The Convex Hull of a Normal Sample

Irene Hueter*†‡
Department of Statistics
Purdue University

August 1992
Revised: August 1993

Abstract

Consider the convex hull of n independent, identically distributed points in the plane. Functionals of interest are the number of vertices N_n, the perimeter L_n and the area A_n of the convex hull. We study the asymptotic behaviour of these three quantities when the points are standard normally distributed. In particular, we derive the variances of N_n, L_n and A_n for large n and prove a central limit theorem for each of these random variables. We enlarge on a method developed by Groeneboom (1988) for uniformly distributed points supported on a bounded planar region. The process of vertices of the convex hull is of central importance. Poisson approximation and martingale techniques are used.

*Research supported by Swiss NSF Grant 20-28927.90

†AMS 1991 *subject classifications. Primary.* 52A22, 60D05, 60J75. *Secondary.* 60F05, 60G44, 60G55

‡*Key words and phrases.* Convex hull, Poisson point process, Markovian jump process, martingales, normal distribution, central limit theorem, variance

1 Introduction

There still is considerable interest in the subject of convex hulls of random vectors, thirty years or so after the first papers were published by Rényi and Sulanke (1963), Efron (1965), Carnal (1970) and Raynaud (1970). Although in recent years important progress has been made [5, 1], it remains a difficult issue to determine the second moment or the limiting distribution of any convex hull functional, even in two dimensions.

Let $Z_1, Z_2, \ldots, Z_n$ be n independently and identically distributed points in the plane. Let N_n, L_n, and A_n be the number of vertices, the perimeter, and the area, respectively, of the convex hull of the set of n points. The method used by Rényi and Sulanke [10] to derive the first moments seems to be too complicated to find the higher moments of the convex hull functionals. More importantly, their method does not offer any help in determining the limiting distribution because the dependence structure between the multivariate extremes is not captured. In a recent paper, Groeneboom [5] develops an elegant method, giving access to variance computations and deriving the asymptotic distribution of N_n. He proves a central limit theorem for N_n for uniformly distributed points in the plane in the two cases of points in an r-polygon and of points in an ellipse, by studying a process running through all the vertices of the convex hull. The key ideas are to approximate the sample points close to the boundary of the convex hull by a Poisson point process and to use certain martingales that involve the jump measure of a Markov process.

The purpose of this note is to show how the approach introduced in [5] may be extended to non-uniform distributions and to other convex hull functionals like the perimeter and the area of the planar convex hull. We will restrict our attention to normally distributed points here. For a more general treatment the reader is referred to Hueter [7], where higher-dimensional, spherically symmetric distributions with exponential tails are investigated, and Hueter [6], where other bivariate distributions are addressed. We will make extensive use of Poisson approximation and martingales. Note that the normal distribution cannot be expected to be the limiting distribution of distributions which die off too slowly, e.g., the ones with slowly varying tail, for which $E[N_n] \sim 4$ as n grows to infinity (see [1]). Curiously, all the known results regarding the variance of N_n, indicate that the orders of magnitude of the variance and the expectation of N_n coincide.

The main results of this work are the following

Theorem 1.1 *Let N_n be the number of vertices of the convex hull of a normal sample of size n. Then, as $n \to \infty$,*

$$\frac{N_n - 2\sqrt{2\pi \ln n}}{\{2\sqrt{2\pi \ln n}\,(1 + \pi c_2)\}^{1/2}} \xrightarrow{L} \mathcal{N}(0,1), \tag{1}$$

where

$$-1/\pi \quad \leq \quad c_2 \leq \sqrt{3}/(2\pi) - 1/6 \approx 0.108998$$

and $\mathcal{N}(0,1)$ denotes the standard normal distribution.

Theorem 1.2 *Let L_n be the perimeter and A_n the area of the convex hull of a sample of size n from a bivariate normal distribution with identity covariance matrix. Then, as $n \to \infty$,*

$$(L_n - 2\pi\sqrt{2\ln n})/\sqrt{4\pi^{3/2}\ln n} \xrightarrow{L} \mathcal{N}(0,1) \tag{2}$$

and

$$(A_n - 2\pi\ln n)/\sqrt{2\pi^{3/2}(\ln n)^2} \xrightarrow{L} \mathcal{N}(0,1). \tag{3}$$

The convex hull of a normal sample has been previously studied by Carnal [2], Raynaud [9] and Dwyer [3], but only for first moments.

The structure of this paper is as follows. Section 2 introduces the process that keeps track of the convex hull vertices. This process will be immediately replaced by another process that is easier to deal with. In Section 3 we derive the jump measure of the process of vertices, which will be of central importance in our analysis, and introduce some martingales that ultimately will relate the expectation of our convex hull functionals to some functionals of the jump measure. The proofs of the main results are deferred to Section 4.

2 Approximation of the vertex process

In our investigations, a key role is played by the process of consecutive vertices in the convex hull. For the definition, we essentially follow Groeneboom [5].

Definition 2.1 *For each $a \in \mathbb{R}$ we define "the vertex process" $W_n(a)$ as the point (X_k, Y_k) of the sample such that $Y_k - aX_k$ is minimal. If there are several of such points, then we take the one with the biggest first coordinate. This happens with probability zero for fixed a.*

The picture of this definition is the following. We collect the convex hull vertices by turning a line around the set of n points in such a way that always at least one point is hit and all the remaining points lie on one side of the "supporting line". While the time parameter runs through $\mathbb{R}$, the "lower half" of the convex hull vertices is recorded. The process $\{W_n(a) : a \in \mathbb{R}\}$ is a pure jump process, has right-continuous paths and is non-Markovian. However, the vertex process is not far from being Markovian in the sense that it can be approximated by a similar, but much simpler process, endowed with the preferable Markov property. Thus, our next task is to find a Poisson point process whose realizations bear as much resemblance as possible to the sample points near the boundary of the convex hull. Then by coupling a Poisson and a Binomial distribution, it is not hard to show that the variational distance between the two distinct vertex processes tends to zero as n tends to infinity.

Let $Z_1, Z_2, \ldots, Z_n$ be n independently and identically distributed points from the standard normal distribution. Since the number of convex hull vertices is left invariant by any affine transformation, the choice of the parameters of the underlying normal distribution only affects the continuous functionals like the perimeter and the area of the convex hull that we will consider. In our analysis, the following quantities will be used repeatedly

$$\begin{aligned} r_0 &= r_0(n) = \sqrt{2\ln n} \qquad (4)\\ r_1 &= r_1(n) = r_0 - \varepsilon_n/2 \\ \text{and} \\ r_2 &= r_2(n) = r_0 + \varepsilon_n/2, \end{aligned}$$

where ε_n is such that $\varepsilon_n \to 0$ and $\varepsilon_n r_0 \to \infty$ as $n \to \infty$. To make the notation less cumbersome we will write r_0, r_1, and r_2 instead of $r_0(n)$, $r_1(n)$, and $r_2(n)$ (It should be clear from the meaning we will attach to those quantities below that they depend on the sample size n). It will be convenient to shift the whole sample $Z_1, Z_2, \ldots, Z_n$ by the vector $(0, r_2)$, i.e. Z_1 is assumed to have the density

$$f(x, y) = \varphi(x)\varphi(r_2 - y),$$

where $(x, y) \in I\!R^2$ and $\varphi(x) = (2\pi)^{-1/2}\exp(-x^2/2)$. Now notice that with high probability the boundary of the convex hull of the sample belongs only to a small region $A_n^* \subset I\!R^2$, for most distributions with small enough tails. For rotationally invariant distributions A_n^* is nothing more than an annulus; in particular, for the normal distribution, A_n^* is enclosed between two spheres whose radius difference converges to zero as $n \to \infty$. We require A_n^* to satisfy the following two conditions as $n \to \infty$:

$$P(W_n(a) \in A_n^*,\ \forall a \in I\!R) \longrightarrow 1 \qquad (5)$$

and

$$P(Z_1 \in A_n^*) \longrightarrow 0.$$

If we choose the region

$$A_n^* = \{(u_1, u_2) \in I\!R^2 : r_1^2 \leq u_1^2 + (r_2 - u_2)^2 \leq r_2^2\}, \qquad (6)$$

then the two requirements given in (5) above hold.

Lemma 2.1 *Let η_n and ξ_n denote the sample point process of size n and the Poisson point process on $I\!R^2$ with intensity measure $\int_\cdot dF$, respectively. There exist processes $\tilde{\eta}_n$ and $\tilde{\xi}_n$ defined on the same probability space such that*

$$\tilde{\eta}_n \stackrel{L}{=} \eta_{n|_{A_n^*}}, \qquad \tilde{\xi}_n \stackrel{L}{=} \xi_{n|_{A_n^*}} \qquad (7)$$

and

$$P(\tilde{\eta}_n \neq \tilde{\xi}_n) \leq 2P(Z_1 \in A_n^*), \qquad (8)$$

where $P(Z_1 \in A_n^*) \sim \exp\{-r_1^2/2\} \sim n^{-1+\varepsilon_n/r_0}$ *as* $n \to \infty$.

(Here " | " stands for "restriction to".) For the proof, see [5], the proof of Lemma 2.2, where a coupling argument leads to an upper bound between the variational distance between the Binomial and the Poisson distribution.(In the case of the normal distribution considered here, the parameters of the relevant Binomial distribution are n and $P(Z_1 \in A_n^*)$ and the parameter of the relevant Poisson distribution is $nP(Z_1 \in A_n^*)$.) □

Unlike the Poisson point process defined in [5], our Poisson point process is *inhomogeneous,* which complicates the moment calculations considerably as we will see later on. In fact, the densities of the process W have unpleasant integrals in their denominators. For uniformly distributed points, the same integrals can easily be solved. Distributions whose tails do not decrease sufficiently quickly, and therefore, do not allow Poisson approximation of the extreme sample points, are the so-called *algebraic-tailed* distributions (For further details, see Carnal [2], Dwyer [3] and Aldous et al. [1]). Typically, their expected number $E[N_n]$ of convex hull vertices does not increase with n (see [2]). In finding the asymptotical distribution of N_n, combinatorical methods in [1] are applied more successfully.

As a result of our progress we may now define the vertex process $\{W(a) : a \in \mathbb{R}\}$ of the convex hull of the Poisson point process ξ_n (Notice that W still depends on n). The condition $P(Z_1 \in A_n^*) \to 0$ and Lemma 2.1 force the variational distance between the vertex processes W_n and W to shrink to zero. It is easily seen that, by the independence of the events of a Poisson point process defined on disjoint sets, the new vertex process W has the Markov property. Note that when the time parameter a of $W(a) = (X(a), Y(a))$ is sufficiently small, i.e. when $|a| \leq (2\varepsilon_n/r_2)^{1/2}$, a can be interpreted as the angle between the vectors $(0, -r_2)$ and $(X(a), -r_2)$. Therefore, in a neighbourhood of the origin, $X(a)/r_2$ grows proportionally to the slope parameter a. This local correspondence between the length of the path followed and the time spent after having started the path at a certain point, say, close to the origin will come up several times in all that follows.

Furthermore, it will be advantageous to our calculations to approximate the outer boundary $\{(u_1, u_2) \in A_n^* : u_1^2 + (r_2 - u_2)^2 = r_2^2\}$ of A_n^* by a parabola in the neighbourhood of the origin, namely by $u_1^2/2r_2 \approx r_2 - (r_2^2 - u_1^2)^{1/2}$. To derive an upper bound for the error that we make by locally using the parabola approximation, consider

$$S_n = \{(u_1, u_2) \in A_n^* : \sqrt{u_1^2 + (r_2 - u_2)^2} \leq r_2,\ |u_1| \leq \sqrt{2r_2\varepsilon_n}\} \tag{9}$$

and

$$B_n = \{(u_1, u_2) \in \mathbb{R}^2 : u_1^2/2r_2 \leq u_2,\ u_1^2 \leq 2r_2\varepsilon_n\}. \tag{10}$$

Let $\xi_{n|_{S_n}}$ and $\xi_{n|_{B_n}}$ be the restrictions of the Poisson point process with intensity $n\int_\cdot dF$ to S_n and B_n, respectively. Let W^S and W^B be the vertex processes defined as in Definition 2.1, but now by the sample replaced by a realization of $\xi_n|_{S_n}$ and $\xi_n|_{B_n}$, respectively.

Lemma 2.2 *Let the point processes* δ_n^S *and* δ_n^B *be defined by*

$$\begin{aligned}\delta_n^S &= \{W^S(a) : 2r_2 a \leq \sqrt{2r_2\varepsilon_n}\}, \\ \delta_n^B &= \{W^B(a) : 2r_2 a \leq \sqrt{2r_2\varepsilon_n}\}.\end{aligned} \tag{11}$$

Then, as $n \to \infty$,

$$P(\delta_n^S \neq \delta_n^B) \leq C(\varepsilon_n/r_2)^{1/2} \exp\{-r_0\varepsilon_n/2\} \tag{12}$$

for some constant $0 < C < \infty$.

Proof. Since in the neighbourhood of the origin the parabola runs below the curve $v = r_2 - (r_2^2 - u^2)^{1/2}$, the error introduced by the parabola approximation is given by

$$\begin{aligned}P((B_n \setminus S_n) \cup (S_n \setminus B_n)) &= P(B_n \setminus S_n) \\ &\leq 2(2\pi r_2)^{-1}\sqrt{2r_2\varepsilon_n}\int_{r_2}^{\infty} se^{-s^2/2}ds \\ &= (2\pi r_2)^{-1}\sqrt{2r_2\varepsilon_n}\exp\{-r_2^2/2\}\end{aligned}$$

Consequently,

$$nP((B_n \setminus S_n) \cup (S_n \setminus B_n)) \leq C(\varepsilon_n/r_2)^{1/2} \exp\{-r_0\varepsilon_n/2\}$$

for some positive finite constant C.

Since $W(0)$ is in A_n^* with a probability tending to one, any "minimal" region containing at least one vertex point of the convex hull of the Poisson point process is no larger than B_n. Any line of slope 0 above the horizontal axis at distance greater than ε_n intersects the circle of radius r_1 centered at $(0, r_2)$. On the other hand, that there is no smaller region containing $W(0)$ with a probability tending to one can be seen as follows. Assume such a region is defined by $\tilde{B}_n = \{(u_1, u_2) \in \mathbb{R}^2 : u_1^2/2r_2 \leq u_2,\ |u_1| \leq a_n\}$ for some $0 < a_n \leq \sqrt{2r_2\varepsilon_n}$. Now the density $f^{W(0)}(x, y)$ of $W(0)$ at (x, y) is proportional to $\varphi(x)\varphi(r_2 - y)\exp\{-nP(A_0)\}$, where $A_0 = \{(u, v) \in \mathbb{R}^2 : u^2/2r_2 \leq v \leq y\}$, for the following reasons. First, the event that there is no point of the Poisson point process below the line of slope zero through (x, y) happens with probability $\exp\{-nP(A_0)\}$, being the probability of no event of the underlying Poisson distribution. Second, the point with the smallest second coordinate has the density $\varphi(x)\varphi(r_2 - y)$. Therefore, the probability $P(W(0) \notin \tilde{B}_n)$ is given by

$$2\int_0^{\varepsilon_n}\int_{a_n}^{\sqrt{2r_2 y}} f^{W(0)}(x, y)dx\,dy.$$

Straightforward integration yields

$$\sim (\Phi(\sqrt{r_2\varepsilon_n}) - \Phi(a_n/\sqrt{2}))/(\Phi(\sqrt{r_2\varepsilon_n}) - 1/2),$$

which tends to zero as $n \to \infty$ iff $a_n \to \sqrt{2r_2\varepsilon_n}$. □

For the rest of the paper, we will always resort to the parabola approximation without further mention.

3 Moments of convex hull functionals

In this section the jump measure of the process W will be derived first. Then some martingales will be studied which relate the expectation of some functionals of the convex hull to the expectation of some functionals of the jump measure. Let η_b be the number of jumps, L_b the sum of all jump lengths of the process $\{W(c) : 0 \le c \le b\}$, and A_b the area of the convex hull of $\{(0, r_2)\} \cup \{W(c) : 0 \le c \le b\}$. For each $a > 0$ and $b > 0$ with $a < b$ define the σ-algebra

$$\mathcal{F}_{a,b} = \sigma\{W(c) : a \le c \le b\}. \tag{13}$$

Define the sector

$$S_0 = \{(u_1, u_2) \in \mathbb{R}^2 : u_2 \ge u_1^2/2r_2\}$$

and let $C_0 = C_0(S_0)$ be the set of continuous functions $g : S_0 \to \mathbb{R}$ with compact support contained in S_0. Furthermore, for each $a > 0$ and $(x, y) \in S_0$ let the linear operator $L_a : C_0 \to C_0$ be defined by

$$\begin{aligned} [L_a g](x,y) &= \lim_{h\downarrow 0} h^{-1} E\{g(W(a+h)) - g(x,y) | W(a) = (x,y)\} \qquad (14)\\ &= n\int_0^{X_0 - x} u\varphi(x+u)\varphi(r_2 - au - y)\{g(x+u, y+au) - g(x,y)\}du, \end{aligned}$$

where $X_0 = ar_2 + (a^2r_2^2 + 2r_2(y - ax))^{1/2}$ is the bigger intersection of the approximating parabola $v = u^2/2r_2$ with the line of slope a through (x, y) and $\varphi(x) = (2\pi)^{-1/2}\exp(-x^2/2)$ denotes the density of the one-dimensional standard normal distribution. From the representation

$$[L_a g](w) = \int_{\mathbb{R}^2} [g(w+z) - g(w)]\, M(w; dz) \tag{15}$$

of the operator L_a with respect to the jump measure $M(W(a); \cdot)$ of the process W, we see that the jump measure of W in w at time a is given by

$$M(a, w; B) = n\int_0^{X_0 - x} u\varphi(x+u)\varphi(r_2 - y - au)\, 1_B(u, au)\, du \tag{16}$$

for any Borel set $B \subset \mathbb{R}^2$.

Lemma 3.1 *For each $g \in C_0$ and $b_1 > 0$ the process*

$$\{g(W(b_2)) - \int_{b_1}^{b_2} [L_c g](W(c))dc : \; b_2 \geq b_1\}$$

is a martingale with respect to the filtration $\{\mathcal{F}_{b_1,b_2} : b_2 \geq b_1\}$.

Proof. Our arguments will establish that for each $a > 0$ and $w = (x, y) \in A_n^*$

$$\lim_{h \downarrow 0} h^{-1} E\{g(W(a+h)) - g(w) | W(a) = w\} = [L_a g](w). \tag{17}$$

For each $(x, y) \in A_n^*$ consider $A_h = \{(u, v) \in I\!R^2 : au + (y - ax) \leq v \leq (a+h)u + (y - ax)\}$. Remember f is the bivariate normal density function. Since the probability of finding more than one point of the Poisson point process ξ_n in A_h is $o(h)$ as $h \downarrow 0$, we get, for each $g \in C_0$,

$$\begin{aligned}
E\{g(W(a \; + \; h)) &- g(w)|W(a) = (x, y) = w\} \\
&= n \iint_{A_h} f(v)\{g(v) - g(w)\}\, dv + o(h) \\
&= nh \int_0^\infty u\varphi(x+u)\varphi(r_2 - au - y)\{g(x+u, y+au) - g(x, y)\}du + o(h) \\
&= nh \int_0^{X_0 - x} u\varphi(x+u)\varphi(r_2 - au - y)\{g(x+u, y+au) - g(x, y)\}du + o(h),
\end{aligned}$$

where the remainder term is $o(h)$ uniformly in a for $b_1 \leq a \leq b_2 < \infty$ and $w \in S_0$. To obtain the second equality we use the uniform continuity of g and f on S_0. For the last equality it remains to be seen that the line of slope a through (x, y) cuts the parabola $v = u^2/2r_2$ in X_0 and that integration of the first coordinate above X_0 is negligible by Lemma 2.2. □

Next we introduce the random counting measure

$$\eta(a, b; B) = \sum_{\substack{W(c) \neq W(c-) \\ a \leq c \leq b}} 1_B(W(c) - W(c-))\,, \tag{18}$$

for $b > a > 0$ and $B \in \mathcal{B}^2$ (the family of Borel sets in $I\!R^2$). Let

$$\eta_a = \eta(0, a; I\!R^2) \tag{19}$$

denote the number of jumps of W within the time interval $[0, a]$. We will make use of the following notation for $b > 0$ and $W(b) = (X(b), Y(b))$:

$$L(W(b)) \;\; = \;\; \int_{I\!R^2} \|z\| M(W(b); dz) \tag{20}$$

$$
\begin{aligned}
L'(W(b)) &= \int_{I\!R^2} \|z\|^2 M(W(b);dz) \\
A(W(b)) &= 2^{-1}(r_2 - Y(b) - b^2 r_2/2) \int_{I\!R^2} \|z\| M(W(b);dz) \\
A'(W(b)) &= 4^{-1}(r_2 - Y(b) - b^2 r_2/2)^2 \int_{I\!R^2} \|z\|^2 M(W(b);dz).
\end{aligned}
$$

Lemma 3.2 *For sufficiently small $b > 0$*

$$\{\int_{I\!R_2} g(z)\,\eta(0,b;dz) - \int_0^b dc \int_{I\!R_2} g(z)\, M(W(c);dz)\}, \tag{21}$$

$$\{\eta_b - \int_0^b M(W(c);I\!R^2)\,dc\}, \tag{22}$$

$$\{\eta_b^2 - \int_0^b (2\eta_c + 1)\, M(W(c);I\!R^2)\,dc\}, \tag{23}$$

$$\{L_b - \int_0^b L(W(c))\,dc\}, \tag{24}$$

$$\{L_b^2 - \int_0^b [2L_c\, L(W(c)) + L'(W(c))]\,dc\}, \tag{25}$$

$$\{A_b - \int_0^b A(W(c))\,dc\} \tag{26}$$

and

$$\{A_b^2 - \int_0^b [2A_c\, A(W(c)) + A'(W(c))]\,dc\} \tag{27}$$

are $\mathcal{F}_{0,b}$-martingales.

Proof. The main ingredients of the proof are the properties of the Poisson point process and a theorem due to Stroock (see [11], Theorem 1.3). Part of our reasoning was already given in [5], Lemma 2.6. For our vertex process W, Stroock's result is as follows: For each bounded Borel measurable function $g : I\!R^2 \to I\!R$ that vanishes in a neighbourhood of the origin, for each $a \in I\!R$ and each $\theta \in I\!R^2$, the process

$$
\begin{aligned}
\exp\Big\{ &< i\theta, W(b) - W(a) > + \int_{I\!R^2} g(z)\,\eta(a,b\,;dz) \\
&- \int_a^b dc \int_{I\!R^2} \{e^{<i\theta,z>+g(z)} - 1\} M(W(c);dz) \Big\}, \quad b \geq a
\end{aligned}
$$

is a $\mathcal{F}_{a,b}$-martingale ($<x,y>$ denotes the inner product on $I\!R^2$). Write $J(\lambda)$ for this martingale when $\theta = 0$, $a = 0$ and when g is replaced by λg. Once differentiating $J(\lambda)$ with respect to λ and setting $\lambda = 0$ lead to the first assertion (21). Then (22) is obtained by letting $g(z) = 1$, (24) and (26) by letting $g(z) = \|z\|$ and $g(z) = (r_2 - Y(b) - b^2 r_2/2) \cdot \|z\|/2$, respectively, and by noticing that $\int_{I\!R^2} \|z\|\eta(0,b;dz) = L_b$ and $2^{-1}(r_2 - Y(b) - b^2 r_2/2) \int_{I\!R^2} \|z\|\eta(0,b;dz) \sim A_b$, respectively.

To verify the second moment claims (23), (25) and (27), it is convenient to use the notation $\eta_{a,b} = \eta_b - \eta_a$, $L(a,b) = L_b - L_a$ and $A(a,b) = A_b - A_a$. From (21) we derive $E[\eta_{b,b+h}|\mathcal{F}_{0,b}] \sim h\, M(W(b); I\!R^2)$. Thus, for each $b > 0$,

$$
\begin{aligned}
E\{\eta_{b+h}^2 - \eta_b^2 | \mathcal{F}_{0,b}\} &= E\{\eta_{b,b+h}^2 + 2\eta_b \eta_{b,b+h} | \mathcal{F}_{0,b}\} \\
&= E\{(1 + 2\eta_b)(hM(W(b); I\!R^2) + R_h) | \mathcal{F}_{0,b}\}, \qquad (28)
\end{aligned}
$$

where $|R_h| = o(h)$, as $h \downarrow 0$, which follows from the properties of the underlying Poisson point process. Also

$$
\begin{aligned}
h\, M(W(b); I\!R^2) \exp\{-hM(W(b); I\!R^2)\} &\leq E\{\eta_{b,b+h}^2 | W(b)\} \\
&\leq h\, M(W(b); I\!R^2) + 2^{-1} h^2 [M(W(b); I\!R^2)]^2.
\end{aligned}
$$

Similarly, it can be found $E[L(b,b+h)|\mathcal{F}_{0,b}] \sim h \int_{I\!R^2} \|z\| M(W(b); dz)$, which implies

$$
\begin{aligned}
E\{L_{b+h}^2 - L_b^2 | \mathcal{F}_{0,b}\} &= E\{2L_b h \int_{I\!R^2} \|z\| M(W(b); dz) \\
&\quad + h(\int_{I\!R^2} \|z\|^2 M(W(b); dz)) + R_h | \mathcal{F}_{0,b}\}, \qquad (29)
\end{aligned}
$$

and

$$
\begin{aligned}
hL'(W(b)) \exp\{-hM(W(b); I\!R^2)\} &\leq E\{L^2(b, b+h) | W(b)\} \\
&\leq hL'(W(b)) + 2^{-1} h^2 L'(W(b))^2 .
\end{aligned}
$$

Now divide both sides of the equalities (28) and (29) by h, pass to the limit, integrate the resulting expressions with respect to the time parameter and pull the expectation out of the integrals.

Finally, the verification of (27) consists of mimicking the above arguments together with relying on the relation $A(b, b+h) \sim L(b, b+h) \cdot (r_2 - Y(b) - b^2 r_2/2)/2$. □

By considering the vertex process $\{W(a) = (X(a), Y(a))\}$ under the transformation

$$
\begin{aligned}
R(a) &= X(a) - a\, r_2 \qquad (30) \\
S(a) &= Y(a) - (X(a)^2 - R(a)^2)/2r_2 \\
T(a) &= (R(a), S(a)),
\end{aligned}
$$

the vertex process can be made stationary. The local infinitesimal generator of the stationary process T becomes, for functions $g \in C_0^1 = C_0^1(S_0)$ (= the set of continuously differentiable real-valued functions with compact support, defined on S_0)

$$
\lim_{h \to 0} h^{-1} E\{g(R(a+h), S(a+h)) \;-\; g(r,s) | (R(a), S(a)) = (r,s)\} \qquad (31)
$$

$$= n\sqrt{2\pi}\varphi(r_2 - s)\int_0^{\sqrt{2r_2 s}-r} u\,\varphi(r+u)\,[g(r+u,s) - g(r,s)]\,du$$
$$-r_2\,\frac{\partial}{\partial r}\,g(r,s) - r\,\frac{\partial}{\partial s}\,g(r,s),$$

and the jump measure in $t = (r, s)$ is

$$M(t; B) = n\sqrt{2\pi}\varphi(r_2 - s)\int_0^{\sqrt{2r_2 s}-r} u\,\varphi(r+u)\,1_B(u,0)\,du. \qquad (32)$$

The above transformation can be imagined as follows. As soon as a jump of W is observed, say, at time a, the coordinate system is oriented such that the process W jumps horizontally from $W(a-)$ to $W(a)$, and the origin is shifted by $(ar_2, a^2r_2/2)$. The deterministic part runs through curves parallel to the parabola $u_2 = u_1^2/2r_2$. The following lemma provides a bridge between the functionals of the stationary and the non-stationary processes.

Lemma 3.3 *Let $W(b) = (x, y)$ for sufficiently small $b > 0$ and $T(b)$ as given in (30). Then*

$$\begin{aligned} L(T(b)) &= L(W(b)) - x(r_2 + y)/\sqrt{x^2+y^2} \\ L'(T(b)) &= L'(W(b)) - 2x(r_2 + y) \\ A(T(b)) &= A(W(b)) - x(r_2^2 - x^2 - 2y^2)/(2\sqrt{x^2+y^2})\,. \end{aligned}$$

Proof. These identities are immediate consequences of (31) applied to $g(x, y) = (x^2 + y^2)^{1/2}$, $g(x, y) = x^2 + y^2$ and $g(x, y) = (x^2 + y^2)^{1/2}(r_2 - y)/2$, respectively. □

Note that $M(T(b); I\!R^2) = M(W(b); I\!R^2)$ for each $b > 0$.

Lemma 3.4 *For sufficiently small $b > 0$*

$$\begin{aligned} E\eta_b &= b \cdot E[M(T(0); I\!R^2)] & (33) \\ E\eta_b^2 &= E\eta_b + 2\int_0^b da \int_0^a E[M^*(T(0); I\!R^2)\, M(T(a-c); I\!R^2)]dc & (34) \\ EL_b &= b \cdot E[L(T(0))] & (35) \\ EL_b^2 &= b \cdot E[L'(T(0))] + 2\int_0^b E[L_a\, L(W(a))]da & (36) \end{aligned}$$

where $M^*(\,\cdot\,;\,\cdot\,)$ is the "backward jump measure" of W.

Proof. Since the process T is stationary, to get (33), just combine (22) and (31) with $g \equiv 1$. In deriving relation (34) we take advantage of the following time reversal argument. Consider the time reversed process $\{(-R(a-c), S(a-c)) : c \geq 0\}$. The process $\{(-R(c), S(c)) : c \leq 0\}$ is obtained from the process $\{(R(c), S(c)) : c > 0\}$ by interchanging the sign of the first coordinate and by moving backwards in time. Clearly, this process has the same distribution as the original process $T(c)$. Denote the jump measure of the time reversed process at c by $M^*(T(c); I\!R^2)$. Then by using the stationarity again, we find (34). Next, in view of (24), (31), Lemma 3.3, and the stationarity of the process T, (35) follows from the fact that the density of $T(0)$ is an even function in the first coordinate and the range of integration is symmetric about 0 for this coordinate. Similarly, (36) can be deduced. □

4 Proofs of the Main Results

Recall N_n, L_n and A_n to be the number of vertices, the perimeter and the area, respectively, of the convex hull of a sample of size n. In order to prove the central limit theorems for N_n, L_n and A_n we need three lemmas, one quite technical, giving the densities needed to compute the moments.

Lemma 4.1 *Let $\mathcal{F}_{a+} = \sigma\{T(c) : c \geq a\}$, $\mathcal{F}_0 = \sigma\{T(c) : c \leq 0\}$, and let $A \in \mathcal{F}_0$ and $B \in \mathcal{F}_{a+}$ for each $a > 0$. Then we have*

$$|P(A \cap B) - P(A)P(B)| \leq \tau_n(a), \tag{37}$$

where

$$\tau_n(a) \leq 4\exp\{-\exp((a^2 r_2/8 - \varepsilon_n) r_0/2) a^3 r_2^2/(12\sqrt{2\pi})\} \longrightarrow 0, \text{ as } n \to \infty. \tag{38}$$

Proof. Recall the notation $T(a) = (R(a), S(a))$ for each $a \geq 0$. If both $S(0)$ and $S(a)$ are smaller than $a^2 r_2/8$, the events $A \in \mathcal{F}_0$ and $B \in \mathcal{F}_{a+}$ are supported on disjoint regions of the underlying Poisson point process. By using the Markov structure and the stationarity of the process $\{T(a) : a \in I\!R\}$, we find the same estimate as given in [5], p.357,

$$|\ P(A \cap B) - P(A)P(B)\ | \quad \leq \quad 4P(S(0) \geq a^2 r_2/8) = e^{-nP(K_a)}$$

where $K_a = \{(u, v) : u^2/2r_2 < v < a^2 r_2/8\}$. With the help of the intermediate value theorem, we obtain, for some value $0 < \gamma = \gamma(a) < 1/2$,

$$\begin{aligned} P(K_a) &= \varphi(r_0 + \varepsilon_n/2 - a^2 r_2 \gamma/8)\, a^3 r_2^2/12 \\ &\geq \varphi(r_0 + \varepsilon_n/2 - a^2 r_2 \gamma/16)\, a^3 r_2^2/12. \end{aligned}$$

The right-hand side of the last inequality increases to ∞ as $n \to \infty$, when multiplied by the factor n if $a > 2\sqrt{2\varepsilon_n/r_2}$. Our assertion is verified. □

Lemma 4.2 *For each $a > 0$ the moment generating functions*

$$\begin{aligned} M_1(\lambda) &= E\exp\{\lambda\eta_a\} \\ M_2(\lambda) &= E\exp\{\lambda L_a\} \end{aligned}$$

and

$$M_3(\lambda) = E\exp\{\lambda A_a\}$$

are finite for all values $0 < \lambda$ in a neighbourhood of the origin.

Proof. The random variable η_a can be bounded by a random variable from a Poisson distribution with parameter $\theta \le nP(A_n^*)a/(2\pi r_0)$, where $P(A_n^*) \sim \exp\{-(r_0 - \varepsilon_n/2)^2/2\}$. Thus,

$$E[e^{\lambda(N_a-1)}] \le \exp\{(e^\lambda - 1)e^{r_0\varepsilon_n/2}\, a/(2\pi r_0)\}$$

which is finite for each $0 < \lambda < \infty$.

To show that the moment generating function of the perimeter length is finite, consider the radius R_n of the circle that contains all the sample points. It is sufficient to prove that $E(e^{\lambda R_n}) < \infty$. We have

$$\begin{aligned} E(e^{\lambda R_n}) &= n\int_0^\infty e^{\lambda y}(1 - e^{-y^2/2})^{n-1}\, y e^{-y^2/2}\, dy \\ &\le n\int_0^\infty y e^{\lambda y} e^{-y^2/2}\, dy, \end{aligned}$$

which is finite for each $0 < \lambda < \infty$. In the same way it may be shown that $M_3(\lambda)$ is finite for all values $0 < \lambda$ in a neighbourhood of the origin. □

By the stationarity of the process T, the distribution of $(T(a), T(b))$ is the same as that one of $(T(0), T(b-a))$ and the distribution of $T(a)$ is the same for each $a \in \mathbb{R}$.

Lemma 4.3 *For each $a > 0$ define*

$$\begin{aligned} K_1 &= \{(u,v) \in \mathbb{R}^2 : u^2/(2r_2) \le v \le \varepsilon_n,\ |u| \le \sqrt{2r_2\varepsilon_n}\,\} \\ K_2(x_1,y_1) &= \{(u,v) \in K_1 : x_1 < ar_2 + u,\ y_1 \le v + au + a^2r_2/2\} \\ A_0 &= A_0(y) = \{(u,v) \in \mathbb{R}^2 : u^2/(2r_2) \le v \le y\} \\ A_1 &= A_0(y_1) \\ A_2 &= A_2(y_2) = \{(u,v) \in \mathbb{R}^2 : ar_2 - \sqrt{2r_2\varepsilon_n} \le u \le ar_2 + \sqrt{2r_2\varepsilon_n}\,, \\ &\qquad u^2/(2r_2) \le v \le au + (y_2 - a^2r_2/2)\}. \end{aligned}$$

Then the densities can be expressed as follows:

(i) For any pair $(x,y) \in K_1$

$$f^{T(0)}(x,y) = N_0 \varphi(x)\varphi(r_2 - y)e^{-nP(A_0)},$$

where N_0 *is such that* $f^{T(0)}$ *is a probability density function.*

(ii) For any pairs $(x_1,y_1) \in K_1$ *and* $(x_2,y_2) \in K_2(x_1,y_1)$

$$\begin{aligned} f^{T(0),T(a)}(x_1,y_1;x_2,y_2) &= N_1 \varphi(x_1)\varphi(r_2 - y_1)\varphi(x_2 + ar_2) \\ &\quad \varphi(r_2 - y_2 - ax_2 - a^2 r_2/2)e^{-nP(A_1 \cup A_2)}, \end{aligned}$$

where N_1 *is defined in (iv) below.*

(iii) $P(T(0) = T(a)|T(0) = (x_1,y_1)) = e^{-nP(A_2 \setminus A_1)}$ *for any pair* $(x_1,y_1) \in K_1$.

(iv) $0 < N_1$ *is such that* $\iint_{K_1} \iint_{K_2(x_1,y_1)} f^{T(0),T(a)}(x_1,y_1;x_2,y_2)\, dx_2\, dy_2\, dx_1\, dy_1$ $+ \iint_{K_1} e^{-nP(A_2 \setminus A_1)}\ f^{T(0)}(x_1,y_1) dx_1\, dy_1 = 1.$

Proof. (i) It suffices to take into consideration the following two cases. Below the horizontal line through (x,y) there is no point of the underlying Poisson point process. This happens with probability $\exp\{-nP(A_0)\}$, being the probability of no event of the Poisson point process. Furthermore, the point with the smallest second coordinate has the density $\varphi((x^2 + (r_2 - y)^2)^{1/2})$, which is proportional to $\varphi(x)\,\varphi(r_2 - y)$.

(ii) We begin with the derivation of the common distribution of $W(0)$ and $W(a)$ and carry out the substitution $(x_2,y_2) \mapsto (x_2 + ar_2, y_2 + ax_2 + a^2 r_2/2)$ in (30). Let $A_1' = A_1'(y_1) = \{(u,v) \in \mathbb{R}^2 : v \le y_1\}$ and $A_2' = A_2'(x_1,y_1;x_2,y_2) = \{(u,v) \in \mathbb{R}^2 : x_1 \le u,\ y_1 \le v \le au + (y_2 - ax_2)\}$. Then A_1' and A_2' both must not contain any point. For $(x_1,y_1) \in \mathbb{R}^2$ such that $x_1 < x_2$ and $y_1 < y_2$ we have

$$f^{W(0),W(a)}(x_1,y_1;x_2,y_2) = N_1' \varphi(x_1)\,\varphi(r_2 - y_1)\,\varphi(x_2)\,\varphi(r_2 - y_2)\,e^{-nP(A_1' \cup A_2')},$$

where N_1' is a normalizing constant. Next

$$f^{T(0),T(a)}(x_1,y_1;x_2,y_2) = f^{W(0),W(a)}(x_1,y_1;x_2 + ar_2, y_2 + ax_2 + a^2 r_2/2),$$

where $(x_1,y_1) \in K_1$, $(x_2,y_2) \in K_2(x_1,y_1)$, $K_1 \times K_2 \subset S_0 \times S_0$ and

$$\begin{aligned} A_2(y_2) &= A_2'(x_1,y_1;x_2 + ar_2, y_2 + ax_2 + a^2 r_2/2) \cap \{(u,v) \in \mathbb{R}^2 : u^2/(2r_2) \le v \\ &\qquad \le au + (\varepsilon_n - a^2 r_2/2), ar_2 - \sqrt{2r_2\varepsilon_n} \le u \le ar_2 + \sqrt{2r_2\varepsilon_n}\}, \end{aligned}$$

which equals A_2. That the intersection of both regions A_1' and A_2' is not empty is equivalent to $ar_2/\sqrt{2r_2} - \sqrt{y_1} < \sqrt{y_2}$. Note that the parabolas $v = y_1 - u^2/2r_2$ and $v = y_2 - (u - ar_2)^2/2r_2$ intersect at the point $(x_0, y_1 - x_0^2/2r_2)$, where $x_0 = ar_2/2 + (y_1 - y_2)/a$.

(iii) Let $(x_1, y_1) = (x, y)$. For $W(0)$ to be equal to $W(a)$ conditioned on the event $W(0) = (x, y)$ means that no point of the Poisson point process lies below the line of slope a through (x, y), when we already know that no point lies below the line of slope 0 through (x, y). Consequently, the additional domain where no point of the Poisson realization falls has probability $P(A_2 \setminus A_1)$. Notice that since the line $au + y - a^2 r_2/2$ forming part of the boundary of $A_2(y)$ contains the point (x, y) only if $x = ar_2/2$ and intersects $v = y$ in $(ar_2/2, y)$, $e^{-nP(A_2 \setminus A_1)}$ does not depend on x.

(iv) When considering two time points simultaneously, there are only two cases to include for the normalizing constant N_1 : either $T(0) \neq T(a)$ or $T(0) = T(a)$. □

Proposition 4.4 *Let η_b be the number of jumps, L_b the sum of all jump lengths of the process $\{W(c) : 0 \leq c \leq b\}$ and A_b the area of the convex hull of $\{(0, r_2)\} \cup \{W(c) : 0 \leq c \leq b\}$. Then, as $n \to \infty$,*

(i)

$$E(\eta_b) \sim b \cdot c_1, \tag{39}$$

$$Var(\eta_b) \sim b \cdot c_1 + b^2 \cdot c_2 \tag{40}$$

and

$$(\eta_b - c_1 b)/(c_1 b + c_2 b^2)^{1/2} \xrightarrow{L} \mathcal{N}(0,1), \tag{41}$$

where $c_1 = 1/\sqrt{\pi} \approx 0.56419$, $c_2 \leq \sqrt{3}/(2\pi) - 1/6 \approx 0.108998$ and $\mathcal{N}(0,1)$ denotes the standard normal distribution.

(ii)

$$E(L_b) \sim b, \tag{42}$$

$$Var(L_b) \sim 4\pi^{-1/2} \cdot b + b^2 (r_2 + \mathcal{O}(1) + \mathcal{O}(r_2 (r_0 \varepsilon_n)^{1/2} \, e^{-(r_0 \varepsilon_n)^{1/2}})) \tag{43}$$

and

$$(L_b - b)/(4\pi^{-1/2} b + r_2 \, b^2)^{1/2} \xrightarrow{L} \mathcal{N}(0,1). \tag{44}$$

(iii)

$$E(A_b) \sim b \cdot r_0/2, \tag{45}$$

$$Var(A_b) \sim \pi^{-1/2} r_0^2 \cdot b + b^2 \cdot r_0^3/4 \tag{46}$$

and

$$(A_b - b\, r_0/2)/(\pi^{-1/2} r_0^2 b + b^2 r_0^3/4)^{1/2} \xrightarrow{L} \mathcal{N}(0,1). \tag{47}$$

Proof. Proof of (i): According to Lemma 3.4 we must compute $E[M(T(0); I\!R^2)]$ and $E[M^*(T(0); I\!R^2) \; M(T(a); I\!R^2)]$, which are rather tedious chores. Thus, the following long stretch of integration serves no purpose other than determining the asymptotic constants accompanying the convergence rates for the moments of η_b and L_b. To shorten the forthcoming lengthy expressions slightly, for each $a \in I\!R$ we write

$$M(T(a)) = M(T(a); I\!R^2) \tag{48}$$

$$M^*(T(a)) = M^*(T(a); I\!R^2)$$

Remember that $\varphi(x) = (2\pi)^{-1/2}\exp(-x^2/2)$ and define $\Phi(x) = \int_{-\infty}^{x}\varphi(y)dy$. First from (32) we derive, for $T(0) = (x, y)$,

$$M((x,y)) = n\sqrt{2\pi}\varphi(r_2 - y)\{\varphi(x) - \varphi(\sqrt{2r_2y}) + x\Phi(x) - x\Phi(\sqrt{2r_2y})\}. \tag{49}$$

Similarly, when walking clockwise, we get the backward jump measure

$$M^*((x,y)) = n\sqrt{2\pi}\varphi(r_2 - y)\{\varphi(x) - \varphi(\sqrt{2r_2y}) + x\Phi(x) - x\Phi(-\sqrt{2r_2y})\}. \tag{50}$$

In finding the expectation of the jump measure with respect to the density $f^{T(0)}(\cdot)$ of $T(0)$, given in Lemma 4.3(i), we first integrate the first coordinate and then apply a "shrinking argument" for the remaining integral. The range $[0, \varepsilon_n]$ of integration of the second coordinate shrinks to a point, as $n \to \infty$. In fact,

$$\begin{aligned}
&E[M((x,y))]\\
&\quad = n\sqrt{2\pi}\,N_0\int_0^{\varepsilon_n} e^{-nP(A_0)}\,\varphi^2(r_2-y)\,dy\int_{-\sqrt{2r_2y}}^{\sqrt{2r_2y}}\varphi(x)\{\varphi(x)\\
&\qquad\quad -\ \varphi(\sqrt{2r_2y}) + x\,\Phi(x) - x\Phi(\sqrt{2r_2y})\}dx\\
&\quad = n\sqrt{2\pi}\{\int_0^{\varepsilon_n} e^{-nP(A_0)}\,\varphi^2(r_2-y)\{\pi^{-1/2}\,[2\Phi(2\sqrt{r_2y})-1]\\
&\qquad\quad -\ 2\varphi(\sqrt{2r_2y})[2\Phi(\sqrt{2r_2y})-1]\}dy\}\\
&\qquad\quad \cdot\ \{\int_0^{\varepsilon_n} e^{-nP(A_0)}\,\varphi(r_2-y)\,[2\Phi(\sqrt{2r_2y})-1]dy\}^{-1}\\
&\quad \sim n\sqrt{2\pi}\,\varphi(r_2-\varepsilon_n/2)\{\frac{1}{\sqrt{\pi}}\,\frac{2\Phi(\sqrt{2r_2\varepsilon_n})-1}{2\Phi(\sqrt{r_2\varepsilon_n})-1}\ -\ 2\varphi(\sqrt{r_2\varepsilon_n})\} \qquad (51)\\
&\quad \sim n\sqrt{2\pi}\,\varphi(r_0)/\sqrt{\pi} + \mathcal{O}(n^{-\varepsilon_n/r_2})\\
&\quad = 1/\sqrt{\pi}\ +\ o(1),
\end{aligned}$$

as $n \to \infty$. To get (51), we use $\varepsilon_n \to 0$, $r_2\varepsilon_n \to \infty$, as $n \to \infty$, and the fact that, in the second equality of the above-displayed calculations, denominator and numerator have the same region of integration. The quotient in (51) containing the $\Phi(\cdot)$'s tends to 1 as $n \to \infty$. Now remember that $r_0 = (2\ln n)^{1/2}$ and $r_2 = r_0 + \varepsilon_n/2$. This shows the first moment claim in (i) of our proposition.

In evaluating the expectation $E[M^*(T(0))M(T(a))]$ with respect to the joint density $f^{T(0),T(a)}(\cdot)$ we split the range of integration according to the following three disjoint events: a) $T(a) \neq T(0)$ and the regions, free of points and bounded by the approximating parabolas, are not disjoint. b) $T(a) \neq T(0)$ and the regions, free of points and bounded by the approximating parabolas, are disjoint. c) $T(a) = T(0)$. Furthermore, the following easy inequality will be useful below

$$\varphi(x + ar_2)\varphi(r_2 - y - ax - a^2r_2/2) \le \varphi(x)\varphi(r_2 - y). \tag{52}$$

Write $Y_c = (ar_2/\sqrt{2r_2} - \sqrt{y_1})^2$ and $Y_s = ar_2/2 + (y_1 - y_2)/a$ and recall the densities given in Lemma 4.3. For sufficiently small $a > 0$, as $n \to \infty$,

$$
\begin{aligned}
&E[M^*(T(0))M(T(a))] \\
&\quad \le \{\iint_{K_1} \iint_{K_2(x_1,y_1)} f^{T(0),T(a)}(x_1,y_1;x_2,y_2) M^*((x_1,y_1))\, M((x_2,y_2)) \\
&\qquad dx_2\, dy_2\, dx_1\, dy_1\} + \iint_{K_1} f^{T(0)}(x,y)\, M^*((x,y))\, M((x,y)) dx\, dy \\
&\quad \le \{\iint_{K_1} \iint_{K_2(x_1,y_1)} e^{-nP(A_2\cup A_1)} \varphi(r_2-y_1)\, \varphi(x_1)\, \varphi(r_2-y_2)\, \varphi(x_2) \\
&\qquad \cdot M^*((x_1,y_1))\, M((x_2,y_2)) dx_2\, dy_2\, dx_1\, dy_1\} \\
&\qquad \cdot \{\iint_{K_1} \iint_{K_2(x_1,y_1)} e^{-nP(A_2\cup A_1)}\, \varphi(r_2-y_1)\, \varphi(x_1)\, \varphi(x_2+ar_2) \\
&\qquad \cdot \varphi(r_2 - y_2 - ax_2 - a^2 r_2/2) dx_2\, dy_2\, dx_1\, dy_1\, \}^{-1} \\
&\qquad + N_0 \iint_{K_1} e^{-nP(A_0)}\, \varphi(r_2-y)\, \varphi(x)\, M^*((x,y))\, M((x,y)) dx\, dy.
\end{aligned}
$$

The first inequality is a consequence of the facts that the region of integration of the second term is larger than need be and that in the second term $\exp -nP(A_2 \setminus A_1)$ is less than 1; the second inequality uses (52) and accounts for the fact that in the denominator of the first term the integral in the case $T(a) = T(0)$ is neglected. The next step is just an integration exercise with respect to the first coordinate, where the assumption $n \to \infty$ must be kept in mind. When integrating with respect to the second coordinate, we once again take advantage of the fact that both terms have the same region of integration in their numerator and denominator. Notice the asymptotic behaviour of the following items: $\varepsilon_n \to 0$, $r_2\varepsilon_n \to \infty$, $Y_s \to \infty$ and $Y_s - ar_2 \to -\infty$ if $a > 1/r_2$. Since $ar_2 \in (0, \sqrt{2r_2\varepsilon_n})$, the domain $0 < a < 1/r_2$ is negligible. Moreover, $e^{-nP(A_0)} \sim 1$ for $y = \varepsilon_n/2$. Thus

$$
\begin{aligned}
&E[M^*(T(0))M(T(a))] \\
&\quad \sim n^2\, 2\pi \{\int_0^{\varepsilon_n} e^{-nP(A_1)}\, \varphi^2(r_2-y_1)\, dy_1 \int_{Y_c}^{\varepsilon_n} e^{-nP(A_2\setminus A_1)} \varphi^2(r_2-y_2) \\
&\qquad \{\pi^{-1/2}[\Phi(\sqrt{2}\, Y_s) - \Phi(-2\sqrt{r_2 y_1})] - [\varphi(\sqrt{2r_2y_1}) + \varphi(Y_s)] \\
&\qquad \cdot [\Phi(Y_s) - \Phi(-\sqrt{2r_2y_1})]\}\{\pi^{-1/2}[\Phi(2\sqrt{r_2y_2}) - \Phi(\sqrt{2}(Y_s - r_2 a))] \\
&\qquad - [\Phi(\sqrt{2r_2y_2}) - \Phi(Y_s - r_2 a)] \cdot [\varphi(Y_s - r_2 a) + \varphi(\sqrt{2r_2y_2})]\} dy_2 \\
&\qquad + \int_0^{\varepsilon_n} e^{-nP(A_1)}\, \varphi^2(r_2-y_1)\{\pi^{-1/2}[2\Phi(2\sqrt{r_2y_1}) - 1] \\
&\qquad - 2\varphi(\sqrt{2r_2y_1})[2\Phi(\sqrt{2r_2y_1}) - 1]\} dy_1
\end{aligned}
$$

$$\cdot \int_0^{Y_c} e^{-nP(A_2)} \varphi^2(r_2 - y_2)\{\pi^{-1/2}[2\Phi(2\sqrt{r_2 y_2}) - 1]$$
$$- 2\varphi(\sqrt{2r_2 y_2})[2\Phi(\sqrt{2r_2 y_2}) - 1]\} dy_2\}$$
$$\cdot \{\int_0^{\varepsilon_n} e^{-nP(A_1)} \varphi(r_2 - y_1)\, dy_1 \int_{Y_c}^{\varepsilon_n} e^{-nP(A_2 \setminus A_1)} \varphi(r_2 - y_2)$$
$$[\Phi(Y_s) - \Phi(-\sqrt{2r_2 y_1})] \cdot [\Phi(\sqrt{2r_2 y_2}) - \Phi(Y_s - r_2 a)] dy_2$$
$$+ \int_0^{\varepsilon_n} e^{-nP(A_1)}\, \varphi(r_2 - y_1)\, [2\Phi(\sqrt{2r_2 y_1}) - 1] dy_1$$
$$\cdot \int_0^{Y_c} e^{-nP(A_2)}\, \varphi(r_2 - y_2)[2\Phi(\sqrt{2r_2 y_2}) - 1] dy_2\}^{-1}$$
$$+ 2\pi n^2 N_0 \int_0^{\varepsilon_n} e^{-nP(A_0)}\, \varphi^3(r_2 - y)\{\sqrt{3}(2\pi)^{-1}\, [2\Phi(\sqrt{3}\sqrt{2r_2 y}) - 1]$$
$$+ 3^{-1}\, [\Phi^3(\sqrt{2r_2 y}) - \Phi^3(-\sqrt{2r_2 y})] - 2^{-1}[\Phi^2(\sqrt{2r_2 y}) - \Phi^2(-\sqrt{2r_2 y})]\} dy$$
$$= 1/\pi + \sqrt{3}/(2\pi) - 1/6 - \mathcal{O}(n^{-\varepsilon_n / r_2}).$$

Now apply (34) to find the upper estimate for $Var(\eta_b)$.

The reasoning verifying the asymptotic normal law for the random variable η_b, when suitably normalized, follows along the lines given in [5] (see Theorem 3.3(ii)) since we also are in the situation where we can construct a *stationary, strongly mixing* sequence of random variables, each with finite p-th moment for $p \geq 1$, by cutting η_b into small pieces corresponding to intervals of equal length. On sufficiently small time intervals, from above we know the first two moments of each piece. Clearly, the exponential decay of the dependence of the vertex process indicated by the mixing coefficient in (38) is fast enough. Therefore, a "blocking system" consisting of alternating big and small blocks successfully comes into play here (For details the reader is referred to [5] or [6]). This finishes the verification of (i) above.

Proof of (ii): First note that all the moments of L_b exist by Lemma 4.2. Recall the notation $T(a) = (R(a), S(a))$. From Lemma 3.4, we know that $EL_b = b \cdot E[L(T(0))]$ and $EL_b^2 = b \cdot E[L'(T(0))] + 2\int_0^b E[L_a\, L(W(a))] da$. In evaluating the integral in the second relation, we want to use Lemma 3.3 and the following elementary facts in order. First of all, observe that the density of $T(0)$ is an even function in the first coordinate, the common density of $T(0)$ and $T(a)$ also is an even function in the first coordinate of $T(0)$, and the ranges of integration are symmetric around zero in the just mentioned coordinates. Then note that $R(a)$ is of the order ar_2, and thus, dominates $S(a) + aR(a) + a^2 r_2/2$ and $S(a) \leq \varepsilon_n$ with high probability for $0 < a < \sqrt{2\varepsilon_n / r_2}$. In summary, we obtain for some $\alpha > 0$, as $n \to \infty$,

$$E[L_b^2] \sim b\, E[L'(T(0))] + b^2 E[L^*(T(0)) L(T(a^*))] + b^2(r_2 + o(\varepsilon_n^{1+\alpha})) E[L^*(T(0))], \quad (53)$$

where $L(\cdot)$ and $L'(\cdot)$ are defined in (20), L^* denotes the analogue of L when time runs backwards, and a^* is some value in $(0, b)$. For instance, we find for $T(0) = (x, y)$

$$\begin{aligned} L(T(0)) &= n\sqrt{2\pi}\,\varphi(r_2 - y)\{(\Phi(\sqrt{2r_2y}) - \Phi(x))(x^2+1) - \sqrt{2r_2y}\,\varphi(\sqrt{2r_2y}) \\ &\quad +2x\varphi(\sqrt{2r_2y}) - x\varphi(x)\}. \end{aligned} \tag{54}$$

Similar arguments as employed for the expected counting functionals above give

$$\begin{aligned} E[L(T(0))] &\sim 1 \\ E[L'(T(0))] &\sim 4/\sqrt{\pi} \\ E[L^*(T(0))L(T(a^*))] &\le 2 - 19/(6\pi\sqrt{3}) \\ E[L^*(T(0))] &\sim 1 + \mathcal{O}(\sqrt{r_0\varepsilon_n}\, e^{-\sqrt{r_0\varepsilon_n}}), \end{aligned}$$

which concludes the moment calculations in (ii) of our proposition.

The proof of the CLT for L_b essentially mimics the one for η_b above because the same conditions are satisfied, being finite moments, strong mixing, and stationarity. The only point to pay attention to is the additional contribution to $L(W(a))$ introduced by the transformation into the stationary vertex process T. However, as Lemma 3.3 tells us, the additional term is asymptotically constant in $W(a) = (x, y)$ and this does not change the story.

Proof of (iii): The verification of the claims about A_b follows the same pattern as the proof of (ii) above. By exploiting the symmetry of the densities again, the moments of A_b may be related to the corresponding moments of L_b. In fact, from Lemma 3.4, we have $EA_b = b \cdot E[A(T(0))] \sim E[L(T(0))]\, r_0/2$ and

$$E[A_b^2] \sim b\, E[L'(T(0))]\, r_0^2/4 + b^2\{E[L^*(T(0))L(T(a^*))]\, r_0^2/4 + E[L^*(T(0))]\, r_0^3/4\}, \tag{55}$$

which completes the proof of the second moment relation in (iii). □

Proof of Theorem 1.1. The basic idea of the proof is to use the facts that in a neighbourhood of the origin W and T run according to the same clock (this obviously is not true for large values of the time parameter), the functionals of W and T have the same moments and the first coordinate divided by r_2 indicates the angle of the sector of the convex hull visited by the path followed (see transformation (30)). One half of the number of convex hull vertices $N_n/2$ corresponds to the number of jumps of the process $\{W_n(a) : a \in I\!R\}$. Recall η_n and ξ_n to be the sample point process and the Poisson point process, respectively, and the processes $\tilde{\eta}_n$ and $\tilde{\xi}_n$ investigated in Lemma 2.1. Let W_n', $\tilde{W}_n$ and W be the vertex processes based on the processes $\eta_{n|_{A_n^*}}$, $\tilde{\eta}_n$ and $\tilde{\xi}_n$, respectively. Moreover, introduce the following notation

$$\begin{aligned} N_n &= 2\#\{W_n(a) : a \in I\!R\} \\ \tilde{N}_n &= 2\#\{\tilde{W}_n(a) : a \in I\!R\} \\ N_n' &= 2\#\{W_n'(a) : a \in I\!R\} \\ \tilde{M}_n &= 2\#\{W(a) : a \in I\!R\} \\ M_n &= \#\{T(a) : a \in [-\pi r_0, \pi r_0]\}. \end{aligned}$$

By Lemma 2.2 and by (30) M_n and $\tilde{M}_n$ have the same distribution. By Lemma 2.1 $\lim_{n\to\infty} P(\tilde{N}_n \neq \tilde{M}_n) = 0$. Also $\tilde{N}_n$ and N'_n follow the same distribution and by (5) $\lim_{n\to\infty} P(N'_n \neq N_n) = 0$. Summarizing our progress yields

$$\lim_{n\to\infty} P(M_n \neq N_n) = 0.$$

Therefore, it suffices to show that

$$(M_n - \alpha_n)/\beta_n \xrightarrow{L} \mathcal{N}(0,1)$$

for suitably chosen norming constants α_n and β_n since then

$$(N_n - \alpha_n)/\beta_n \xrightarrow{L} \mathcal{N}(0,1)$$

follows.

Next we partition the interval $[-\pi r_0, \pi r_0]$ into m_n big blocks of equal length $2a_n$, where $2a_n$ equals the length of the interval where the parabola approximates the outer boundary of the annulus A_n^*, and where we have already studied the jump behaviour of the vertex process locally, namely $a_n = \sqrt{2r_2\varepsilon_n} \to \sqrt{2r_0\varepsilon_n}$. As yet, ε_n is not determined, but only subject to the requirements $\varepsilon_n \to 0$ and $\varepsilon_n r_0 \to \infty$ as $n \to \infty$. We wish to choose ε_n as small as possible. Since $E[L_n]/E[N_n] \sim \sqrt{\pi}$ (see Carnal [2]), we may choose

$$2a_n \sim \sqrt{\pi} \tag{56}$$

as the infimum of all sequences $\{a_n\}$ such that $a_n \to \infty$ as $n \to \infty$. On each interval of length increasing to ∞ as n grows to ∞, there are infinitely many expected convex hull vertices. Thus, we may apply (41) with $b = 2a_n$ and find

$$(M_n - 2m_n c_1 a_n)/\{m_n(2a_n c_1 + 4a_n^2 c_2)\}^{1/2} \xrightarrow{L} \mathcal{N}(0,1), \tag{57}$$

equivalently,

$$(M_n - 2\pi c_1\sqrt{2\ln n})/\{2\pi\sqrt{2\ln n}\,(c_1 + \sqrt{\pi}\,c_2)\}^{1/2} \xrightarrow{L} \mathcal{N}(0,1).$$

Finally, from $b = \mathcal{O}(1)$, it follows that $c_2 > -1/\pi$. □

Proof of Theorem 1.2. The proof mimics the above-mentioned proof of Theorem 1.1 (We do not repeat the arguments here). From Proposition 4.4 we see

$$E[L_n] \sim 2\pi r_0\, E[L(T(0))] \sim 2\pi r_0$$

and

$$Var[L_n] \sim m_n[(2a_n)4/\sqrt{\pi} + (2a_n)^2(r_0 + o(r_0))] \sim 2\pi^{3/2} r_0^2,$$

where we resort to using the relations $m_n = \pi r_0/a_n$ and $a_n \sim \sqrt{\pi}/2$ from (56). Similarly,

$$E[A_n] \sim 2\pi r_0 \, E[A(T(0))] \sim \pi r_0^2$$

and

$$Var[A_n] \sim r_0^4 \pi^{3/2}/2.$$

□

References

[1] Aldous, D. J., Fristedt, J., Griffin, P. S. and Pruitt, W. E. (1991). *The number of extreme points in the convex hull of a random sample.* J. Appl. Prob. 28, 287-304.

[2] Carnal, H. (1970). *Die konvexe Hülle von n rotationssymmetrisch verteilten Punkten.* Z. Wahrscheinlichkeitstheorie verw. Geb. 15, 168-176.

[3] Dwyer, R. A. (1991). *Convex hulls of samples from spherically symmetric distributions.* Discrete Appl. Math. 31, 113-132.

[4] Efron, B. (1965). *The convex hull of a random set of points.* Biometrika 52, 331-343.

[5] Groeneboom, P. (1988). *Limit theorems for convex hulls.* Prob. Theory Rel. Fields 79, 327-368.

[6] Hueter, I. (1992). *The convex hull of n random points and its vertex process.* Doctoral Dissertation, University of Berne.

[7] Hueter, I. (1993). *Limit theorems for the convex hull of random points in higher dimensions.* Preprint.

[8] Ibragimov, I. A. and Linnik, Y. V. (1971). *Independent and stationary sequences of random variables.* Groningen: Wolters Noordhoff.

[9] Raynaud, H. (1970). *Sur l'enveloppe convexe des nuages de points aléatoires dans* $\mathbb{R}^n$. J. Appl. Prob. 7, 35-48.

[10] Rényi, A. and Sulanke, R. (1963). *Über die konvexe Hülle von n zufällig gewählten Punkten.* Z. Wahrscheinlichkeitstheorie verw. Geb. 2, 75-84.

[11] Stroock, D. W. (1975). *Diffusion processes associated with Lévy generators.* Z. Wahrscheinlichkeitstheorie verw. Geb. 32, 209-244.