Gaussian Convex Hull Peels

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Abstract

Convex hull peeling of a finite point set provides a natural means to rank points in $\mathbf{R}^d$ and is appealing in multivariate statistics. The convex peels are consecutively obtained by repeatedly performing: (i) construct the convex hull of the point set and (ii) eliminate the points from the point set that span the hull. We examine the number $N_{n,i}$ of vertices of the i-th convex hull peel of Poisson points with n expected points from a normal or other spherically symmetric distributions in $\mathbf{R}^d$ for $d \geq 2$ and show that the large n expectation and variance of $N_{n,i}$, for $i \geq 1$ below a certain threshold, remain constant as functions of i up to an error that depends on i. This sharply contrasts the behavior known for a uniform sample on a polygon or circular disk, where these two moments are strictly increasing functions of i.

1 Introduction

Convex hull peeling is a commonly practiced procedure to eliminate outliers of a data set so as to enable robust statistical inference [9] and provides a natural notion of order statistics of a point set in $\mathbf{R}^d$. The *convex peels* or *convex layers* of $\mathcal{Z}_n = \{Z_1, Z_2, \ldots, Z_n\}$ in $\mathbf{R}^d$ are consecutively constructed by repeating the two steps (i) form the convex hull of the point set and (ii) remove the vertices of the convex hull from the point set. For a formal definition, see Section 3. $O(n \log n)$ time algorithms to build the convex hull peels have been studied in the literature ([7], [2]) along with their performances yet, in spite of a several decades long fascination with random convex hulls (early works are [4], [15], [16], [1], and [14]), up to present, not much at all is known about the behavior of the convex peels in the mathematical and statistical literature. This note is devoted to one of the principal probabilistic questions on convex hull peeling, namely, the properties of the number $N_{n,i}$ of points peeled from a random point set $\mathcal{Z}_n$ at the i-th stage, equivalently, the number of vertices of the i-th convex peel. We peel $\mathcal{Z}_n$ from the outside. Thus, the first peel is just the convex hull of $\mathcal{Z}_n$, the smallest convex set which contains $\mathcal{Z}_n$.

*City University of New York, Baruch College, and University of Massachusetts Boston. Partially supported by PSC-CUNY grants #60032-33-34 and #66503-00-35.

Date: December 26, 2004.

AMS 2000 Math. Subject Classification: *Primary:* 60D05. *Secondary:* 60F05, 60G70, 52A22.

Key words and phrases: convex hulls, convex peels, peeling, random sets, trimming.

It was shown in HUETER [12] for a uniform sample of size n from a polygon or a circular disk that the expectation $\mathbf{E}N_{n,i}$ and the variance $\mathrm{Var}N_{n,i}$ for every $i \geq 2$ "not too large" and for large n are asymptotic to $(\frac{2}{3}\log n)^{i-1}\,\mathbf{E}N_{n,1}$ and $(\frac{2}{3}\log n)^{2(i-1)}\,\mathrm{Var}N_{n,1}$, respectively, and thus, both are increasing functions of i. In contrast, we here uncover the phenomenon that these two moments of $N_{n,i}$ for large n remain constant as functions of i up to an error, which depends on i, for Poisson points with n expected points (for the definition of the Poisson model, see Section 3) from a normal or certain spherically symmetric distributions in $\mathbf{R}^d$ for $d \geq 2$ which decay exponentially. In view of more notation that is required, we postpone to state the results for the latter class to Section 3. Note that throughout $a_n \sim b_n$ will mean that $\lim_{n\to\infty} a_n/b_n = 1$.

THEOREM 1 *Let $N_{n,i}$ denote the number of vertices of the i-th convex layer of a Poisson point process in $\mathbf{R}^d$ for $d \geq 2$ of intensity measure $n \times$ (the standard normal measure) and write $N_n = N_{n,1}$. For every integer $i \geq 1$ not too large (obeying hypothesis (3.8) below) and as $n \to \infty$,*

$$\begin{aligned} \mathbf{E}N_{n,i+1} &\sim \mathbf{E}N_n\,(1 + i\,\mathbf{E}N_n/n) \sim \mathbf{E}N_n, \\ \mathrm{Var}N_{n,i+1} &\sim \mathrm{Var}N_n\,(1 + c(\mathcal{N},d)\,i\,\mathbf{E}N_n/n) \sim \mathrm{Var}N_n \end{aligned}$$

for a uniformly bounded constant $c(\mathcal{N},d)$, where $c_1(\mathcal{N},d) \leq \mathbf{E}N_n/(\log n)^{(d-1)/2} \leq c_2(\mathcal{N},d) \leq 2\sqrt{d-1}\,(2\pi)^{(d-1)/2}/\Gamma(\frac{d}{2})$ and $c_3(\mathcal{N},d) \leq \mathrm{Var}N_n/(\log n)^{(d-1)/2} \leq c_4(\mathcal{N},d)$ for every sufficiently large n and positive finite constants $c_k(\mathcal{N},d)$ for $k = 1,\ldots,4$, independent of n.

For a proof, see Corollary 1 in Section 3. As usual with convex hulls, the case $d = 1$ is not interesting (and not difficult). The essential non-increase of the two moments of $N_{n,i}$ in i strike us as a slight surprise. Since the density of the points, say, of a normal sample increases while we peel from the outside towards the inside of the point cluster, intuitively, one would expect $N_{n,i}$ to grow with i.

A moment's reflections reveal that each of the convex peels of a sample of size n is a random polygon and, except for the first peel, each convex peel $\mathcal{C}_{n,i}$ constitutes the convex hull of a subset of the sample that is supported on the previous peel $\mathcal{C}_{n,i-1}$, a random polygon. A closer look at the asymptotic expressions for the expectation of $N_n = N_{n,1}$, dating back to [15], [16], shows that $\mathbf{E}N_n \sim \frac{2}{3}\,r\log n$ for a uniform sample from a polygon with $r \geq 3$ vertices, while $\mathbf{E}N_n \sim 2\sqrt{2\pi\log n}$ for a normal sample in the plane. In other words, in the former case, the formula for $\mathbf{E}N_n$ depends on the geometry of the boundary of the support of the sample, whereas in the latter case, the formula only depends on n. Hence, a naïve but incorrect explanation for the formula on $\mathbf{E}N_{n,i}$ is that, while we remove points from a normal sample in the peeling steps, the formula for $\mathbf{E}N_n$ applies with n being replaced by the smaller random count of the points that remain. This argument neglects that each convex peel $\mathcal{C}_{n,i}$ for $i \geq 2$ is formed of points with a different and randomly varying support and that the support is no longer the unbounded plane but a random polygon. The contents of this paper is to prove that the two latter obstructions play no role when it comes to $\mathbf{E}N_{n,i}$ and $\mathrm{Var}N_{n,i}$. We also remark that we only deal with the Poisson point process here but do not prove the convergence of the variance $\mathrm{Var}N_{n,i}$ as $n \to \infty$ in the Poisson point process models to the one in the sample point process models.

Asymptotic expressions for $\mathbf{E}N_{n,i}$ and $\mathrm{Var}N_{n,i}$ are valuable in applying Stein's method to $N_{n,i}$ in connection with gaining the rate of convergence in the normal approximation. We will follow this avenue in a separate paper. Progress on $N_{n,1}$ in this direction appears in HUETER [13] for certain rotationally invariant distributions in $\mathbf{R}^d$ for $d \geq 2$.

For more insights into peeling methods, we refer the interested reader to GREEN [6] who reviews and illustrates by pictures convex hull peeling, ellipse peeling, and Tukey peeling among others and to HUETER [12], where background on peeling is listed in more detail.

The rest of the paper is organized as follows. Section 2 surveys the approach taken up in [10, 11]. Section 3 applies this method to the number of vertices of the convex hull peels and presents and proves the main results. At the end, the perimeter and area of the convex layers are briefly addressed.

2 Background: Random Convex Hull and its Vertex Process

We first explain a method in the Gaussian context that the author adopted in [10, 11] to derive large-sample moments of the number of vertices of the convex hull of a Gaussian sample and certain spherically symmetric samples in the plane. This work builds on the approach that GROENEBOOM introduced in [8] for uniform samples from a polygon or a circular disk in the plane and that has been extended to other convex hull functionals in [10] and to higher dimensions in [11]. To illustrate the techniques in one of the simpler cases, we assume that $\mathcal{Z}_n = \{Z_1, Z_2, \ldots, Z_n\}$ are n independent and identically distributed points, drawn from the bivariate standard normal density, and for technical reasons, all shifted by a suitable vector C_* (to be specified below). We are interested in the number $N_n = N_{n,1}$ of vertices of the convex hull $\mathcal{C}_n = \mathcal{C}_{n,1}$ of $\mathcal{Z}_n$, the smallest convex set which contains $\mathcal{Z}_n$. More specifically, our purpose is to find large n expressions for the expectation $\mathbf{E}N_n$ and the variance $\mathrm{Var}N_n$. Since N_n is invariant under affine transformations of $\mathbf{R}^2$, the choice for the center point C_* of our sample has no effect on (the distribution of) N_n.

In order to count the number of vertices of $\mathcal{C}_n$, our strategy is to turn a line l_a of slope a in $(-\infty, \infty)$ around the sample in such a way that, at each moment, at least one sample point lies on the line and all remaining points lie on one side of the line (thus, one halfplane bounded by l_a contains no points). At each slope a, we mark the point with the largest x-coordinate which lies on the line. Each marked point is among those that span $\mathcal{C}_n$. This hence provides us with a tool to count the vertices of $\mathcal{C}_n$ by counting the number of jumps of a stochastic process, which we define as follows: For each $a \in \mathbf{R}$, we define the *vertex process* $\tilde{W}_n(a)$ of $\mathcal{C}_n$ to be the point $Z_k = (X_k, Y_k)$ of $\mathcal{Z}_n$ with minimal $Y_k - aX_k$. If there are several such points, an event that happens with probability zero, then $\tilde{W}_n(a)$ denotes the point with the largest first coordinate among those points. The process $\{\tilde{W}_n(a) : a \in \mathbf{R}\}$ is a pure jump process, has right-continuous paths and is non-Markovian. Yet, if we replace the sample point process by a suitable Poisson point process, then the vertex process exhibits the Markov property and is more accessible to analysis. We will choose an approximation of the sample point process by a Poisson point process in a suitable fashion such that their numbers of convex hull vertices are close with high probability for large n.

For this purpose, we will rely on the notation $r_0(n) = \sqrt{2\log n}$, $r_1(n) = r_0(n) - \varepsilon(n)/2$, and $r_2(n) = r_0(n) + \varepsilon(n)/2$ for $\varepsilon(n) = \kappa \log\log n / r_0(n)$ and some constant $\kappa \in (3, \infty)$, independent of n. Choose $C_* = (0, r_2(n))$ and let B_s denote a circular disk of radius s centered at $C_* = (0, r_2(n))$. Thus, we assume that each Z_i has density function $\varphi(x)\,\varphi(r_2(n) - y)$, where $\varphi(x) = e^{-x^2/2}/\sqrt{2\pi}$. It is an exercise to show (for instance, see [13], Lemma 2) that the annulus $A^*(n) = B_{r_2(n)} \setminus B_{r_1(n)}$ enjoys the properties $\lim_{n\to\infty} \mathbf{P}(Z_1 \in A^*(n)) = 0$ and $\lim_{n\to\infty} \mathbf{P}(\partial\mathcal{C}_n \subset A^*(n)) = 1$, even more, $\partial\mathcal{C}_n \subset B^c_{r_1(n)}$ almost surely as $n \to \infty$, where $\partial\mathcal{C}_n$ denotes the boundary of $\mathcal{C}_n$ and $B^c_{r_1(n)}$ denotes the complement of $B_{r_1(n)}$. If $W_n(a)$ denotes the vertex process (as defined above of the sample) of a Poisson point process on $\mathbf{R}^2$ with intensity equal to $n\times$ normal measure, centered at C_*, then $\{W_n(a) : a \in \mathbf{R}\}$ is a Markov process. By relying on the infinitesimal generator of $W_n(a)$ and elementary properties of the Poisson point process, one derives (see [10], page 861) that the jump measure of the process W_n in the point $w = (x, y)$ at time a is given by

$$M(W_n(a); F) = M(a, w; F) = n \int_0^\infty u\,\varphi(x + u)\,\varphi(r_2(n) - y - au)\,1_F(u, au)\,du \tag{2.1}$$

for any Borel set F in $\mathbf{R}^2$, where 1_F denotes the indicator function of F. Let η_b denote the *number of jumps* of $\{W_n(c) : c \in [0, b]\}$ for $b > 0$. Via martingale arguments, it was established in [10], Lemma 3.2, that

$$\begin{aligned} \mathbf{E}\eta_b &= \mathbf{E}\int_0^b M(W_n(c); \mathbf{R}^2)\,dc, \\ \mathbf{E}\eta_b^2 &= \mathbf{E}\int_0^b (2\eta_c + 1)\,M(W_n(c); \mathbf{R}^2)\,dc \end{aligned} \tag{2.2}$$

for small enough $b > 0$. If we transform $W_n(a) = (X_n(a), Y_n(a))$ by $R_n(a) = X_n(a) - ar_2(n)$ and $S_n(a) = Y_n(a) - [X_n(a)^2 - R_n(a)^2]/2r_2(n)$, then $\{T_n(a) = (R_n(a), S_n(a)) : a \in \mathbf{R}\}$ becomes a stationary process and (2.2) becomes (see [10], Lemma 3.4)

$$\begin{aligned} \mathbf{E}\eta_b &= b\,\mathbf{E}[M(T_n(0); \mathbf{R}^2)], \\ \mathbf{E}\eta_b^2 &= \mathbf{E}\eta_b + 2\int_0^b da \int_0^a \mathbf{E}[M^*(T_n(0); \mathbf{R}^2)\,M(T_n(a - c); \mathbf{R}^2)]\,dc \end{aligned} \tag{2.3}$$

for small enough $b > 0$, where $M^*(\cdot;\cdot)$ denotes the backward jump measure of W_n. Upon a few calculations, when $T_n(0) = (x, y)$,

$$\begin{aligned} M(T_n(0); \mathbf{R}^2) &= n\sqrt{2\pi}\,\varphi(r_2(n) - y)\,\{\varphi(x) - \varphi(\sqrt{2r_2(n)y}) + x\,\Phi(x) - x\,\Phi(\sqrt{2r_2(n)y})\}, \\ M^*(T_n(0); \mathbf{R}^2) &= n\sqrt{2\pi}\,\varphi(r_2(n) - y)\,\{\varphi(x) - \varphi(\sqrt{2r_2(n)y}) + x\,\Phi(x) - x\,\Phi(-\sqrt{2r_2(n)y})\}, \end{aligned}$$

where $\Phi(z) = \int_{-\infty}^{z} \varphi(s)\,ds$. Recall that $a_n \sim b_n$ means $\lim_{n\to\infty} a_n/b_n = 1$. In view of the stationarity of the process T_n and some coupling arguments between the sample point process and the Poisson point process on $A^*(n)$, it can be shown ([10], page 874) that

$$\mathbf{E}N_n \sim 2\pi r_0(n)\,\mathbf{E}[M(T_n(0);\mathbf{R}^2)] = 2\pi\sqrt{2\log n}\,\mathbf{E}[M(T_n(0);\mathbf{R}^2)]. \tag{2.4}$$

A number of calculations reveal ([10], page 870) that the leading term of $\mathbf{E}[M(T_n(0);\mathbf{R}^2)]$ does not depend on n and, in light of $n\sqrt{2\pi}\,\varphi(r_0(n)) = 1$,

$$\mathbf{E}[M(T_n(0);\mathbf{R}^2)] \;\sim\; n\sqrt{2\pi}\,\varphi(r_0(n))/\sqrt{\pi} + O(n^{-\varepsilon(n)/r_2(n)}) \sim 1/\sqrt{\pi}. \tag{2.5}$$

Similarly, the bounds on $\mathbf{E}[M^*(T_n(0);\mathbf{R}^2)\,M(T_n(a-c);\mathbf{R}^2)]$, reported in [10], do not depend on n. In summary, we collect

$$\begin{aligned} \mathbf{E}N_n &\sim 2\sqrt{2\pi\log n} \\ c_1\sqrt{\log n} \le \mathrm{Var}N_n &\le c_2\sqrt{\log n} \end{aligned} \tag{2.6}$$

for two positive finite constants c_1 (not specified) and $c_2 = 2\sqrt{2\pi}\,(1+\frac{\sqrt{3}}{2}-\frac{\pi}{6}) \approx 6.729929\cdots$. While the result on $\mathrm{Var}N_n$ was new in [10], the expression for the first moment $\mathbf{E}N_n$ appeared before in RÉNYI AND SULANKE [16], CARNAL [1], and RAYNAUD [14].

Unlike for uniformly distributed points on a polygon or a disk, when the formulae [8] for $\mathbf{E}N_n$ and $\mathrm{Var}N_n$ depend on the geometry of the boundary of the support, since the points of a bivariate Gaussian sample are supported in the entire unbounded plane, the formulae [10] for $\mathbf{E}N_n$ and $\mathrm{Var}N_n$ only depend on n for a normal sample of large size n. This contrast could vanish if we regard the number $N_{n,2}$ or $N_{n,i}$ of vertices of the second or i-th convex peel of the samples ($i \ge 2$) because, in either the uniform or normal models, the convex hull is formed of a set of points which are supported on a (random) polygon, thus, on a region with a geometric (random) boundary. As we will see in our next section, these opposite behaviors between the uniform and the normal models persist for the consecutive sample convex hull peels, at least up to a certain number of convex layers. Both, the expectation and variance of $N_{n,i}$ for large n are strictly increasing functions of i when the sample is uniform, whereas $\mathbf{E}N_{n,i}$ and $\mathrm{Var}N_{n,i}$ for large n remain constant in i up to a negligible error, which depends on i, for Poisson points of n expected points from a normal distribution. We like to point out that here we restrict our attention to the Poisson model, while, in [12], the results apply to the binomial as well as the Poisson point process.

3 Convex Hull Peels and the Vertex Process

This section describes how the particulars of Section 2 are applied to infer about the $(i+1)$th peel, given $\mathcal{C}_{n,j}$ for $1 \le j \le i$. Let us give the formal definition of the convex hull peels with their number of vertices. Consider the convex hull $\mathcal{C}_n = \mathcal{C}_{n,1}$ of $\mathcal{Z}_n$, the smallest convex set that contains $\mathcal{Z}_n$, with number $N_n = N_{n,1}$ of vertices (or extremes) and vertex set

$\mathcal{Y}_{n,1} = \mathcal{Y}_n$. Then $\mathcal{C}_n$ as well defines the first convex hull peel or convex layer of $\mathcal{Z}_n$. Let us continue to precise the entire (finite) sequence of convex peels of $\mathcal{Z}_n$, which induce a discrete-time stochastic process of convex peels $\{\mathcal{C}_{n,i}\}_{1\le i\le \mathcal{D}_n=\mathcal{D}(\mathcal{Z}_n)}$. Let the i-th *convex hull peel* or *convex layer* $\mathcal{C}_{n,i}$ for $2 \le i \le \mathcal{D}_n$ be defined recursively by

$$\begin{aligned}\mathcal{C}_{n,i} &= \text{convex hull}(\mathcal{Z}_n \setminus \bigcup_{j=1}^{i-1} \mathcal{Y}_{n,j}), \qquad (3.7)\\ N_{n,i} &= \text{number of vertices of } \mathcal{C}_{n,i}\end{aligned}$$

with set of vertices (or extremes) $\mathcal{Y}_{n,i}$ of $\mathcal{C}_{n,i}$, where $\mathcal{D}(\mathcal{Z}_n)$ is the largest positive integer i for which $\mathcal{Z}_n \setminus \cup_{j=1}^{i-1} \mathcal{Y}_{n,j} \ne \emptyset$, called the *depth* of the point set $\mathcal{Z}_n$ relative to convex hull peeling. Take the convex hull of a single point to be the point itself. Furthermore, take $\mathcal{C}_{n,0}$ to be $\mathbf{R}^d$ or the support of the sampling distribution of $\mathcal{Z}_n$. If the points of $\mathcal{Z}_n$ are randomly sampled from an absolutely continuous distribution in $\mathbf{R}^d$, then with probability 1, each edge of any convex layer $\mathcal{C}_{n,i}$ contains exactly two points of $\mathcal{Z}_n$. Our results about the $N_{n,i+1}$ will be restricted to those i below a particular threshold index.

For every $\delta, \delta' > 0$, define the random index $i_1^{\delta}(n) = \sup\{i \ge 1 : n - \sum_{k=1}^{i} N_{n,k} \ge n^{\delta}\}$ and the deterministic threshold value $i_2^{\delta'}(n) = \sup\{i \ge 1 : i \le n^{1-\delta'}/\mathbf{E}N_n\}$. Choose $\delta, \delta' > 0$ as small as possible such that for every $1 \le i < i_2^{\delta'}(n)$ and as $n \to \infty$,

$$\mathbf{E}N_{n,i+1} \sim \mathbf{E}(N_{n,i+1} \,|\, 1 \le i \le i_1^{\delta}(n)) \qquad \mathrm{Var}N_{n,i+1} \sim \mathrm{Var}(N_{n,i+1} \,|\, 1 \le i \le i_1^{\delta}(n)). \quad (3.8)$$

We say that $i \ge 1$ is *not too large,* if the statements in (3.8) are satisfied with the minimal choices of δ and δ'. For the sequel, we will resort to the short phrase 'i is not too large' when refering to hypothesis (3.8). The conditions in (3.8) assure that, when basing the derivations of the expectation and variance of $N_{n,i+1}$ only on those realizations of $\mathcal{Z}_n$ or $N_{n,1}, N_{n,2}, \ldots$ for which $i \le i_1^{\delta}(n)$, we obtain the asymptotic (in n) expressions for $\mathbf{E}N_{n,i+1}$ and $\mathrm{Var}N_{n,i+1}$. Moreover by definition, $i < i_2^{\delta'}(n)$ guarantees that $i\,\mathbf{E}N_n/n = o(1)$ as $n \to \infty$. We remark that, if i is sufficiently small, then $\mathbf{P}(i > i_1^{\delta}(n))$ will be overwhelmingly close to zero and the conditions in (3.8) are satisfied. This is controlled by the choice of δ'. With the estimates for $\mathbf{E}N_n$ and $\mathrm{Var}N_n$, which we come across here, it is not difficult to verify that the threshold index $i_2^{\delta'}(n)$ well exceeds some positive power of n. In addition, we like to stress that implementing the first precaution, the index $i_1^{\delta}(n)$ may be overly cautious since the contributions to $\mathbf{E}N_{n,i+1}$ and $\mathrm{Var}N_{n,i+1}$ from $\mathcal{Z}_n$ with $n - \sum_{k=1}^{i} N_{n,k} < n^{\delta}$ are of smaller and negligible order as compared to the leading terms of $\mathbf{E}N_{n,i+1}$ and $\mathrm{Var}N_{n,i+1}$.

Before we are ready to state our first result, let us precise the Poisson model and mention a few basic properties. Let $\mathcal{Z}_n = \{Z_1, Z_2, \ldots, Z_N\}$ denote a realization of a Poisson point process in $\mathbf{R}^d$ of intensity measure $nf(\cdot)$, where f denotes the common absolutely continuous distribution in $\mathbf{R}^d$ of the points in $\mathcal{Z}_n$ and N is a Poisson random variable with parameter or expected value n, independently of $Z_1, Z_2, \ldots$. Thus, for any Borel set A in $\mathbf{R}^d$, the cardinality of the intersection of A with the Poisson point process in $\mathbf{R}^d$ is a Poisson variable with parameter $n\int_A f(x)\,dx$, and for any k disjoint Borel sets $A_1,\ldots,A_k$ in $\mathbf{R}^d$, the cardinalities of the intersections of A_j with the Poisson point process in $\mathbf{R}^d$ are mutually independent variables.

THEOREM **2 (Normal points in the plane)** *Let $N_{n,i}$ denote the number of vertices of the i-th convex layer $\mathcal{C}_{n,i}$ of a Poisson point process in $\mathbf{R}^2$ of intensity measure $nf(\cdot)$, where f denotes the bivariate standard normal density function. For every integer $i\geq 1$ not too large (obeying (3.8)) and as $n\to\infty$,*

$$\begin{aligned}\mathbf{E}N_{n,i+1} &\sim \mathbf{E}N_n\,(1+i\,\mathbf{E}N_n/n)\sim 2\sqrt{2\pi\log n}(1+2i\sqrt{2\pi\log n}/n)\sim\mathbf{E}N_n,\\ \mathrm{Var}N_{n,i+1} &\sim \mathrm{Var}N_n\,(1+c(\mathcal{N})\,i\,\mathbf{E}N_n/n)\sim\mathrm{Var}N_n\,(1+2i\,c(\mathcal{N})\sqrt{2\pi\log n}/n)\sim\mathrm{Var}N_n\end{aligned}$$

for a uniformly bounded constant $c(\mathcal{N})$, where $c_1\leq \mathrm{Var}N_n/\sqrt{\log n}\leq c_2$ for every large enough n and two positive finite constants c_1 and $c_2=2\sqrt{2\pi}\,(1+\frac{\sqrt{3}}{2}-\frac{\pi}{6})\approx 6.729929\cdots$.

Proof. The derivations of $\mathbf{E}(N_{n,i+1}\,|\,\mathcal{C}_{n,1},\ldots,\mathcal{C}_{n,i})$ and $\mathrm{Var}(N_{n,i+1}\,|\,\mathcal{C}_{n,1},\ldots,\mathcal{C}_{n,i})$ lie at the core of our proof. At the outset, given $\mathcal{C}_{n,j}$ for $1\leq j\leq i$, we set

$$r_{t,i+1}(n) \;=\; r_t(n-\sum_{k=1}^{i}N_{n,k}) \tag{3.9}$$

for $t=0,1,2$, where the $r_t(n)$ were defined in the paragraph prior to (2.1). We take note that, given $\mathcal{C}_{n,j}$ for $1\leq j\leq i$ or just $\mathcal{C}_{n,i}$, the random polygon $\mathcal{C}_{n,i+1}$ is the convex hull of the points that lie in the interior $\mathrm{int}(\mathcal{C}_{n,i})$ of $\mathcal{C}_{n,i}$ (these points lie in the interior of $\mathcal{C}_{n,i}$ with probability one since any of them lies on the boundary of $\mathcal{C}_{n,i}$ with probability zero) and each point among these has a conditional density equal to $\varphi(x)\,\varphi(y)/\mathbf{P}(Z_1\in\mathrm{int}(\mathcal{C}_{n,i}))$. For technical reasons, as earlier, we will shift the density of each Z_k and engage the conditional density $\varphi(x)\,\varphi(r_{2,i+1}(n)-y)/\mathbf{P}(Z_1\in\mathrm{int}(\mathcal{C}_{n,i}))$. As to the logistics of the definition of $r_{2,i+1}(n)$, we observe (but do not worry) that the points, conditioned to lie in $\mathcal{C}_{n,i}$, cannot lie in $\mathcal{C}^c_{n,i}\cap B_{r_{2,i+1}(n)}$. The quantity $r_{2,i+1}(n)$ will turn out to play no role in the final results.

With $A^*_{i+1}(n)=\mathcal{C}_{n,i}\setminus B_{r_{1,i+1}(n)}$ for $i\geq 1$ and $A^*_1(n)=A^*(n)$, it follows from the properties of $A^*(n)$ that, given $\mathcal{C}_{n,1},\ldots,\mathcal{C}_{n,i}$, the boundary $\partial\mathcal{C}_{n,i+1}\subset A^*_{i+1}(n)$ almost surely as $n\to\infty$. Given $\mathcal{C}_{n,j}$ for $1\leq j\leq i$, the vertex process $\tilde{W}_{n,i+1}(a)$ of the convex peel $\mathcal{C}_{n,i+1}$ is defined as in Section 2, by relying on the subsample $\mathcal{Z}_n\cap\mathrm{int}(\mathcal{C}_{n,i})$. Similarly, given $\mathcal{C}_{n,j}$ for $1\leq j\leq i$, we define the vertex process $W_{n,i+1}(a)$ which is based on the Poisson point process on $\mathbf{R}^2$, restricted to $\mathcal{C}_{n,i}$, with intensity equal to $(n-\sum_{k=1}^{i}N_{n,k})/\mathbf{P}(Z_1\in\mathrm{int}(\mathcal{C}_{n,i}))\times$ normal measure, centered at C_* and restricted to $\mathcal{C}_{n,i}$. This Markovian jump process $W_{n,i+1}(a)$ has conditional jump measure, given $\mathcal{C}_{n,j}$ for $1\leq j\leq i$, at the point $w=(x,y)$ and at time a

$$\begin{aligned}\{M(W_{n,i+1}(a);F)\,|\,\mathcal{C}_{n,1},\ldots,\mathcal{C}_{n,i}\} \;&=\; (n-\sum_{k=1}^{i}N_{n,k})\,\mathbf{P}(Z_1\in\mathrm{int}(\mathcal{C}_{n,i}))^{-1} \\ &\qquad \int_0^\infty u\,\varphi(x+u)\,\varphi(r_{2,i+1}(n)-y-au)\,1_F(u,au)\,du\end{aligned} \tag{3.10}$$

for any Borel set F in $\mathbf{R}^2$. Let $\eta_{b,i+1}$ denote the *number of jumps* of $\{W_{n,i+1}(c) : c \in [0,b]\}$ for $b > 0$. Then by virtue of parallel arguments invoked to demonstrate (2.2), we collect

$$\begin{aligned} \mathbf{E}(\eta_{b,i+1}\,|\,\mathcal{C}_{n,1},\ldots,\mathcal{C}_{n,i}) &= \mathbf{E}[\int_0^b M(W_{n,i+1}(c);\mathbf{R}^2)\,dc\,|\,\mathcal{C}_{n,1},\ldots,\mathcal{C}_{n,i}], \qquad (3.11)\\ \mathbf{E}(\eta^2_{b,i+1}\,|\,\mathcal{C}_{n,1},\ldots,\mathcal{C}_{n,i}) &= \mathbf{E}[\int_0^b (2\eta_{c,i+1}+1)\,M(W_{n,i+1}(c);\mathbf{R}^2)\,dc\,|\,\mathcal{C}_{n,1},\ldots,\mathcal{C}_{n,i}] \end{aligned}$$

for small enough $b > 0$. If we transform $W_{n,i+1}(a) = (X_{n,i+1}(a), Y_{n,i+1}(a))$ by $R_{n,i+1}(a) = X_{n,i+1}(a) - ar_{2,i+1}(n)$ and $S_{n,i+1}(a) = Y_{n,i+1}(a) - [X_{n,i+1}(a)^2 - R_{n,i+1}(a)^2]/2r_{2,i+1}(n)$, then $\{T_{n,i+1}(a) = (R_{n,i+1}(a), S_{n,i+1}(a)) : a \in \mathbf{R}\}$ is a stationary process and

$$\begin{aligned} \mathbf{E}(\eta_{b,i+1}\,|\,\mathcal{C}_{n,1},\ldots,\mathcal{C}_{n,i}) &= b\,\mathbf{E}[M(T_{n,i+1}(0);\mathbf{R}^2)\,|\,\mathcal{C}_{n,1},\ldots,\mathcal{C}_{n,i}], \qquad (3.12)\\ \mathbf{E}(\eta^2_{b,i+1}\,|\,\mathcal{C}_{n,1},\ldots,\mathcal{C}_{n,i}) &= \mathbf{E}(\eta_{b,i+1}\,|\,\mathcal{C}_{n,1},\ldots,\mathcal{C}_{n,i}) + 2\int_0^b da \int_0^a \mathbf{E}[M^*(T_{n,i+1}(0);\mathbf{R}^2)\\ &\qquad \cdot M(T_{n,i+1}(a-c);\mathbf{R}^2)|\,\mathcal{C}_{n,1},\ldots,\mathcal{C}_{n,i}]\,dc \end{aligned}$$

for small enough $b > 0$, where $M^*(\cdot;\cdot)$ denotes the backward jump measure of $W_{n,i+1}$. We do not repeat the calculations to deduce an expression for $\mathbf{E}[M(T_{n,i+1}(0);\mathbf{R}^2)\,|\,\mathcal{C}_{n,1},\ldots,\mathcal{C}_{n,i}]$ and $\mathbf{E}[M^*(T_{n,i+1}(0);\mathbf{R}^2)\,M(T_{n,i+1}(a-c);\mathbf{R}^2)|\,\mathcal{C}_{n,1},\ldots,\mathcal{C}_{n,i}]$, which follow much along the same lines presented in [10], proof of Proposition 4.4. Let us state the results, as $n \to \infty$,

$$\begin{aligned} &\mathbf{E}[M(T_{n,i+1}(0);\mathbf{R}^2)\,|\,\mathcal{C}_{n,1},\ldots,\mathcal{C}_{n,i}]\\ &\qquad \sim (n - \sum_{k=1}^{i} N_{n,k})\,\mathbf{P}(Z_1 \in \mathrm{int}(\mathcal{C}_{n,i}))^{-1}\,\sqrt{2\pi}\,\varphi(r_0(n - \sum_{k=1}^{i} N_{n,k}))/\sqrt{\pi}\\ &\qquad\quad + O(n^{-\varepsilon(n-\sum_{k=1}^{i} N_{n,k})/r_2(n-\sum_{k=1}^{i} N_{n,k})})\\ &\qquad \sim 1/[\sqrt{\pi}\,\mathbf{P}(Z_1 \in \mathrm{int}(\mathcal{C}_{n,i}))]. \qquad (3.13) \end{aligned}$$

An explanation of (3.13) in words is that, in formula (2.5), n is replaced by $n - \sum_{k=1}^{i} N_{n,k}$ and both, the conditional jump measures $M(\cdot;\cdot)$ and $M^*(\cdot;\cdot)$ of $T_{n,i+1}$, given $\mathcal{C}_{n,1},\ldots,\mathcal{C}_{n,i}$, get attached a factor $\mathbf{P}(Z_1 \in \mathrm{int}(\mathcal{C}_{n,i}))^{-1}$. Hence, with the expansion $\sum_{j=0}^{\infty} x^j$ of the function $1/(1-x)$ for $|x| < 1$ in mind, as $n \to \infty$,

$$\begin{aligned} \mathbf{E}(N_{n,i+1}\,|\,\mathcal{C}_{n,1},\ldots,\mathcal{C}_{n,i}) &\sim 2\pi\,r_{0,i+1}(n)\,\mathbf{E}[M(T_{n,i+1}(0);\mathbf{R}^2)\,|\,\mathcal{C}_{n,1},\ldots,\mathcal{C}_{n,i}]\\ &\sim 2\sqrt{\pi}\,r_{0,i+1}(n)\,[1 + \mathbf{P}(Z_1 \notin \mathrm{int}(\mathcal{C}_{n,i}))]\\ &= 2\sqrt{\pi}\,r_0(n - \sum_{k=1}^{i} N_{n,k})\,[1 + \mathbf{P}(Z_1 \notin \mathrm{int}(\mathcal{C}_{n,i}))]\\ &\sim 2\sqrt{\pi}\,r_0(n)\,[1 + \mathbf{P}(Z_1 \notin \mathrm{int}(\mathcal{C}_{n,i}))]\\ &\sim \mathbf{E}N_n\,[1 + \mathbf{P}(Z_1 \notin \mathrm{int}(\mathcal{C}_{n,i}))], \qquad (3.14) \end{aligned}$$

where, in the second last line, we use that i is not too large and we recall $r_0(n) = \sqrt{2\log n}$, thus, $r_0(n - \sum_{k=1}^{i} N_{n,k}) = r_0(n)\,(1 + o(1))$ as $n \to \infty$ ($r_0(n)$ is slowly varying at ∞). Also, as we next see, $\mathbf{P}(Z_1 \notin \mathrm{int}(\mathcal{C}_{n,i})) \sim o(1)$ as $n \to \infty$, which justifies the approximation in the second line of (3.14). In addition, if we realize that $\mathbf{P}(Z_1 \in \mathrm{int}(\mathcal{C}_{n,i})) = \mathbf{P}(Z_1 \in \mathcal{C}_{n,i})$, then employing the reasoning in EFRON [4], page 335, at (3.7) to get hold of the identity $\mathbf{P}(Z_1 \notin \mathcal{C}_{n,1}) = \mathbf{E}N_{n+1}/(n+1) \sim \mathbf{E}N_n/n$ for the sample point process, which (as we verify) carries over to the Poisson point process, together with conditioning, we arrive at the relation

$$\mathbf{P}(Z_1 \in \mathcal{C}_{n,i-1} \setminus \mathcal{C}_{n,i}) \sim \frac{\mathbf{E}N_{n,i}}{n - \sum_{j=1}^{i-1} \mathbf{E}N_{n,j}} \, (1 - \mathbf{P}(Z_1 \notin \mathcal{C}_{n,i-1})) \tag{3.15}$$

and continue to obtain

$$\begin{aligned} \mathbf{P}(Z_1 \notin \mathrm{int}(\mathcal{C}_{n,i})) &= \mathbf{P}(Z_1 \notin \mathrm{int}(\mathcal{C}_{n,i-1})) + \mathbf{P}(Z_1 \in \mathcal{C}_{n,i-1} \setminus \mathcal{C}_{n,i}) \qquad (3.16)\\ &\sim \mathbf{P}(Z_1 \notin \mathrm{int}(\mathcal{C}_{n,i-1})) + \frac{\mathbf{E}N_{n,i}}{n - \sum_{j=1}^{i-1} \mathbf{E}N_{n,j}} \, (1 - \mathbf{P}(Z_1 \notin \mathcal{C}_{n,i-1}))\\ &\sim \mathbf{P}(Z_1 \notin \mathrm{int}(\mathcal{C}_{n,i-1})) + \frac{\mathbf{E}N_{n,i}}{n} \, (1 + n^{-1} \sum_{j=1}^{i-1} \mathbf{E}N_{n,j}) \, (1 - \mathbf{P}(Z_1 \notin \mathcal{C}_{n,i-1})), \end{aligned}$$

where in the last line we applied the expansion of $1/(1-x)$ to $x = n^{-1} \sum_{j=1}^{i-1} \mathbf{E}N_{n,j}$. Iterating the last line at display (3.16) yields

$$\mathbf{P}(Z_1 \notin \mathrm{int}(\mathcal{C}_{n,i})) \sim \sum_{k=1}^{i} \frac{\mathbf{E}N_{n,k}}{n} \, (1 - \{n^{-1} \sum_{j=1}^{k-1} \mathbf{E}N_{n,j}\}^2) \tag{3.17}$$

for i not too large. By virtue of (3.14), we assemble

$$\begin{aligned} \mathbf{E}(N_{n,i+1} \,|\, \mathcal{C}_{n,1}, \ldots, \mathcal{C}_{n,i}) &\sim \mathbf{E}N_n \, [1 + \sum_{k=1}^{i} \frac{\mathbf{E}N_{n,k}}{n} \, (1 - \{n^{-1} \sum_{j=1}^{k-1} \mathbf{E}N_{n,j}\}^2)] \qquad (3.18)\\ &\sim \mathbf{E}(N_{n,i} \,|\, \mathcal{C}_{n,1}, \ldots, \mathcal{C}_{n,i-1}) + \mathbf{E}N_n \, \frac{\mathbf{E}N_{n,i}}{n} \, (1 - \{n^{-1} \sum_{j=1}^{i-1} \mathbf{E}N_{n,j}\}^2). \end{aligned}$$

Taking expectations on both sides of (3.18) leads to

$$\mathbf{E}N_{n,i+1} \sim \mathbf{E}N_{n,i} \, [1 + \frac{\mathbf{E}N_n}{n} \, (1 + o(1))]. \tag{3.19}$$

Finally, via an easy induction over i or iteration of the relation in (3.19), we obtain

$$\mathbf{E}N_{n,i+1} \sim \mathbf{E}N_n \, (1 + i \, \mathbf{E}N_n/n) \sim \mathbf{E}N_n$$

as $n \to \infty$ since i is not too large and $i \, \mathbf{E}N_n/n = o(1)$. This verifies our claims on $\mathbf{E}N_{n,i+1}$.

The adaptations to the derivation of $\mathrm{Var}N_n$ required to deduce $\mathrm{Var}\,(N_{n,i+1} \,|\, \mathcal{C}_{n,1}, \ldots, \mathcal{C}_{n,i})$ are in quite close vicinity of those indicated above. We do not repeat the patterns. We arrive at $\mathrm{Var}(N_{n,i+1} \,|\, \mathcal{C}_{n,1}, \ldots, \mathcal{C}_{n,i}) \sim \mathrm{Var}N_n \, (1 + c(\mathcal{N}) \, i \, \mathbf{E}N_n/n) \sim \mathrm{Var}N_n$ as $n \to \infty$

for a uniformly bounded constant $c(\mathcal{N})$. At last, as an appeal to the identity $\mathrm{Var}X = \mathbf{E}[\mathrm{Var}(X\,|\,Y)] + \mathrm{Var}[\mathbf{E}(X\,|\,Y)]$ for two random variables X and Y, we get

$$\begin{aligned}\mathrm{Var}N_{n,i+1} &= \mathbf{E}[\mathrm{Var}(N_{n,i+1}\,|\,\mathcal{C}_{n,1},\ldots,\mathcal{C}_{n,i})] + \mathrm{Var}[\mathbf{E}(N_{n,i+1}\,|\,\mathcal{C}_{n,1},\ldots,\mathcal{C}_{n,i})] \\ &\sim \mathbf{E}[\mathrm{Var}N_n\,(1+c(\mathcal{N})\,i\,\mathbf{E}N_n/n)] + \mathrm{Var}[\mathbf{E}N_n\,(1+i\,\mathbf{E}N_n/n)] \\ &= \mathrm{Var}N_n\,(1+c(\mathcal{N})\,i\,\mathbf{E}N_n/n) \sim \mathrm{Var}N_n\end{aligned}$$

as $n \to \infty$ since $\mathrm{Var}[\mathbf{E}N_n\,(1+i\,\mathbf{E}N_n/n)] = 0$.

Recalling (2.6) or from [10], Theorem 1.1, page 856, that $c_1 \le \mathrm{Var}N_n/\sqrt{\log n} \le c_2$ for every large enough n and two positive finite constants c_1 and $c_2 = 2\sqrt{2\pi}\,(1+\frac{\sqrt{3}}{2}-\frac{\pi}{6}) \approx 6.729929\cdots$ finishes our proof. □

The results of Theorem 2 extend to higher dimensions as well as a wider class of spherically symmetric distributions that tail off exponentially. For that matter, let us now assume that $\mathcal{Z}_n$ is sampled from an absolutely continuous distribution in $\mathbf{R}^d$ for $d \ge 2$ with radial component R whose upper tail probability function $F_R(x) = \mathbf{P}(\|Z_1\| > x)$ satisfies

$$x = L(\frac{1}{F_R(x)}) \tag{3.20}$$

for an increasing function L that *varies slowly at* ∞ (for each $\lambda > 0$, $\lim_{x\to\infty} L(\lambda x)/L(x) = 1$) and obeys certain smoothness conditions. For this purpose, we express $L(x)$ in the form $L(x) = a(x)\,\exp\{\int_1^x \epsilon(t)/t\,dt\}$ with $a(x) \to a_0 \notin \{0,\infty\}$ and $\epsilon \to 0$ as $x \to \infty$ (see FELLER [5], Corollary 9.9, page 282) and, if we assume $a(x) \equiv 1$ and write $s = 1/F_R(x)$, we collect

$$x = L(s) = \exp\{\int_1^s \epsilon(t)/t\,dt\}. \tag{3.21}$$

If the function ν is defined by $0 < \nu(u) = \epsilon(L^{-1}(u)) = \epsilon(1/F_R(u))$, where $\nu(u) \to 0$ as $u \to \infty$, then, as in CARNAL [1], DWYER [3], and HUETER [11], [13], we impose the *smoothness conditions* on ν that (i) $\nu(x)$ is decreasing for large x, (ii) $x\,\nu'(x)\,\log(\nu(x)) = o(1)$ as $x \to \infty$, and (iii) $\nu(x)\,\log x = o(1)$ as $x \to \infty$. These conditions, which are satisfied by a large family of distributions, assure that ϵ be slowly varying at ∞. For an example, suppose that $F_R(r) = c_0\,\exp\{-r^k\}$ for $k > 0$ and a constant c_0. Then for large n and r, we have $L(n) \sim (\log n)^{1/k}$, $\epsilon(n) \sim (k\,\log n)^{-1}$, and $\nu(r) \sim k^{-1}\,r^{-k}$. Notice that $L(n)\,\epsilon(n)^{1/2} \to 0$ iff $k > 2$ and $L(n)\,\epsilon(n)^{1/2} \to \infty$ iff $k < 2$. The normal distribution has $k = 2$ and $L(n) \sim \sqrt{2\,\log n}$ for large n.

One of our main contributions is the following theorem.

THEOREM **3 (Rotationally invariant points in $\mathbf{R}^d$)** *Let $N_{n,i}$ denote the number of vertices of the i-th peel $\mathcal{C}_{n,i}$ of a Poisson point process in $\mathbf{R}^d$ for $d \ge 2$ of intensity measure $nf(\cdot)$, where f denotes the density function of a rotationally invariant, exponentially-tailed distribution in $\mathbf{R}^d$, as described above, with $\nu(x)$ satisfying conditions $(i)-(iii)$ above such* *that $L(n)\,\epsilon(n)^{1/2} \not\to \infty$ as $n \to \infty$. For every integer $i \geq 1$ not too large (see hypothesis (3.8)) and as $n \to \infty$,*

$$\begin{aligned} \mathbf{E}N_{n,i+1} &\sim \mathbf{E}N_n\,(1 + i\,\mathbf{E}N_n/n) \sim \mathbf{E}N_n, \\ \mathrm{Var}N_{n,i+1} &\sim \mathrm{Var}N_n\,(1 + c(\mathcal{R})\,i\,\mathbf{E}N_n/n) \sim \mathrm{Var}N_n \end{aligned}$$

for a uniformly bounded constant $c(\mathcal{R})$, where $c_1(\mathcal{R}) \leq \epsilon(n)^{(d-1)/2}\,\mathbf{E}N_n \leq c_2(\mathcal{R})$ and $c_3(\mathcal{R}) \leq \epsilon(n)^{(d-1)/2}\,\mathrm{Var}N_n \leq c_4(\mathcal{R})$ for every sufficiently large n and positive finite constants $c_k(\mathcal{R})$ for $k = 1, \ldots, 4$, independent of n.

Proof. The proof rests on the same technique and reasoning that we have employed in the proof of Theorem 2 and on the results that we recast from [11]. More accurately, again as we have observed earlier at (3.13) in the proof of Theorem 2, when calculating the conditional expectation and variance of $N_{n,i+1}$, given $\mathcal{C}_{n,1}, \ldots, \mathcal{C}_{n,i}$, the conditional jump measures $M(\cdot;\cdot)$ and $M^*(\cdot;\cdot)$ of $T_{n,i+1}$ have attached a factor $\mathbf{P}(Z_1 \in \mathrm{int}(\mathcal{C}_{n,i}))^{-1}$ and n is replaced by $n - \sum_{k=1}^{i} N_{n,k}$. With this in mind, we arrive at, as $n \to \infty$,

$$\mathbf{E}(N_{n,i+1}\,|\,\mathcal{C}_{n,1}, \ldots, \mathcal{C}_{n,i}) \sim \mathbf{E}(N_{n-\sum_{k=1}^{i} N_{n,k}})\,[1 + \mathbf{P}(Z_1 \notin \mathrm{int}(\mathcal{C}_{n,i}))] \tag{3.22}$$

(see the reasoning in (3.13) and (3.14)). Theorem 1 in HUETER [11] tells us that $c_1(\mathcal{R}) \leq \epsilon(n)^{(d-1)/2}\,\mathbf{E}N_n \leq c_2(\mathcal{R})$ for every sufficiently large n and positive finite constants $c_k(\mathcal{R})$ for $k = 1, 2$, independent of n. We recall that the function ϵ is slowly varying at ∞. This together with our assumption that i not be too large implies that $\mathbf{E}(N_{n-\sum_{k=1}^{i} N_{n,k}}) \sim \mathbf{E}N_n$ as $n \to \infty$, and hence,

$$\mathbf{E}(N_{n,i+1}\,|\,\mathcal{C}_{n,1}, \ldots, \mathcal{C}_{n,i}) \sim \mathbf{E}N_n\,[1 + \mathbf{P}(Z_1 \notin \mathrm{int}(\mathcal{C}_{n,i}))], \tag{3.23}$$

a relation being identical to the one displayed in (3.14). Following the lines in the proof of Theorem 2, we find that $\mathbf{E}N_{n,i+1} \sim \mathbf{E}N_n\,(1 + i\,\mathbf{E}N_n/n) \sim \mathbf{E}N_n$ as $n \to \infty$.

In order to verify the claim that $\mathrm{Var}N_{n,i+1} = \mathrm{Var}N_n\,(1 + c(\mathcal{R})\,i\,\mathbf{E}N_n/n)$, the arguments run in parallel to those detailed at the end of the proof of Theorem 2.

Finally, teaming up these findings with Theorem 1, [11], which asserts that $c_3(\mathcal{R}) \leq \epsilon(n)^{(d-1)/2}\,\mathrm{Var}N_n \leq c_4(\mathcal{R})$ for every sufficiently large n and positive finite constants $c_k(\mathcal{R})$ for $k = 3, 4$, independent of n, completes our proof. □

In $\mathbf{R}^2$, the asymptotic expression $\mathbf{E}N_n \sim 2\,(\pi/\epsilon(n))^{1/2}$ as $n \to \infty$ is due to CARNAL [1], and in higher dimensions ($d \geq 2$), DWYER [3] proved that, for all sufficiently large n, $\mathbf{E}N_n \leq d^{1/2}\,(8\,\pi\,d/(d-1))^{(d-1)/2}\,\epsilon(n)^{-(d-1)/2}$, prior to the results on $\mathbf{E}N_n$ and $\mathrm{Var}N_n$ in HUETER [11].

Theorem 3 has the following

COROLLARY **1 (Normal points in $\mathbf{R}^d$)** *Let $N_{n,i}$ denote the number of vertices of the i-th convex layer $\mathcal{C}_{n,i}$ of a Poisson point process in $\mathbf{R}^d$ for $d \geq 2$ of intensity measure $nf(\cdot)$, where f denotes the standard normal density function in $\mathbf{R}^d$. For every integer $i \geq 1$ not too large (see hypothesis (3.8)) and as $n \to \infty$,*

$$\begin{aligned} \mathbf{E}N_{n,i+1} &\sim \mathbf{E}N_n\,(1 + i\,\mathbf{E}N_n/n) \sim \mathbf{E}N_n, \\ \mathrm{Var}N_{n,i+1} &\sim \mathrm{Var}N_n\,(1 + c(\mathcal{N}, d)\,i\,\mathbf{E}N_n/n) \sim \mathrm{Var}N_n \end{aligned}$$

for a uniformly bounded constant $c(\mathcal{N}, d)$, where $c_1(\mathcal{N}, d) \leq \mathbf{E}N_n/(\log n)^{(d-1)/2} \leq c_2(\mathcal{N}, d) \leq 2\sqrt{d-1}\,(2\pi)^{(d-1)/2}/\Gamma(\frac{d}{2})$ and $c_3(\mathcal{N}, d) \leq \mathrm{Var}N_n/(\log n)^{(d-1)/2} \leq c_4(\mathcal{N}, d)$ for every sufficiently large n and positive finite constants $c_k(\mathcal{N}, d)$ for $k = 1, \ldots, 4$, independent of n.

Proof. Reconciling the statements of Theorem 3 with the results in [11], Theorem 2, proves all our claims. □

We conclude this article with results pertaining to the area $A_{n,i}$ and perimeter $L_{n,i}$ of $\mathcal{C}_{n,i}$. Since the approach in [10] that heavily relies on the jump process of consecutive vertices of the convex hull also obtains the moments of $A_n = A_{n,1}$ and $L_n = L_{n,1}$, the ideas and technique adopted here as well enable us to infer about the moments of $A_{n,i}$ and $L_{n,i}$. Our subsequent result is valid for both models, the sample point process and the Poisson point process, because the first moments were established [10] to be asymptotically equal.

THEOREM **4** *Let $L_{n,i}$ and $A_{n,i}$ denote the perimeter and area, respectively, of the i-th convex layer $\mathcal{C}_{n,i}$ of a standard normal sample of size n in the plane. For every integer $i \geq 1$ not too large (see hypothesis (3.8)) and as $n \to \infty$,*

$$\begin{aligned} \mathbf{E}L_{n,i+1} &\sim \mathbf{E}L_n\,(1 + i\,\mathbf{E}N_n/n) \sim 2\pi\,\sqrt{2\log n}\,(1 + 2i\,\sqrt{2\pi\log n}/n) \sim \mathbf{E}L_n, \\ \mathbf{E}A_{n,i+1} &\sim \mathbf{E}A_n\,(1 + i\,\mathbf{E}N_n/n) \sim 2\pi\,\log n\,(1 + 2i\,\sqrt{2\pi\log n}/n) \sim \mathbf{E}A_n. \end{aligned}$$

Proof. We do not mimic the calculations, which are in [10]. See the proof of Theorem 1.2, stated on page 856. In essence, when calculating the conditional expectations of $L_{n,i+1}$ and $A_{n,i+1}$, given $\mathcal{C}_{n,1}, \ldots, \mathcal{C}_{n,i}$, the adjustments that are needed consist in replacing n by $n - \sum_{k=1}^{i} N_{n,k}$ and an extra factor $\mathbf{P}(Z_1 \in \mathrm{int}(\mathcal{C}_{n,i}))^{-1}$ to $\mathbf{E}L_n$ and $\mathbf{E}A_n$. Then proceeding along the lines of the proof of Theorem 2 immediately yields the advertized. □

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