Self–Affine Carpets on the Square Lattice

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Abstract

We explore the "Hausdorff dimension at infinity" for self–affine carpets defined on the square lattice. This notion of dimension (due to Barlow and Taylor), which is the correct notion from a probabilistic perspective, differs for these sets from more "naive" indices of fractal dimension.

1 Introduction

The most widely used notion of dimension for "fractal" sets in the lattice $\mathbf{Z}^d$ is the discrete Minkowski (or "box") dimension. For any subset A of the lattice, the *discrete Minkowski dimension* $\dim_M(A)$ is defined by

$$\dim_M(A) = \lim_{R\to\infty} \frac{\log \#\{A \cap B_R(\mathbf{0})\}}{\log R}$$

if the limit exists, where # denotes cardinality and $B_R(\mathbf{0})$ is a ball of radius R centered at the origin. Recently, Barlow and Taylor ([1],[2]) introduced an analogue

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to Hausdorff dimension for sets in $\mathbf{Z}^d$. (Consult [2] for a discussion of several indices of discrete fractal dimension and further references.)

The following definition of "Hausdorff dimension at infinity" is equivalent to the one described for $\mathbf{Z}^d$ in [2], but is more direct and intuitive; for concrete calculations, however, one must usually partition the space into shells as in [2] – see the next section.

A continuous, increasing function $\varphi : [0, \infty) \to [0, \infty)$ satisfying $\varphi(0) = 0$ will be called a *gauge function.* Given a subset A of a discrete metric space, and an arbitrary "base point" $\mathbf{0}$ in the space, define the *Hausdorff content* $\mathcal{H}^*_\varphi(A)$ in the gauge φ by

$$\mathcal{H}^*_\varphi(A) = \inf \sum_{i=1}^{\infty} \varphi\left(\frac{\operatorname{diam}(U_i)}{\operatorname{diam}(U_i \cup \{\mathbf{0}\})}\right), \tag{1}$$

where the infimum is over all countable covers $\bigcup_{i=1}^{\infty} U_i \supset A$. Observe that finiteness of this quantity does not depend on the choice of the base point $\mathbf{0}$. If $\varphi(t) = t^s$, we write $\mathcal{H}^*_s$ for $\mathcal{H}^*_\varphi$. The *Hausdorff dimension at infinity* $\dim_H(A)$ of A is the infimum of the exponents $s > 0$ such that $\mathcal{H}^*_s(A)$ is finite.

Of course, the dimension indices $\dim_H(A)$ and $\dim_M(A)$ only depend on the geometry of A "at infinity", i.e., they are unaffected if A is changed inside a finite ball. An important advantage of the index $\dim_H$ over the simpler $\dim_M$ was described in [2] in connection with random walk hitting probabilities. Namely, suppose a random walk $\{S_n\}$ on the d-dimensional lattice has a Green function satisfying $G(0, x) \sim c|x|^{\alpha-d}$ as $|x| \to \infty$. Then $S_n \in A$ for infinitely many n a.s. if $\dim_H(A) > d - \alpha$, but *not* if $\dim_H(A) < d - \alpha$. However, to make the case for the dimension index $\dim_H$ persuasive, more natural examples of sets where the two dimension indices disagree are needed, as the examples given in [2] seem somewhat artificial. Describing such examples is the goal of this note.

A natural class of sets that turn out to be sensitive to distinct notions of dimension are the lattice analogues of the "self–affine carpets" studied by McMullen [6] (under the name "general Sierpiński carpets") and by Bedford [3].

We begin by defining the *self–affine carpets* on the square lattice. The idea is to first partition the whole square lattice into "approximate shells" and, within each shell, to mimic the construction of the carpets in the plane.

For integers $m \le n$ let $\mathcal{D} \subset \{0, 1, \ldots, m-1\} \times \{0, 1, \ldots, n-1\}$ be a nonempty, finite alphabet and let

$$\alpha \;=\; \frac{\log m}{\log n} \le 1. \tag{2}$$

Furthermore, for each $N \ge 0$ define

$$\mathcal{C}_N \;=\; \{(x,y) \in \mathbf{Z}^2 :\; 0 \le x < n^{\lfloor \alpha N \rfloor} \text{ and } 0 \le y < m^N\} \tag{3}$$

and the approximate shell $\mathcal{S}_N$ in the first quadrant of $\mathbf{Z}^2$

$$\mathcal{S}_N \;=\; \mathcal{C}_{N+1} \setminus \mathcal{C}_N. \tag{4}$$

For each $(x,y) \in \mathcal{S}_N$ write

$$\begin{cases} x = \sum_{\nu=0}^{\lfloor \alpha N \rfloor} x_\nu n^{\lfloor \alpha N \rfloor - \nu} & \text{with } 0 \le x_\nu < n \\ \\ y = \sum_{\nu=0}^{N} y_\nu m^{N-\nu} & \text{with } 0 \le y_\nu < m. \end{cases} \tag{5}$$

Note that $x_0 + y_0 > 0$. Fix $\mathcal{D}$ as above, and denote by π the projection map to the y-axis, so that $\pi(\mathcal{D}) \subset \{0, 1, \ldots, m-1\}$. For any N and any lattice point $(x,y) \in \mathcal{S}_N$, we let $(x,y) \in \Lambda = \Lambda(\mathcal{D})$ if and only if $(x_\nu, y_\nu) \in \mathcal{D}$ for each $\nu \le \lfloor \alpha N \rfloor$ and $y_\nu \in \pi(\mathcal{D})$ for each $\lfloor \alpha N \rfloor < \nu \le N$. Taking the union over all $N \ge 1$ defines the self-affine carpet $\Lambda \subset \mathbf{Z}^2$.

When $n = m$, the set Λ is, roughly speaking, “self–similar”, so one expects the Hausdorff dimension at infinity $\dim_H(\Lambda)$ and the discrete Minkowski dimension $\dim_M(\Lambda)$ to coincide, but typically they will differ when $m < n$. (We will be more precise shortly.)

Next, we describe some parameters of the self-affine carpets that will appear in the dimension formulae below. Let

$$a(i) \;=\; \sum_{j=0}^{n-1} 1_{\mathcal{D}}(i,j) \tag{6}$$

be the number of elements in row i in the pattern induced by the alphabet $\mathcal{D}$. Note that

$$\#\pi(\mathcal{D}) \;=\; \sum_{i=0}^{m-1} 1_{\{a(i) > 0\}}, \tag{7}$$

is the number of nonempty rows in the pattern.

In this note we prove the following discrete version of the results of McMullen [6] and Bedford [3].

Theorem 1.1 *Let* $dim_H(\Lambda)$ *and* $dim_M(\Lambda)$ *denote the Hausdorff dimension at infinity and the discrete Minkowski dimension of the set* Λ*. Then*

$$dim_H(\Lambda) \;=\; \log_m \sum_{i=0}^{m-1} a(i)^{\alpha} \tag{8}$$

and

$$dim_M(\Lambda) \;=\; \log_m \#\pi(\mathcal{D}) \,+\, \log_n \frac{\#\mathcal{D}}{\#\pi(\mathcal{D})}\,. \tag{9}$$

The second assertion is straightforward, and is included only for comparison — observe that the Hausdorff dimension at infinity $\dim_H(\Lambda)$ and the discrete Minkowski dimension $\dim_M(\Lambda)$ agree if and only if either $m = n$ or all the positive values among the row counts $a(i)$ are equal.

Although the dimension results for the carpets on the square lattice match the results in [6] exactly, the proofs must be changed, as an appeal to the strong law of large numbers is no longer possible due to the finite resolution; instead, we apply a large deviation bound.

In contrast to self–similar sets in Euclidean space, "self–similar" sets on the square lattice have infinite Hausdorff content in their dimension. Thus, in the discrete setting the Hausdorff content of a self–*affine* carpet in its dimension does not provide an indicator of the difference between Hausdorff and Minkowski dimensions (compare [4] and [7]).

The remainder of this note is organized as follows. The next section replaces the definition of Hausdorff content given above by a version adapted to the construction of Λ. In Section 3 useful probability measures are constructed, following [6]. The lower bound for Hausdorff dimension is proved in Section 4 using some basic large deviation estimates; the upper bound is in Section 5.

2 Approximate Squares

To compute the Hausdorff content of the self–affine carpets, we partition $\mathbf{Z}^2$ into the shells $\mathcal{S}_N$, and in turn, break these up into at most mn large approximate squares. These squares (like the shells) have the property that their distance from the origin is comparable to their size.

The shell $\mathcal{S}_N$ is a union of "approximate squares" $\hat{Q}_N$ of height m^N, each N-*shell square* $\hat{Q}_N$ being determined by fixing the coordinates x_0 and y_0 in the representation (5). Next, we partition the shell squares into smaller approximate squares. Let $(x, y) \in \mathcal{S}_N$ have an expansion (5). Then we let $Q_k(x, y)$ consist of all points $(x', y') \in \mathbf{Z}^2$ such that x and x' disagree at most in the $\lfloor \alpha k \rfloor$ least significant digits in their base n-expansion, while y and y' differ at most in the k least significant digits in their base m-expansion. Observe that the height and width of the approximate square $Q_k(x, y)$ differ by at most a factor of n for every $0 \leq k \leq N$. Moreover, for each $k \geq 0$ all m^k-approximate squares are pairwise disjoint.

For $0 \leq l \leq N$ define the set $\mathcal{U}_l$ of approximate squares contained in $\mathcal{S}_N$ of height no larger than m^l:

$$\mathcal{U}_l \quad = \quad \{Q_k(x, y) \subset \mathcal{S}_N\}_{k \leq l} \, .$$

For any gauge function φ, define

$$\mathcal{H}^N_\varphi(\Lambda \cap \hat{Q}_N) = \min \sum_{i=1}^{r} \varphi\left(\frac{\text{height}(U_i)}{\text{height}(\hat{Q}_N)}\right), \tag{10}$$

where the minimum is over all coverings $\{U_i\}$ of $\Lambda \cap \hat{Q}_N$ such that $U_i \in \mathcal{U}_N$. If $\varphi(t) = t^\beta$, we write $\mathcal{H}^N_\beta$ for $\mathcal{H}^N_\varphi$. The *Hausdorff dimension at infinity* $\dim_H(\Lambda)$ of Λ is given by

$$\dim_H(\Lambda) = \inf\{s : \ \mathcal{H}_s(\Lambda) = \sum_{N=0}^{\infty} \sum_{\hat{Q}_N \in \mathcal{S}_N} \mathcal{H}^N_s(\Lambda \cap \hat{Q}_N) < \infty\}. \tag{11}$$

This definition is easily seen to be equivalent to the definitions given in [2] and in our introduction (Section 1), since any set of diameter m^k can be covered by a bounded number of approximate squares Q_k.

3 A Probability Measure on the N-shell squares

To compute the lower and upper bounds for the Hausdorff dimension of Λ, we equip each N-shell square in the lattice with a probability measure.

Specify an N-shell square $\hat{Q}_N \subset \mathcal{S}_N$ by picking some allowed pair (x_0, y_0). Using the representation (5), any probability vector $\mathbf{p} = \{p(d) : \ d \in \mathcal{D}\}$ defines a probability measure $\mu_\mathbf{p}$ on $\hat{Q}_N$ which is the image of the product measure $\mathbf{p}^\mathbf{N}$ on $\mathcal{D}^\mathbf{N}$.

We restrict attention to probability vectors $\mathbf{p}$ such that for each $d \in \mathcal{D}$ the value $p(d)$ depends only on the *second* coordinate of d. We then write $p(\pi(d)) := p(d)$ for every $d \in \mathcal{D}$. More precisely, $\mu_{\mathbf{p}}$ makes the digits (x_ν, y_ν) for $\nu = 1, \dots, \lfloor \alpha N \rfloor$ i.i.d. random variables with law $\mathbf{p}$, while the digits $y_{\lfloor \alpha N \rfloor + 1}, \dots, y_N$ are i.i.d. with distribution $(a(0)p(0), \dots, a(m-1)p(m-1))$, which is the projection of $\mathbf{p}$ to the y-axis. Consequently, if x and y have the representations (5), then

$$\begin{aligned} \mu_{\mathbf{p}}[Q_k(x,y)] &= \prod_{\nu=1}^{\lfloor \alpha(N-k) \rfloor} p(x_\nu, y_\nu) \prod_{\nu=\lfloor \alpha(N-k) \rfloor + 1}^{N-k} a(y_\nu) p(x_\nu, y_\nu) \qquad (12) \\ &= \prod_{\nu=1}^{N-k} p(y_\nu) \prod_{\nu=\lfloor \alpha(N-k) \rfloor + 1}^{N-k} a(y_\nu), \qquad (13) \end{aligned}$$

since $Q_k(x,y)$ is obtained from the N-shell square $\hat{Q}_N$ by specifying the digits (x_ν, y_ν) for $\nu = 1, \dots, \lfloor \alpha(N-k) \rfloor$ and also $y_{\lfloor \alpha(N-k) \rfloor + 1}, \dots, y_{N-k}$.

Taking logarithms to base m,

$$\log_m \mu_{\mathbf{p}}[Q_k(x,y)] = \sum_{\nu=1}^{N-k} \log_m p(y_\nu) \; + \sum_{\nu=\lfloor \alpha(N-k) \rfloor + 1}^{N-k} \log_m a(y_\nu) \qquad (14)$$

The $\mu_{\mathbf{p}}$-expectation of this sum is

$$-(N-k)\xi_{\mathbf{p}} + O(1), \qquad (15)$$

where

$$\xi_{\mathbf{p}} \;\; := \;\; -\sum_{d \in \mathcal{D}} p(d) \log_m p(d) - (1-\alpha) \sum_{d \in \mathcal{D}} p(d) \log_m a(d). \qquad (16)$$

and the error term in (15) is at most $B := \log_m n$.

4 A Lower Bound for the Hausdorff Content

For every $\epsilon > 0$, $A > 0$ and all integers $N > k \geq 0$ let

$$\begin{aligned} \mathcal{G}_\epsilon(k) \;\; = \;\; \{ \, Q_k(x,y) \in \mathcal{U}_N : \; & \mu_{\mathbf{p}}[Q_k] > 0 \;\text{ and} \\ & \log_m \mu_{\mathbf{p}}[Q_k] + (N-k)\, \xi_{\mathbf{p}} \leq (N-k)\epsilon + A + B \}. \qquad (17) \end{aligned}$$

denote the set of "good" m^k-approximate squares; here $B = \log_m n$ and A is a large parameter. The following standard large deviation estimate will be helpful.

Lemma 4.1 (Hoeffding [5]) *Assume that* $\{S_k\}_{k\geq 1}$ *are the partial sums of a sequence of independent random variables with zero expectation. Let* B *denote a bound on the absolute values of the increments. Then for every* $\epsilon > 0$ *and* $k \geq 1$,

$$\mathbf{P}(S_k \geq \epsilon k) \leq e^{-c_1 \epsilon^2 k}, \tag{18}$$

where $c_1 = B^{-2}/2$.

Next we show that in any disjoint cover of the set $\Lambda \cap \hat{Q}_N$, most of the measure $\mu_{\mathbf{p}}$ is carried by the "good" squares.

Lemma 4.2 *Let* $\mathcal{V} \subset \mathcal{U}_N$ *be a cover of* $\Lambda \cap \hat{Q}_N$. *Then there exists some constant* $c_1 > 0$ *such that, for every* $\epsilon > 0$ *and* $A > 0$,

$$\sum_{k=0}^{N} \sum_{Q_k \in \mathcal{V} \cap \mathcal{G}_\epsilon(k)} \mu_{\mathbf{p}}[Q_k] \geq 1 - \frac{1}{c_1 \epsilon^2} \exp\{-2c_1 A \epsilon\}. \tag{19}$$

Proof. Since $\{y_\nu\}_{\nu \geq 1}$ are i.i.d. random variables with respect to $\mu_{\mathbf{p}}$, we can apply Lemma 4.1 to the partial sums $\log_m \mu_{\mathbf{p}}[Q_k]$, after centering (subtracting their expectations). If

$$\log_m \mu_{\mathbf{p}}[Q_k] + (N-k)\xi_{\mathbf{p}} > (N-k)\epsilon + A + B\,.$$

then the centered sums mentioned above are greater than $(N-k)\epsilon + A$. Therefore

$$\sum_{\substack{Q_k \in \mathcal{U}_N \\ Q_k \notin \mathcal{G}_\epsilon(k)}} \mu_{\mathbf{p}}[Q_k] \ \leq\ \exp\{-c_1[\epsilon^2(N-k) + 2A\epsilon]\}.$$

Consequently,

$$\sum_{k=1}^{N} \sum_{\substack{Q_k \in \mathcal{V} \\ Q_k \notin \mathcal{G}_\epsilon(k)}} \mu_{\mathbf{p}}[Q_k] \ \leq\ e^{-2c_1 A\epsilon} \frac{1}{c_1 \epsilon^2}\,.$$

This proves our claim.

□

Proposition 4.3 $dim_H(\Lambda) \geq \xi_{\mathbf{p}}$ *for any probability vector* $\mathbf{p}$, *where* $\xi_{\mathbf{p}}$ *is defined in (16).*

Proof. It suffices to show that for any $\beta < \xi_{\mathbf{p}}$ there is a positive $C_2 = C_2(\beta)$ such that for all N and all N-shell squares $\hat{Q}_N$, the Hausdorff content satisfies $\mathcal{H}^N_\beta(\Lambda \cap \hat{Q}_N) \geq C_2$.

Given $\beta < \xi_{\mathbf{p}}$, choose a positive $\epsilon < \xi_{\mathbf{p}} - \beta$. Let $\mathcal{V} \subset \mathcal{U}_N$ be any cover of $\Lambda \cap \hat{Q}_N$ by approximate squares. An appropriate choice of the constant $A > 0$ together with Lemma 4.2 guarantee that

$$\sum_{k \leq N} \sum_{Q_k \in \mathcal{V} \cap \mathcal{G}_\epsilon(k)} \mu_{\mathbf{p}}[Q_k] \geq \frac{1}{2}. \tag{20}$$

Each $Q_k \in \mathcal{G}_\epsilon(k)$ satisfies

$$\mu_{\mathbf{p}}[Q_k] \ \leq \ m^{-(N-k)\beta}\, m^{A+B}, \tag{21}$$

and consequently, using (20) we get

$$\begin{aligned} \sum_{Q \in \mathcal{V}} \left(\frac{\text{height}(Q)}{\text{height}(\hat{Q}_N)} \right)^\beta &\geq \sum_{k=1}^{N} \sum_{\mathcal{V} \cap \mathcal{G}_\epsilon(k)} m^{-(N-k)\beta} \\ &\geq m^{-A-B} \sum_{k=1}^{N} \sum_{\mathcal{V} \cap \mathcal{G}_\epsilon(k)} \mu_{\mathbf{p}}[Q_k] \geq \frac{1}{2} m^{-A-B}. \end{aligned}$$

Thus $\mathcal{H}^N_\beta(\Lambda \cap \hat{Q}_N) \geq \frac{1}{2} m^{-A-B}$ for all N, which implies that $\mathcal{H}_\beta(\Lambda) = \infty$. Since this holds for all $\beta < \xi_{\mathbf{p}}$, we conclude that $\dim_H(\Lambda) \geq \xi_{\mathbf{p}}$. □

A routine argument easily shows that

$$\xi_{\mathbf{p}} = \sum_{d \in \mathcal{D}} p(d) \log_m p(d)^{-1} + \sum_{d \in \mathcal{D}} p(d) \log_m a(d)^{\alpha - 1}$$

is maximal for

$$p(d) \ = \ \frac{1}{Z}\, a(d)^{\alpha-1}, \tag{22}$$

where $Z = \sum_{d \in \mathcal{D}} a(d)^{\alpha-1}$, which yields

$$\xi_* := \max_{\mathbf{p}} \xi_{\mathbf{p}} = \log_m Z\,. \tag{23}$$

Corollary 4.4 $dim_H(\Lambda) \geq \xi_* \ = \ \log_m Z\,.$

5 An Upper Bound for the Hausdorff Content

The following lemma is a quantitative version of an argument used by McMullen [6].

Lemma 5.1 *Let* $0 < \alpha < 1$*, and let* B *and* ϵ *be positive finite constants. Then there exists some* $\theta \in (0,1)$ *with the following property: For all sufficiently large* N*, if a finite sequence* $\{f(j)\}_{j=0}^{N}$ *satisfies* $f(0) = 0$ *and*

$$|f(j) - f(j-1)| \leq B \ \textit{ for all } \ 1 \leq j \leq N, \tag{24}$$

then there is an integer $k \in [\theta N, N]$ *such that*

$$\frac{f(k)}{k} - \frac{f(\lfloor \alpha k \rfloor)}{\alpha k} \geq -\epsilon\,. \tag{25}$$

Proof. Let $\theta \in (0,1)$ and let $\{f(j)\}_{j=0}^{N}$ be a finite sequence satisfying $f(0) = 0$ and (24). Interpolating linearly, we extend f to a Lipschitz function defined on the whole interval $[0, N]$. Consider the telescoping sum

$$\sum_j \left\{ \frac{f(\alpha^{-j})}{\alpha^{-j}} - \frac{f(\alpha^{1-j})}{\alpha^{1-j}} \ : \ \theta N + 1 \leq \alpha^{-j} \leq N \right\} \geq -2B. \tag{26}$$

(Using the inequality $|f(t)| \leq Bt$.) By taking θ small we can make the number of summands on the left arbitrarily large. Specifically, if $\theta < \alpha^{4B/\epsilon+2}$ then for large N the number of summands on the lefthand side of (26) is at least $4B/\epsilon$, thus, there exists a $t \in [\theta N + 1, N]$ such that $\frac{f(t)}{t} - \frac{f(\alpha t)}{\alpha t} \geq -\epsilon/2$. Finally, $k = \lfloor t \rfloor$ satisfies (25) provided N is sufficiently large. □

Proposition 5.2 *Let* $dim_H(\Lambda)$ *denote the Hausdorff dimension at infinity of* Λ*. Then* $dim_H(\Lambda) \leq \xi_*$*, where* ξ_* *is given in (23).*

Proof. In all that follows, we will use the probability vector $\mathbf{p}$ defined in (22).

Fix an N-shell square $\hat{Q}_N$. Let $(x, y) \in \hat{Q}_N$ and define $R_j(x, y) = \sum_{\nu=1}^{j} \log_m a(y_\nu)$. Replacing k by $N - k$ in (14) and using the probability vector $\mathbf{p}$ in (22), we obtain

$$\begin{aligned} \log_m \mu_{\mathbf{p}}[Q_{N-k}(x,y)] + k \log_m Z &= \left((\alpha - 1) \sum_{\nu=1}^{k} + \sum_{\nu=\lfloor \alpha k \rfloor + 1}^{k} \right) \log_m a(y_\nu) \\ &= \alpha\, R_k(x,y) - R_{\lfloor \alpha k \rfloor}(x,y). \end{aligned} \tag{27}$$

Equivalently,

$$\frac{1}{\alpha k}\log_m \mu_{\mathbf{p}}[Q_{N-k}] + \frac{1}{\alpha}\log_m Z \;=\; \frac{R_k(x,y)}{k} - \frac{R_{\lfloor \alpha k\rfloor}(x,y)}{\alpha k}. \tag{28}$$

Observe that the ratios $R_k(x,y)/k$ are bounded by $B = \log_m n$.

Let $\epsilon > 0$. By Lemma 5.1, there exists some $\theta \in (0,1)$ such that if N is sufficiently large, then for every $(x,y) \in \Lambda \cap \hat{Q}_N$ there is an integer $k \in [\theta N, N]$ satisfying

$$\frac{1}{k}\log_m \mu_{\mathbf{p}}[Q_{N-k}] + \log_m Z \ge -\alpha\epsilon \ge -\epsilon\,. \tag{29}$$

For each point $(x,y) \in \hat{Q}_N$, we find the *smallest* k in $[\theta N, N]$ for which $Q_{N-k}(x,y)$ satisfies (29). As (x,y) runs over the N-shell square $\hat{Q}_N$, these squares form a *disjoint* cover $\mathcal{V}$ of $\hat{Q}_N$, and

$$\sum_{\theta N \le k \le N}\;\sum_{Q_{N-k}\in\mathcal{V}} m^{-k(\xi_*+2\epsilon)} \;\le\; \sum_{\theta N \le k \le N}\;\sum_{Q_{N-k}\in\mathcal{V}} m^{-k\epsilon}\mu_{\mathbf{p}}[Q_{N-k}] \;\le\; m^{-\epsilon\theta N}.$$

Summing over N, it follows that $\mathcal{H}_{\xi_*+2\epsilon}(\Lambda) < \infty$. Since $\epsilon > 0$ is arbitrary, this shows that $\dim_H(\Lambda) \le \xi_*$.

□

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