On the Displacement Exponent of the Self-Avoiding Walk in Three and Higher Dimensions

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Abstract

This paper proves that the self-avoiding walk (SAW) in $\mathbf{Z}^d$ has its lower and upper displacement exponents between $\frac{1}{4}+\frac{1}{d}$ and $\frac{1}{2}+\frac{1}{2d}$ for $d \geq 2$ and at exactly 1 for $d = 1$. The problem on the value of the displacement exponent of the SAW, in three dimensions especially, dates back to work (FLORY (1949) [3]) on linear polymer chains in chemical physics and is surrounded by cascades of open conjectures on the SAW, mostly for $d = 2$. While little is mathematically known about the SAW for $d = 2$ or $d = 3$, as early as in the 1980ies, Monte Carlo simulations produced confidence intervals for the displacement exponent and other critical exponents. Our results extend the ones for $d = 2$, presented in [9], and are based on the method which involves the weakly self-avoiding cone process in a certain shape of the point process of self-intersections introduced there.

1 Introduction

This work derives upper and lower bounds on the root mean square displacement exponent of the *self-avoiding walk* (SAW) in the d-dimensional hypercubic lattice $\mathbf{Z}^d$ for all $d \geq 1$. Understanding the self-avoiding walk from a mathematical point of view is one of the major open long-standing problems at the interface of probability, statistical physics, and physical chemistry. The problem on the displacement exponent of the SAW, for $d = 3$ in particular, has its roots in a 1949-problem [3] of chemist and 1974-Nobel prize winner PAUL FLORY [5] who observed that the end-to-end distance of a long linear polymer chain in three dimensions must have a power of the chain length larger than 1/2 and should approach 0.6. A reconsideration of his arguments [3, 4, 5] by FISHER [2] later led to the *Flory formula* $3/(d+2)$ for $1 \leq d \leq 4$ for the root mean square displacement exponent of the SAW, which turned out to be surprisingly accurate. It was confirmed for $d = 2$ (HUETER [9]), apparently

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is correct for $d = 1$, and is strongly believed to be correct for $d = 4$ and barely overshooting for $d = 3$. In the mathematical physics literature, it is quite enthusiastically believed that, in the latter case, the value of the displacement exponent of the SAW is not rational.

The SAW serves as a model for linear polymer molecules. Polymers are of interest to chemists and physicists and are the fundamental building schemes in biological systems. A polymer is a long chain of monomers which are joined to one another by chemical bonds. These polymer molecules arrange themselves randomly with the restriction of no overlap. This repelling force drives the polymers to be more diffusive than a simple random walk. Numerous stones wait to be uncovered from a mathematically rigorous perspective since very little is known about the 2-, 3- and 4-dimensional polymers or self-avoiding walk. To document the appeal and fascination of the subject, there is a nearly unlimited and continually and rapidly growing set of simulations together with heuristic arguments and a zoo of "numerical artifacts" that lend themselves to a landscape of conjectures. The literature has devoted much attention to this theme. We refer the reader to other references for a survey (e.g. consult MADRAS AND SLADE [13] and LI, MADRAS, AND SOKAL [12] for a useful review of the developments at the simulation end).

This paper presents some answers to the question on the average distance between the two ends of a long polymer. Our results as well pertain to the asymptotic expected distance of the weakly self-avoiding walk from its starting point up through a large step size.

Consider the weakly self-avoiding walk in $\mathbf{Z}^d$ starting at the origin. More precisely, if $J_n = J_n(\cdot)$ denotes the *number* of *self-intersections* or the *self-intersection local time* (SILT) of a symmetric simple random walk $S_0 = \mathbf{0}, S_1, \ldots, S_n$ in the d-dimensional integer lattice starting at the origin, that is,

$$J_n = J_n(S_0, S_1, \ldots, S_n) = \sum_{0 \le i < j \le n} 1_{\{S_i = S_j\}}, \tag{1.1}$$

and if $\beta \ge 0$ denotes the *self-intersection parameter,* then the *weakly self-avoiding walk* is the stochastic process, induced by the probability measure

$$\mathbf{Q}_n^\beta = \frac{\exp\{-\beta J_n\}}{\mathbf{E}\exp\{-\beta J_n\}} \tag{1.2}$$

on the set of simple random walk paths of length n, where $\mathbf{E}$ stands for the expectation relative to the random walk. In other words, $J_n = r$ self-intersections are penalized by the factor $\exp\{-\beta r\}$. The measure $\mathbf{Q}_n^\beta$ may be looked at as a measure on the set of all simple random walks of length n which weighs relative to the number of self-intersections. This restraint walk is also being called the *Domb-Joyce model* in the literature (see LAWLER [11], p. 170) While when setting $\beta = 0$ we recover the simple random walk (SRW), letting $\beta \to \infty$ well mimics the SAW. The SAW in $\mathbf{Z}^d$ visits each site of its path *exactly* once because its path is a SRW-path *without* self-intersections.

We shall study the expected distance of the weakly SAW from its starting point after n steps, as measured by the Euclidean length and the root mean square displacement at the n-th step. Let $\mathbf{E}_\beta = \mathbf{E}_{\mathbf{Q}_n^\beta}$ denote expectation under the measure $\mathbf{Q}_n^\beta$. Thus, $\mathbf{E}_0$ denotes expectation wrt. to the SRW. Also, write $S_n = (X_n^1, X_n^2, \ldots, X_n^d)$ for every integer $n \geq 0$. Objects of interest to us are the expectation $\mathbf{E}_\beta$ of the *distance*

$$\chi_n = \|S_n\| = \{ \sum_{k=1}^{d} (X_n^k)^2 \}^{1/2} \tag{1.3}$$

of the walk from the starting point $\mathbf{0}$, the *mean square displacement* (MSD) $\mathbf{E}_\beta \chi_n^2$, and the *root mean square displacement* (RMSD) $(\mathbf{E}_\beta \chi_n^2)^{1/2}$ of the weakly SAW. The RMSD exponent of the weakly SAW and the SAW, respectively, may be defined by

$$\begin{aligned} \nu_\beta(d) &= \lim_{n\to\infty} \frac{\ln \mathbf{E}_\beta(\chi_n^2)}{2 \ln n} \\ \nu_\infty(d) &= \lim_{n\to\infty} \lim_{\beta\to\infty} \frac{\ln \mathbf{E}_\beta(\chi_n^2)}{2 \ln n} \end{aligned} \tag{1.4}$$

if the limits exist – otherwise we regard the upper and lower exponents $\overline{\nu}_\beta(d), \underline{\nu}_\beta(d), \overline{\nu}_\infty(d)$, and $\underline{\nu}_\infty(d)$ via lim sup and lim inf – . We now state our main results.

THEOREM **1 (RMSD Exponent of the (Weakly) SAW)** *The weakly self-avoiding walk with $\beta > 0$ and the self-avoiding walk in $\mathbf{Z}^d$ for $d \geq 2$ have*

$$\frac{1}{4} + \frac{1}{d} \leq \underline{\nu}_\beta(d), \overline{\nu}_\beta(d), \underline{\nu}_\infty(d), \overline{\nu}_\infty(d) \leq \frac{d+1}{2d} \tag{1.5}$$

and $\nu_\beta(1) = \nu_\infty(1) = 1$. Moreover, there are some constants $0 < \rho_3 = \rho_3(d, \beta) \leq \rho_4 = \rho_4(d) < \infty$ (ρ_4 : uniform in β and ρ_3 : uniform in β as $\beta \to \infty$) such that, for $d \geq 2$,

$$\rho_3(d,\beta) \leq \liminf_{n\to\infty} n^{-(\frac{1}{2}+\frac{2}{d})} \mathbf{E}_\beta(\chi_n^2) \leq \limsup_{n\to\infty} n^{-\frac{d+1}{d}} \mathbf{E}_\beta(\chi_n^2) \leq \rho_4(d). \tag{1.6}$$

See Corollary 3 for a proof when $d \geq 2$ and Theorem 3 when $d = 1$. Spelled out, the lower bound $1/4 + 1/d$ on the RMSD exponent for the first few $d \geq 2$ is $3/4, 7/12 = 0.5833..., 1/2, \ldots$. Moreover in each case, the exponent of the asymptotic expected distance exhibits the same bounds as the RMSD exponent of the walk.

THEOREM **2 (Distance Exponent of the (Weakly) SAW)** *The lower and upper distance exponents of the weakly self-avoiding walk with $\beta > 0$ and the self-avoiding walk in $\mathbf{Z}^d$ for $d \geq 2$ satisfy the inequalites indicated in* (1.5) *and equal* 1 *for* $d = 1$.

In addition, there are some constants $0 < \rho_1 = \rho_1(d, \beta) \leq \rho_2 = \rho_2(d) < \infty$ (ρ_2 : uniform in β and ρ_1 : uniform in β as $\beta \to \infty$) such that, for $d \geq 2$,

$$\rho_1(d,\beta) \leq \liminf_{n\to\infty} n^{-(\frac{1}{4}+\frac{1}{d})} \mathbf{E}_\beta(\chi_n) \leq \limsup_{n\to\infty} n^{-\frac{d+1}{2d}} \mathbf{E}_\beta(\chi_n) \leq \rho_2(d). \tag{1.7}$$

The proof is in Corollary 2 for $d \geq 2$ and in Theorem 3 for $d = 1$. HUETER [9] proves the results in the two-dimensional context. We like to emphasize that the proof of the bounds on the exponents for $d \geq 3$ is less subtle and difficult than their counterpart for $d = 2$ since the underlying space in which the walk moves is considerably larger in three and higher dimensions. Not only has the proof a different touch but also the reasons why the bounds arise are different for $d \geq 3$. Our results here and in [9] are novel for $d = 2, 3$, and 4 and $\beta \in (0, \infty]$, while the result on the RMSD exponent of the SAW for $d \geq 5$ is in HARA AND SLADE [7, 8] (the authors prove that the continuum limit is Gaussian for $d \geq 5$) and the one on the RMSD exponent of the weakly SAW for $d = 1$ can be found in GREVEN AND DEN HOLLANDER [6]. Of course, the result on the one-dimensional SAW is quite obvious. BRYDGES AND SPENCER [1] establish that the scaling limit of the weakly SAW is Gaussian for sufficiently small $\beta > 0$ and $d \geq 5$.

A couple of Monte Carlo simulations were performed as early as in the 1980ies to estimate the value of the RMSD exponent of the SAW for $d = 2, 3$ (for more references and details on this, see MADRAS AND SOKAL [14]). The produced 95%-confidence intervals appear to center around the value 0.59... and would suggest a value slightly larger than $7/12 = 0.58333...$. A more recent simulation (PRELLBERG, [16]) generated an interval around 0.5874.... Just for $d = 4$, a logarithmic correction $(\ln n)^{1/4}$ associated with $\mathbf{E}_\beta \chi_n^2$ is being predicted (visit e.g. LAWLER [11], p. 167).

Our approach is to first deduce bounds on the expected displacement of the weakly SAW and to transfer these to the SAW. As in [9], our strategy is to engage a walk – called a "weakly self-avoiding cone process" – that is penalized according to the number of self-intersections in a particular cone which contains the endpoint of the walk, conditioned on the "shape" of the point process of self-intersections contained in all cones (see Definitions 1 and 2 below for the terminology), and, to compare its expected distance with the one of the weakly SAW. The advantage of using this alternative process is that its expected distance is accessible by means of rather precise calculations because we condition on the shape. Since the distribution of the shape is not known for the weakly SAW, however, we cannot closely relate the expected distances of the two processes for $d \geq 3$.

The paper is organized as follows. Section 2 specifies the SRW-paths that are significant from a weakly SAW's point of view. Section 3 recalls the notions of shape of the underlying point process of self-intersections and of the weakly self-avoiding cone process. Section 4 calculates asymptotic mean distances of this process and links those to the ones of the weakly SAW. Section 5 discusses the transfer of the distance and MSD exponents to the SAW. Finally, Section 6 takes care of the one-dimensional weakly SAW.

2 SILT that is Typical to the Weakly SAW

In the remainder of the paper, we will exploit the information that is contained in the intersections that the weakly SAW discourages but does not forbid as the SAW. In low dimensions, the weakly SAW pays attention to the SRW-paths that exhibit a smaller number of self-intersections than is expected for the SRW. This effect is most emphasized in dimension 1. Paths that have about $\mathbf{E}_0 J_n$ self-intersections are not important from the perspective of a weakly SAW. While a weakly SAW-path of length n will turn out to have expected SILT of order n in all dimensions, the SRW is forced to intersect itself more frequently, at least in dimensions 1 and 2. We begin to review the average $\mathbf{E}_0 J_n$ for the SRW and to derive the range for J_n that is significant from the point of view of the weakly SAW with $\beta > 0$. A favorite exercise in a probability course is as follows. By invoking the Fubini theorem and the Local Central Limit theorem, we obtain for all sufficiently large even n,

$$\begin{aligned} \mathbf{E}_0 J_n &= \sum_{0 \le i < j \le n} \mathbf{P}_0(S_i = S_j) \qquad (2.1) \\ &= (1 + o(1)) \sum_{0 \le i < j \le n/2} 2\,(d/(2\pi(j-i)))^{d/2} \\ &= (1+o(1)) \begin{cases} (\frac{2}{\pi^{1/2}})\, n^{3/2} & d = 1, \\ \frac{1}{\pi}\, n \ln n & d = 2, \\ c_d\, n & d \ge 3 \end{cases} \end{aligned}$$

for some positive finite constants c_d, where we used the $o(\cdot)$ notation, that is, write $f(n) = o(g(n))$ as $n \to \infty$ for two real-valued functions f and g if $\lim_{n\to\infty} f(n)/g(n) = 0$.

We now restate the results in Propositions 1 and 2 of [9], Section 2. Since the arguments of proof in [9] are carried out in the time space, as opposed to the state space, they as well apply for $d \neq 2$. When reading the proofs in [9], written for $d = 2$, the reader might want to replace the number '4' by '$2d$', the number of nearest neighboring sites of each lattice site, and rely on $\omega_0(d)$ rather than $\omega_0(2)$ (denoted by ν_0 in [9]), where $\omega_0(d)$ denotes the logarithm of the *connective constant* or the exponent of the number of SAW-paths. At the outset, the following inequalities are worth noting. Recalling the definition of the measure $\mathbf{Q}_n^\beta$ in (1.2), if A_1, A_2 denote any two subintervals of $[0, n^2]$, we obtain

$$\begin{aligned} \mathbf{Q}_n^\beta(J_n \in A_1) &= \frac{\mathbf{E}_0(\exp\{-\beta J_n\}\, 1_{\{J_n \in A_1\}})}{\mathbf{E}_0(\exp\{-\beta J_n\})} \\ &\le \frac{\mathbf{E}_0(\exp\{-\beta J_n\}\, 1_{\{J_n \in A_1\}})}{\mathbf{E}_0(\exp\{-\beta J_n\}\, 1_{\{J_n \in A_2\}})}\,. \qquad (2.2) \end{aligned}$$

As a consequence of (2.2), if we show that

$$\mathbf{E}_0(\exp\{-\beta J_n\}\, 1_{\{J_n \in A_1\}}) \quad < \quad e^{-\tau_* n} \cdot \mathbf{E}_0(\exp\{-\beta J_n\}\, 1_{\{J_n \in A_2\}}) \qquad (2.3)$$

for some subinterval A_2 of $[0, n^2]$ and real number τ_*, then it follows that

$$\mathbf{Q}_n^\beta(J_n \in A_1) < e^{-\tau_* n}. \qquad (2.4)$$

In words, in order to derive exponential decay in n as in (2.4), it suffices to find a subset of SRW-paths that contributes strictly more to $\mathbf{E}_0 \exp\{-\beta J_n\}$ than the set of paths with $J_n \in A_1$. In connection with an upper bound on J_n, it in fact turns out that the set of all self-avoiding paths satisfies this requirement (see [9]). In order to establish a likely upper bound on J_n, we set $A_1 = (Bn, n^2]$ for some real $B > 0$ and $A_2 = \{0\}$. To this end, it is enough to dervive a lower bound on $\mathbf{P}_0(J_n = 0) = \mathbf{E}_0(\exp\{-\beta J_n\}\, 1_{\{J_n=0\}})$ and to figure out for which B it is strictly larger than $\mathbf{E}_0(\exp\{-\beta J_n\}\, 1_{\{J_n > Bn\}})$. While we here omit the details precised in [9], we summarize the conclusion.

Proposition **1** (Proposition 1 of [9], **Upper Bound on J_n**) *Let $d \geq 1, \beta > 0$, and $\omega_0(d)$ denote the exponent of the number of self-avoiding walks. There is some constant $\tau_u > 0$, independent of β, such that for every $B > B_*(\beta, d) = (\ln(2d) - \omega_0(d))/\beta > 0$ and every integer $n \geq 1$,*

$$\mathbf{Q}_n^\beta(J_n > Bn) < e^{-\tau_u n}.$$

On the other hand, the paths with J_n of order less than n are not significant either. To come up with an asymptotic likely lower bound on J_n, we set $A_1 = [0, n^{1-\delta}]$ for some suitable $\delta > 0$ and demonstrate (2.3) for an appropriate set A_2 and constant τ_* that does not depend on δ. In [9], $A_2 = [0, bn]$ for suitably small $b > 0$ and the set of paths with $J_n \leq n^{1-\delta}$ is modified to a set of paths with $J_n \leq bn$ by introducing of order n repetitions of steps to each path. Because there are considerably more positions to choose the positions of the repetitions from than positions of repetitions for large n, this scheme gives rise to a new set of paths whose size is of strictly larger exponential order than the unmodified set of paths, the paths with $J_n \leq n^{1-\delta}$ (if b is suitably small). A more complete account of arguments is given in [9]. We collect the conclusion we will take advantage of later on.

Proposition **2** (Proposition 2 of [9], **Lower Bound on J_n**) *Let $d \geq 1$ and $\beta > 0$. There is a constant $\zeta_*(\beta) > 0$ (made precise in [9]) such that for $b_* = \zeta_*(\beta)/\beta > 0$, for every suitably small $\delta > 0$, every $b < b_*$, and every sufficiently large n,*

$$\mathbf{Q}_n^\beta(J_n \leq n^{1-\delta}) < e^{-\tau_l n}$$

for a constant $\tau_l = \tau_l(\beta) > 0$, uniformly in δ.

A consequence of Propositions 1 and 2 is the following

Corollary **1** (**Likely Interval for J_n**) *Let $d \geq 1$ and $\beta > 0$. There are some constants $0 < b_1 \leq b_2 < \infty$ and $\tau = \tau(\beta) > 0$, where βb_2 is a positive number independent of β and βb_1 may depend on β in such a way that βb_1 tends to zero as $\beta \to \infty$, such that for every sufficiently large n,*

$$\mathbf{Q}_n^\beta(J_n \notin [b_1 n, b_2 n]) < e^{-\tau n}.$$

In the sequel, we will frequently assume that

$$J_n \in [b_1 n, b_2 n] \tag{2.5}$$

for all sufficiently large n and be aware of error terms, which are negligible in view of Corollary 1. We remark that comparing (2.1) with $\mathbf{E}_\beta J_n \in [b_1 n, b_2 n]$, which is immediate from Corollary 1, along with a few observations on b_1 [9] reveals that, for $d \leq 2$, the expectation $\mathbf{E}_0 J_n$ is of larger order in n than $\mathbf{E}_\beta J_n$, whereas for $d = 3, 4$ and β below a certain threshold, $\mathbf{E}_0 J_n < \mathbf{E}_\beta J_n$.

3 Point Process of Self-Intersections

For the rest of the paper, we shall omit discussion of the obvious case $\beta = 0$. A process which is intimately related to the weakly SAW is a process which suppresses self-intersections in a particular cone that contains the endpoint of the walk, named weakly self-avoiding cone process in [9]. Before we state the definitions in Sections 3.1 and 3.2 and proceed to the calculations in Section 4, we begin to explain the prime ideas and give some intuition.

Heuristics. We will partition the space $\mathbf{R}^d$ into a deterministic set of cones for each step size n of the walk and classify these cones according to how many self-intersections they contain. Some will carry a more typical number of self-intersections than others – for $d = 2$ for instance, "typical" means of order $n^{1/2}$ – . Clearly, the event that a cone is less typical will depend on the realized path. We will see that, for $d \geq 3$, the space becomes large in the sense that the cones which contribute most of the self-intersections of the weakly SAW have cardinality of order *strictly* less than the order of the total number of cones. The measure associated with the weakly self-avoiding cone process penalizes the walk less when the walk is in a cone that carries fewer than the typical number of self-intersections, while it penalizes the walk more when the walk is in a cone that carries more than the typical number of self-intersections. It is reasonable to think that a walk which mostly visits the latter class of cones tends to get pushed at a larger distance from its starting point than a walk which mostly visits the former class of cones and that the power γ of the asymptotic order n^γ in n of the expected distance from the starting point at time n is strictly larger for the latter than for the former walk. This can be made precise, and in fact, γ can attain each value in $[1/2, 1]$ under the related measure for $d = 2$ [9] if no restrictions are imposed on J_n and the shape of the self-intersection points. The principal question is: 'Which weakly self-avoiding cone process has the same asymptotic order (in n) of the expected distance as the weakly SAW ?'

For $d = 2$, there is a definite answer (see [9]), namely, the only possible choice of a weakly self-avoiding cone process that stays in a useful relation to the weakly SAW (in that one of the two measures is not singular with respect to the other), especially, has the same order of expected distance as the weakly SAW is the walk whose expected distance is controlled by the expected distance in cones that carry of asymptotic order $n^{1/2}$ self-intersections. For $d \geq 3$, we have partial answers in the sense that we find upper and lower bounds. The difficulty is that not enough is known about the distribution of the shape of the point process of self-intersections of the weakly SAW. To reach bounds on the expected distance of the weakly SAW, we will consider the extreme cases for the value of the shape.

3.1. Point Process of Self-Intersections and Cones. To get ready to refer to the particulars of the point process of self-intersection points with $J_n \in [b_1 n, b_2 n]$ and its associated cones in the next subsection, we first recall the notation and language set in [9].

Let $\Phi = \Phi_n = \{x_1, x_2, \ldots\}$ denote the point process of *self-intersection points* of the SRW in $\mathbf{Z}^d$ when $J_n \in [b_1 n, b_2 n]$ in such a way that $|\Phi| = J_n$. We count each point x_i of Φ with multiplicity, exactly as many times as there are self-intersections of the SRW at x_i in the sense of the defining sum in (1.1). Note that Φ depends on n, b_1, and b_2, thus, on β. This random sequence of points Φ in $\mathbf{Z}^d$ may also be interpreted as a random measure. Observe that $\mathbf{E}_0\Phi$ is σ-finite. Let N_Φ denote the set of all point sequences, generated by Φ, $\mathcal{N}_\Phi$ the point process σ-algebra generated by N_Φ, and $\varphi \in N_\Phi$ denote a realization of Φ. Formally, Φ is a measurable mapping from the underlying probability space into $(N_\Phi, \mathcal{N}_\Phi)$ that induces a distribution on $(N_\Phi, \mathcal{N}_\Phi)$, the distribution $\mathbf{P}_\Phi$ of the point process Φ. By virtue of the σ-finiteness of $\mathbf{E}_0\Phi$, the measure $\mathbf{P}_\Phi$ is a probability measure. Also, let $\mathbf{E}_\Phi$ denote expectation relative to $\mathbf{P}_\Phi$. For more details, e.g. consult Stoyan, Kendall, and Mecke [17], see Chapter 4, p. 99.

At the core of the connection between the weakly SAW and the related process lies the question as to how to assign the self-intersection points to cones which are positioned at the origin in $\mathbf{Z}^d$. A "cone of points" of Φ in $\mathbf{Z}^d$ is induced by the points of Φ closest to a certain line L from a *test set* of lines. Thus, let $\mathcal{V} = \mathcal{V}_n$ denote a deterministic set of half-lines – which we shall call *lines*, for ease – that emanate from the origin and the intersection points of which with the d-dimensional unit sphere are equally spaced out on the sphere, i.e. uniformly and regularly distributed over the sphere (for $d = 2$, the lines are equally spaced around a circle). We fix some constants $0 < v_1 < v_2 < \infty$ (v_1: independent of β, see the proof of Proposition 3 for details) and assume that the cardinality

$$|\mathcal{V}| = v_n\, n^{1-1/d} \tag{3.1}$$

for $v_1 \le v_n \le v_2$ for all sufficiently large n. We like to point out that a suitable choice in (3.1) to renormalize Φ is helpful in comparing the expected distances of the weakly SAW and the related process. Next, we regard the restriction of the process Φ to a fiber, in other words, all points that lie closer to a line L than to any other line in $\mathcal{V}$. For any $L \in \mathcal{V}$, define the *cone* $\mathcal{C}_L$ of points by

$$\mathcal{C}_L \quad = \quad \{x_i \in \Phi : \mathrm{dist}(x_i, L) \le \mathrm{dist}(x_i, L') \text{ for all } L \ne L' \in \mathcal{V}\} \tag{3.2}$$

with a systematic way to break ties when equality $\mathrm{dist}(x_i, L) = \mathrm{dist}(x_i, L')$ arises for two lines L and L'. Thus, each point of Φ belongs to exactly one $\mathcal{C}_L$ and $|\mathcal{C}_L|$ indicates the SILT of the walk in the cone that contains the line L. To classify every line of $\mathcal{V}$ relative to the SILT that its associated cone holds up, for any suitably small $\delta > 0$, each $0 \le r \le 1 - \delta$, and for any constants $0 < a_1 < a_2 < \infty$, define the random sets of lines

$$\mathcal{L}_{r,\delta} \quad = \quad \mathcal{L}_{r,\delta}(\Phi) = \{L \in \mathcal{V} :\, 2|\mathcal{C}_L| \in [a_1 n^r, a_2 n^{r+\delta}]\} \tag{3.3}$$

$$
\begin{aligned}
\mathcal{L}_r &= \mathcal{L}_r(\Phi) = \{L \in \mathcal{V} :\, 2|\mathcal{C}_L| \in [a_1 n^r, a_2 n^r]\} \\
\mathcal{L}_{1/2\pm} &= \mathcal{L}_{1/2\pm}(\Phi) = \{L \in \mathcal{V} :\, 2|\mathcal{C}_L| \in [a_1 n^{1/2-\delta}, a_2 n^{1/2+\delta}]\} \\
\mathcal{L}_- &= \mathcal{L}_-(\Phi) = \{L \in \mathcal{V} :\, 2|\mathcal{C}_L| \in [0, a_1 n^{1/2-\delta})\} \\
\mathcal{L}_+ &= \mathcal{L}_+(\Phi) = \{L \in \mathcal{V} :\, 2|\mathcal{C}_L| \in (a_2 n^{1/2+\delta}, 2b_2 n]\} \\
\mathcal{L}_{1/d\pm} &= \mathcal{L}_{1/d\pm}(\Phi) = \{L \in \mathcal{V} :\, 2|\mathcal{C}_L| \in [a_1 n^{1/d-\delta}, a_2 n^{1/d+\delta}]\} \\
\mathcal{L}_-^* &= \mathcal{L}_-^*(\Phi) = \{L \in \mathcal{V} :\, 2|\mathcal{C}_L| \in [0, a_1 n^{1/d-\delta})\} \\
\mathcal{L}_+^* &= \mathcal{L}_+^*(\Phi) = \{L \in \mathcal{V} :\, 2|\mathcal{C}_L| \in (a_2 n^{1/d+\delta}, 2b_2 n]\} \\
\mathcal{L}_\emptyset &= \mathcal{L}_\emptyset(\Phi) = \{L \in \mathcal{V} :\, |\mathcal{C}_L| = 0\}
\end{aligned}
$$

with the convention that

$$
\begin{aligned}
\mathcal{L}_{0,\delta} &= \mathcal{L}_{0,\delta}(\Phi) = \{L \in \mathcal{V} :\, 2|\mathcal{C}_L| \in [0, a_2 n^\delta]\} \\
\mathcal{L}_{1-\delta,\delta} &= \mathcal{L}_{1-\delta,\delta}(\Phi) = \{L \in \mathcal{V} :\, 2|\mathcal{C}_L| \in [a_1 n^{1-\delta}, 2b_2 n]\}.
\end{aligned}
$$

We will choose a_1 and a_2 such that $a_1\beta$ and $a_2\beta$ are positive numbers that do not depend on β and n.

3.2. Weakly Self-Avoiding Cone Process relative to r-Shaped Φ. For what follows, recall from (1.3) that χ_n denotes the Euclidean distance of the walk from the origin after n steps. Assume that $h : \mathbf{R} \times N_\Phi \to \mathbf{R}_+$ denotes a nonnegative measurable real-valued function and $\mathcal{L}_*(\Phi)$ denotes any subset of lines in $\mathcal{V}$. Since $\mathbf{E}_0\Phi$ is σ-finite, we may disintegrate relative to the probability measure $\mathbf{P}_\Phi$

$$
\mathbf{E}_\Phi\left(\sum_{L\in\mathcal{L}_*(\Phi)} h(L,\Phi)\right) = \int \sum_{L\in\mathcal{L}_*(\varphi)} h(L,\varphi)\, d\mathbf{P}_\Phi(\varphi) \tag{3.4}
$$

(consult also Kallenberg [10], p. 83, and Stoyan, Kendall, and Mecke [17], p. 99. For a discussion of examples of Palm distributions [15] of $\mathbf{P}_\Phi$, the reader is referred to Appendix A of Hueter [9]) and apply formula (3.4) with

$$
h(L,\Phi) = \frac{\exp\{-\beta|\mathcal{C}_L|\}}{|\mathcal{L}_{1/2}(\Phi)|}, \tag{3.5}
$$

with $\mathbf{P}_{\Phi|\chi_n}(\varphi|x)$ in place of $\mathbf{P}_\Phi(\varphi)$ and $\mathcal{L}_* = \mathcal{L}_{1/2}$ to define the numbers $a_x = a_x(\mathcal{L}_{1/2})$ by

$$
\begin{aligned}
\exp\{-\beta a_x n^{1/2}/2\} &= \mathbf{E}_{\Phi|\chi_n}(\,|\mathcal{L}_{1/2}(\Phi)|^{-1} \sum_{L\in\mathcal{L}_{1/2}(\Phi)} e^{-\beta|\mathcal{C}_L|} | \chi_n = x) \qquad (3.6)\\
&= \int_{\mathbf{Z}^d} |\mathcal{L}_{1/2}(\varphi)|^{-1} \sum_{L\in\mathcal{L}_{1/2}(\varphi)} e^{-\beta|\mathcal{C}_L|}\, d\mathbf{P}_{\Phi|\chi_n}(\varphi|x)
\end{aligned}
$$

for $0 \le x \le n$, where we set $\sum_{L\in\mathcal{L}_{1/2}} = 0$ if $\mathcal{L}_{1/2} = \emptyset$. This definition of $a_x(\mathcal{L}_{1/2})$ is to be understood in that the function $a_x(\mathcal{L}_{1/2})$ takes the value defined in (3.6) when χ_n is close to x and x is a possible value of χ_n, and, the values of $a_x(\mathcal{L}_{1/2})$ in between are interpolations between the values just defined. Also, the number $a_x n^{1/2}/2$ may be interpreted as a "typical" SILT in the class of cones associated with $\mathcal{L}_{1/2}$, provided that $\chi_n = x$. In a similar fashion, for any subset $\mathcal{L}$ of $\mathcal{L}_{r,\delta} \subset \mathcal{V}$ and for $\mathcal{L} = \mathcal{L}_{r,\delta}$ in particular, define the numbers $a_1 \leq a_x = a_x(\mathcal{L}) \leq a_2$ by

$$\exp\{-\beta a_x(\mathcal{L}) n^r/2\} \quad = \quad \mathbf{E}_{\Phi|\chi_n}(\,|\mathcal{L}(\Phi)|^{-1} \sum_{L\in\mathcal{L}(\Phi)} e^{-\beta|\mathcal{C}_L|}|\chi_n = x) \tag{3.7}$$

for $0 \leq x \leq n$, where we set $\sum_{L\in\mathcal{L}} = 0$ if $\mathcal{L} = \emptyset$ and, as in (3.6), mean that the function $a_x(\mathcal{L})$ gets interpolated between possible values x of χ_n. Shortly, we will turn our attention to estimating the quotients of the expectations

$$\mathbf{E}_0(\chi_n \exp\{-\beta a_{\chi_n} n^r/2\}) \quad = \quad \mathbf{E}_0(\chi_n\,|\mathcal{L}_{r,\delta}(\Phi)|^{-1} \sum_{L\in\mathcal{L}_{r,\delta}(\Phi)} e^{-\beta|\mathcal{C}_L|}),$$

$$\mathbf{E}_0(\exp\{-\beta a_{\chi_n} n^r/2\}) \quad = \quad \mathbf{E}_0(\,|\mathcal{L}_{r,\delta}(\Phi)|^{-1} \sum_{L\in\mathcal{L}_{r,\delta}(\Phi)} e^{-\beta|\mathcal{C}_L|}). \tag{3.8}$$

We next define the shape of Φ and the weakly self-avoiding cone process relative to Φ in a certain shape [9]. The former indicates which classes of cones contribute a significant portion of points to Φ, while the latter represents a process that suppresses self-intersections in the cone that contains the endpoint S_n of the walk.

Definition **1 (Shape of Φ)** *Let $\rho, \delta > 0$ be suitably small. We say that Φ has* shape r *or is* r-shaped *(or $\mathcal{L}_{r,\delta}$* contributes to J_n essentially*) if*

$$\sum_{L\in\mathcal{L}_{r,\delta}} |\mathcal{C}_L| \geq \frac{1}{2}\, J_n^{1-\rho} \tag{3.9}$$

with the convention that multiple shapes are allowed. In particular, when $r = 1/2$, then we as well say that Φ has circular shape *or is* circular.

The shape of Φ is well-defined, that is, there exists $0 \leq r \leq 1-\delta$ which satisfies the inequality in (3.9) (see [9]). We remark that, because there are at most $b_2 n$ self-intersections and any cone $\mathcal{C}_L$ with $L \in \mathcal{L}_{r,\delta}$ carries at least $a_1 n^r/2$ of them, we gather that $|\mathcal{L}_{r,\delta}| \leq (2b_2/a_1)\, n^{1-r}$ and, in the reverse direction, a lower bound when Φ has shape r, thanks to (2.5),(3.3), and (3.9). Together

$$\frac{(b_1)^{1-\rho}}{a_2}\, n^{1-r-\rho-\delta} \quad \leq \quad |\mathcal{L}_{r,\delta}| \leq \frac{2b_2}{a_1}\, n^{1-r}. \tag{3.10}$$

Since we choose a_1 and a_2 such that βa_1 and βa_2 are independent of β and since βb_2 is so, too (see Corollary 1), it follows that b_2/a_1 and the upper bound on $|\mathcal{L}_{r,\delta}|$ are independent of β.

DEFINITION **2 (Weakly Self-Avoiding Cone Process relative to r-Shaped Φ)** *Let $\delta > 0$. Define the* weakly self-avoiding cone process (WSACP) relative to Φ in shape r *by the d-dimensional process induced by the conditional probability measure*

$$\mathbf{Q}_n^{\beta,\mathcal{V},r,s} = \kappa_{n,s} \frac{\exp\{-\beta|\mathcal{C}_{L'}|\}}{\mathbf{E}_0(\,|\mathcal{L}_{s,\delta}(\Phi)|^{-1}\sum_{L\in\mathcal{L}_{s,\delta}(\Phi)} e^{-\beta|\mathcal{C}_L|})} \tag{3.11}$$

on the set of SRW-paths of length n, given that the shape of Φ is r and the line L' closest to the endpoint S_n lies in $\mathcal{L}_{s,\delta}$, where $\mathcal{L}_{s,\delta}$ and the denominator in (3.11) *are spelled out in* (3.3) *and* (3.8), *respectively. Let $\mathbf{E}_{\beta,\mathcal{V},*(r)}$ denote expectation of the unconditional process (not conditioned on L') relative to Φ in shape r. The normalizing constants $\kappa_{n,s}$ are chosen such that $\mathbf{E}_{\beta,\mathcal{V},*(r)}(1) = 1$, that is, the unconditional measure is a probability measure.*

In many cases, an appropriate choice of the parameters $\kappa_{n,s}$ relates the WSACP to another process that suppresses self-intersections on subsets of $\mathbf{Z}^d$. The more is known about the shape of the latter process, the more effective the relationship to the WSACP will be. We remark that we will rely on Definition 2 with $\mathcal{L}_{s,\delta}$ replaced by other subsets of $\mathcal{V}$. In the subsequent discussion, we shall frequently make use of the short notation $\mathcal{L}$ for $\mathcal{L}(\Phi)$ and $\sum_{\mathcal{L}}$ for $\sum_{L\in\mathcal{L}(\Phi)}$. Let $\mathbf{P}_{\Phi,r}$ denote the conditional SRW-measure, given that $J_n \in [b_1 n, b_2 n]$ and the shape of Φ equals r. If Φ has shape r, keeping in mind that L' denotes the line in $\mathcal{V}$ closest to S_n, then adding over all lines in $\mathcal{V}$ and taking expectation yields

$$\mathbf{E}_{\beta,\mathcal{V},*(r)}(1|L' \in \mathcal{L}_{s,\delta}(\Phi)) \;=\; \kappa_{n,s}\cdot \frac{\mathbf{E}_0(|\mathcal{V}|^{-1}\sum_{\mathcal{L}_{s,\delta}} e^{-\beta|\mathcal{C}_L|})}{\mathbf{P}_{\Phi,r}(L'\in\mathcal{L}_{s,\delta}(\Phi))\;\mathbf{E}_0(|\mathcal{L}_{s,\delta}(\Phi)|^{-1}\sum_{\mathcal{L}_{s,\delta}} e^{-\beta|\mathcal{C}_L|})}\,.$$

Let $\mathcal{P}(\mathcal{V})$ denote any partition of $\mathcal{V}$. For later use, we therefore arrive at

$$\begin{aligned}
\mathbf{E}_{\beta,\mathcal{V},*(r)}(1) &= \sum_{\mathcal{L}_i\in\mathcal{P}(\mathcal{V})} \mathbf{P}_{\Phi,r}(L'\in\mathcal{L}_i(\Phi))\cdot \mathbf{E}_{\beta,\mathcal{V},*(r)}(1|L'\in\mathcal{L}_i(\Phi)) \\
&= \sum_{\mathcal{L}_i\in\mathcal{P}(\mathcal{V})} \mathbf{P}_{\Phi,r}(L'\in\mathcal{L}_i)\cdot\kappa_{n,i}\cdot \frac{\mathbf{E}_0(|\mathcal{V}|^{-1}\sum_{\mathcal{L}_i} e^{-\beta|\mathcal{C}_L|})}{\mathbf{P}_{\Phi,r}(L'\in\mathcal{L}_i)\;\mathbf{E}_0(|\mathcal{L}_i|^{-1}\sum_{\mathcal{L}_i} e^{-\beta|\mathcal{C}_L|})} \\
&= \sum_{\mathcal{L}_i\in\mathcal{P}(\mathcal{V})} \kappa_{n,i}\cdot \frac{\mathbf{E}_0(|\mathcal{V}|^{-1}\sum_{\mathcal{L}_i} e^{-\beta|\mathcal{C}_L|})}{\mathbf{E}_0(|\mathcal{L}_i|^{-1}\sum_{\mathcal{L}_i} e^{-\beta|\mathcal{C}_L|})}
\end{aligned} \tag{3.12}$$

for some normalizing constants $\kappa_{n,i} = \kappa_{n,i}(r)$ that depend on the shape r of Φ. Parallel lines to those in (3.12) lead to

$$\begin{aligned}
\mathbf{E}_{\beta,\mathcal{V},*(r)}(\chi_n) &= \sum_{\mathcal{L}_i\in\mathcal{P}(\mathcal{V})} \mathbf{P}_{\Phi,r}(L'\in\mathcal{L}_i)\cdot \mathbf{E}_{\beta,\mathcal{V},*(r)}(\chi_n|L'\in\mathcal{L}_i) \\
&= \sum_{\mathcal{L}_i\in\mathcal{P}(\mathcal{V})} \kappa_{n,i}\cdot \frac{\mathbf{E}_0(\chi_n\,|\mathcal{V}|^{-1}\sum_{\mathcal{L}_i} e^{-\beta|\mathcal{C}_L|})}{\mathbf{E}_0(|\mathcal{L}_i|^{-1}\sum_{\mathcal{L}_i} e^{-\beta|\mathcal{C}_L|})}.
\end{aligned} \tag{3.13}$$

We will engage formula (3.13) when $r = 1/d$ and $r = 1/2$, the extreme cases for the value of the shape of Φ in the case of a weakly SAW, to provide an upper bound and a lower bound on $\mathbf{E}_\beta(\chi_n)$, respectively. To this end, it will suffice to find an upper bound on $\mathbf{E}_{\beta,\mathcal{V},*(1/d)}(\chi_n)$ and a lower bound on $\mathbf{E}_{\beta,\mathcal{V},*(1/2)}(\chi_n)$.

4 Expected Distances

In the first part of this section, we turn our attention to bound $\mathbf{E}_{\beta,\mathcal{V},*(1/d)}(\chi_n)$ from above and to reason that $\mathbf{E}_\beta(\chi_n)$ has a comparable upper bound as $\mathbf{E}_{\beta,\mathcal{V},*(1/d)}(\chi_n)$ for large enough n, whereas, in the second part, we will investigate to bound both expectations from below.

Proposition **3 (Distance of the WSACP in Shape 1/d, Upper Bound)** *Let $\beta > 0$ and $d \geq 1$. There is a finite constant ω_2 (specified in* (4.15) *below), independent of n and β, such that for all sufficiently large n,*

$$\mathbf{E}_{\beta,\mathcal{V},*(1/d)}(\chi_n) \quad \leq \quad \omega_2\, n^{(d+1)/(2d)}\,. \tag{4.1}$$

Proof. Formula (3.13) reveals that the two issues consist in estimating the $\kappa_{n,\cdot}$ and the quotients. The second task will be deferred to the second half of this proof.

Fix suitably small $\delta/2 > \rho > 0$. Assume that Φ has shape $1/d$. Choose the partition $\mathcal{P}(\mathcal{V}) = \mathcal{L}_{1/d\pm} \cup \mathcal{L}^*_- \cup \mathcal{L}^*_+$ (see (3.3)). Associate with those three sets the constants $\kappa_{n,1/d\pm}, \kappa^*_{n,-}$, and $\kappa^*_{n,+}$ (depending on the shape $1/d$), which we now estimate. Observe that $\mathcal{P}(\mathcal{V})$ is an approximate partition of $\mathcal{V}$ but the slight overlap in the interval ends will not have any effect on the ultimate estimates of $\mathbf{E}_{\beta,\mathcal{V},*(1/d)}(\chi_n)$ that we will come up with since we will drop all terms but one. Because the terms which have the smallest $|\mathcal{C}_L|$ dominate in the sum $\sum_{\mathcal{L}_+} e^{-\beta|\mathcal{C}_L|}$ and those emerge from the class of cones that potentially has the largest cardinality (consult (3.10)), in view of (3.3), (3.10) along with $v_1 n^{1-1/d} \leq |\mathcal{V}| \leq v_2 n^{1-1/d}$ for sufficiently large n from (3.1), we obtain $|\mathcal{L}^*_+| \leq (2b_2/a_1) n^{1-1/d-\delta}$ and

$$\kappa^*_{n,+} \cdot \frac{\mathbf{E}_0(|\mathcal{V}|^{-1}\sum_{\mathcal{L}^*_+} e^{-\beta|\mathcal{C}_L|})}{\mathbf{E}_0(|\mathcal{L}^*_+|^{-1}\sum_{\mathcal{L}^*_+} e^{-\beta|\mathcal{C}_L|})} \leq \kappa^*_{n,+}\,(\frac{2b_2}{a_1 v_1})\,n^{-\delta}. \tag{4.2}$$

In addition, since $\mathcal{L}^*_-$ contains at most as many lines as $\mathcal{V}$, another calculation brings about

$$\kappa^*_{n,-} \cdot \frac{\mathbf{E}_0(|\mathcal{V}|^{-1}\sum_{\mathcal{L}^*_-} e^{-\beta|\mathcal{C}_L|})}{\mathbf{E}_0(|\mathcal{L}^*_-|^{-1}\sum_{\mathcal{L}^*_-} e^{-\beta|\mathcal{C}_L|})} \leq \kappa^*_{n,-}. \tag{4.3}$$

Combining (3.3), (3.10) and considerations analogous to those that provided us with (3.10) leads to

$$((b_1)^{1-\rho}/a_2)\,n^{1-1/d-\rho-\delta} \leq |\mathcal{L}_{1/d\pm}| \leq \min(2b_2/a_1, v_2)\,n^{1-1/d+\delta}. \tag{4.4}$$

Arranging (4.4) with the bounds on $|\mathcal{V}|$ in (3.1) and $|\mathcal{L}_{1/d\pm}|/|\mathcal{V}| \le 1$ gives

$$\frac{(b_1)^{1-\rho}}{a_2 v_2}\, n^{-\rho-\delta} \le \frac{\mathbf{E}_0(|\mathcal{V}|^{-1}\sum_{\mathcal{L}_{1/d\pm}} e^{-\beta|\mathcal{C}_L|})}{\mathbf{E}_0(|\mathcal{L}_{1/d\pm}|^{-1}\sum_{\mathcal{L}_{1/d\pm}} e^{-\beta|\mathcal{C}_L|})} \le \min(\frac{2b_2}{a_1 v_1}, 1) \tag{4.5}$$

for sufficiently large n. Therefore, we may choose $0 < \kappa^*_{n,+} \le 1$ and $\kappa^*_{n,-}$ as a function that decreases to zero with n arbitrarily slowly. In that event, we conclude from (3.12) in company with (4.2), (4.3), and (4.5) that for all sufficiently large n,

$$\max(\frac{a_1 v_1}{2b_2}, 1) \le \kappa_{n,1/d\pm} \le \frac{a_2 v_2}{(b_1)^{1-\rho}}\, n^{\rho+\delta}. \tag{4.6}$$

The bounds on $\kappa_{n,1/d\pm}$ in (4.6) along with $\kappa_{n,1/d\pm} \to \kappa_{n,1/d}$ as $\delta \to 0$ imply that

$$\max(\frac{a_1 v_1}{2b_2}, 1) \le \kappa_{n,1/d} \le \frac{a_2 v_2}{(b_1)^{1-\rho}}\, n^{\rho}. \tag{4.7}$$

We now borrow a few results from [9]. Let $\mathbf{P}_{\chi_n}$ denote the probability distribution of the distance χ_n of the SRW (see Section 3, [9], for a discussion on $\mathbf{P}_{\chi_n}$). Recall $a_x = a_x(\mathcal{L}_{1/2+r})$ from (3.7). For each $r \in [-1/2, 1/2]$, define

$$\begin{aligned} q_r(x) &= \exp\{-\beta a_x n^{1/2+r}/2\} \\ g_r(n) &= \int_0^n (a_x(\mathcal{L}_{1/2+r}))^{1/2} q_r(x)\, d\mathbf{P}_{\chi_n}(x). \end{aligned} \tag{4.8}$$

In the proof of Proposition 4 [9], for each $r \in [-1/2, 1/2]$, as $n \to \infty$, the identities

$$\begin{aligned} \mathbf{E}_0(\chi_n\, |\mathcal{L}_{1/2+r}|^{-1} \sum_{\mathcal{L}_{1/2+r}} e^{-\beta|\mathcal{C}_L|}) &= \beta^{1/2}\, K_r(n)\, n^{3/4+r/2} g_r(n)(1+o(1)) \\ \mathbf{E}_0(|\mathcal{L}_{1/2+r}|^{-1} \sum_{\mathcal{L}_{1/2+r}} e^{-\beta|\mathcal{C}_L|}) &= \int_0^n q_r(x)\, d\mathbf{P}_{\chi_n}(x) = h_r(n) \end{aligned} \tag{4.9}$$

were derived, where the constants $K_r(n) \le M < \infty$, uniformly in r. Here, M is independent of $\beta > 0$ (visit Lemma 1, [9]). Clearly, for each $r \in [-1/2, 1/2]$, we have $g_r(n)/h_r(n) \le a_2$, where a_2 is as specified in the definitions in (3.3). We set $1/2 + r = 1/d$, equivalently, $r = 1/d - 1/2$ and apply the estimates in (4.9). In light of (3.1) and (3.10), as $n \to \infty$,

$$\frac{\mathbf{E}_0(\chi_n\, |\mathcal{V}|^{-1}\sum_{\mathcal{L}_{1/d}} e^{-\beta|\mathcal{C}_L|})}{\mathbf{E}_0(|\mathcal{L}_{1/d}|^{-1}\sum_{\mathcal{L}_{1/d}} e^{-\beta|\mathcal{C}_L|})} \le (1+o(1)) M (a_2\beta)^{1/2}\, n^{1/2+1/(2d)} \min(\frac{2b_2}{a_1 v_1}, 1). \tag{4.10}$$

The very same arguments as presented in the proof of Proposition 4 in [9] with the updated cardinality of $\mathcal{V}$ imply that, as $n \to \infty$,

$$\frac{\mathbf{E}_0(\chi_n\, |\mathcal{V}|^{-1}\sum_{\mathcal{L}^*_-} e^{-\beta|\mathcal{C}_L|})}{\mathbf{E}_0(|\mathcal{L}^*_-|^{-1}\sum_{\mathcal{L}^*_-} e^{-\beta|\mathcal{C}_L|})} = o(n^{(d+1)/(2d)}), \tag{4.11}$$

$$\frac{\mathbf{E}_0(\chi_n\, |\mathcal{V}|^{-1}\sum_{\mathcal{L}^*_+} e^{-\beta|\mathcal{C}_L|})}{\mathbf{E}_0(|\mathcal{L}^*_+|^{-1}\sum_{\mathcal{L}^*_+} e^{-\beta|\mathcal{C}_L|})} = o(n^{(d+1)/(2d)}), \tag{4.12}$$

where the power of n in the expression on the lefthand side of (4.11) and (4.12), respectively, is strictly less than $n^{(d+1)/(2d)}$. We do not reproduce the arguments here.

As a consequence of (3.13), (4.10), (4.11), and (4.12) together with our choice of the parameters $\kappa_{n,1/d\pm}, \kappa^*_{n,-}$, and $\kappa^*_{n,+}$, we obtain for all sufficiently large n,

$$\mathbf{E}_{\beta,\mathcal{V},*(1/d)}(\chi_n) \quad \leq \quad (1+o(1))\,\kappa_{n,1/d\pm}\, M\,(a_2\beta)^{1/2}\,\min(\frac{2b_2}{a_1v_1},1)\,n^{(d+1)/(2d)}. \qquad (4.13)$$

Hence, since $\delta, \rho > 0$ were arbitrary, in view of (4.7), for all sufficiently large n,

$$\mathbf{E}_{\beta,\mathcal{V},*(1/d)}(\chi_n) \quad \leq \quad \omega_2(\beta)\,n^{(d+1)/(2d)}, \qquad (4.14)$$

where we put

$$\omega_2(\beta) \quad = \quad 2M\,(a_2\beta)^{1/2}(a_2v_2/b_1)\min(2b_2/(a_1v_1),1) < \infty. \qquad (4.15)$$

Finally, it remains to be seen that $\omega_2(\beta)$ is independent of $\beta > 0$. As we remarked on the foot of definition (3.3), $a_1\beta$ and $a_2\beta$ do not depend on $\beta > 0$, and it follows from Corollary 1 that βb_2 does not either. Also, recall that M does not. Hence, if we choose v_1 independent of β and v_2 such that v_2a_2/b_1 is a positive finite number independent of β, then it is straightforward that $\omega_2(\beta)$ is independent of β. This completes our proof. □

Proposition **4** **(Distance of the Weakly SAW, Upper Bound)** *Let $\beta > 0$ and $d \geq 1$. There is a finite constant $\rho_2(d)$ (independent of β) such that for all sufficiently large n,*

$$\mathbf{E}_{\beta}(\chi_n) \quad \leq \quad \rho_2(d)\,n^{(d+1)/(2d)}. \qquad (4.16)$$

Proof. The idea of proof, written in the proof of Proposition 6, [9], in the special case $d = 2$, rests on the following. If the weakly SAW were to visit the cones associated with lines in $\mathcal{L}_{r,\delta}$ for some $r > t$ with a probability that is bounded away from zero for some arbitrarily large n, then, for some suitably small $\varepsilon > 0$, the weakly SAW would visit cones associated with lines in $\mathcal{L}_{s,\delta}$ for $s \leq 1/d$ at distance from the origin at least $n^{1/2+t/2+\varepsilon/2}$ with a probability that is bounded away from zero for some arbitrarily large n. Yet the estimate of the latter probability brings along a contradiction as long as $t \geq 1/d$ (see [9] for the details). This construction is possible for all r with $|\mathcal{L}_{r,\delta}| = o(|\mathcal{V}|)$ as $n \to \infty$, which guarantees the existence of lines of $\mathcal{V}$ in $\mathcal{L}_s$ for some $s \leq 1/d$. Hence, the cutoff value of t is $1/d$. Replacing the $1/2$ by $t \geq 1/d$ and mimicking the proof of Proposition 6 of [9] line by line in combination with Proposition 3 accomplishes the assertion in (4.16). □

Proposition **5** **(Distance of the WSACP in Shape 1/2, Lower Bound)** *Let $\beta > 0$ and $d \geq 2$. There is a positive constant $\omega_1(\beta)$ (as specified in (4.23) below), uniform in β as $\beta \to \infty$, such that for all sufficiently large n,*

$$\mathbf{E}_{\beta,\mathcal{V},*(1/2)}(\chi_n) \quad \geq \quad \omega_1(\beta)\,\max(n^{1/4+1/d}, n^{1/2}). \qquad (4.17)$$

Proof. For the proof in the case $d = 2$, the reader is referred to Proposition 4 of [9]. Henceforth, we now assume that $d \geq 3$.

Again, fix suitably small $\delta, \rho > 0$. Assume that Φ has shape 1/2. Choose the partition $\mathcal{P}(\mathcal{V}) = \mathcal{L}_+ \cup \mathcal{L}_{1/2\pm} \cup (\mathcal{L}_- \cap \mathcal{L}_+^*) \cup \mathcal{L}_{1/d\pm} \cup (\mathcal{L}_-^* \setminus \mathcal{L}_\emptyset) \cup \mathcal{L}_\emptyset$ (see (3.3) for the definitions). With each of these sets, we associate the corresponding constant $\kappa_{n,\cdot}$. We will bound the sum in (3.13) from below by the sum over the terms indexed by $\mathcal{L}_-$ and $\mathcal{L}_{1/2\pm}$, and ultimately, by $\mathcal{L}_\emptyset$ and $\mathcal{L}_{1/2\pm}$.

Reviewing the inequalities in the proof of Proposition 3 and working along the same lines, we choose $\kappa_{n,1/d\pm}$ as in (4.6), $\tilde{\kappa} \leq \kappa_{n,-}, \kappa_{n,1/2\pm} \leq 1$ uniformly in n for some $\tilde{\kappa} > 0$, chosen independently of β, and the remaining relevant $\kappa_{n,\cdot}$ accordingly such that $\mathbf{E}_{\beta,\mathcal{V},*(1/2)}(1) = 1$ (see (3.12)). By virtue of (3.13),

$$\begin{aligned}\mathbf{E}_{\beta,\mathcal{V},*(1/2)}(\chi_n) \quad &\geq \quad \kappa_{n,1/2\pm} \cdot \frac{\mathbf{E}_0(\chi_n\,|\mathcal{V}|^{-1}\sum_{\mathcal{L}_{1/2\pm}} e^{-\beta|\mathcal{C}_L|})}{\mathbf{E}_0(|\mathcal{L}_{1/2\pm}|^{-1}\sum_{\mathcal{L}_{1/2\pm}} e^{-\beta|\mathcal{C}_L|})} \\ &\quad + \kappa_{n,-} \cdot \frac{\mathbf{E}_0(\chi_n\,|\mathcal{V}|^{-1}\sum_{\mathcal{L}_-} e^{-\beta|\mathcal{C}_L|})}{\mathbf{E}_0(|\mathcal{L}_-|^{-1}\sum_{\mathcal{L}_-} e^{-\beta|\mathcal{C}_L|})}\,. \end{aligned} \tag{4.18}$$

Now, since $|\mathcal{L}_{1/2\pm} \cup \mathcal{L}_+| \leq \min(2b_2/a_1, v_2)\, n^{1/2+\delta}$ thanks to (3.3) and (3.10) and since $d \geq 3$, we have $|\mathcal{L}_-| = |\mathcal{V} \setminus (\mathcal{L}_{1/2\pm} \cup \mathcal{L}_+)| = |\mathcal{V}| - o(|\mathcal{V}|)$ as $n \to \infty$. Therefore, $|\mathcal{L}_-|/|\mathcal{V}| = 1 - o(1)$ as $n \to \infty$. Combining this with the estimates in (4.9), we collect

$$\frac{\mathbf{E}_0(\chi_n\,|\mathcal{V}|^{-1}\sum_{\mathcal{L}_-} e^{-\beta|\mathcal{C}_L|})}{\mathbf{E}_0(|\mathcal{L}_-|^{-1}\sum_{\mathcal{L}_-} e^{-\beta|\mathcal{C}_L|})} \quad \geq \quad (1-o(1))\,\frac{\mathbf{E}_0(\chi_n\,|\mathcal{L}_\emptyset|^{-1}\sum_{\mathcal{L}_\emptyset} e^{-\beta|\mathcal{C}_L|})}{\mathbf{E}_0(|\mathcal{L}_\emptyset|^{-1}\sum_{\mathcal{L}_\emptyset} e^{-\beta|\mathcal{C}_L|})} \tag{4.19}$$

for all sufficiently large n. A routine calculation provides

$$\frac{\mathbf{E}_0(\chi_n\,|\mathcal{L}_\emptyset|^{-1}\sum_{\mathcal{L}_\emptyset} e^{-\beta|\mathcal{C}_L|})}{\mathbf{E}_0(|\mathcal{L}_\emptyset|^{-1}\sum_{\mathcal{L}_\emptyset} e^{-\beta|\mathcal{C}_L|})} \geq \gamma_0\, n^{1/2} \tag{4.20}$$

for some positive constant γ_0 which does not depend on β. Therefore, we took care of the second term in (4.18) and achieve the lower bound, for all sufficiently large n,

$$\frac{\mathbf{E}_0(\chi_n\,|\mathcal{V}|^{-1}\sum_{\mathcal{L}_-} e^{-\beta|\mathcal{C}_L|})}{\mathbf{E}_0(|\mathcal{L}_-|^{-1}\sum_{\mathcal{L}_-} e^{-\beta|\mathcal{C}_L|})} \geq (1-o(1))\,\gamma_0\, n^{1/2}. \tag{4.21}$$

Next, our assumption that Φ has shape 1/2 guarantees the assertion of Lemma 2 in [9]. The first term on the righthand side of (4.18) is bounded below in the proof of Proposition 4 of [9] (consult (3.45), (3.50), and (3.52), [9]). Specifically, together with (3.1), (3.3), and (3.10), we get as $n \to \infty$,

$$\frac{\mathbf{E}_0(\chi_n\,|\mathcal{V}|^{-1}\sum_{\mathcal{L}_{1/2\pm}} e^{-\beta|\mathcal{C}_L|})}{\mathbf{E}_0(|\mathcal{L}_{1/2\pm}|^{-1}\sum_{\mathcal{L}_{1/2\pm}} e^{-\beta|\mathcal{C}_L|})} \quad \geq \quad n^{1/d+1/4-\rho-3\delta/2}\,(1+o(1))\,\gamma_*\,(a_1\beta)^{1/2}\,\frac{(b_1)^{1-\rho}}{a_2 v_2}\,,$$

where $\gamma_* > 0$ is a constant independent of β as $\beta \to \infty$ (see Proposition 3, [9]). Finally, putting each of the pieces (4.18) and (4.21) together with our last display and in company with $\tilde{\kappa} \leq \kappa_{n,-}, \kappa_{n,1/2\pm}$, we reach

$$\mathbf{E}_{\beta,\mathcal{V},*(1/2)}(\chi_n) \quad \geq \quad (\tilde{\kappa}/2)(n^{1/4+1/d-\rho-3\delta/2}\,\gamma_*\,(a_1\beta)^{1/2}\,\frac{(b_1)^{1-\rho}}{a_2 v_2} + n^{1/2}\,\gamma_0) \tag{4.22}$$

for all sufficiently large n. Since $\delta, \rho > 0$ were arbitrary, if we write

$$\omega_1(\beta) \quad = \quad (\tilde{\kappa}/2)\,\min(\gamma_0, \gamma_*\,(a_1\beta)^{1/2}\,(b_1/(a_2 v_2))) > 0, \tag{4.23}$$

then we summarize, as announced in (4.17), for all large enough n,

$$\mathbf{E}_{\beta,\mathcal{V},*(1/2)}(\chi_n) \quad \geq \quad \omega_1(\beta)\max(n^{1/4+1/d}, n^{1/2}).$$

To finish up, it remains to be seen that $\omega_1(\beta)$ above is independent of $\beta > 0$ as $\beta \to \infty$. Again, since $a_1\beta$ and γ_0 do not depend on $\beta > 0$, γ_* is independent of β as $\beta \to \infty$, as we mentioned earlier, and we chose $\tilde{\kappa}$ independent of β and v_2 such that $v_2 a_2/b_1$ is independent of β (Proposition 3), it indeed is immediate that $\omega_1(\beta)$ does not depend on β as $\beta \to \infty$. This finishes our proof. □

Proposition **6** **(Distance of the Weakly SAW, Lower Bound)** *Let $\beta > 0$ and $d > 1$. There is a constant $0 < \rho_1(d,\beta)$ (uniform in β as $\beta \to \infty$) such that for all large enough n,*

$$\rho_1(d,\beta)\,n^{1/4+1/d} \quad \leq \quad \mathbf{E}_\beta(\chi_n). \tag{4.24}$$

Proof. If the lines in $\mathcal{L}_{1/2\pm}$ are spaced out in $\mathbf{R}^d$ such that there are of order $|\mathcal{V}|$ lines in $\mathcal{L}_s$ for $s < 1/2$ along with their associated cones between any two lines in $\mathcal{L}_{1/2\pm}$, then the weakly SAW would have no larger probability to have S_n in $\mathcal{L}_{1/2\pm}$ than in any other case when the lines in $\mathcal{L}_{1/2\pm}$ are not maximally spaced out, in view of Corollary 1. In fact, in the latter case, $\mathcal{V}$ may be chosen smaller such that its size is of strictly smaller order of magnitude (in n) than $n^{1-1/d}$. But this does not decrease $\mathbf{E}_{\beta,\mathcal{V},*(1/2)}(\chi_n)$ (see (3.12)) and $\kappa_{n,1/2\pm}$. Hence, in the former case, and thus, in the latter case as well, the weakly SAW has $\kappa_{n,1/2\pm}$ bounded away from zero as $n \to \infty$, as does the WSACP in the proof of Proposition 5. Consequently, by virtue of Proposition 5, there is a constant $0 < \rho_1(d,\beta)$ that is proportional to $\omega_1(\beta)$, defined in Proposition 5, such that $\mathbf{E}_\beta(\chi_n) \geq \rho_1(d,\beta)\,n^{1/4+1/d}$ for all sufficiently large n. This completes our proof. □

Propositions 4 and 6 establish the bounds on the distance exponent of the weakly SAW for $d \geq 2$ that we advertized in Theorem 2.

5 Distance Exponent of the Self-Avoiding Walk

A number of arguments towards uniform bounds in β as $\beta \to \infty$ demonstrates that the bounds on the distance exponent of the weakly SAW extend to the SAW.

Corollary **2** *Let $\beta > 0$ and $d \geq 2$. There are some constants $0 < \rho_1 = \rho_1(d, \beta) \leq \rho_2 = \rho_2(d) < \infty$ (ρ_2: independent of β and ρ_1: independent of β as $\beta \to \infty$) such that*

$$\rho_1(d, \beta) \leq \liminf_{n \to \infty} n^{-(1/4+1/d)} \, \mathbf{E}_\beta(\chi_n) \leq \limsup_{n \to \infty} n^{-(d+1)/(2d)} \, \mathbf{E}_\beta(\chi_n) \leq \rho_2(d).$$

Furthermore, the distance exponent of the self-avoiding walk lies in $[1/4 + 1/d, (d+1)/(2d)]$ for $d \geq 2$.

Proof. The proof is identical to the one of Corollary 2 of [9] with $n^{3/4}$ replaced by $n^{1/4+1/d}$ and $n^{(d+1)/(2d)}$ in connection with the lower bound and upper bound, respectively, relying on Propositions 4 and 6 above. □

Aside from when $d = 1$, which we postpone to Theorem 3, this completes the proof of Theorem 2. The next result accomplishes Theorem 1, except for the case $d = 1$, which will be in Theorem 3, too.

Corollary **3** *Let $\beta > 0$ and $d \geq 2$. There are some constants $0 < \rho_3 = \rho_3(d, \beta) \leq \rho_4 = \rho_4(d) < \infty$ (ρ_4: independent of β and ρ_3: independent of β as $\beta \to \infty$) such that*

$$\rho_3(d, \beta) \leq \liminf_{n \to \infty} n^{-(1/2+2/d)} \, \mathbf{E}_\beta(\chi_n^2) \leq \limsup_{n \to \infty} n^{-(d+1)/(d)} \, \mathbf{E}_\beta(\chi_n^2) \leq \rho_4(d).$$

Moreover, $1/4 + 1/d \leq \underline{\nu}_\infty(d), \overline{\nu}_\infty(d), \underline{\nu}_\beta(d), \overline{\nu}_\beta(d) \leq (d+1)/(2d)$.

Proof. We shall argue in the setting of the weakly SAW and point out that parallel lines to the ones invoked before allow to extend the results about the MSD exponent to the SAW. First, the lower bound follows from the inequality $\mathbf{E}_\beta \chi_n^2 \geq (\mathbf{E}_\beta \chi_n)^2$ and Corollary 2.

Consequently, it suffices to verify the upper bound on $\mathbf{E}_\beta \chi_n^2$. For any suitably small $\epsilon > 0$, we set $M_\epsilon = (\rho_2(d) \, n^{(d+1)/(2d)})^{1+\epsilon}$ and follow the lines in the proof of Corollary 3 of [9] to end up with

$$\mathbf{E}_\beta(\chi_n^2) \quad \leq \quad (1 + o(1)) \, (M_\epsilon^2 + 1)$$

as $n \to \infty$. Since $\epsilon > 0$ is arbitrary, we let $\rho_4(d) = \rho_2(d)^2$, which ends the proof. □

We mention the following corollary of Theorem 2 of this paper and Theorem 3 of [9].

Corollary **4** *Let $\beta > 0, d \geq 2$, and let P_n denote the perimeter of the convex hull of the walk $S_0, S_1, \ldots, S_n$. Then P_n satisfies all statements in Corollaries 2 and 3 with χ_n replaced by P_n and different constants ρ_2 and ρ_4.*

6 One-Dimensional Weakly SAW

This paragraph handles the case $d = 1$. We note in passing that the one-dimensional SAW is not interesting. In light of Propositions 3 and 4 or the observation that $\mathbf{E}_\beta(\chi_n) \leq n$, it is enough to show a lower bound on $\mathbf{E}_\beta(\chi_n)$.

Proposition **7 (Distance of the WSACP in Shape 1**) *Let $\beta > 0$ and $d = 1$. There is a positive constant $\omega_3(\beta)$ (specified in (6.7) below), which depends on β and may tend to zero as $\beta \to \infty$, such that for all sufficiently large n,*

$$\mathbf{E}_{\beta,\mathcal{V},*(1)}(\chi_n) \quad \geq \quad \omega_3(\beta)\, n\,.$$

Proof. We let $\mathcal{V}$ consist of the two half-lines that emanate from the origin. Thus, we have $|\mathcal{V}| = 2$. There is at least one line L in $\mathcal{V}$ with $|\mathcal{C}_L| \geq J_n/2$. Therefore, Φ has shape 1. Define $\mathcal{L}_1 = \mathcal{L}_1(\Phi) = \{L \in \mathcal{V} : 2|\mathcal{C}_L| \in [b_1 n, 2b_2 n]\}$. Thus, we set $a_1 = b_1$ and $a_2 = 2b_2$ (compare with (3.3)). From (3.12), we collect

$$\mathbf{E}_{\beta,\mathcal{V},*(1)}(1) \quad \geq \quad \kappa_{n,1} \cdot \frac{\mathbf{E}_0(|\mathcal{V}|^{-1} \sum_{\mathcal{L}_1} e^{-\beta|\mathcal{C}_L|})}{\mathbf{E}_0(|\mathcal{L}_1|^{-1} \sum_{\mathcal{L}_1} e^{-\beta|\mathcal{C}_L|})} \tag{6.1}$$

for a normalizing constant $\kappa_{n,1}$. We conclude from (3.12) and $1/2 \leq |\mathcal{L}_1|/|\mathcal{V}| \leq 1$ (because $\mathcal{L}_1$ contains at least one of the two lines in $\mathcal{V}$) that a possible choice is $1/2 \leq \kappa_{n,1} \leq 2$. In addition, from (3.13),

$$\mathbf{E}_{\beta,\mathcal{V},*(1)}(\chi_n) \quad \geq \quad \kappa_{n,1} \cdot \frac{\mathbf{E}_0(\chi_n\, |\mathcal{V}|^{-1} \sum_{\mathcal{L}_1} e^{-\beta|\mathcal{C}_L|})}{\mathbf{E}_0(|\mathcal{L}_1|^{-1} \sum_{\mathcal{L}_1} e^{-\beta|\mathcal{C}_L|})}. \tag{6.2}$$

Recall the numbers $a_x(\mathcal{L}_1)$ from (3.7) with $r = 1$. They take values in $[a_1, a_2]$. Define

$$\begin{aligned} \mu_x(1) \quad &= \quad (\beta a_x)^{1/2}\, n \\ q_1(x) \quad &= \quad \exp\{-\beta \frac{a_x}{2} n\} \end{aligned} \tag{6.3}$$

for every $n \geq 0$, $\beta > 0$, and x in $[0, n]$. Similarly as in (3.4), [9], since the a_x are bounded in x, for suitably small $\varepsilon \geq 0$ and for $\gamma > 0$, we may define

$$\begin{aligned} \hat{r}_1 \quad &= \quad \hat{r}_1(\varepsilon, \gamma) = \sup\{x \in [0, n] : x \leq \gamma \mu_x(1) n^{-\varepsilon}\} \\ \hat{r}_2 \quad &= \quad \hat{r}_2(\gamma) = \hat{r}_1(0, \gamma). \end{aligned} \tag{6.4}$$

Since Φ has shape 1, the $a_x(\mathcal{L}_1)$ satisfy a hypothesis analogous to Condition D in (3.5), [9], specifically, the $a_x(\mathcal{L}_1)$ obey the following condition for $\varepsilon = 0$ and $\gamma > 0$:

Condition $\tilde{\mathbf{D}}$. For any $\beta > 0$ and any suitably small $\varepsilon \geq 0$, there exist some $\gamma > 0$ and $\omega_* > 0$ such that

$$\int_{\hat{r}_1}^{n} x\, q_1(x)\, d\mathbf{P}_{\chi_n}(x) = \omega_n \int_0^{\hat{r}_1} x\, q_1(x)\, d\mathbf{P}_{\chi_n}(x) \tag{6.5}$$

with $\omega_n \geq \omega_*$ for all sufficiently large n.

This can be seen as the analogous assertion in Lemma 2, [9], proven in Appendix B in [9], working along exactly the same arguments, where $\mathcal{L}_{1/2}, r_1, r_2$, $\mu_x, q(x)$, and $\kappa_{n,1/2}$ are replaced by $\mathcal{L}_1, \hat{r}_1, \hat{r}_2$, $\mu_x(1), q_1(x)$, and $\kappa_{n,1}$, respectively. The idea is to take advantage of the observation that the set of paths that end at two sites which are at distance 2 apart and whose SILT exhibits the same order of magnitude in n have comparable sizes in n. Pulling the estimates in (4.9) with $r = 1/2$ together with Condition $\tilde{\mathrm{D}}$, accompanied by $|\mathcal{L}_1|/|\mathcal{V}| \geq 1/2$, yields

$$\frac{\mathbf{E}_0(\chi_n\,|\mathcal{V}|^{-1}\sum_{\mathcal{L}_1} e^{-\beta|\mathcal{C}_L|})}{\mathbf{E}_0(|\mathcal{L}_1|^{-1}\sum_{\mathcal{L}_1} e^{-\beta|\mathcal{C}_L|})} \quad \geq \quad (1+o(1))\,(\beta a_1)^{1/2}\,(K_{1/2}/2)\,n \tag{6.6}$$

as $n \to \infty$, where $K_{1/2} \geq 2\gamma_1$ for some positive real number γ_1 which is independent of β as $\beta \to \infty$. As a consequence of (6.2), (6.6), and our choice $1/2 \leq \kappa_{n,1}$, we arrive at

$$\mathbf{E}_{\beta,\mathcal{V},*(1)}(\chi_n) \quad \geq \quad \omega_3(\beta)\,n \tag{6.7}$$

for all sufficiently large n, where we write $\omega_3(\beta) = (\beta a_1)^{1/2}\,\gamma_1/4 > 0$, which depends on β and may tend to zero as $\beta \to \infty$ because $\beta a_1 = \beta b_1$ does. This achieves our proof. □

Theorem **3 (Distance of the Weakly SAW in Z)** *Let $\beta > 0$ and $d = 1$. There is a constant $0 < \rho_5(\beta)$ such that for all large enough n,*

$$\rho_5(\beta)\,n \quad \leq \quad \mathbf{E}_\beta(\chi_n). \tag{6.8}$$

Additionally, $\rho_5(\beta) \leq \liminf_{n\to\infty} n^{-1}\,\mathbf{E}_\beta(\chi_n)$ and $\rho_5(\beta)^2 \leq \liminf_{n\to\infty} n^{-2}\,\mathbf{E}_\beta(\chi_n^2)$. Moreover, $\nu_\beta(1) = 1$.

Proof. With probability at least 1/2, the weakly SAW has S_n in $\mathcal{L}_1$. Hence, $\kappa_{n,1}$ is bounded away from zero for all large enough n for the weakly SAW. Consequently, in view of Proposition 7, there is a constant $\rho_5(\beta) > 0$ that is proportional to $\omega_3(\beta)$ such that $\mathbf{E}_\beta(\chi_n) \geq \rho_5(\beta)\,n$, for all sufficiently large n, as advertized in (6.8).

The claim on $\liminf_{n\to\infty} n^{-1}\,\mathbf{E}_\beta(\chi_n)$ now is obvious. Also, the claim pertaining to $\liminf_{n\to\infty} n^{-2}\,\mathbf{E}_\beta(\chi_n^2)$ follows from $\mathbf{E}_\beta(\chi_n^2) \geq (\mathbf{E}_\beta(\chi_n))^2$. It is apparent that $\nu_\beta(1) = 1$. This completes our proof. □

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