The Convex Hull of a Self–Similar Measure

Irene Hueter
University of Florida, Gainesville

January 31, 2003

Abstract

Let $X_1, X_2, \ldots$ be i.i.d. random points in $\mathbb{R}^2$ with distribution ν, and let N_n denote the number of points spanning the convex hull of $X_1, X_2, \ldots, X_n$. We obtain $\gamma_1\, n^{1/3} \leq \mathbf{E}(N_n) \leq \gamma_2\, n^{1/3}\,(\log n)^{2/3}$ for some positive constants γ_1, γ_2 and sufficiently large n under the assumption that ν is a certain self-similar measure on the unit disk. Our main tool consists in a geometric application of the renewal theorem. Exactly the same approach can be adopted to prove the analogous result in $\mathbb{R}^d$.

1 Introduction

Let $X_1, X_2, \ldots$ be i.i.d. points sampled from some probability distribution, and let N_n denote the number of vertices of the convex hull of $X_1, \ldots, X_n$. The asymptotic behaviour of N_n as well as other functionals of the convex hull in two or higher dimensions has been well-studied and has attracted numerous authors' attention in the case when the points are drawn from a distribution that is absolutely continuous with respect to Lebesgue measure. However, the problem has not been addressed to the author's knowledge if the probability distribution is allowed to be supported on a *fractal.* Without surveying the results and their various interesting applications in the absolutely continuous case (see [5] for reference to the literature), we prefer to motivate and discuss the singular case at some length.

Convex hulls arise in connection with multivariate extreme value theory. Clearly, the convex hull vertices represent some curious class of extreme points of a cloud of points. Since by now it is a fact that, in physics, biology, medicine or other sciences, many objects under study are categorized as being of fractal nature, it is natural to examine the convex hull of points sampled from a fractal set. If we collect n points from DLA, from a path of

[1]Mathematics Subject Classification: 28A80, 52A07, 60D05.

[2]Key words and phrases: Convex hull, renewal theorem, self-similar.

some diffusion or planar Brownian motion, say, in some uniform way, how many extreme points will we see in average as a function of n ? If we can observe some species of animals or plants only along pieces of the boundary of a region, or in fact, the animals' range is restricted to some subregions because of existing thickets, swamps or poisonous ground, how do we estimate the total number of animals living in this region ? Should the thickets and swamps be accounted for, too, so as to estimate accurately ?

Our goal is to present a simple example of a fractal set on a disk, that shares some of the properties of the above-described situations, and to give bounds on the growth rate of the expectation $\mathbf{E}(N_n)$ as a function of n for large n. To find the approximate rate of increase of $\mathbf{E}(N_n)$ is all we can hope for when working with an underlying iterated function system as we will. Note that the approach taken does not readily generalize to answer the same questions for Brownian motion, any self–similar measure, or any self–affine stochastic process. This requires further study. It is worthwhile mentioning that our construction allows many holes in the support, as such, may well mimic *non-convex* supports and might shed some light upon the necessity of the stringent condition of convexity. Convexity needs to be assumed for all results about N_n to hold in the absolutely continuous case.

In the continuous setting, closest to our construction in the singular case, is the uniform distribution on the unit disk studied by Rényi and Sulanke [9]

$$\mathbf{E}(N_n) \sim \Gamma(\frac{5}{3})(\frac{2}{3\pi})^{1/3}\, 2\pi\, n^{1/3},$$

where $a_n \sim b_n$ means $\lim_{n\to\infty} a_n/b_n \to 1$. Assuming absolute continuity when counting the convex hull vertices prevents three or more points from falling on any line in the plane with positive probability. The singular measure at hand will not pose a paradox since its construction will still guarantee this property for the points spanning the convex hull. The rate $n^{1/3}$ in the result above is maximal on a disk among all radially symmetric and unimodal distributions [2], and, typically smaller for other distributions. A measure spreading out the points less evenly often decreases the exponential rate. The exponent tends to be sensitive to the underlying geometry of the boundary of the support of the measure. Therefore, it is natural to expect the exponent to depend on the degree of distortion of the uniform distribution to the singular distribution. This phenomenon of different qualitative behaviour in the absolutely continuous versus singular case is being observed for related discrete geometric functionals, e.g. for the traveling salesman problem (see [7]). Surprisingly, it will turn out that the order of magnitude of $\mathbf{E}(N_n)$ remains at least $n^{1/3}$. (We postpone the statement of the main result to Section 3, until after having described the self–similar measure.) This result might reflect upon the fact that the assumption of self–similarity of the measure forces a high degree of organization upon the random points, which is comparable to assuming a uniform distribution. However, despite the seeming analogy of the two results, the proof requires an approach that is decidedly different from the one in the absolutely continuous case. We observe that our results translate to higher-dimensions in a straightforward way. As we will see, our setting covers the uniform distribution on the unit disk.

Let ν be the self–similar measure (defined in Section 2, (4)), $\mathbf{K}$ the support of ν, and let $\mathcal{C}_n$ denote the convex hull of the first n points chosen according to ν. We shall exploit the connection (see Lemma 3.2 below) between the measure $\nu(\mathbf{K} \setminus \mathcal{C}_n)$ of the subset of $\mathbf{K}$ outside the convex hull and a deterministic set $S(\varepsilon)$, being the union of all 'ε-caps' of $\mathbf{K}$. The 'ε-cap region' has been studied by Bárány and Larman [1] for the uniform distribution on a convex compact region and by Massé [8] for general absolutely continuous distributions in the Euclidean space $I\!R^d$. The expectation $\mathbf{E}(N_{n+1})$ can be recovered by using the Efron identity [3]

$$\mathbf{E}[\nu(\mathbf{K} \setminus \mathcal{C}_n)] = \frac{\mathbf{E}(N_{n+1})}{n+1}, \tag{1}$$

which holds for any positive integer n. In fact, $\nu(\mathbf{K} \setminus \mathcal{C}_n)$ measures the speed at which $\mathcal{C}_n$ approaches the boundary of the support. Notice that the convex hull of random points is a less tractable object than many other Euclidean functionals because the convex hull does not relate to the convex hull of a subset of the random points or to the convex hull on certain subregions in any amenable way. The problem cannot be partitioned into subproblems. However, the key observation is that there is a deterministic object, the region $S(\varepsilon)$, containing the boundary of the convex hull with high probability, which enjoys a nice scaling property. In fact, the probability of $S(\frac{1}{n})$ is close to the first moment $\mathbf{E}(N_n/n)$. Thus, our major concerns center around the study of the pattern of the caps, being the elements of the set $S(\varepsilon)$. Moreover, the region $S(\varepsilon)$ is of interest in its own as $S(\varepsilon)$ or some appropriate function of $S(\varepsilon)$ provide a measure for the speed at which the convex hull approaches the support (see [1, 8]).

This study was motivated by the desire to understand the random convex hull on self–similar sets such as the Sierpiński Gasket or Carpets. It is worthwhile mentioning that a standard argument shows for the uniform product measure on the Sierpiński Gasket and on any 'non-degenerate' (self–similar or self–affine) Carpet, that the expectation $\mathbf{E}(N_n)$ has asymptotic order $\log n$ like in the case of the uniform distribution on any r-polygon, the reason being that 'most' convex hull vertices lie in the corners of the support. In the very corner, the self–similar measure is not distinguishable from any measure proportional to the Lebesgue measure. By a degenerate set we mean an invariant set of probability measure 1, falling on a union of lines or points, which has $N_n = n$, $N_n = 1, 2$ or some other positive integer.

The organization of the paper is as follows. The statement of the main result is postponed to Section 3. In Section 2 we will construct the self–similar measure and introduce some more notation. In Section 3, we will give the definition of the 'caps' and study some of their properties. Finally, Section 4 is devoted to the proof of our result.

2 Self–Similar Probability Distributions

Consider the unique [6] nonempty, compact subset $\tilde{\mathbf{K}}$ of $(0, 2\pi] \subset I\!R$ satisfying $\tilde{\mathbf{K}} = \bigcup_{i=1}^{m} \psi_i(\tilde{\mathbf{K}})$, where $m > 1$ and the ψ_i are contractive similarity transformations of $(0, 2\pi]$, each with similarity ratio $r_i \in (0, 1)$ (i.e., for any points $x, y \in I\!R$, we have $|\psi_i(x) - \psi_i(y)| = r_i|x - y|$). Furthermore, suppose the self–similar set $\tilde{\mathbf{K}}$ satisfies the *Strong Open Set Condition* (SOSC), that is, there exists a bounded open set V such that $V \cap \tilde{\mathbf{K}} \neq \emptyset$ and the images $\psi_i(V)$ are pairwise disjoint subsets of V, i.e. $\psi_i(V) \cap \psi_j(V) = \emptyset$ for each $i \neq j$ and $\bigcup_{i=1}^{m} \psi_i(V) \subset V$. Now we shall wrap the set $\tilde{\mathbf{K}} \subset (0, 2\pi]$ around the unit disk. More precisely, identify the interval $(0, 2\pi]$ with the boundary of the unit disk. Thus, each point in $\tilde{\mathbf{K}}$ is mapped 1-to-1 onto a point of the boundary of the unit disk in $I\!R^2$. Location and orientation of $\tilde{\mathbf{K}}$ can be chosen arbitrarily. Denote the image of $\tilde{\mathbf{K}}$ under this mapping by $\tilde{\mathbf{D}}$. Finally, connect each point in $\tilde{\mathbf{D}}$ to the origin and call the union of these unit vectors by $\mathbf{K}$. Let φ denote the map which takes $\tilde{\mathbf{K}}$ to $\mathbf{K}$:

$$\mathbf{K} = \varphi(\tilde{\mathbf{K}}). \tag{2}$$

Observe that, while the invariant set $\tilde{\mathbf{K}}$ is self–similar, the set $\mathbf{K}$ is *not*. However, the endowed probability measure will be as we will see shortly.

Before we turn to construct a self–similar measure living on $\mathbf{K}$, we need some more notation. Each element of the set $\mathbf{K}$ may be represented by a (one-sided) infinite sequence of symbols from the alphabet $\{1, 2, \ldots, m\}$. For each integer $n > 0$, let $\Sigma_n = \{\mathbf{i} = i_1 i_2 \ldots i_n,\ i_j \in \{1, 2, \ldots, m\}\}$ denote the set of sequences of length n. For any infinite word $i_1 i_2 i_3 \ldots,$ define

$$\begin{aligned} \tilde{K}_{i_1 i_2 \ldots i_n} &= \psi_{i_1} \circ \psi_{i_2} \circ \ldots \circ \psi_{i_n}(\tilde{\mathbf{K}}) \\ K_{i_1 i_2 \ldots i_n} &= \varphi(\tilde{K}_{i_1 i_2 \ldots i_n}) \end{aligned} \tag{3}$$

and

$$k_{i_1 i_2 \ldots} = \bigcap_{n=1}^{\infty} K_{i_1 i_2 \ldots i_n}.$$

Note that $\mathbf{K} \supset K_{i_1} \supset K_{i_1 i_2} \supset \ldots$ is a nested sequence of nonempty (compact) sets. Thus, the intersection is nonempty, in fact, a unit vector in $I\!R^2$. Let ν be a probability measure on $\mathbf{K}$. Say ν is *self–similar* if there exist positive real numbers $p_1, p_2, \ldots, p_m$ such that $p_1 + p_2 + \ldots + p_m = 1$ and

$$\nu(K_{i_1 i_2 \ldots i_n}) = p_{i_1} p_{i_2} \cdot \ldots \cdot p_{i_n} \tag{4}$$

for all sequences $i_1 i_2 \ldots i_n \in \Sigma_n$. The measure ν depends on the geometry of $\mathbf{K}$ through the contraction ratios r_i.

To avoid cumbersome indices, let us agree on the following: For any two finite sequences $\mathbf{i} = i_1 i_2 \ldots i_n$ and $\mathbf{j} = j_1 j_2 \ldots j_l$, write

$$\begin{aligned} K_{\mathbf{i}} &= K_{i_1 i_2 \ldots i_n} \\ K_{\mathbf{ij}} &= K_{i_1 i_2 \ldots i_n j_1 j_2 \ldots j_l} \\ r_{\mathbf{i}} &= r_{i_1} r_{i_2} \ldots r_{i_n} \\ p_{\mathbf{i}} &= p_{i_1} p_{i_2} \ldots p_{i_n}, \end{aligned}$$

with this short notation being used at other obvious instances not specified here. Moreover, the measure ν will have the property

$$\nu(\mathbf{K}) = \nu(K_1) + \nu(K_2) + \ldots + \nu(K_m).$$

Say two sequences $i_1 i_2 \ldots i_n$ and $i'_1 i'_2 \ldots i'_{n'}$ are *incomparable* if they do *not* agree in each of the first $(n \wedge n')$ entries. Let us apply the notation in (3) to a bounded open set $\tilde{V} \in (0, 2\pi]$ such that V is a subset of the unit circle. Suppose that now V is a bounded open set satisfying $V \cap \mathbf{K} \neq \emptyset$, $V_i \cap V_j = \emptyset$ for each $i \neq j$ and $\bigcup_{i=1}^{m} V_i \subset V$. Clearly, such a set V exists since φ is a continuous bijection. The next result is borrowed from Lalley ([7], Lemma 1, p.5).

Proposition 2.1 *Let $p_1, p_2, \ldots, p_m$ be positive real numbers satisfying $p_1 + p_2 + \ldots + p_m = 1$. There exists a unique probability measure $\nu = \nu(p_1, p_2, \ldots, p_m)$ satisfying $\nu(K) = 1$ and*

$$\nu(K_{i_1 i_2 \ldots i_n}) = p_{i_1} p_{i_2} \ldots p_{i_n} \tag{5}$$

for every finite sequence $i_1 i_2 \ldots i_n \in \Sigma_n$. Moreover, the measure ν satisfies

$$\nu(K_{i_1 i_2 \ldots i_n} \cap K_{i'_1 i'_2 \ldots i'_{n'}}) = 0 \tag{6}$$

for all finite sequences $\mathbf{i} = i_1 i_2 \ldots i_n$ and $\mathbf{i}' = i'_1 i'_2 \ldots i'_{n'}$ that do not *agree in each of the first $n \wedge n'$ entries (i.e. $\mathbf{i}$ and $\mathbf{i}'$ are incomparable).*

Proof. First, observe that for two incomparable sequences $\mathbf{i} \in \Sigma_n$ and $\mathbf{i}' \in \Sigma_{n'}$

$$V_{\mathbf{i}} \cap V_{\mathbf{i}'} = \emptyset. \tag{7}$$

A consequence of the construction of $\mathbf{K}$ is that the origin is contained in $K_{\mathbf{i}} \cap K_{\mathbf{i}'}$. The intersection of the sets $K_{\mathbf{i}}$ and $K_{\mathbf{i}'}$ might even be much larger than one point. However, $\nu(K_{\mathbf{i}} \cap K_{\mathbf{i}'}) = 0$ as we show next.

Note that $\bigcap_n \overline{V}_{i_1 i_2 \ldots i_n}$ is a single unit vector. This vector must be $k_{i_1 i_2 \ldots}$ because $\mathbf{K} \cap V \neq \emptyset$, thus for each n, $K_{i_1 i_2 \ldots i_n} \cap \overline{V}_{i_1 i_2 \ldots i_n} \neq \emptyset$. Consequently, each $k_{i_1 i_2 \ldots}$ is an element of $\overline{V}$, implying $\mathbf{K} \subset \overline{V}$ and

$$K_{\mathbf{i}} \subset \overline{V}_{\mathbf{i}} \tag{8}$$

for any integer n and any sequence $\mathbf{i} \in \Sigma_n$. Recall that $\mathbf{K} \cap V \neq \emptyset$. Hence, there is a $k_{j_1 j_2 \ldots} \in V$. Since the sets $K_{j_1 j_2 \ldots j_n}$ shrink to $k_{j_1 j_2 \ldots}$ as $n \to \infty$ and V is open, for sufficiently large l and $\mathbf{j} = j_1 j_2 \ldots j_l$,

$$K_{\mathbf{j}} \subset V. \tag{9}$$

Fix this sequence $\mathbf{j} \in \Sigma_l$ and choose any two incomparable sequences $\mathbf{i} \in \Sigma_n$ and $\mathbf{i'} \in \Sigma_{n'}$. It follows from (7), (8) and (9) that

$$K_{\mathbf{ij}} \cap K_{\mathbf{i'}} = \emptyset. \tag{10}$$

Suppose ν is a probability measure satisfying equation (5). Since all $p_k > 0$, obviously, for every sequence $\mathbf{i} \in \Sigma_n$

$$\nu(K_{\mathbf{i}}) = \nu(\sum K_{\mathbf{ijh}}),$$

where the summation is over all finite sequences $\mathbf{h} \in \Sigma_s$ for some integer $s \geq 0$. Observe that for any such sequence $K_{\mathbf{ijh}} \subset K_{\mathbf{i}}$. Therefore, equation (10) implies that (6) must hold unless $\mathbf{i}$ and $\mathbf{j}$ are comparable. Hence, ν assigns probability 1 to the set of elements in $\mathbf{K}$ with *unique* representations $k_{i_1 i_2 \ldots}$.

Assume Z is a random element of $\mathbf{K}$ with probability measure ν, where ν satisfies (5). Then Z has a unique representation $k_{I_1 I_2 \ldots}$, where $I_1, I_2, \ldots$ are random variables with values in $\{1, 2, \ldots, m\}$. But equation (5) forces $I_1 I_2, \ldots$ be i.i.d. with

$$\mathbf{P}(I_w = k) = p_k, \tag{11}$$

for each $k = 1, 2, \ldots, m$. Thus, ν is uniquely determined. Finally, it remains to be shown that there is a probability measure on $\mathbf{K}$ satisfying equation (5). Let $I_1, I_2, \ldots$ be i.i.d. with distribution given by condition (11), let $Z = k_{I_1, I_2, \ldots}$ and let ν be the distribution of Z. In particular, ν satisfies equation (6). It follows that

$$\begin{aligned} \nu(K_{i_1 i_2 \ldots i_n}) &= \mathbf{P}(I_1 = i_1, I_2 = i_2, \ldots, I_n = i_n) \\ &= p_{i_1} p_{i_2} \ldots p_{i_n} \end{aligned}$$

for any finite sequence $i_1 i_2 \ldots i_n$ as required. □

Observe that we allow the uniform distribution as a special case of ν, more precisely, we have $r_i \equiv p_i \equiv \frac{1}{m}$, $\mathbf{K}$ is the unit disk and the images V_i satisfy $V_i \cap V_j = \emptyset$ for any $i \neq j$ and $\overline{V_i} \cap \overline{V_j} \neq \emptyset$ for some $i \neq j$. It is important to notice that the vertices of $\mathcal{C}_n$ are not necessarily preserved as vertices under the transformations $\{\varphi \circ \psi_i \circ \varphi^{-1}\}$. Consequently, there is no affine transformation taking the measure ν to the uniform distribution on the unit disk or vice versa. Therefore, the results on the convex hull of random points picked according to the uniform distribution cannot be manipulated to the extent to be of any help in our setting. Another remark is in order here. In fact, it suffices for $\tilde{\mathbf{K}}$, equivalently for $\mathbf{K}$, to satisfy the Open Set Condition (OSC), i.e. the images $\psi_i(V)$ are pairwise disjoint subsets of V for some open bounded set V since for self–similar sets in Euclidean space the OSC is equivalent to the SOSC [10].

3 Cap Region

Let ν be the self–similar probability measure on the limiting set $\mathbf{K} \subset I\!R^2$ defined in (2) in the previous section. For every $0 \le \varepsilon \le 1$, define the ε-*cap region* $S(\varepsilon)$ of $\mathbf{K}$ as the union of ε-caps, i.e.

$$S(\varepsilon) = \{x \in \mathbf{K} : \inf \nu(H_x) \le \varepsilon\}, \tag{12}$$

where the infimum is over all closed halfplanes H_x containing the point x. Note that the set $S(\varepsilon)$ is closed and the closure of the component of the complement of $S(\varepsilon)$ containing the origin is a convex region: pick any two points x and y from the boundary of $S(\varepsilon)$, facing the origin, and join them by a line segment. No point z on this line segment can lie in the interior of $S(\varepsilon)$ by definition (12). The set $S(\varepsilon)$ has been introduced by Bárányi and Larman in [1] for the uniform distribution defined on a convex compact subset of $I\!R^d$, and, was analyzed by Massé in [8] for arbitrary absolutely continuous distributions in Euclidean space.

Let $T(\varepsilon)$ denote the boundary of $S(\varepsilon)$ facing the origin, more precisely,

$$T(\varepsilon) = \partial S(\varepsilon) \cap D^o,$$

where $\partial S(\varepsilon)$ denotes the boundary of $S(\varepsilon)$ and D^o denotes the interior of the unit disk. Note that the set $T(\varepsilon)$ consists of countably many points. In the sequel, $\mathbf{E}$ shall denote the expectation with respect to the measure ν.

Theorem 3.1 *Let $X_1, X_2, \ldots, X_n$ be n independent and identically distributed points according to the self–similar measure ν with support $\mathbf{K}$ contained in the unit disk. Let $\mathcal{C}_n$ be the convex hull of $X_1, X_2, \ldots, X_n$, and let N_n denote the number of vertices of $\mathcal{C}_n$. Moreover, for every $\varepsilon \in (0,1)$, let $S(\varepsilon)$ be the ε-cap region defined in (12). Then there exists some constant $\beta = \beta(\nu) > 0$, depending on the measure ν, such that for every $\varepsilon \in (0,1)$, for arbitrarily small $\delta > 0$ and for sufficiently large n, we have*

$$\lim_{\varepsilon \to 0} \nu(S(\varepsilon))/\varepsilon^{2/3} \quad = \quad \beta \tag{13}$$

and

$$e^{-1}\beta n^{1/3} \quad \le \quad \mathbf{E}(N_{n+1}) \le \beta\,(2+\delta)^{2/3}\,(\ln n)^{2/3}\,n^{1/3}. \tag{14}$$

For the first claim see Proposition 4.4 below. The second statement follows from identity (1) and setting $\varepsilon = 1/n$ and $\varepsilon = (2+\delta)\,\ln n/n$ in Lemma 3.2 below.

Remarks:

1. In (13), the value β depends on ν, and in turn, ν depends on $p_1, p_2, \ldots, p_m$ and on $r_1, r_2, \ldots, r_m$ through the unknown bounded open set V stated in the SOSC on $\mathbf{K}$. However, the exponent $2/3$ in (13) is uniform over all choices of ν.

2. In both inequalities in Lemma 3.2 below, the second term of the upper bound is independent of the choice of ν. In many regularly behaving cases, $\mathbf{E}(N_n)$ increases at the rate $n\nu(S(\frac{1}{n}))$, thus, the lower bound indicates the correct order. In particular, this is true for the uniform distribution [9]. However, our technique of proof does not imply that the lower bound provides the correct rate of increase. It is conceivable that there is another "upper function" for $\mathbf{E}(N_n)$ or an additional term coming in for a singular ν but not for the uniform distribution.

We will break up the proof of the theorem into several pieces, which fill the remainder of this paper. The following result was proved by Massé [8] for absolutely continuous distributions.

Lemma 3.2 *For each $0 < \varepsilon < 1$ and $n > 2$, the following inequalities hold*

$$\nu(S(\varepsilon))\,(1-\varepsilon)^n \quad \leq \quad \mathbf{E}[\nu(\mathbf{K}\setminus\mathcal{C}_n)] \leq \nu(S(\varepsilon)) \,+\, n[(1-\varepsilon)^{n-1} + \varepsilon^{n-1}],$$

in particular for each $\alpha \in (0,1)$,

$$e^{-1}\frac{1}{n} \leq e^{-1}\,\nu(S(\frac{1}{n})) \quad \leq \quad \mathbf{E}[\nu(\mathbf{K}\setminus\mathcal{C}_n)] \leq \nu(S(\frac{1}{n^\alpha})) \,+\, R_\alpha(n)$$

where

$$R_\alpha(n) \quad = \quad n[\exp\{-\frac{n-1}{n^\alpha}\} \,+\, \frac{1}{n^{\alpha(n-1)}}].$$

Proof. Write $u(x) = \inf \nu(H_x)$ for each $x \in \mathbf{K}$. So, $u(x) \in [0,1]$. For any $x \in \mathbf{K}$,

$$[1-u(x)]^n \leq \nu(x \notin \mathcal{C}_n)$$

since for any H_x such that $u(x) = \inf \nu(H_x)$

$$\nu(X_i \notin H_x \text{ for each } 1 \leq i \leq n) \leq \nu(x \notin \mathcal{C}_n).$$

On the other hand, suppose that $x \notin \mathcal{C}_n$. It follows there exists a point $X_* \in \{X_1, X_2, \ldots, X_n\}$ such that one of the halfplanes bounded by the line passing through x and X_* contains $\{X_i\}_{i=1}^n$. Denote this halfplane by H. It follows that

$$\nu(H)^{n-1} + (1-\nu(H))^{n-1} \leq u(x)^{n-1} + (1-u(x))^{n-1}$$

because for $x \in H$ and for fixed positive real m and varying $s \in [0,1]$ the sum $s^m + (1-s)^m$ is maximal if either s or $1-s$ is minimal.

Now, if Z is a point picked randomly from the measure ν, then

$$\mathbf{E}[\nu(\mathbf{K}\setminus\mathcal{C}_n)] \leq \mathbf{E}[1_{S(\varepsilon)^c}(Z)\, n\,(\varepsilon^{n-1} + (1-\varepsilon)^{n-1}) \,+\, 1_{S(\varepsilon)}(Z)],$$

where $1_{S(\varepsilon)}$ denotes the indicator function of $S(\varepsilon)$ and $S(\varepsilon)^c$ denotes the complement of $S(\varepsilon)$. This immediately gives the upper bound of $\mathbf{E}[\nu(\mathbf{K}\setminus\mathcal{C}_n)]$. Similarly, the lower bound of $\mathbf{E}[\nu(\mathbf{K}\setminus\mathcal{C}_n)]$ is obtained. □

Fix $\varepsilon \in (0,1)$. Choose a point $x \in T(\varepsilon)$ and denote by H_x a halfplane for which the infimum in (12) is no larger than ε. Define the *ε-cap*

$$C_x(\varepsilon) = \mathbf{K} \cap H_x. \tag{15}$$

Observe that $\nu(C_x(\varepsilon)) = \varepsilon$ by a continuity argument. In fact, the term cap is justified only by the shape of $C_x(\varepsilon)$. The set $C_x(\varepsilon)$ is convex. However, keep in mind that $C_x(\varepsilon)$ can be a rather thin set (this is the reason for dealing with the filled version of $C_x(\varepsilon)$ below). Call x the *basepoint* of the ε-cap $C_x(\varepsilon)$. Moreover, if D denotes the unit circle in $\mathbb{R}^2$, let the *width* $\eta = \eta(C_x(\varepsilon))$ of $C_x(\varepsilon)$ be the length of the line segment in $H_x \cap D$ lying on the line bisecting $H_x \cap D$. Obviously, the 'center' of $C_x(\varepsilon)$ on the bounding line and the basepoint of the cap might differ, and, the center might lie outside $\mathbf{K}$. It is important to notice here that, for given ε, basepoint x and cap $C_x(\varepsilon)$, its width is uniquely determined. $\varepsilon = \nu(C_x(\varepsilon))$ is a continuous, strictly increasing function in $\eta(C_x(\varepsilon))$ as can easily be seen by recalling definition (12). A key role in our calculations plays the relation between the probability ε of the cap $C_x(\varepsilon)$ and its width η. It will turn out, though, that we get around the explicit evaluation of $\nu(C_x(\varepsilon))$ in terms of $\eta(C_x(\varepsilon))$.

4 Renewal Theorem

Instead of studying certain cap coverings of $S(\varepsilon)$, an approach taken up in [1], we are going to give a rescaling argument involving the ε-caps. For this purpose, let us partition the set $S(\varepsilon)$ into n-th generation cells for each n. For each finite sequence $\mathbf{i} = i_1 i_2 \dots i_n$ define the cell

$$Q_{\mathbf{i}}(\varepsilon) = Q_{i_1 i_2 \dots i_n}(\varepsilon) = K_{i_1 i_2 \dots i_n} \cap S(\varepsilon). \tag{16}$$

Similarly, as we noticed about the caps, the cells might be thin sets. Define the *width* $w(Q_{\mathbf{i}}(\varepsilon))$ of the cell $Q_{\mathbf{i}}(\varepsilon)$ as the maximal vertical diameter of the cell $Q_{\mathbf{i}}(\varepsilon)$. Observe that ε is a strictly increasing function in $w(Q_{\mathbf{i}}(\varepsilon))$. By the definition of the probability measure ν, we have

$$\nu(Q_{\mathbf{i}j}(\varepsilon)) = p_j\, \nu(Q_{\mathbf{i}}(\varepsilon)) \tag{17}$$

for each index $j \in \{1, 2, \dots, m\}$ each $n \geq 1$ and each finite sequence $\mathbf{i} \in \Sigma_n$. The following result is a consequence of the fact that there exist some sequences of caps exhibiting a similar pattern. Several approximations, that need to be addressed because the self-similarity of the measure and the self–similarity of the subsectors might not perfectly move in line, keep the result and its proof from looking much simpler.

Proposition 4.1 *For any $\varepsilon \in (0,1)$ and arbitrary $\gamma = \gamma(\varepsilon) > 0$, choose some sufficiently large n, some finite n-sequence $\mathbf{i}$, and some cell $Q_{\mathbf{i}}(\varepsilon)$. Then there exist some $h = h(\mathbf{i})$ and some $\mathbf{j} = j_1 j_2 \dots j_h \in \Sigma_h$ such that*

$$w(Q_{\mathbf{i}}(\varepsilon')) \;\; = \;\; p_{\mathbf{j}}\, w(Q_{\mathbf{i}}(\varepsilon))$$

implies

$$\varepsilon' = \varepsilon\{[p_{\mathbf{j}} - \frac{q_h}{\eta}]^{3/2} + v(\eta, p_{\mathbf{j}}, q_h)\} \tag{18}$$

for some $q_h < \gamma$, *where* η *is the width of some* ε*-cap with base point in* $Q_{\mathbf{i}}(\varepsilon)$ *and*

$$v(\eta, p_{\mathbf{j}}, q_h) = C\frac{q_h}{\eta^{5/2}}\sqrt{p_{\mathbf{j}} - q_h/\eta} + o(\frac{q_h}{\eta^{5/2}}\sqrt{p_{\mathbf{j}} - q_h/\eta}) + o((p_{\mathbf{j}} - q_h/\eta)^{9/4}\eta^{3/4}) \tag{19}$$

for some real C.

Proof. There are two approximations going into this proof, the first being the parabola approximation to the boundary of the unit circle, and the second and major complication of the proof bearing from using a special sequence of caps. The main idea of proof can be sketched as follows. Choose a cell $Q_{\mathbf{i}}(\varepsilon)$ of generation n and some $x \in T(\varepsilon) \cap Q_{\mathbf{i}}(\varepsilon)$. The point x defines a cap $C_x(\varepsilon)$ of probability ε. Then map the cell $Q_{\mathbf{i}}(\varepsilon)$ onto a subcell $Q_{\mathbf{i}}(\varepsilon')$ with predetermined width $p_{\mathbf{j}}\, w(Q_{\mathbf{i}}(\varepsilon))$. Hence, the probability ε' is being forced. Roughly speaking, we are dealing with the question by how much the probability of a cap decreases if the cap is rescaled by shrinking its width by a certain factor. Our major business consists in deriving an approximate expression to the ratio ε'/ε.

Given $\varepsilon \in (0,1)$, choose some $\gamma > 0$. Pick any cell $Q_{\mathbf{i}}(\varepsilon)$ with $\mathbf{i} = i_1 i_2 \dots i_n$ and n sufficiently large. Choose a point $x \in T(\varepsilon) \cap Q_{\mathbf{i}}(\varepsilon)$ and identify x with the base point of some ε-cap $C_x(\varepsilon)$ of width η. Suppose $x = k_{s_1 s_2 s_3 \dots}$. Next we will find a smaller cap by mapping the cell $Q_{\mathbf{i}}(\varepsilon)$ onto the subcell $Q_{\mathbf{i}}(\varepsilon_1')$: Let $p_{j_1} w(Q_{\mathbf{i}}(\varepsilon))$ be the width of the cell $Q_{\mathbf{i}}(\varepsilon_1')$ for some $\varepsilon_1' \in (0,1)$ and write x_1' for $T(\varepsilon_1') \cap k_{s_1 s_2 s_3 \dots}$. Observe the point x_1' is in $Q_{\mathbf{i}}(\varepsilon_1')$ and is the base point for some cap $C_{x_1'}(\varepsilon_1')$, where $\varepsilon_1' < \varepsilon$. Since the line bounding the cap $C_{x_1'}(\varepsilon_1')$ needs not be parallel to the line bounding the cap $C_x(\varepsilon)$ and there is no apparent connection between those two caps, we will construct a different cap $C_{x_1^*}(\varepsilon_1^*)$ with the property that $C_{x_1^*}(\varepsilon_1^*)$ is a scaled down copy of $C_x(\varepsilon)$. The similarity of the caps gives a handle on the ratio $\varepsilon_1^*/\varepsilon$ of their probabilities. Observe that, generally, the two smaller caps $C_{x_1'}(\varepsilon_1')$ and $C_{x_1^*}(\varepsilon_1^*)$ might be of considerably different sizes. However, by iterating the construction and producing two sequences of caps, the probabilities of such caps will be arbitrarily close after sufficiently many steps. More precisely, the construction stops when one of the caps achieved by mapping one subcell to a smaller subcell is 'close enough' to one of the caps which are similar to each other.

First, obvious repetition of the above argument brings along a sequence $\{Q_{\mathbf{i}}(\varepsilon_t')\}_{\{t\geq 1\}}$ of cells of widths shrunk by a factor $\{p_{j_1 j_2 \dots j_t}\}$. Now by the similarity of the subsectors, there are (infinitely many) points, being the center of a cap, that is a scaled down copy of $C_x(\varepsilon)$, after possible rotation and shifting, and in fact, having a bounding halfline which gives the infimum of the measure in definition (12): Suppose $K_{a_1 a_2 \dots a_l}$ is the smallest subsector containing $C_x(\varepsilon)$, i.e. for each a_{l+1}, we have $C_x(\varepsilon) \not\subset K_{a_1 a_2 \dots a_{l+1}}$. Choose a_{l+1} such that $Q_{\mathbf{i}}(\varepsilon) \subset K_{a_1 a_2 \dots a_{l+1}}$, which is possible since n has been chosen large enough, and find the center of a smaller cap $C_{x_1^*}(\varepsilon_1^*)$ in $K_{a_1 a_2 \ldots a_{l+1}}$, such that the 'diameter' of the cap has been shrunk by a factor r_{l+1} and $C_{x_1^*}(\varepsilon_1^*)$ is a scaled down copy of $C_x(\varepsilon)$. Write η_1^* for the width of $C_{x_1^*}(\varepsilon_1^*)$. Next choose a_{l+2} such that $Q_{\mathbf{i}}(\varepsilon) \subset K_{a_1 a_2 \ldots a_{l+2}}$ and find a smaller cap of width η_2^* in the same way. Repeating this argument produces a strictly decreasing sequence $\eta_1^* > \eta_2^* > \eta_3^* > \ldots > 0$ of positive reals corresponding to widths of caps since each ratio $r_w \in (0,1)$.

Recall that η is the width of $C_x(\varepsilon)$. For any $\gamma > 0$ and any $j_1 j_2 \ldots$ there exists a positive integer h, such that $\eta\, p_{j_1} p_{j_2} \ldots p_{j_h}$ lies within distance γ of $\eta_{h^*}^*$ for some positive integer h^*. To see this, consider the differences $\{p_{j_1} p_{j_2} \ldots p_{j_s} - p_{j_1} p_{j_2} \ldots p_{j_{s+1}}\}$ and $\{\eta_s^* - \eta_{s+1}^*\}$ bounded above by $\{p_{j_1} p_{j_2} \ldots p_{j_s}\}$ and η_s^*, respectively. Therefore, each of the two difference sequences is strictly decreasing with limit zero. Write

$$\begin{aligned} \eta^* &= \eta_{h^*}^* \\ \mathbf{j} &= j_1 j_2 \ldots j_h \\ q_h &= \eta\, p_{\mathbf{j}} - \eta^* < \gamma \end{aligned}$$

and denote by x^* the base point of the cap of width η^*. Hence, the last subcell constructed is $Q_{\mathbf{i}}(\varepsilon_h')$ of width $p_{j_1} p_{j_2} \ldots p_{j_h} w(Q_{\mathbf{i}}(\varepsilon))$. Arguing in the same manner as we did to find $C_x(\varepsilon)$, given the cell $Q_{\mathbf{i}}(\varepsilon)$, we can find a cap $C_{x'}(\varepsilon')$ of probability ε' and width η', given the cell $Q_{\mathbf{i}}(\varepsilon_h')$, which has probability close to the probability of the cap of width η^* by using the following approximations.

In the neighbourhood of the origin, we can approximate the circle of radius 1 centered at $(0,1)$ by the parabola $x^2/2$ since, for sufficiently small $|x|$,

$$\frac{x^2}{2} = 1 - \sqrt{1 - x^2} - o(x^3),$$

where we use the notation $g(x) = o(x)$ if $g(x)/x \to 0$ as $x \to 0$. Suppose H_x is the halfplane for which the infimum in definition (12) is achieved. The probability ε equals the probability of $\bigcup_{\mathbf{k} \in \Sigma_n} K_{\mathbf{k}} \cap H_x$. Because n is large, this probability boils down to the calculation of the area of the union of some rectangles between a horizontal line and the parabola $x^2/2$ in a neighbourhood of the origin. First turn the cap such that the line supporting the cap is horizontal. Also, we may assume that the first coordinate of x is sufficiently close to 0. Then notice that the subsectors $K_{\mathbf{k}}$ cross this line approximately vertically. Thus, ε is very close to the Lebesgue measure of the subsectors $K_{\mathbf{k}}$ crossing the supporting line of the cap vertically, i.e. the union of those rectangles between the horizontal coordinate-axis in $\mathbb{R}^2$ and a circle of radius 1 centered at $(0, -1+\eta)$, whence, to the integral

$$I_n(x,\eta) = \sum_{\mathbf{k} \in \Sigma_n \text{ and } K_{\mathbf{k}} \in H_x} \int_{x-\sqrt{2\eta}}^{x+\sqrt{2\eta}} 1_{K_{\mathbf{k}}}(y) \frac{p_{\mathbf{k}}}{r_{\mathbf{k}}} (\eta - \frac{y^2}{2})\, dy, \tag{20}$$

where $1_{K_{\mathbf{k}}}$ denotes the indicator function of $K_{\mathbf{k}}$. Integration yields

$$I_n(x,\eta) = \sum_{\mathbf{k}\in\Sigma_n \text{ and } K_{\mathbf{k}}\in H_x} \frac{p_{\mathbf{k}}}{r_{\mathbf{k}}}\, t_{\mathbf{k}}\, \eta^{3/2} \tag{21}$$

for some reals $t_{\mathbf{k}} = t_{\mathbf{k}}(x)$. Furthermore,

$$\varepsilon = I_n(x,\eta) + o(\eta^{9/4}) \tag{22}$$

by (20) and (21). Taylor expansion of the function I_n at the point η^* gives

$$\begin{aligned} I_n(x',\eta') &= I_n(x^*,\eta^*) + Cq_h\sqrt{\eta^*} + o(q_h\sqrt{\eta^*}) \\ &= I_n(x^*,\eta^*) + C\frac{q_h}{\eta}\sqrt{p_{\mathbf{j}} - q_h/\eta} + o(\frac{q_h}{\eta}\sqrt{p_{\mathbf{j}} - q_h/\eta}) \end{aligned} \tag{23}$$

for some real constant C. A moment's thought reveals that, by construction of the caps $C_{x_s^*}(\varepsilon_s^*)$, the quotient of $I_n(x,\eta)$ and $I_n(x^*,\eta^*)$ does not depend on x and x^* and

$$I_n(x^*,\eta^*) = I_n(x,\eta)[p_{\mathbf{j}} - q_h/\eta]^{3/2}. \tag{24}$$

Finally, by (21), (23), (24) and easy algebra

$$\begin{aligned} \frac{\varepsilon'}{\varepsilon} &= \frac{I_n(x,\eta)[p_{\mathbf{j}} - q_h/\eta]^{3/2} + o((\eta^*)^{9/4}) + C\frac{q_h}{\eta}\sqrt{p_{\mathbf{j}} - q_h/\eta} + o(\frac{q_h}{\eta}\sqrt{p_{\mathbf{j}} - q_h/\eta})}{I_n(x,\eta) + o(\eta^{9/4})} \\ &= [p_{\mathbf{j}} - q_h/\eta]^{3/2} + v(\eta, p_{\mathbf{j}}, q_h), \end{aligned}$$

where

$$v(\eta, p_{\mathbf{j}}, q_h) = C\frac{q_h}{\eta^{5/2}}\sqrt{p_{\mathbf{j}} - q_h/\eta} + o(\frac{q_h}{\eta^{5/2}}\sqrt{p_{\mathbf{j}} - q_h/\eta}) + o((p_{\mathbf{j}} - q_h/\eta)^{9/4}\eta^{3/4}).$$

This completes our derivation. □

Lemma 4.2 *For any $\varepsilon \in (0,1)$ and arbitrary $\gamma = \gamma(\varepsilon) > 0$, choose some sufficiently large n. Then there exist some $k = k(\gamma)$ and some $\mathbf{j} = j_1 j_2 \ldots j_k \in \Sigma_k$ such that for each cell $Q_{\mathbf{i}}(\varepsilon)$ for $\mathbf{i} \in \Sigma_n$,*

$$w(Q_{\mathbf{i}}(\varepsilon')) \quad = \quad p_{\mathbf{j}}\, w(Q_{\mathbf{i}}(\varepsilon))$$

implies

$$\varepsilon' = \varepsilon\{[p_{\mathbf{j}} - \frac{q_k}{\eta}]^{3/2} + v(\eta, p_{\mathbf{j}}, q_k)\}$$

for some $q_k(\mathbf{i},\mathbf{j}) < \gamma$, where η is the width of some ε-cap and $v(\eta, p_{\mathbf{j}}, q_k)$ as defined in (19).

Proof. There are exactly m^n cells $Q_{\mathbf{i}}(\varepsilon) \subset S(\varepsilon)$. Let k be the smallest common multiplier of $\{h(\mathbf{i})\}_{\{\mathbf{i}\in\Sigma_n\}}$. Our claim follows from the similarity of the special sequence of caps $C_{x_s^*}(\varepsilon_s^*)$ as constructed in the previous proposition. □

Instead of allowing cells $Q_{\mathbf{i}}(\varepsilon)$ with $\mathbf{i} \in \Sigma_n$ only in the previous lemma, obviously, we can prove the same result for cells in any cutset of the set of all infinite sequences from the alphabet, where the cutset is contained in a sphere of fixed radius. Let us define the function

$$g(p_{\mathbf{j}}, q_k, \eta) = [p_{\mathbf{j}} - \frac{q_k}{\eta}]^{3/2} + v(\eta, p_{\mathbf{j}}, q_k). \tag{25}$$

This function is continuous in all its arguments.

Corollary 4.3 *For every sequence $\mathbf{i} \in \Sigma_n$, every $0 < \varepsilon < 1$ and every $\gamma > 0$, there exist a $k = k(\gamma)$ and some $\mathbf{j} = j_1 j_2 \dots j_k$ such that*

$$\nu(Q_{\mathbf{i}}(\varepsilon)) \quad = \quad \sum_{\mathbf{j}\in\Sigma_k} \nu(Q_{\mathbf{i}}(\varepsilon\, g(p_{\mathbf{j}}, q_k, \eta))),$$

where $q_k = q_k(\mathbf{i}, \mathbf{j}) < \gamma$ and g is defined in (25).

Proof. First observe that, for any two reals $0 < \varepsilon,\ \varepsilon' < 1$, the ratio of the widths of $Q_{\mathbf{i}}(\varepsilon)$ and $Q_{\mathbf{i}}(\varepsilon')$ and the ratio of the probabilities of $Q_{\mathbf{i}}(\varepsilon)$ and $Q_{\mathbf{i}}(\varepsilon')$ are approximately equal for sufficiently large n, since $Q_{\mathbf{i}}(\varepsilon)$ is close to a rectangle, i.e. a vertical cutset of the subsector $K_{\mathbf{i}}$. Thus, the probability of $Q_{\mathbf{i}}(\varepsilon)$ equals the probability of $K_{\mathbf{i}}$, multiplied by the width $w(Q_{\mathbf{i}}(\varepsilon))$. Now, let $\mathbf{j} = j_1 j_2 \dots j_k$ and q_k as found in Lemma 4.2. Since the p_j form a probability vector, by using Lemma 4.2 and equation (17), we see that

$$\begin{aligned} \nu(Q_{\mathbf{i}}(\varepsilon)) &= \nu(Q_{\mathbf{i}}(\varepsilon)) \sum_{\mathbf{j}\in\Sigma_k} p_{\mathbf{j}} \\ &= \sum_{\mathbf{j}\in\Sigma_k} p_{\mathbf{j}}\, \nu(Q_{\mathbf{i}}(\varepsilon)) \\ &= \sum_{\mathbf{j}\in\Sigma_k} \nu(Q_{\mathbf{i}}(\varepsilon([p_{\mathbf{j}} - \frac{q_k}{\eta}]^{3/2} + v(\eta, p_{\mathbf{j}}, q_k)))), \end{aligned}$$

as required. □

Proposition 4.4 *Let θ be the unique real such that $\sum_{\mathbf{j}\in\Sigma_k}^{m} p_{\mathbf{j}}^{\frac{3}{2}\theta} = 1$. Then there exists a constant $0 < \beta < \infty$ such that*

$$\lim_{\varepsilon\to 0} \nu(S(\varepsilon))/\varepsilon^{\theta} = \beta. \tag{26}$$

Proof. For each real $0 < \varepsilon < 1$, define the function $G(\varepsilon) = \nu(S(\varepsilon))$. Next choose $\varepsilon_* > 0$ sufficiently small such that each n emerging from the two inequalities (27) below will be large enough as required by Lemma 4.2 and Corollary 4.3. Observe, that η is a function of ε (and of the r_i and p_i). Now, let γ be such that $\gamma/\eta^{5/2} = h(\varepsilon)$ and h is an arbitrarily small function with $h(\varepsilon) \to 0$ as $\varepsilon \to 0$ (e.g. $h(\varepsilon) = \exp\{-1/\varepsilon\}$). Let $k = k(\gamma)$, the sequence $\mathbf{j} = j_1 j_2 \dots j_k$ and the reals q_k have the meanings attached as mentioned in Lemma 4.2 and Corollary 4.3. Then for each $n \geq 0$, ($n = 0$ provides the empty sequence)

$$G(\varepsilon) = \sum \nu(Q_{\mathbf{i}}(\varepsilon))$$

where the summation ranges over all finite sequences $\mathbf{i} = i_1 i_2 \dots i_{(n-k+1)}$ satisfying

$$-\frac{3}{2}\sum_{j=1}^{n-k} \ln p_{i_j} < \ln(\frac{\varepsilon}{\varepsilon_*}) \leq -\frac{3}{2}\sum_{j=1}^{n} \ln p_{i_j} \tag{27}$$

It follows from Corollary 4.3 that there is a function $\tilde{v}(\cdot)$ and some $\tilde{q}_k$ (both independent of $\mathbf{i}$) such that the function $G(\varepsilon)$ can be expressed as a functional equation:

$$G(\varepsilon) = \sum_{\mathbf{j}\in\Sigma_k} G(\varepsilon([p_{\mathbf{j}} - \frac{\tilde{q}_k}{\eta}]^{3/2} + \tilde{v}(\eta, p_{\mathbf{j}}, \tilde{q}_k))) + \tilde{R}(\varepsilon), \tag{28}$$

where

$$\begin{aligned}
\tilde{R}(\varepsilon) &= \sum_{\mathbf{j}\in\Sigma_k} \nu(S(\varepsilon([p_{\mathbf{j}} - \frac{\tilde{q}_k}{\eta}]^{3/2} + \tilde{v}(\eta, p_{\mathbf{j}}, \tilde{q}_k)))) \cdot 1_{\tilde{A}(\varepsilon_*)}(\varepsilon), \\
\tilde{A}(\varepsilon_*) &= \{s > 0 :\ \varepsilon_* \geq s([p_{\mathbf{j}} - \frac{\tilde{q}_k}{\eta}]^{3/2} + \tilde{v}(\eta, p_{\mathbf{j}}, \tilde{q}_k)) \\
&\qquad > \varepsilon_*([p_{\mathbf{j}} - \frac{\tilde{q}_k}{\eta}]^{3/2} + \tilde{v}(\eta, p_{\mathbf{j}}, \tilde{q}_k))\ \}.
\end{aligned}$$

The function $\tilde{R}$ is bounded, has finitely many discontinuities, and depends on γ and ε_*. Since the iteration in (28) occurs on a bounded interval once ε and ε_* were chosen, the function $[p_{\mathbf{j}} - \frac{\tilde{q}_k}{\eta}]^{3/2} + \tilde{v}(\eta, p_{\mathbf{j}}, \tilde{q}_k)$ can be bounded above and below by an expression independent of ε and will be arbitrarily close to $p_{\mathbf{j}}^{3/2}$ by using an appropriate γ. This shows that the functional equation for G can be rewritten as

$$G(\varepsilon) = \sum_{\mathbf{j}\in\Sigma_k} G(\varepsilon((p_{\mathbf{j}} - \delta_{1\mathbf{j}})^{3/2} + \delta_{2\mathbf{j}})) + R(\varepsilon), \tag{29}$$

for some reals $\delta_{1\mathbf{j}}$ and $\delta_{2\mathbf{j}}$, where

$$\begin{aligned}
R(\varepsilon) &= \sum_{\mathbf{j}\in\Sigma_k} \nu(S(\varepsilon((p_{\mathbf{j}} - \delta_{1\mathbf{j}})^{3/2} + \delta_{2\mathbf{j}}))) \cdot 1_{A(\varepsilon_*)}(\varepsilon) \\
A(\varepsilon_*) &= \{s >:\ \varepsilon_* \geq s((p_{\mathbf{j}} - \delta_{1\mathbf{j}})^{3/2} + \delta_{2\mathbf{j}}) > \varepsilon_*((p_{\mathbf{j}} - \delta_{1\mathbf{j}})^{3/2} + \delta_{2\mathbf{j}})\ \}.
\end{aligned}$$

Similarly, the function R is bounded, has finitely many discontinuities, and depends on γ and ε_*. Set $\varepsilon = e^{-t}$ for $t > 0$. The last displayed equation can be written as a renewal equation:

$$e^{\theta t}G(e^{-t}) \;=\; \sum_{\mathbf{j}\in\Sigma_k} e^{\theta t}\, G(e^{-t+\ln[(p_{\mathbf{j}}-\delta_{1\mathbf{j}})^{3/2}+\delta_{2\mathbf{j}}]}) + e^{\theta t}R(e^{-t}). \tag{30}$$

Define $H(t) = e^{\theta t}G(e^{-t})$. Then

$$\begin{aligned} H(t) &= \sum_{\mathbf{j}\in\Sigma_k} ((p_{\mathbf{j}}-\delta_{1\mathbf{j}})^{3/2}+\delta_{2\mathbf{j}})^{\,\theta}\, H(t-\ln[(p_{\mathbf{j}}-\delta_{1\mathbf{j}})^{3/2}+\delta_{2\mathbf{j}}]) + e^{\theta t}R(e^{-t}) \\ &= \int_0^\infty H(t+s)\mu(ds) + e^{\theta t}R(e^{-t}) \end{aligned}$$

for $t > 0$. The probability measure μ on $(0,\infty)$ puts mass $((p_{\mathbf{j}}-\delta_{1\mathbf{j}})^{3/2}+\delta_{2\mathbf{j}})^{\,\theta}$ at $-\ln[(p_{\mathbf{j}}-\delta_{1\mathbf{j}})^{3/2}+\delta_{2\mathbf{j}}]$. Usually, the renewal theorem comes with two different cases, the nonarithmetic and the arithmetic case. However, in our setting $\delta_{1\mathbf{j}}$ or $\delta_{2\mathbf{j}}$ may always be chosen such that the *nonarithmetic* case arises. In this case, the renewal theorem for $I\!R$ ([4], Ch.11) and the fact that R is Riemann integrable imply that

$$\lim_{t\to\infty} H(t) = \int e^{\theta s}\, R(e^{-s})ds \,/\, \overline{\mu} = \beta, \tag{31}$$

where $\overline{\mu} = \lim_{k\to\infty}\sum_{\mathbf{j}\in\Sigma_k} ((p_{\mathbf{j}}-\delta_{1\mathbf{j}})^{3/2}+\delta_{2\mathbf{j}})^{\,\theta}\cdot\ln[(p_{\mathbf{j}}-\delta_{1\mathbf{j}})^{3/2}+\delta_{2\mathbf{j}}] > 0$, for some positive constant β. To see this, choose a sequence $\varepsilon_v \to 0$ as $v\to\infty$ and for each ε_v some ε_{*v} sufficiently small such that the differences $|H(t_{v+1}) - H(t_v)|$ are sufficiently small and the integral is finite. Moreover, observe that $\delta_{1\mathbf{j}}$, $\delta_{2\mathbf{j}} \to 0$ as $k\to\infty$. We conclude that

$$\lim_{\varepsilon\to 0} G(\varepsilon)/\varepsilon^{\theta} = \beta.$$

□

Hence,

$$\lim_{\varepsilon\to 0} \nu(S(\varepsilon))/\varepsilon^{2/3} = \beta$$

since $m > 1$ and $\theta = 2/3$ is the unique solution.

It can be readily seen that an exactly analogous construction of self–similar measure is possible in the unit ball in $I\!R^d$. The halfplanes in definition (12) are replaced by halfspaces. The exponent of the limiting expression for the probability of the cap region stems from the geometry of the support, i.e. the parabola approximation to the boundary of the unit ball. If $\mathbf{K}_d$ denotes the limiting set in the unit ball and ν_d denotes the self–similar probability measure on $\mathbf{K}_d$, then we can prove the following result.

Theorem 4.5 *Let $X_1, X_2, \ldots, X_n$ be n independent and identically distributed points according to the self–similar measure ν_d with support $\mathbf{K}_d$ contained in the unit d-ball. Let $\mathcal{C}_n$ be the convex hull of $X_1, X_2, \ldots, X_n$ and let N_n denote the number of vertices of $\mathcal{C}_n$. Moreover, for every $\varepsilon \in (0,1)$, let $S(\varepsilon)$ be the ε-cap region analogous to the one in (12).*

Then there exists some constant $\beta_d > 0$, depending on the measure ν_d, such that for every $\varepsilon \in (0,1)$, for arbitrarily small $\delta > 0$ and for sufficiently large n, we have

$$\lim_{\varepsilon \to 0} \nu(S(\varepsilon)) = \beta_d \, \varepsilon^{2/(d+1)}$$

and

$$e^{-1} \beta_d \, n^{(d-1)/(d+1)} \leq \mathbf{E}(N_{n+1}) \leq \beta_d \, (2+\delta)^{2/(d+1)} \, (\ln n)^{2/(d+1)} \, n^{(d-1)/(d+1)}.$$

References

[1] I. Bárány and D.G. Larman (1988). *Convex bodies, economic cap coverings, random polytopes.* Mathematika **35**, 274–291.

[2] L. Devroye (1991). *On the oscillation of the expected number of extreme points of a random set.* Stat. Prob. Letters **11**, 281–286.

[3] B. Efron (1965). *The convex hull of a random set of points.* Biometrika **52**, 331–343.

[4] W. Feller (1971). *An Introduction to Probability Theory and its Applications.* Vol. 2, 2nd edition, Wiley, New York.

[5] I. Hueter (1994). *The convex hull of a normal sample.* Adv. Appl. Prob. **26**, 855–875.

[6] J. Hutchinson (1981). *Fractals and self–similarity.* Indiana U. Math. J. **30**, 713–747.

[7] S.P. Lalley (1990). *Travelling salesman with a self–similar itinerary.* Probab. Engin. Inform. Sciences. **4**, 1–18.

[8] B. Massé (1993). *Principes d'invariance pour la probabilité d'un dilaté de l'enveloppe convexe d'un échantillon.* Ann. Inst. Henri Poincaré, **29(1)**, 37–55.

[9] A. Rényi and R. Sulanke (1963). *Über die konvexe Hülle von n zufällig gewählten Punkten.* Z. Wahrscheinlichkeitstheorie verw. Geb. **2**, 75–84.

[10] A. Schief (1994). *Separation properties for self–similar sets.* Proc. AMS **122**, 111–115.

Department of Mathematics
University of Florida
358 Little Hall
PO Box 118105
Gainesville, FL 32611-8105