raised in [13] pertaining to the behaviour of the isotropic contact process at the weak/strong survival threshold. Particularly, for the isotropic process on  $\mathcal{T}_{d+1}$ , we collect  $\beta(\lambda_2) = 1/\sqrt{d}$ .

To give the reader an idea why equality in (3.2) may make sense for the isotropic case, we present some heuristic argument on  $\mathcal{T}_{d+1}$ . Analytic building blocks to prove (3.2) will follow. The *heuristic* reasoning goes as follows: Count the number of infection trails that reach distance  $n$  and return to the root of the tree at some time. The expected cardinality of this set would have asymptotic exponential order  $(d\beta^2)^n$  if we neglect the fact that possibly it is easier for the infection to move back to the root from distance  $n$  provided that some vertices on the return path may not have recovered yet. Therefore, the estimate  $(d\beta^2)^n$  is a lower bound. The analysis deals with the control of the dependencies to show that the lower bound is an upper bound as well. On the one hand, this expected number would grow exponentially large as  $n \rightarrow \infty$  if  $d\beta^2 > 1$ , and thereby, the contact process would surely survive locally. On the other hand, the expected number would decrease exponentially fast to zero as  $n \rightarrow \infty$  if  $d\beta^2 < 1$ , and whence, the contact process might not survive locally. Hence, setting this asymptotic order of the expected number of infection trails returning to the root from distance  $n$  equal 1 right at the threshold to local survival yields the value  $1/\sqrt{d}$  for  $\beta(\lambda_2)$ .

The *analytical* counterparts are the following. There will be a second leading eigenvalue  $\theta_2$  of a matrix that describes the size of the set of all infection trails that reach distance  $n$  for large  $n$  and return to the root of the tree. As a consequence of the Hölder inequality, this eigenvalue satisfies the inequality

$$\theta_1(\lambda)^2 \leq d\theta_2(\lambda),$$

with  $\theta_2(\lambda) = 1$  at  $\overline{\mathcal{R}}_2 \cap \overline{\mathcal{R}}_3$  (see Proposition 2 later). In company with some estimates gotten from the Gibbs variational principle (see at the end of Section 5 as well), strict inequality will be in force in the last display for the anisotropic case and equality for the isotropic one.

After this digression, we turn to define the anisotropic particulars. For any vertex  $x \in \mathcal{T}$ , define

$$u_x = u_x(\lambda) = \mathbf{P}(x \in A_t \text{ for some } t > 0).$$

Thus, this probability is a function of vertex  $x$  (and not merely of the distance of the vertex). It is easily seen that we maintain

$$\mathbf{E}|\mathcal{Y}_n| = \sum_{x \in \mathcal{G}_n} u_x(\lambda).$$

By the same argument as earlier along with the homogeneity of the vertices from the perspective of the contact process, when concatenating any two addresses  $x$  and  $y$  in  $\mathcal{T}$  (so that no reduction occurs, i.e. such that  $|xy| = |x| + |y|$ ), we have  $u_{xy} \geq u_x u_y$ . A

supermultiplicativity argument demonstrates that, for every  $a \in \mathcal{A}_+$  and each vertex  $x = aa \dots a \in \mathcal{G}_n$ ,

$$\lim_{|x|=n \rightarrow \infty} u_x(\lambda)^{1/n} = \beta_a = \beta_a(\lambda) \quad (3.3)$$

exists and that  $u_x(\lambda) \leq \beta_a(\lambda)^n$  for all  $n \geq 0$ . Since the remainder of this paper will revolve around weak survival, we will assume that  $d \geq 2$ . More precisely, in the subsequent discussion in which we focus on the case  $d \geq 2$ , it is assumed without saying that there is some  $\epsilon > 0$  such that at least two infection rates  $\lambda_i, \lambda_j \geq \epsilon$  for two distinct  $i, j \in \mathcal{A}_+$ . If  $\lambda$  is restricted to those  $\lambda$ , then it can be shown that there is some  $\rho = \rho(\epsilon) < 1$  such that, for  $\lambda \in \mathcal{R}_3^c$ , each  $\beta_i(\lambda) \leq \rho < 1$ . In words, each  $\beta_i$  is bounded away from 1 on the complement of strong survival.

Even though periodic words  $z$  give rise to limits  $u_z^{1/|z|}$  as  $|z| \rightarrow \infty$ , in general, those limits  $u_z^{1/|z|}$  as  $|z| \rightarrow \infty$  do not exist. However, the entire collection  $\{u_x\}_{x \in \mathcal{T}}$  jointly converges in some sense, the reason being the geometric decay of the  $u_x(\lambda)$ . The following result that we borrow from [6], Section 2, is essential towards proving this convergence.

**Lemma 1** *For each  $\lambda \in \mathcal{R}_3^c$  and each  $x \in \mathcal{G}_n$ , there is some constant  $0 < \gamma < 1$  such that*

$$u_x(\lambda) \leq [\max_{a \in \mathcal{A}_+} \beta_a(\lambda)]^n \leq \gamma^n.$$

**Proof.** The proof proceeds by induction over the distance of the vertices from the root. Fix  $\lambda$ . Without loss of generality, we may assume that  $\lambda_b \geq \lambda_i$  for every  $i \in \mathcal{A}$ . Let  $y_n = bb \dots b$  with  $|y_n| = n$ . For  $n = 1$ , clearly,  $u_{y_1}(\lambda) = u_b(\lambda) \geq u_i(\lambda)$  for every  $i \in \mathcal{A}$  by the definition of the functions  $u_x(\lambda)$  and the one of the rules of infection of the contact process. Now assume that  $u_{y_k}(\lambda) \geq u_x(\lambda)$  for every  $x \in \mathcal{G}_k$  and every  $k = 1, 2, \dots, n-1$ . Thus, no  $x \in \mathcal{G}_{n-1}$  has larger probability than  $y_{n-1}$  to ever be infected. But upon infection of  $y_{n-1}$ , no vertex in  $\mathcal{G}_n$  has larger probability to be infected than  $y_n$  (by the same means as a few lines earlier). Hence,  $u_{y_n}(\lambda) \geq u_x(\lambda)$  for every  $x \in \mathcal{G}_n$ . Since this argument is valid for every integer  $n > 0$ , the desired results follow from the fact that  $u_{y_n}(\lambda) \leq \beta_b(\lambda)^n$  for every  $n \geq 0$ . It remains to recognize that the statement about the second inequality follows from the remark preceding Lemma 1. This ends the proof.  $\square$

In order to describe the behaviour of the transition probabilities  $u_{xy}/u_x$  of the infection for any  $x, y \in \mathcal{T}_{2d}$  and  $|y|$  sufficiently large, the conditional probability that  $xy$  is ever infected, given the event that  $x$  is ever infected, we shall construct some “first transition” matrices to capture the event of “first passage” of the infection to a vertex. Since, for fixed  $x$ , these quotients depend on  $x$ , while in the isotropic case they depend only on  $|x|$ , those matrices will depend on  $x$ . In light of the geometric decay explained in Lemma 1, we can construct some Hölder continuous Perron-Frobenius matrices, and thus, potential functions on the set of all doubly infinite reduced words  $\Sigma$  with associated Gibbs measures. Once a probability measure is defined on  $\Sigma$ , it will make sense to define a “typical” matrix relative to this measure so as to have access to some distinguished matrix that is uniformly defined for the entire tree  $\mathcal{T}$ .

## 4 Potential Functions and Gibbs Measures

Let  $\sigma : \Sigma \rightarrow \Sigma$  denote the forward shift on  $\Sigma$ , the set of all doubly infinite reduced words, that is,  $\sigma(x_k x_{k+1} \dots) = x_{k+1} x_{k+2} \dots$  for every  $k$ . Note that  $\sigma$  is Lipschitz continuous. For any function  $f : \Sigma \rightarrow \mathbf{R}$  and for every  $n \geq 1$ , define the partial sums

$$S_n f = f + f \circ \sigma + f \circ \sigma^2 + \dots + f \circ \sigma^{n-1}.$$

Define the  $n$ -cylinder sets

$$\Gamma(i) = \Gamma(i_1 i_2 \dots i_n) = \{\xi = (x_k)_{k=-\infty}^{\infty} \in \Sigma : x_j = i_j, 1 \leq j \leq n\}$$

for  $i = i_1 i_2 \dots i_n \in \mathcal{G}_n$ . In spirit of BOWEN [2], for any Hölder continuous function  $f$  on  $\Sigma$ , there is a real constant  $P(f)$ , some constants  $0 < C_1 \leq C_2 < \infty$ , and a *unique*  $\sigma$ -invariant probability measure  $\mu_f$  on the Borel sets of  $\Sigma$  such that for each  $i = i_1 i_2 \dots i_n \in \mathcal{G}_n$ ,

$$C_1 \leq \frac{\mu_f(\Gamma(i))}{\exp\{S_n f(j) - nP(f)\}} \leq C_2 \quad (4.1)$$

for every  $j \in \Gamma(i)$ . The measure  $\mu_f$  is called the *Gibbs state* (or Gibbs measure) with *potential function*  $f$  and the constant  $P(f)$  is called the *pressure* of  $f$ . The usefulness of Gibbs measures is reflected upon the rather precise exponential estimates (4.1). Importantly, this expression simplifies further when we choose the pressure equal zero, which we manage to do if  $f$  is negative. Namely, if  $f < 0$ , then there exists a unique constant  $\delta > 0$  such that

$$P(\delta f) = 0. \quad (4.2)$$

It follows that the Gibbs state  $\mu_{\delta f}$  (up to multiplicative constants) of an  $n$ -cylinder equals the exponential  $\exp\{\delta S_n f\}$  of the additive functional  $S_n f$ , which we may choose freely. The pressure function comes with other good qualities. For example, the function  $a \rightarrow P(af)$  is continuous (RUELLE [22]). For every Hölder continuous  $f$  and for every integer  $n \geq 1$ , the pressure functional satisfies  $P(S_n f) = nP(f)$ . For more background the reader is referred to BOWEN [2]. Next, we mention an example (Example I) in another context in which Gibbs states bear some natural meaning and we motivate their roles for the contact process on  $\mathcal{T}$  in Example II prior to discussing the relevant details in Section 5.

**Example I: Self-Affine Fractals and their Gibbs States.** Let  $A_1, A_2, \dots, A_K$  be a set of contractive, affine, invertible self-mappings of  $\mathbf{R}^2$  and let  $\tilde{\Lambda}$  be the unique compact subset in  $\mathbf{R}^2$  that satisfies

$$\tilde{\Lambda} = \bigcup_{i=1}^K A_i(\tilde{\Lambda}),$$

called the *self-affine* set with affinities  $A_1, A_2, \dots, A_K$ . If  $A_i = T_i + \alpha_i$  for some translation vectors  $\alpha_i$ , thus,  $T_i$  is identified with an invertible, strictly contractive  $2 \times 2$  matrix, say, with strictly positive entries and  $\alpha(T_j)$  denotes half the length of the major axis of the

ellipse  $T_j(B)$ , where  $B$  is the unit circle, then we may define a potential function  $\varphi : \cup_{n=0}^{\infty} \{1, 2, \dots, K\}^n \rightarrow \mathbf{R}$  by

$$\alpha(T_{i_1} T_{i_2} \dots T_{i_n}) = \exp\{S_n \varphi(\mathbf{i})\}$$

for  $\mathbf{i} = i_1 i_2 \dots$  (see HUETER AND LALLEY [9]). The function  $\varphi < 0$  can be shown to be Hölder continuous with some exponent on the appropriate completion of the infinite code space. Hence, there is a number  $\delta > 0$  such that the pressure  $P(\delta\varphi) = 0$  and a unique  $\sigma$ -invariant probability measure  $\mu_{\delta\varphi}$  on the Borel sets of  $\{1, 2, \dots, K\}^{\mathbf{N}}$  that satisfies equation (4.1) with  $P(\delta\varphi) = 0$ . If  $\pi : \{1, 2, \dots, K\}^{\mathbf{N}} \rightarrow \tilde{\Lambda}$  denotes the projection of the sequence space onto the self-affine set defined by  $\pi(\mathbf{i}) = \lim_{n \rightarrow \infty} A_{i_1} A_{i_2} \dots A_{i_n} y$ , where  $\mathbf{i} = i_1 i_2 \dots \in \{1, 2, \dots, K\}^{\mathbf{N}}$  and  $y$  is any point in  $\mathbf{R}^2$ , then, under some checkable conditions on the  $T_i$ , the image under the projection  $\pi$  of the Gibbs state  $\mu_{\delta\varphi}$  is the *unique* ergodic shift-invariant probability measure on  $\tilde{\Lambda}$  that has *full* Hausdorff dimension (see Theorem 1.2 in HUETER AND LALLEY [9]). The latter quality to carry such a measure having maximal and full dimension is a desirable property for a set in connection with a dynamical system. Moreover,  $\bar{\varphi} = \int \varphi \mu_{\delta\varphi}$  represents the exponential rate of half the length of the major axis of “typical” ellipses relative to  $\mu_{\delta\varphi}$ .

**Example II: Contact Process and its Gibbs States.** In case of the contact process, some potential function  $\varphi = \varphi_\rho : \Sigma \rightarrow \mathbf{R}$  will be defined as the logarithm of the leading eigenvalue of some Hölder continuous matrix associated with the vertices of the tree  $\mathcal{T}$  (consult Section 5). This will lead to Gibbs states and some typical matrix relative to the distinguished Gibbs state with zero pressure. In summary, the point of this principle is that some naturally arising quantities in a system, in our case, transition matrices can be brought into play to construct probability measures that help to better understand the dynamics of a process that evolves over time, which in turn will hint at the typical behaviour, in our case at the typical matrices of the process. We postpone further discussion to Section 5.

Yet another use of the concept of Gibbs states is their roles as stationary measures for the contact process. Consider the limit points  $\omega = \omega_1 \omega_2 \dots \in \Omega$  in the weak survival phase. Define the distribution  $\mu_n$  under  $\mathbf{P}$  of the process  $\omega_n, \omega_{n+1}, \dots$ , that is, for any Borel set  $V \subset \Omega$ ,

$$\mu_n(V) = \mathbf{P}((\omega_n, \omega_{n+1}, \dots) \in V). \quad (4.3)$$

In [6], Section 3, a function  $b : \Sigma \rightarrow \mathbf{R}$  was constructed and it was shown that there are some constants  $0 < c_1 \leq c_2 < \infty$  and  $0 < \alpha < 1$  such that the function  $\varphi : \Sigma \rightarrow \mathbf{R}$  defined by  $\varphi(\dots x_1 x_2 \dots) = \log b_{x_1}$  satisfies, for every  $x \in \Sigma$ ,

$$\mathbf{P}(\omega_j = x_j \text{ for each } 1 \leq j \leq m) = C \exp\{S_m \varphi(x)\} (1 + O(\alpha^m)) \quad (4.4)$$

for some constant  $C$  that may be bounded by  $c_1 \leq C \leq c_2$  with  $c_i$  independent of  $x$  and  $m$ . Also, there are some constants  $0 < C_1 \leq C_2 < \infty$  and a unique  $\sigma$ -invariant probability

measure  $\mu_\varphi$  on the Borel sets of  $\Sigma$  such that

$$C_1 \leq \frac{\mu_\varphi(\Gamma(x_1 x_2 \dots x_m))}{\exp\{S_m \varphi(j) - mP(\varphi)\}} \leq C_2$$

for every  $j$  in the cylinder set  $\Gamma(x_1 x_2 \dots x_m)$ . Since it turns out that  $\varphi < 0$ , there exists a unique  $\delta > 0$  such that the pressure  $P(\delta\varphi) = 0$ .

**Theorem 4 (Hueter [6])** *For  $\lambda \in \mathcal{R}_2$  and every  $n \geq 1$ , the measure  $\mu_n$  is absolutely continuous with respect to  $\mu_\varphi$  and  $\mu_n \xrightarrow{\mathcal{D}} \mu_\varphi$  as  $n \rightarrow \infty$ . Furthermore,  $P(\varphi) = 0$ .*

Thus, the stochastic process  $\omega_1, \omega_2, \dots$  is asymptotically stationary, that is, the joint distribution of  $\omega_n, \omega_{n+1}, \dots$  converges to that one of a stationary process as  $n \rightarrow \infty$ . The limiting process is a Gibbs state, hence, isomorphic to a Bernoulli shift, but in general non-Markovian. From (4.4), we see that each stationary distribution decays exponentially in the distance  $m$  from the root. The rate at which the measure of a cylinder set  $\Gamma(x_1 x_2 \dots x_m)$  decays depends on the relative frequency of the generators in the reduced word  $x_1 x_2 \dots x_m$  and the order of the letters. It is easy to show that, at least in the symmetric case, the distribution is *spherically symmetric* if and only if the contact process is isotropic.

## 5 First-Passage and Backscatter Matrices

After presenting some background and ideas on the isotropic process in Section 3, it is hoped that we properly motivated the usefulness of the tools that we shall introduce to study the anisotropic process and mimic some objects traditionally applied to deal with random walks. For every  $\lambda$ , define

$$r_u = r_u(\lambda) = \inf\{r > 0 : \sum_{m=0}^{\infty} \sum_{x \in \mathcal{G}_m} u_x(\lambda)^r < \infty\}. \quad (5.1)$$

This exponent  $r_u(\lambda)$  takes some finite value for  $\lambda \in \text{int}(\mathcal{R}_1 \cup \mathcal{R}_2)$  and is nondecreasing in each  $\lambda_j$  because  $u_x(\lambda)$  is nondecreasing in each  $\lambda_j$ . For instance, it is easy to see that, for  $\lambda \notin \mathcal{R}_1$ , we must have  $r_u(\lambda) \geq 1$ . Some intuitive values yet subtle to prove (see [6]) are: for  $d > 1$ , we obtain  $r_u(\lambda) = 1$  for  $\lambda \in \overline{\mathcal{R}}_1 \cap \overline{\mathcal{R}}_2$  and  $r_u(\lambda) = 2$  for  $\lambda \in \overline{\mathcal{R}}_2 \cap \overline{\mathcal{R}}_3$ . Another easy but crucial fact [6] is that for each  $t > r_u(\lambda)$ , we have  $\sum_{m=0}^{\infty} \sum_{x \in \mathcal{G}_m} u_x(\lambda)^t < \infty$ . Next, for each  $x_1 x_2 \dots x_{k-1} \in \mathcal{G}_{k-1}$  and for all  $i, j \in \mathcal{A}$  with  $i \neq x_{k-1}^{-1}$ , define

$$\mathcal{F}_{ij} = \mathcal{F}(x_1 \dots x_{k-1}, i, j) = \{x_{k+1} \dots x_{n-1} \in \mathcal{G}_{n-k-1} : x_{k+1} \neq i^{-1}, x_{n-1} \neq j^{-1}\} \quad (5.2)$$

and let  $\mathcal{F}(x_1 \dots x_{k-1}, i, j)$  be the empty set if  $i = x_{k-1}^{-1}$ . Then, for each real  $\rho > 0$ , all integers  $n - 1 > k \geq 1$ , and each  $x_1 x_2 \dots x_{k-1} \in \mathcal{G}_{k-1}$ , define the  $2d \times 2d$  matrix  $H_\rho(n; x_1 x_2 \dots x_{k-1}; \lambda)$  by

$$(H_\rho(n; x_1 x_2 \dots x_{k-1}; \lambda))_{ij} = \sum_{x_{k+1} x_{k+2} \dots x_{n-1} \in \mathcal{F}_{ij}} \left( \frac{u_{x_1 x_2 \dots x_n}(\lambda)}{u_{x_1 x_2 \dots x_{k-1}}(\lambda)} \right)^\rho \quad (5.3)$$

for all  $i, j \in \mathcal{A}$ , thus, equal zero for  $i = x_{k-1}^{-1}$ . Note that for  $\rho = 1, k = 1$ , the  $(i, j)$ -entry of  $H_1(n; 1; \lambda)$  equals the expected number of vertices at distance  $n$  from the root 1 that are ever to be infected whose word representation begins in  $i$  and ends in  $j$ . Similarly, for  $\rho = 2, k = 1$ , the  $(i, j)$ -entry of  $H_2(n; 1; \lambda)$  is a lower bound for the expected number of vertices at distance  $n$  from the root whose word representation begins in  $i$  and ends in  $j$  that are ever infected and upon infection are ever to send the infection back to the root. Observe that for every  $\lambda \notin \mathcal{R}_3$  and each  $\rho > 0$ , the matrix  $H_\rho(n; x; \lambda)$  is an aperiodic, irreducible, and nonnegative matrix, thus, a Perron-Frobenius matrix. Therefore, the Perron-Frobenius theorem tells us that  $H_\rho(n; x; \lambda)$  has a largest positive eigenvalue. It can be proven [6] that for every  $\lambda \in \text{int}(\mathcal{R}_1 \cup \mathcal{R}_2)$  and each real  $\rho > r_u(\lambda)$ , the matrix  $x \rightarrow H_\rho(n; x; \cdot)$  is Hölder continuous (with some exponent) on  $\Sigma$ .

Now we are ready to define some *potential function*  $\varphi_\rho(x) = \varphi_{\rho; \lambda}(x) : \Sigma \rightarrow \mathbf{R}$  for every  $\rho > r_u(\lambda)$ . In [6], Proposition 5.1 of LALLEY [11] was applied to define for every  $x = \dots x_1 x_2 \dots \in \Sigma$ , every integer  $k > 0$ , and sufficiently large  $n$ ,

$$H_\rho(n; x_1 x_2 \dots x_{k-1}; \lambda) V(\sigma^{n-k} x) = e^{S_{n-k} \varphi_\rho(x)} V(x), \quad (5.4)$$

where  $V(x)$  is the right eigenvector associated with the matrix  $H_\rho(n; x_1 \dots x_{k-1}; \lambda)$  (for more details, see [6]). The Hölder continuity of  $\varphi_\rho = \varphi_{\rho; \lambda}$  immediately follows from the Hölder continuity of  $V$  and the map  $x \rightarrow H_\rho(n; x_1 \dots x_{k-1}; \lambda)$ . To facilitate notation, we will often just write  $\varphi_\rho$ . Thus,  $H_\rho(n; x_1 x_2 \dots x_{k-1}; \lambda)$  has top eigenvalue  $\exp\{S_{n-k} \varphi_\rho(x)\}$ . Observe that, for every  $x \in \Sigma$ , the functions  $S_n \varphi_\rho(x)$ , and thus,  $H_\rho(n+k; x_1 \dots x_{k-1}; \cdot)$ , depend on the relative frequencies of the generators in the reduced word  $x_1 x_2 \dots x_{k-1}$  and the *order* of the letters as well. One shows that  $r_u(\lambda) < 2$  for  $\lambda \in \text{int}(\mathcal{R}_1 \cup \mathcal{R}_2)$ . Next  $H_2(n; x_1 x_2 \dots x_{k-1}; \lambda)$  may be expressed [6] as the  $(n-k)$ -th power of a  $2d \times 2d$  Perron-Frobenius matrix  $B_{2, n, x_1 \dots x_{k-1}}(\lambda)$  which (a) preserves the lead eigenvalue, and (b) has a certain form that specifies the allowed transitions. More precisely, for any positive number  $\rho$ , define a  $2d \times 2d$  matrix  $B_\rho = B_\rho(\lambda) = B_{\rho, n, x_1 \dots x_{k-1}}(\lambda)$ , indexed by elements of  $\mathcal{A}$ , by

$$\begin{aligned} (B_\rho(\lambda))_{ij} &= b_j(\lambda)^\rho \quad \text{if } j \neq i^{-1}, \\ &= 0 \quad \quad \quad \text{if } j = i^{-1} \end{aligned} \quad (5.5)$$

where the  $b_j = b_j(\lambda)$  satisfy the equation

$$\sum_{i \in \mathcal{A}} \frac{a_i^\rho}{\alpha_\rho + a_i^\rho} = 1 \quad (5.6)$$

with

$$a_j^2 = b_j^2, \quad \rho = 2, \quad \text{and } \alpha_2 = \exp\{S_{n-k} \varphi_2(x)/(n-k)\}.$$

Thus,  $B_2^{n-k}$  has lead eigenvalue  $\exp\{S_{n-k} \varphi_2(x)\}$  and both  $B_1$  and  $B_2$  are Perron-Frobenius matrices. The matrix entries  $b_j$  or  $b_j^2$  depend on  $n$  and the sequence  $x_1 x_2 \dots x_{k-1}$ . If  $\delta =$

$\delta_\rho > 0$  is such that the pressure  $P(\delta\varphi_\rho) = 0$ , then we may define the expectation of  $\varphi_\rho$  with respect to the probability measure  $\mu_{\delta\varphi_\rho}$  by

$$\bar{\varphi}_\rho = \int \varphi_\rho d\mu_{\delta\varphi_\rho} \quad (5.7)$$

and let  $\bar{b}_j^2 = \bar{b}_j(\lambda)^2$  denote the matrix entries corresponding to  $\bar{\varphi}_{2;\lambda}$  for  $\lambda \in \text{int}(\mathcal{R}_1 \cup \mathcal{R}_2)$ . For any positive number  $\rho$ , define the matrix  $M_\rho = M_\rho(\lambda)$  to be the  $2d \times 2d$  matrix, indexed by elements of  $\mathcal{A}$ , whose entries are given by

$$\begin{aligned} (M_\rho(\lambda))_{ij} &= \bar{b}_j(\lambda)^\rho & \text{if } j \neq i^{-1}, \\ &= 0 & \text{if } j = i^{-1}. \end{aligned} \quad (5.8)$$

We will refer to  $M_1(\lambda)$  and  $M_2(\lambda)$  as the *first-passage matrix* and the *backscatter matrix*, respectively. These two matrices will play a distinguished role, for instance, they are important in the proofs of Theorems 3. Since by construction, for  $\rho > 0$  and  $\lambda \notin \mathcal{R}_3$ ,  $M_\rho(\lambda)$  is an aperiodic, irreducible, and nonnegative matrix, thus, a Perron-Frobenius matrix, the Perron-Frobenius theorem lets us conclude that  $M_\rho(\lambda)$  has a largest positive eigenvalue  $\theta_\rho = \theta(\rho; \lambda)$ . In fact,

$$\begin{aligned} \theta_1 = \theta_1(\lambda) &= \exp \bar{\varphi}_1 \\ \theta_2 = \theta_2(\lambda) &= \exp \bar{\varphi}_2, \end{aligned} \quad (5.9)$$

where the first identity has an elementary proof and the second identity immediately follows from the definition of  $M_2$ . The matrix  $B_2$  is constructed such that the entries of  $M_\rho$  satisfy the equation

$$\sum_{i \in \mathcal{A}} \frac{\bar{b}_i(\lambda)^\rho}{\theta_\rho(\lambda) + \bar{b}_i(\lambda)^\rho} = 1. \quad (5.10)$$

In the isotropic case,  $\theta_1(\lambda)$  coincides with the function  $\theta_1(\lambda)$  introduced earlier in Section 2, and indeed, one shows [6] that

$$\lim_{n \rightarrow \infty} (\mathbf{E}|\mathcal{Y}_n|)^{1/n} = \theta_1(\lambda).$$

Importantly, note that  $\theta_1$  coincides with  $\theta_1$  in Theorem 3. The functions  $\theta_\rho(\lambda)$  share the same continuity and strict monotonicity properties with the function  $\beta(\lambda)$  (strict monotonicity along “directions of increase” for  $\lambda$ , see [6]). The eigenvalues  $\theta_1$  and  $\theta_2$  are of special concern. While  $\theta_1$  controls the growth of  $\mathbf{E}|\mathcal{Y}_n|$ , the function  $\theta_2$  controls how many infection trails are expected to reach distance  $n$  and return to the root vertex. The two functions  $\theta_1$  and  $\theta_2$  may be employed to recognize the phases of the contact process:

$$\begin{aligned} \theta_1(\lambda) < 1 &\Leftrightarrow \lambda \in \text{int}(\mathcal{R}_1) \\ \theta_1(\lambda) > 1 &\Leftrightarrow \lambda \notin \mathcal{R}_1 \\ \theta_2(\lambda) < 1 &\Rightarrow \lambda \notin \mathcal{R}_3 \\ \theta_2(\lambda) > 1 &\Leftrightarrow \lambda \in \mathcal{R}_3. \end{aligned} \quad (5.11)$$

In fact, it was shown [6] that  $\theta_1(\lambda) = 1$  for  $\lambda \in \overline{\mathcal{R}}_1 \cap \overline{\mathcal{R}}_2$  and  $\theta_2(\lambda) = 1$  for  $\lambda \in \overline{\mathcal{R}}_2 \cap \overline{\mathcal{R}}_3$  (see Proposition 2 below).

A few words pertaining to the ingredients to the proof of Theorem 3 will conclude this section. The principal statements are the upper and lower bounds for the Hausdorff dimension of the limit set in (3.1) and the inequality and instance of equality in (3.2). First, an observation that is essential to prove the upper bound in (3.1) is that the sets  $\mathcal{Y}_m$  provide a sequence of open covers of the limit set  $\Lambda$ , in particular, if  $\mathcal{E}_x$  denotes the set of all ends of  $\mathcal{T}$  that pass through  $x$ , then  $\Lambda \subset \cup_{x \in \mathcal{Y}_m} \mathcal{E}_x$ . Second, to prove the lower bound in (3.1), one engages in considering a certain labelled Galton-Watson tree, embedded in the set of vertices of  $\mathcal{T}$  that are ever infected, whose limit set is a subset contained in  $\Lambda$  and that has a Hausdorff dimension which approaches the required bound. Finally, inequality (3.2) is a consequence of connecting the Cauchy-Schwarz inequality with the matrices  $M_1$  and  $M_2$  in the right fashion, of the values of the lead eigenvalues  $\theta_1$  and  $\theta_2$  at the second phase transition, whereas the moment of equality in (3.2) emerges from the Gibbs variational principle [2], Theorem 1.22, applied to the potential functions  $\varphi_1$  and  $\varphi_2$ .

## 6 Characterization of the Weak/Strong Survival Threshold

The anisotropic branching random walk [10] and the contact process on a graph are both well described by functions of a model parameter that capture the event “first passage” of any particle to a site. Studying the latter process is quite a bit more challenging than the former. Incidentally, on a good number of infinite, locally finite graphs of non-Euclidean geometry, the two models exhibit a discontinuous phase transition, as the underlying parameter varies over the survival regime of the population. Perhaps amazingly, both processes meet again at the same equation which characterizes this phase separation even though the “first passage functions” associated with each of the two models are interrelated differently.

The following fundamental relations hold for the symmetric contact process (see [6], Section 5).

**Proposition 1** *For  $\lambda \notin \mathcal{R}_3$ , every  $i \in \mathcal{A}_+$  and  $x = x_1 x_2 \dots x_{n-1} i \in \mathcal{G}_n$ , there is some  $D = D_\lambda(x)$  (independent of  $i$ ) so that*

$$b_i(\lambda) = \lambda_i D + b_i(\lambda)^2 D \lambda_i + 2b_i(\lambda) D \sum_{j \in \mathcal{A}_+ - \{i\}} \lambda_j b_j(\lambda), \quad (6.1)$$

where the  $b_j = b_j(\lambda)$  are defined in (5.5) and (5.6) and  $C_1 \leq D_\lambda(x) \leq C_2(n+1)$  for some positive finite constants  $C_1$  and  $C_2$ , independent of  $x, n$ , and  $\lambda$ .

Observe that the system of identities (6.1) may differ for different  $\lambda$  since the coefficients  $D_\lambda(x)$  are not necessarily the same for different  $\lambda$ . It is worthwhile mentioning that it is not difficult to show that, if the contact process is symmetric, then  $\bar{b}_i(\lambda) = \bar{b}_j(\lambda)$  for all  $i, j \in \mathcal{A}_+$  if and only if  $\lambda_i = \lambda_j$  for all  $i, j \in \mathcal{A}_+$ . Parallel relationships were found for the symmetric

anisotropic (branching) random walk on  $\mathcal{T}_{2d}$  (see e.g. HUETER AND LALLEY [10]). Here is a brief summary. Let us consider a *random walk* on  $\mathcal{T}_{2d}$ , starting at the root vertex, with a one-step distribution  $\{p_i\}_{i \in \mathcal{A} \cup \{1\}}$  that is *nearest neighbour, symmetric, nondegenerate, and aperiodic*. In other words, we assume that  $p_i > 0$  iff  $i \in \mathcal{A} \cup \{1\}$  and  $p_i = p_{i^{-1}}$  for each  $i \in \mathcal{A}$ . For  $d \geq 2$ , this random walk is transient. If for  $x = x_1 x_2 \dots x_k$ , we let  $T_x$  denote the time when the random walk first reaches  $x$  (set  $T_x = \infty$  on the event that the random walk never visits  $x$ ), let

$$F_x(z) = \mathbf{E} z^{T_x} 1_{\{T_x < \infty\}} \quad (6.2)$$

for  $|z| \leq 1$  and let  $F_x(z)$  to be the direct analytic continuation of the function in (6.2) for  $|z| > 1$ , then, by the Markov property, the random walk can be seen to satisfy

$$F_i(z) = p_1 z F_i(z) + p_i z + p_i z F_i(z)^2 + 2F_i(z) z \sum_{j \in \mathcal{A}_+ - \{i\}} p_j F_j(z). \quad (6.3)$$

At the singularity of the generating functions  $F_i$ , that is, at the radius of convergence  $R$ , we derive [10]

$$\sum_{i \in \mathcal{A}} \frac{F_i(R)^2}{1 + F_i(R)^2} = 1, \quad (6.4)$$

an equation which is formally independent of the  $p_i$ . One verifies a palette of nice properties of the  $F_x$  for  $x \neq \{\text{root}\}$ , for example, each  $F_i(R) < 1$ , the functions  $F_x$  are multiplicative, i.e.  $F_{x_1 x_2 \dots x_k} = \prod_{i=1}^k F_{x_i}$ , and the  $F_x$  control the expected number of visits by particles of the *branching random walk* to  $\mathcal{G}_n$  which evolves as follows. Start with one particle at the root at time zero. Each particle lives for one unit of time. At the end of its lifetime, a particle creates a geometrically distributed number of offspring  $\geq 1$  with mean  $\lambda > 1$  and places them at its current vertex. After this fission, all particles move randomly to new vertices according to the step distribution  $\{p_i\}_{i \in \mathcal{A} \cup \{1\}}$ . All random choices in this evolution are assumed to be mutually independent.

Relations (6.1) and (6.3) offer a bridge between the contact process and the random walk and yield a promising handle to characterize the singularity that occurs for the contact process. Indeed, (6.4) may be expressed in the language of the contact process so as to become common ground for the two processes. We return to the contact process and collect the equation at the threshold to strong survival.

**Proposition 2 (Hueter [6])** *For  $\lambda \in \overline{\mathcal{R}}_2 \cap \overline{\mathcal{R}}_3$  and  $\{b_i(\lambda) = b_i(\lambda, x)\}_{i \in \mathcal{A}_+}$ ,*

$$\sum_{i \in \mathcal{A}} \frac{b_i(\lambda)^2}{1 + b_i(\lambda)^2} = 1, \quad (6.5)$$

*in particular,  $\theta_2(\lambda) = 1$ , where  $\theta_2(\lambda)$  is defined in (5.9).*

**Proof.** We sketch some ideas of proof [6] to derive the equation at the singularity, call it  $\mathcal{D}_c$ , which can be proven to be identical  $\overline{\mathcal{R}}_2 \cap \overline{\mathcal{R}}_3$  (omitted here). To identify the singularity,

we will rely on the fact that system (6.3), which is identical to system (6.1), is singular at  $z = R$ . The derivation is written out in terms of the  $b_i$ . If  $\{H_i(\{b_k\}_{k \in \mathcal{A}_+})\}_{i \in \mathcal{A}_+}$  denotes the difference between the lefthand and righthand sides of the fundamental equations, then the Jacobian matrix of the  $d \times d$  system of equations is formally given by  $(dH_i/db_j)_{i,j \in \mathcal{A}_+}$ . Apply the (complex) Implicit Function theorem (for the singular case):

$$\begin{aligned} 1 &= db_i/db_i = 2 \sum_{k \in \mathcal{A}_+} \lambda_k D b_k \\ 0 &= db_i/db_j = 2\lambda_j D b_i. \end{aligned} \quad (j \neq i)$$

Thus, the Jacobian matrix may be written as  $I - K(\lambda)$ , where  $I$  is the identity and  $K$  has entries

$$\begin{aligned} K(\lambda)_{ij} &= 2 \sum_{k \in \mathcal{A}_+} \lambda_k D b_k && \text{if } j = i, \\ &= 2\lambda_j D b_i && \text{if } j \neq i. \end{aligned}$$

Some elementary reasoning on the spectrum of  $K(\lambda)$  implies that we are interested in the eigenvalue equation  $K(\lambda)v = v$  for  $\lambda \in \mathcal{D}_c$ . Set  $s = 2 \sum_{j \in \mathcal{A}_+} \lambda_j D v_j > 0$ . Rewriting  $K(\lambda)v = v$  yields

$$\begin{aligned} v_i &= v_i [2 \sum_{j \in \mathcal{A}_+} \lambda_j D b_j] + 2b_i \sum_{j \in \mathcal{A}_+ - \{i\}} \lambda_j D v_j \\ &= v_i [2 \sum_{j \in \mathcal{A}_+ - \{i\}} \lambda_j D b_j] + 2b_i \sum_{j \in \mathcal{A}_+} \lambda_j D v_j \\ &= 2v_i \sum_{j \in \mathcal{A}_+ - \{i\}} \lambda_j D b_j + b_i s. \end{aligned}$$

Multiplying each side by  $b_i$  and substituting some arrangement of equation (6.1), i.e.

$$b_i - \lambda_i D - \lambda_i D b_i^2 = 2b_i \sum_{j \in \mathcal{A}_+ - \{i\}} \lambda_j D b_j,$$

gives

$$\begin{aligned} v_i b_i &= v_i [b_i - \lambda_i D - \lambda_i D b_i^2] + b_i^2 s \\ \iff v_i D \lambda_i &= s b_i^2 / (1 + b_i^2). \end{aligned}$$

Now, we recall that  $s = 2 \sum_{j \in \mathcal{A}_+} \lambda_j D v_j$  and that  $b_i(\lambda) = b_{i-1}(\lambda)$  to see (6.5). In particular, (6.5) holds when the  $\bar{b}_i$  are in place of the  $b_i$ . Hence, by (5.10), this yields  $\theta_2(\lambda) = 1$ .  $\square$

We end this section and our discussion about the anisotropic contact process on  $\mathcal{T}_{2d}$  with a result that describes the growth of  $A_t$  over space *and time*. The upshot is that the contact process moves at linear distance with time as does the isotropic contact process (see LALLEY [12]).

Let  $r_t$  and  $R_t$  denote the *smallest* and *largest* distances among sites in  $A_t$  at time  $t > 0$ , respectively, and let  $N_n(ns)$  denote the number of vertices in  $A_{ns}$  at distance  $n$  and at time  $ns$ . Furthermore, similarly as we earlier defined  $\bar{\varphi}_1$ , let  $\bar{\Phi}_s$  denote the logarithm of the lead eigenvalue of some P-F matrix that, roughly speaking, measures the behaviour of the infection transitions at time scale  $s$ .

**Theorem 5 (Hueter [6], Right- and Leftmost Particles Move Linearly in Time)**  
*Let  $d > 1$  and  $\lambda$  in  $\overline{\mathcal{R}}_2$ . Then there exist smallest and largest solutions  $0 < s_1 \leq s_2 < \infty$  of “some eigenvalue equation”  $\bar{\Phi}_s = 0$ . Almost surely on the event of survival,*

$$\begin{aligned}\lim_{t \rightarrow \infty} r_t/t &= 1/s_2, \\ \lim_{t \rightarrow \infty} R_t/t &= 1/s_1.\end{aligned}$$

Moreover, for each  $s > 0$  such that  $\bar{\Phi}_s > 0$ , almost surely on the event of survival,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log N_n(ns) = \bar{\Phi}_s.$$

## 7 Contact Process on Subperiodic Trees

Little is known on the contact process on less regular trees than the homogeneous ones, left alone on trees that are complicated behaved. Even though it is not too difficult to set up exotic irregular trees that force the contact process to behave unexpectedly, for instance, such that the contact process gets trapped, and thus, exhibits only one phase transition (one survival phase), recognizing some geometric characteristics of irregular trees and graphs that are responsible for a *second* (often discontinuous) phase transition is desirable. It is hoped that knowing the roots for a phase transition of this process may enhance understanding the nature of phase transition in this and related percolation structure and threshold models. Let us glance at a well-behaved class of irregular trees  $\mathcal{T}$ , the *subperiodic* and *superperiodic trees* (see LYONS AND PERES [18] for a more detailed treatment). This class of trees is quite curious because the growth can be changed in a continuous fashion. It is natural to ask the question whether the growth may be the crucial geometric indicator for weak survival, and if so, which value may be the cutoff point.

**Definition 1** *Let  $\mathcal{T}(v)$  denote the subtree of  $\mathcal{T}$  that has  $v$  as its root. Let  $N \geq 0$ . A tree  $\mathcal{T}$  is called  $N$ -subperiodic (respectively,  $N$ -periodic) if for every  $v \in \mathcal{T}$  there exists an adjacency-preserving injection (respectively, bijection)  $f : \mathcal{T}(v) \rightarrow \mathcal{T}(f(v))$  with  $|f(v)| \leq N$ . A tree is subperiodic (periodic) if there is some  $N$  for which it is  $N$ -subperiodic ( $N$ -periodic).*

This definition is borrowed from [18]. Observe that for any subperiodic tree  $\mathcal{T}$ , the limit  $\lim_{n \rightarrow \infty} |\mathcal{G}_n|^{1/n}$  exists, denotes the *growth rate*  $gr\mathcal{T}$  of  $\mathcal{T}$  ( $\mathcal{G}_n$  denotes the set of all vertices at distance  $n$  from the root) and equals the branching number of the tree (It is a result of FURSTENBERG [3] (see [18], Chapter 2) that subperiodicity suffices for equality of  $gr\mathcal{T}$

and the branching number of  $\mathcal{T}$ , and especially, for the existence of  $gr\mathcal{T}$  as a limit). A subperiodic tree may have many infinite paths (reaching the set of ends) that are isomorphic to  $\mathbf{Z}_+$ . On  $\mathbf{Z}_+$ , though, the symmetric process has  $\lambda_1 = \lambda_2$ . Hence, a priori, the actual scenario is uncertain. However, it turns out that not only is the infection more likely to follow paths that have most subtrees attached to them but also most of the paths around enjoy this feature.

Clearly, the definition of the isotropic contact process with infection parameter  $\lambda > 0$  on  $\mathcal{T}$  is very much the same as the one presented in Section 1.2 on homogeneous trees.

**Theorem 6 (Hueter [7])** *The isotropic contact process with parameter  $\lambda > 0$  on any subperiodic tree of growth  $gr\mathcal{T} > 1$  has  $\lambda_1 < \lambda_2$ . The critical contact process at  $\lambda_1$  on any such tree dies out and at  $\lambda_2$  survives weakly.*

The idea of proof is to regard the *transport of infection* and to compare the expected infection transported away from the initial infected site with the one transported back to the initial infected site. In the next result, that describes the dimension of the limit set,  $\tilde{\beta} = \tilde{\beta}(\lambda)$  may be interpreted as the conditional probability of the event that a fixed “generic” vertex (i.e. one that “carries the right growth”) at large distance from the root is ever infected given that its parent is ever infected. Equivalently,  $\log \tilde{\beta}$  may be thought of as the asymptotic exponential rate of the expected fraction of infected vertices at fixed large distance.

**Theorem 7 (Hueter [7])** *Almost surely on the event of survival, the limit set  $\Lambda$  of the weakly supercritical contact process on any subperiodic tree of  $gr\mathcal{T} > 1$  has Hausdorff dimension (in metric  $d_\alpha$ )*

$$\delta_H(\Lambda) = -\frac{\log(gr\mathcal{T} \cdot \tilde{\beta})}{\log \alpha}.$$

*Furthermore, this Hausdorff dimension is a strictly increasing and continuous function in  $\lambda$  on  $[\lambda_1, \lambda_2)$  and discontinuous from the right at  $\lambda_2$ .*

As a consequence, the limit set of the contact process on any subperiodic tree of growth  $d$  has the same Hausdorff dimension as the one on a homogeneous tree [13] of growth  $d$ . Analogous results to Theorems 6 and 7 as well hold for any superperiodic tree with *finite* upper growth rate, defined as the  $\limsup |\mathcal{G}_n|^{1/n}$ , in which case the function  $\tilde{\beta} = \tilde{\beta}(\lambda)$  is defined by

$$\tilde{\beta} = (gr\mathcal{T})^{-1} \lim_{n \rightarrow \infty} (\mathbf{E}|\mathcal{Y}_n|)^{1/n}. \tag{7.1}$$

## 8 Open Questions

We list a number of related questions and issues for further study.

1. Can the contact process exhibit weak survival on any subperiodic tree of growth  $gr\mathcal{T} = 1$  ?

2. Is weak survival possible on any *amenable* graph ?

A result of HUETER [8] states that the isotropic contact process on every infinite, locally finite, connected transitive graph  $\mathcal{G}$  of growth  $> 1$  has  $0 < \lambda_1 < \lambda_2 < \infty$ . Thus, there is weak survival on transitive nonamenable such graphs. To place a word of caution, in contrast, PEMANTLE AND STACEY [21] found an example of a nonamenable tree of bounded degree without weak survival phase.

3. What can be said about the behaviour of the contact process on random trees or *random graphs* ?

What about the contact process in *random environment* ?

4. How do the results change when certain vertices of a tree or graph turn *immune* against infection after having previously been infected ?
5. The contact process has strong ties with computer virus propagation on networks, which has been analyzed mathematically surprisingly little. Reasons for this connection are as follows. Many computer viruses spread much as biological infections described by the contact process do, nodes of a network recover as well, some nodes are healthier than others, and often attempts of infection are not successful.

What are the connections between the spread of computer viruses on networks on the one hand and the contact process and branching random walk on graphs on the other hand ? What can we learn from one process about the other ?

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