Recovering a Family of Two-dimensional Gaussian Variables from the Minimum Process

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Abstract

Suppose that $\{(X_t, Y_t) : t > 0\}$ is a family of two independent Gaussian random variables with means $m_1(t)$ and $m_2(t)$ and variances $\sigma_1^2(t)$ and $\sigma_2^2(t)$. If at every time $t > 0$ the first and second moment of the *minimum process* $X_t \wedge Y_t$ are known, are the parameters governing these four moment functions uniquely determined ? We answer this question in the *negative* for a large class of Gaussian families including the "Brownian" case. Except for some degenerate situation where one variance function dominates the other, in which case the recovery of the parameters is fully successful, the second moment of the minimum process does not provide any additional clues on identifying the parameters.

1 Introduction

Let $\{(X_t, Y_t) : t > 0\}$ be a family of two independent Gaussian random variables with mean $(m_1(t), m_2(t))$ and variances $(\sigma_1^2(t), \sigma_2^2(t))$, where the time-dependent means and variances are real-valued functions. Consider the *minimum process* $X_t \wedge Y_t$ of X_t and Y_t for every $t > 0$. Does knowledge of the minimum process uniquely determine the parameters of the processes X_t and Y_t ?

If we restrict our attention to the class of mean and variance functions $m_1(t) = \mu_1 t^\gamma$, $m_2(t) = \mu_2 t^\gamma$, $\sigma_1(t) = \sigma_1 t^\omega$ and $\sigma_2(t) = \sigma_2 t^\omega$ for fixed γ and ω and *unknown* parameters μ_i and σ_i $(i = 1, 2)$ and if $\mathbf{E}$ denotes the expectation of (X_t, Y_t), then we have the following

Theorem 1 *Suppose that* $\mathbf{E}X_t \wedge Y_t$ *is known for every* $t > 0$*. Then* (a) *if* $\omega \neq \gamma$*, the parameters* μ_1, μ_2, *and* $\sigma_1^2 + \sigma_2^2$ *can be uniquely determined* (μ_1 *and* μ_2 *up to permutation*), *and* (b) *if* $\omega = \gamma$, none *of the latter three parameters can be uniquely identified. Furthermore, knowing* $\mathbf{E}(X_t \wedge Y_t)^2$ *for every* $t > 0$ *does not give any additional clue on* σ_1 *or* σ_2 *in* (a).

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The proof is in Propositions 1 and 2 and Lemma 2. Observe that case (a) includes the case where X_t and Y_t are two independent Brownian motions, started from the origin, with drifts μ_1 and μ_2 and scaling σ_1 and σ_2. As a consequence of situation (a), the first two moments of the minimum process $X_t \wedge Y_t$ remain unaffected by changing $\sigma_1 = 3$ and $\sigma_2 = 4$ to $\sigma_1 = \sqrt{24}$ and $\sigma_2 = 1$. Prior to looking beyond the special choice of mean and variance functions above and commenting on the aspect of the time dependence, we like to say a few words about how we came across this question.

If we consider random variables instead of stochastic processes and specify the *distribution* of the minimum random variable, the question is related to a problem, known as the Anderson–Ghurye–problem in the statistical literature [1, 2, 6] and motivated by a supply-demand problem (for more details, see[1, 2]) in econometrics or a reliability problem in complex systems [3]. The following theorem holds [2, 3, 5]: Suppose that X_1 and X_2 are two bivariate normally distributed random variables. Then the distribution of the minimum random variable uniquely determines the parameters of X_1 and X_2. In fact, even more generally, when examining the minimum of n non-singular normal random vectors with correlation, an affirmative answer has been given in [6, 7]. The integer n is unique, too. A crucial ingredient to the proofs was the property of unique factorization of the distribution, i.e. whether the distribution of the maximum can be expressed uniquely as a product of the individual distribution functions [4]. This property is known to hold for the Cauchy family as well [5]. In contrast, this feature fails for two independent exponentially distributed random variables with parameters λ_1 and λ_2 (which may be seen by writing down the distribution function of the minimum random variable).

Now, given that in the Gaussian family the knowledge of the distribution of the minimum random variable is sufficient, one might ask whether less, say, the knowledge of the *first* or *first two moments* of the minimum random variable suffices and uniquely determines the parameters of the individual random variables.

In this note, we study this question for two independent Gaussian processes X_t and Y_t with means $m_1(t)$ and $m_2(t)$ and variances $\sigma_1^2(t)$ and $\sigma_2^2(t)$ by exploiting the gained information provided by the *dependence on time.* Two Brownian motions with drifts and scale parameters provide a special example thereof, called "Brownian" case shortly. From a different angle, the problem may be viewed as, given a family of two affinely transformed Gaussian variables, the transformations applied being time-dependent and unknown to the observer, to recover the parameters of the transformations from the resulting Gaussian family. It turns out that the answer to this identification problem depends on the choice of functions $m_i(t)$ and $\sigma_i(t)$ and may be negative or positive, the former being more common.

Let us summarize our main results. In the class we examined, the recovery never is completely successful unless the variance functions run on different scales (degenerate case), that is, $\sigma_i(t) = \sigma_i t^{\omega_i}$ for $i = 1, 2$ with $\omega_1 \neq \omega_2$. In the latter setup, the second moment of $X_t \wedge Y_t$ combined with the first allow to identify σ_1 and σ_2 (see the Remark at the end of Section 3). In general, the individual variances are lost (Theorem 1) while the "joint" variance $\sigma_1^2 + \sigma_2^2$ of the two processes gets revealed. The least transparent situation arises when all of the four functions $m_i(t)$ and $\sigma_i(t)$ are of the same order of magnitude, thus, $\gamma = \omega$, since in that case the time dependence gets lost. We notice that introducing correlation between the two Gaussian components and attempting to recover it as well might enrich the discussion yet is not needed to explain the story because we already found a negative answer in the independent case. On another note on the theme of for which processes do the moments of the minimum (or maximum) process characterize the parameters of the individual processes, we turn to the Poisson family. Here, identifying the parameters is entirely successful.

Theorem 2 *Let $N_t \sim Poiss(\lambda_1 t)$ and $M_t \sim Poiss(\lambda_2 t)$ be two independent Poisson point processes with intensities $\lambda_1, \lambda_2 > 0$. Suppose that the function $\mathbf{E} N_t \wedge M_t$ is known for every $t > 0$, where $\mathbf{E}$ denotes expectation with respect to the joint distribution of N_t and M_t. Then the parameters λ_1 and λ_2 can be uniquely determined.*

For a proof, see Proposition 4.

The paper is organized as follows. Section 2 gives an explicit formula for the first moment of $X_t \wedge Y_t$. Section 3 is devoted to the (partial) identification of parameters. Section 4 collects the second moment of $X_t \wedge Y_t$. The Poisson case is deferred to Section 5.

2 First Moment of the Minimum Process

We begin with obtaining the first moment of the minimum process $\{X_t \wedge Y_t : t > 0\}$ of the family $\{(X_t, Y_t) : t > 0\}$ of two *independent* Gaussian random variables with mean $(m_1(t), m_2(t))$ and variances $(\sigma_1^2(t), \sigma_2^2(t))$, where $m_1, m_2 \colon \mathbf{R} \to \mathbf{R}$ and $\sigma_1, \sigma_2 \colon \mathbf{R} \to \mathbf{R}_+$ are differentiable functions (In Section 3, a more specific class of mean and variance functions will be addressed). Thus, at any time $t > 0$, the random variables X_t and Y_t are mutually independent. The pair (X_s, Y_s) may be a realization of a Gaussian process $\{(X_t, Y_t) : t > 0\}$ at time s. Let $\mathbf{E}$ denote the expectation of the random variable (X_t, Y_t) (Whence, we will omit explicit reference to the dependence of the expectation on t). Define the standard normal density function φ and its distribution function Φ by

$$\begin{aligned} \varphi(z) &= (2\pi)^{-1/2} \exp(-z^2/2) \\ \Phi(z) &= \int_{-\infty}^{z} \varphi(x) dx \end{aligned}$$

for every real number z. Finally, define the "standardized" mean difference $\alpha(t)$ of the two random variables X_t and Y_t by

$$\alpha(t) = \frac{m_2(t) - m_1(t)}{\sqrt{\sigma_1^2(t) + \sigma_2^2(t)}}. \tag{2.1}$$

Lemma 1 *For every $t > 0$, the expectation of the minimum process is*

$$\mathbf{E} X_t \wedge Y_t = [\Phi(\alpha(t)) - \frac{1}{2}](m_1(t) - m_2(t)) + \frac{1}{2}(m_1(t) + m_2(t)) - \sqrt{\sigma_1^2(t) + \sigma_2^2(t)}\varphi(\alpha(t)).$$

Proof. Collecting the expectation with respect to the bivariate (Gaussian) density f_{X_t,Y_t} of (X_t, Y_t) is a routine calculation in which some care is needed. Observe that $f_{X_t,Y_t}(x,y) = [\sigma_1(t)\sigma_2(t)]^{-1}\varphi((x-m_1(t))/\sigma_1(t))\varphi((y-m_2(t))/\sigma_2(t))$. We obtain

$$\begin{aligned}
\mathbf{E}X_t \wedge Y_t &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{x} y f_{X_t,Y_t}(x,y)\,dy + \int_{-\infty}^{\infty} x\,dx \int_{x}^{\infty} f_{X_t,Y_t}(x,y)\,dy \\
&= [\sigma_1(t)\sigma_2(t)]^{-1} \int_{-\infty}^{\infty} \varphi((x-m_1(t))/\sigma_1(t))\,dx \int_{-\infty}^{x} y\,\varphi((y-m_2(t))/\sigma_2(t))\,dy \\
&\quad + [\sigma_1(t)\sigma_2(t)]^{-1} \int_{-\infty}^{\infty} x\,\varphi((x-m_1(t))/\sigma_1(t))\,dx \int_{x}^{\infty} \varphi((y-m_2(t))/\sigma_2(t))dy \\
&= \int_{-\infty}^{\infty} \varphi(r)\,dr \int_{-\infty}^{(\sigma_1(t)r+m_1(t)-m_2(t))/\sigma_2(t)} \varphi(s)\,(m_2(t)+\sigma_2(t)s)\,ds \\
&\quad + \int_{-\infty}^{\infty} \varphi(r)\,(m_1(t)+\sigma_1(t)r)\,dr \int_{(\sigma_1(t)r+m_1(t)-m_2(t))/\sigma_2(t)}^{\infty} \varphi(s)\,ds \\
&= m_2(t) \int_{-\infty}^{\infty} \varphi(r)\,\Phi((\sigma_1(t)r+m_1(t)-m_2(t))/\sigma_2(t))\,dr \\
&\quad + m_1(t) \int_{-\infty}^{\infty} \varphi(r)\,[1-\Phi((\sigma_1(t)r+m_1(t)-m_2(t))/\sigma_2(t))]\,dr \\
&\quad - \sigma_2(t) \int_{-\infty}^{\infty} \varphi(r)\,\varphi((\sigma_1(t)r+m_1(t)-m_2(t))/\sigma_2(t))\,dr \\
&\quad + \sigma_1(t) \int_{-\infty}^{\infty} r\,\varphi(r)\,[1-\Phi((\sigma_1(t)r+m_1(t)-m_2(t))/\sigma_2(t))]\,dr, \qquad (2.2)
\end{aligned}$$

where the two substitutions $r = (x-m_1(t))/\sigma_1(t)$ and $s = (y-m_2(t))/\sigma_2(t)$ provided in the third equality of the display. Write the shorthands m_1, m_2, σ_1 and σ_2 for $m_1(t), m_2(t), \sigma_1(t)$ and $\sigma_2(t)$. Then by the rotational symmetry of the bivariate standard normal distribution, by completing the square in the exponents, and in view of the fact that $\varphi(s)$ has derivative $-s\varphi(s)$, in few steps, we find

(i) $\int_{-\infty}^{\infty} \varphi(r)\,\Phi((\sigma_1 r + m_1 - m_2)/\sigma_2)\,dr = 1 - \Phi(\alpha(t))$,

(ii) $\int_{-\infty}^{\infty} \varphi(r)\,\varphi((\sigma_1 r + m_1 - m_2)/\sigma_2)\,dr = \varphi(\alpha(t))\,\sigma_2/\sqrt{\sigma_1^2+\sigma_2^2}$,

(iii) $\int_{-\infty}^{\infty} r\,\varphi(r)\,[1-\Phi((\sigma_1 r + m_1 - m_2)/\sigma_2)]\,dr = -\varphi(\alpha(t))\,\sigma_1/\sqrt{\sigma_1^2+\sigma_2^2}$.

Combining the three latter identities with (2.2) establishes the advertized expression for $\mathbf{E}X_t \wedge Y_t$. □

Next we will deal with the question whether the expression in Lemma 1 characterizes the parameters of the individual processes X_t and Y_t.

3 Identification of Parameters

For the purpose of attempting to recover the means and the variances of the two-dimensional Gaussian variable (X_t, Y_t), consider the general "Gaussian" case, where each of the mean functions $m_i(t)$ and variance functions $\sigma_i^2(t)$ $(i = 1, 2)$ is proportional to some power of t. A special but illuminating example (presented later) is the "Brownian" case, where for each t the distribution of (X_t, Y_t) coincides with the one of (X'_t, Y'_t), with $\{(X'_t, Y'_t) : t > 0\}$ being a planar Brownian motion with some drift and scaling.

Gaussian Case. Since in identifying the parameters of the Gaussian family, the time dependence will be utilized to explore the behaviour of some functions as t or $1/t$ approaches zero, our selection of the mean and variance functions is representative for many others not discussed here, in particular, analytic mean and variance functions (because their Taylor approximations fall into our target class). To wit, similar ideas to those explained below may be helpful to study the identification problem for the Gaussian family with other governing functions. Suppose that these four functions are described by

$$\begin{aligned} m_1(t) &= \mu_1 t^{\gamma_1} \qquad & m_2(t) = \mu_2 t^{\gamma_2} \\ \sigma_1(t) &= \sigma_1 t^{\omega_1} \qquad & \sigma_2(t) = \sigma_2 t^{\omega_2}. \end{aligned} \tag{3.1}$$

We also assume that there is agreement on the values of γ_i and ω_i $(i = 1, 2)$ (for instance, by invoking log-log plots upon the graphs of functions), and thus, that these values are known and fixed for the discussion that follows. Consequently,

$$\begin{aligned} \sigma^2(t) &= \sigma_1^2(t) + \sigma_2^2(t) = \sigma_1^2 t^{2\omega_1} + \sigma_2^2 t^{2\omega_2} \\ \alpha(t) &= (\mu_2 t^{\gamma_2} - \mu_1 t^{\gamma_1})/\sigma(t). \end{aligned} \tag{3.2}$$

A natural assumption is that both coordinates of the Gaussian family run on the same time scales, that is, $\gamma_1 = \gamma_2$ and $\omega_1 = \omega_2$. Otherwise, one of the means (or variances) is dominating the other mean (or variance), which can be regarded as a degenerate case. We shall examine the latter case later. First, assume that $\gamma = \gamma_1 = \gamma_2$ and $\omega = \omega_1 = \omega_2$, with this meaning of γ and ω for the rest of the paper. Then, by Lemma 1,

$$\begin{aligned} \mathbf{E} X_t \wedge Y_t &= [\Phi(\alpha(t)) - \frac{1}{2}](\mu_1 - \mu_2) t^{\gamma} + (\mu_1 + \mu_2) t^{\gamma}/2 \\ &\quad - (\sigma_1^2 + \sigma_2^2)^{1/2}\, t^{\omega}\, \varphi(\alpha(t)). \end{aligned} \tag{3.3}$$

We distinguish the two cases $\gamma = \omega$ (Proposition 1 below) and $\gamma \neq \omega$ (Proposition 2 below). In the first case, if we write $\alpha_* = (\mu_2 - \mu_1)/\sqrt{\sigma_1^2 + \sigma_2^2}$, identity (3.3) simplifies to

$$\begin{aligned} \mathbf{E} X_t \wedge Y_t &= t^{\omega}\{\Phi(\alpha_*)(\mu_1 - \mu_2) + \mu_2 - \sqrt{\sigma_1^2 + \sigma_2^2}\varphi(\alpha_*)\} \\ &= t^{\omega}\{-\sqrt{\sigma_1^2 + \sigma_2^2}[\Phi(\alpha_*)\alpha_* + \varphi(\alpha_*)] + \mu_2\}. \end{aligned} \tag{3.4}$$

Proposition 1 *Assume that $m(t) = \mathbf{E} X_t \wedge Y_t$ is given for every $t > 0$. In case $\gamma = \omega$, the parameters μ_1, μ_2, and $\sigma_1^2 + \sigma_2^2$ can* not *be uniquely identified (nor σ_1 or σ_2).*

Proof. To verify the statement, it is enough to show that the same value of the $\{\}$-bracket in (3.4) is assigned to distinct triples $(\mu_1, \mu_2, \sigma_1^2 + \sigma_2^2)$. Write $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$ and

$h(\mu_1, \mu_2, \sigma) = -\sigma[\Phi(\alpha_*)\alpha_* + \varphi(\alpha_*)] + \mu_2$, where we recall $\alpha_* = (\mu_2 - \mu_1)/\sigma$. The first derivative of h with respect to σ is equal to $-\varphi(\alpha_*) < 0$, the first derivative of h with respect to μ_1 is equal to $\Phi(\alpha_*)$, which is > 0 except for $\alpha_* = -\infty$, and, the first derivative of h with respect to μ_2 is equal to $1 - \Phi(\alpha_*)$, which is > 0 except for $\alpha_* = +\infty$. Hence, the function h is strictly decreasing in σ and strictly increasing in μ_1 and μ_2 as long as μ_1 and μ_2 take finite values. In other words, in view of the continuity of h in each variable, for each triple (μ_1, μ_2, σ), one finds $\mu_i' > \mu_i$ ($i = 1$ or 2) and $\sigma' > \sigma$ such that $h(\mu_1, \mu_2, \sigma) = h(\mu_1', \mu_2', \sigma')$. But this means that the value of the function h does *not* uniquely determine the parameters μ_1, μ_2, and σ, as desired. □

The situation quickly improves if the scales γ and ω are different.

Proposition 2 *Suppose that $m(t) = \mathbf{E}X_t \wedge Y_t$ is given for every $t > 0$. Firstly, let $\gamma > \omega$. Then the parameters $\mu_1 \wedge \mu_2$, $\mu_1 \vee \mu_2$, and $\sigma_1^2 + \sigma_2^2$ can be uniquely identified as follows:*

$$\begin{aligned}
\sqrt{\sigma_1^2 + \sigma_2^2} &= -[\varphi(0)]^{-1} \lim_{t\to 0} t^{-\omega}\, m(t) = -\sqrt{2\pi} \lim_{t\to 0} t^{-\omega}\, m(t) \qquad (3.5)\\
\mu_1 + \mu_2 &= 2 \lim_{t\to 0} t^{-\gamma} \{m(t) + \sqrt{\sigma_1^2 + \sigma_2^2}\, t^{\omega}\, \varphi(\alpha(t))\}\\
\mu_1 \wedge \mu_2 &= \lim_{t\to\infty} t^{-\gamma}\, m(t).
\end{aligned}$$

Secondly, let $\gamma < \omega$. Then the parameters $\mu_1 \wedge \mu_2$, $\mu_1 \vee \mu_2$, and $\sigma_1^2 + \sigma_2^2$ can be uniquely identified as described in (3.5) *if each limit* $\lim_{t\to 0}$ ($\lim_{t\to\infty}$) *is replaced by* $\lim_{t\to\infty}$ ($\lim_{t\to 0}$).

Proof. Observe that, if $\gamma > \omega$, then $\lim_{t\to 0} \alpha(t) = 0$ and $\lim_{t\to\infty} \alpha(t) = \infty$ ($= -\infty$) if $\mu_2 > \mu_1$ ($\mu_2 < \mu_1$). On the other hand, if $\gamma < \omega$, then $\lim_{t\to\infty} \alpha(t) = 0$ and $\lim_{t\to 0} \alpha(t) = \infty$ ($= -\infty$) if $\mu_2 > \mu_1$ ($\mu_2 < \mu_1$). A small number of elementary exercises that take advantage of (3.3) establishes all limits, and thus, verifies all claims. □

Observe that merely the pair (μ_1, μ_2) can be recovered, that is, μ_1 and μ_2 up to permutation but not, for instance, μ_1. Moreover, the scale parameters σ_1 and σ_2 cannot be identified either since σ_1 or σ_2 cannot be expressed by means of the function $m(t)$ (see Lemma 1), which is a function of the sum $\sigma_1^2 + \sigma_2^2$ of squares.

Example: Brownian Case. Consider the mean and scaling functions $m_i(t)$ and $\sigma_i(t)$ ($i = 1, 2$) with $\gamma = \gamma_1 = \gamma_2 = 1$ and $\omega = \omega_1 = \omega_2 = 1/2$, which covers the case of $\{X_t : t > 0\}$ and $\{Y_t : t > 0\}$ being two independent real-valued Brownian motions, started at the origin, with drifts μ_1 and μ_2 and scale parameters σ_1 and σ_2. Hence, $\gamma > \omega$, and, we recover $\mu_1 \wedge \mu_2$, $\mu_1 \vee \mu_2$, and $\sigma_1^2 + \sigma_2^2$ by proceeding as explained in (3.5).

Remark: Degenerate Case. Suppose that one of the equalities $\omega_1 = \omega_2$ and $\gamma_1 = \gamma_2$ fails. More specifically, consider the situation when $\gamma = \gamma_1 = \gamma_2 > \omega_1 > \omega_2$. One may suspect that in this case it might be easier to identify σ_1 and σ_2. Taking a closer look will show that indeed a single parameter can be recovered from the expectation of $X_t \wedge Y_t$, namely, σ_2, whereas the information on the other scale parameter σ_1 gets buried in the expression of the first moment of $X_t \wedge Y_t$: we have $\sigma_2 = -(\varphi(0))^{-1} \lim_{t\to 0} t^{-\omega_2} m(t)$. The remaining limits of this type are infinite. If $\gamma < \omega_1$ and $\omega_1 > \omega_2$, then $\sigma_1 = -(\varphi(0))^{-1} \lim_{t\to\infty} t^{-\omega_1} m(t)$, whereas σ_2 cannot be identified at this point (the second moment will help, however, see Section 4). The remaining case studies are left to the interested reader.

4 Second Moment of the Minimum Process

To answer the question as to whether the second moment of the process $X_t \wedge Y_t$ carries additional clues about separating the parameters σ_1 and σ_2, we continue to calculate $\mathbf{E}(X_t \wedge Y_t)^2$. To circumvent the difficulties arising when analyzing this second moment, we adjust the expression for $\mathbf{E}(X_t \wedge Y_t)^2$, after calculating $\mathbf{E}X_t \wedge Y_t$, so that the means are set to be equal zero, i.e. $\mu_1 = \mu_2 = 0$. The next result offers a decomposition of $\mathbf{E}(X_t \wedge Y_t)^2$ and connects the second moment of the minimum process of X_t and Y_t to the second moment of the minimum process of two independent Gaussian variables U_t and V_t with zero means yet with the same variance functions as X_t and Y_t.

Proposition 3 *Suppose that $\mathbf{E}X_t \wedge Y_t$ and $\sigma_1^2(t) + \sigma_2^2(t)$ are known for every $t > 0$. Then for every $t > 0$, the second moment $\mathbf{E}(X_t \wedge Y_t)^2$ can be recovered from $\mathbf{E}(U_t \wedge V_t)^2$, where U_t and V_t are two independent Gaussian variables with zero means and the same variance functions $\sigma_1^2(t)$ and $\sigma_2^2(t)$ as X_t and Y_t. Specifically, if $m(t) = \mathbf{E}X_t \wedge Y_t$ is the expression given in Lemma* 1 *and if the functions m_0 and h are defined by*

$$\begin{aligned} m_0(t) &= -\{\sigma_1^2(t) + \sigma_2^2(t)\}^{1/2}\, \varphi(0) = -\{(\sigma_1^2(t) + \sigma_2^2(t))/(2\pi)\}^{1/2}, \\ h(t) &= m(t) - m_0(t) \\ &= [\Phi(\alpha(t)) - \frac{1}{2}](m_1(t) - m_2(t)) \,+\, \frac{1}{2}(m_1(t) + m_2(t)) \end{aligned}$$

for every $t > 0$, then we have

$$\mathbf{E}U_t \wedge V_t = m_0(t) \tag{4.1}$$
$$\mathbf{E}(X_t \wedge Y_t)^2 = \mathbf{E}(U_t \wedge V_t)^2 + h(t)[2m_0(t) + h(t)]. \tag{4.2}$$

Proof. First note that if $\mu_1 = \mu_2 = 0$, then $m_1(t) = m_2(t) = 0$ and $\alpha(t) = 0$ for all $t > 0$. Hence, (4.1) immediately follows. If $\alpha(t) = 0$ for X_t and Y_t, then all claims are obvious. Hence, suppose now that $\alpha(t) \neq 0$ for X_t and Y_t.

To verify (4.2), we will shift both processes X_t and Y_t by the same function $h(t)$ so that the shifted processes have minimum process with expectation equal $m_0(t)$. By definition, $\mathbf{E}X_t \wedge Y_t = m(t) = m_0(t) + h(t)$. Since $(X_t - h(t)) \wedge (Y_t - h(t)) = X_t \wedge Y_t - h(t)$, it follows that the processes on both sides of the equation have the same moments. If we define

$$R_t = X_t - h(t) \qquad\qquad S_t = Y_t - h(t),$$

then $\mathbf{Var}(R_t \wedge S_t) = \mathbf{Var}(X_t \wedge Y_t)$ and $\mathbf{E}R_t \wedge S_t = m(t) - h(t) = m_0(t)$. In other words,

$$\mathbf{E}[(X_t \wedge Y_t)^2] - [m_0(t) + h(t)]^2 = \mathbf{E}[(R_t \wedge S_t)^2] - [m_0(t)]^2,$$

equivalently,

$$\mathbf{E}[(X_t \wedge Y_t)^2] = \mathbf{E}[(R_t \wedge S_t)^2] + h(t)[2m_0(t) + h(t)].$$

Since $U_t \wedge V_t$ and $R_t \wedge S_t$ have the same variance as $X_t \wedge Y_t$, and, $U_t \wedge V_t$ and $R_t \wedge S_t$ have the same mean, we find that

$$\mathbf{E}[(U_t \wedge V_t)^2] = \mathbf{E}[(R_t \wedge S_t)^2]. \tag{4.3}$$

This establishes (4.2). To see the first statement, it remains to notice that, if indeed the first moment $\mathbf{E}X_t \wedge Y_t$ and $\sigma_1^2(t) + \sigma_2^2(t)$ are both known for every $t > 0$, then $h(t)$ and $m_0(t)$ are known, and thus, the righthand side of

$$\mathbf{E}[(X_t \wedge Y_t)^2] = \mathbf{E}[(U_t \wedge V_t)^2] + h(t)[2m_0(t) + h(t)]$$

is known as soon as we find an expression for $\mathbf{E}[(U_t \wedge V_t)^2]$. This finishes our proof. □

Consequently, if the two mean parameters μ_1 and μ_2 are not both zero, then, after retrieving the parameters μ_1, μ_2, and $\sigma_1^2(t) + \sigma_2^2(t)$, we subtract $h(t)[2m_0(t) + h(t)]$ from $\mathbf{E}(X_t \wedge Y_t)^2$, which we assume to be known for every $t > 0$ to make sure that then $\mu_1 = \mu_2 = 0$.

Lemma 2 *Suppose that $\mu_1 = \mu_2 = 0$. For every $t > 0$, the second moment of $X_t \wedge Y_t$ is*

$$\mathbf{E}[(X_t \wedge Y_t)^2] = \frac{1}{2}(\sigma_1^2(t) + \sigma_2^2(t)). \tag{4.4}$$

Proof. Keep in mind that $m_1(t) = m_2(t) = 0$ for every $t > 0$. We proceed similarly as in the proof of Lemma 1 and end up with

$$\begin{aligned} \mathbf{E}[(X_t \wedge Y_t)^2] &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{x} y^2 f_{X_t,Y_t}(x,y)\, dy + \int_{-\infty}^{\infty} x^2\, dx \int_{x}^{\infty} f_{X_t,Y_t}(x,y)\, dy \\ &= \sigma_2^2(t) \int_{-\infty}^{\infty} \varphi(r)\, [-\varphi(\sigma_1(t)r/\sigma_2(t))\, \sigma_1(t)r/\sigma_2(t) + \Phi(\sigma_1(t)r/\sigma_2(t))]\, dr \\ &\quad + \sigma_1^2(t) \int_{-\infty}^{\infty} r^2\, \varphi(r)\, [1 - \Phi(\sigma_1(t)r/\sigma_2(t))]\, dr, \end{aligned}$$

where we relied on the identity $\int_{-\infty}^{a} s^2 \varphi(s)\, ds = -a\varphi(a) + \Phi(a)$. Combining the last display with the three elementary identities

(i) $\int_{-\infty}^{\infty} r\, \varphi(r)\, \varphi(\sigma_1(t)r/\sigma_2(t))\, dr = 0$,

(ii) $\int_{-\infty}^{\infty} \varphi(r)\, \Phi(\sigma_1(t)r/\sigma_2(t))\, dr = 1/2$,

(iii) $\int_{-\infty}^{\infty} r^2\, \varphi(r)\, [1 - \Phi(\sigma_1(t)r/\sigma_2(t))]\, dr = 1/2$

leads to the formula in (4.4). □

Hence, whenever $\gamma_i = \gamma \neq \omega = \omega_i$, the minimum process has the expectation and variance the same for both cases $\sigma_1 = 6$ and $\sigma_2 = 6$ and $\sigma_1 = 1$ and $\sigma_2 = \sqrt{71}$, say. In the nondegenerate case with $\gamma \neq \omega$, the previous lemma answers the question in the *negative* as to whether the second moment is helpful at identifying the individual parameters σ_1 and σ_2 since $\sigma_1^2(t) + \sigma_2^2(t)$ is previously known from $\mathbf{E}X_t \wedge Y_t$ if $\omega \neq \gamma$. This ends the proof of Theorem 1. On the other hand, if σ_1 (or σ_2) has been identified by means of $\mathbf{E}X_t \wedge Y_t$ (for instance, in the degenerate case), then, in light of Proposition 3, the second moment of $X_t \wedge Y_t$ immediately yields the value of σ_2 (or σ_1). Indeed, $\mathbf{E}[(X_t \wedge Y_t)^2] = (\sigma_1^2 t^{2\omega_1} + \sigma_2^2 t^{2\omega_2})/2$. Thus, if $\omega_1 > \omega_2$, then $\mathbf{E}[(X_t \wedge Y_t)^2]t^{-2\omega_2} \to \sigma_2^2/2$ as $t \to 0$ and $\mathbf{E}[(X_t \wedge Y_t)^2]t^{-2\omega_1} \to \sigma_1^2/2$ as $t \to \infty$. Observe that if $\omega = \gamma$, then Proposition 3 does not apply. However, the second moment of the minimum process is given by relation (4.2).

5 Poisson Point Processes

Since in the Gaussian family generally the individual variances cannot be recovered from the first two moments of the minimum process, it is interesting to address this issue of identification in the Poisson family, where the variance is equal to the mean. It is natural to let the time scaling be the same for both processes. For the sake of simplicity, suppose that the time scale is linear (other choices are dealt with similarly). Consider two independent Poisson point processes N_t and M_t with (positive) intensity rates λ_1 and λ_2 for every $t > 0$, that is,

$$N_t \sim \text{Poiss}(\lambda_1 t) \qquad M_t \sim \text{Poiss}(\lambda_2 t). \tag{5.1}$$

Then the expectation of the minimum process $N_t \wedge M_t$ with respect to the joint distribution of N_t and M_t is given by

$$\begin{aligned} \mathbf{E}N_t \wedge M_t &= \sum_{x=1}^{\infty}\sum_{y=1}^{x} y e^{-(\lambda_1+\lambda_2)t} (\lambda_1 t)^x (\lambda_2 t)^y/(x!\, y!) \\ &\quad + \sum_{y=1}^{\infty}\sum_{x=1}^{y-1} x e^{-(\lambda_1+\lambda_2)t} (\lambda_1 t)^x (\lambda_2 t)^y/(x!\, y!) \\ &= e^{-(\lambda_1+\lambda_2)t}\{(\lambda_1 t)(\lambda_2 t) + (\lambda_1 t)^2(\lambda_2 t)/2 + (\lambda_1 t)(\lambda_2 t)^2/2 + R_4(t)\} \\ &= e^{-(\lambda_1+\lambda_2)t}\lambda_1\lambda_2\{t^2 + (\lambda_1+\lambda_2)t^3/2 + R_4(t)\}, \end{aligned} \tag{5.2}$$

where, for every integer $k > 0$, $R_k(t) = c_k t^k + c_{k+1}t^{k+1} + c_{k+2}t^{k+2} + \ldots$ for some reals c_k, which may change from line to line. The first moment of $N_t \wedge M_t$ suffices to uniquely identify the two parameters λ_1 and λ_2.

Proposition 4 *Suppose that the function $v(t) = \mathbf{E}N_t \wedge M_t$ is known for every $t > 0$. Then the parameters λ_1 and λ_2 are uniquely determined via the two following identities:*

$$\begin{aligned}\lambda_1\lambda_2 &= \lim_{t\to 0} t^{-2}\, v(t)\\ \lambda_1+\lambda_2 &= -2(\lambda_1\lambda_2)^{-1} \lim_{t\to 0} t^{-3}\{v(t)-\lambda_1\lambda_2 t^2\}.\end{aligned}$$

Proof. Observe that $R_k(0) = 0$ for every integer $k > 0$. Finding $\lambda_1\lambda_2$ and $\lambda_1 + \lambda_2$ is a routine exercise in calculus as well as solving $a = \lambda_1\lambda_2$ and $b = \lambda_1 + \lambda_2$ for λ_1 and λ_2 in terms of a and b by means of the quadratic formula. □

This proves Theorem 2.

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