Incentive Efficient Price Systems in Large Insurance Economies with Adverse Selection*

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Abstract

We decentralize incentive efficient allocations in large adverse selection economies by introducing a Walrasian market for mechanisms, that is, for menus of contracts. Facing a budget constraint, informed individuals choose lotteries over mechanisms, while firms supply (slots at) mechanisms at given prices. An equilibrium requires that firms cannot favorably change, or cut, prices. We show that an equilibrium exists and is incentive efficient. We provide a way to compute an equilibrium as the solution to a recursive programming problem that selects the incentive efficient outcome preferred by the highest type within an appropriately defined set. For Rothschild and Stiglitz economies, this is the only equilibrium outcome.

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1 Introduction

In a seminal paper, Prescott and Townsend (1984a, hereafter PT; and 1984b) first laid out the question of whether constrained optimal allocations could be decentralized through prices, linking explicitly the problem of mechanism design to the description of a Walrasian market.

In this paper we continue this line of investigation, and introduce a Walrasian market for mechanisms, i.e., for platforms where a large number of agents meet and trade. A mechanism is a menu of contracts, one for each type of individuals in the population, and crucially may allow cross-subsidization across types participating the mechanism. As the baseline story, we will use a Rothschild and Stiglitz (1976; hereafter RS) insurance economy, where contracts are policies specifying insurance premium and coverage levels, and mechanisms are insurance policy menus.

Individuals choose lotteries\(^1\)\(^2\) over mechanisms, while firms supply (slots at) mechanisms. Lotteries are priced linearly: the price of a lottery is the average price of its mechanisms. The price of a mechanism can be thought of as the fee for a slot at the mechanism, i.e., for a net trade (or contract) among the ones in the menu. For given prices, individuals buy budget feasible lotteries, while firms realize the revenues from the sale of their offered mechanisms. Once lotteries are chosen, their mechanism outcome realizes, and agents choose which contract to commit to, here summarized by final state-contingent consumption bundles. Only one mechanism can be entered ex post, so that contracts are exclusive in the sense of PT. However, all sorts of mechanisms can be traded, or entered, at the ex-ante stage.

For this market structure, we introduce a notion of equilibrium which, beyond the standard requirements of optimization, rational expectations, and market clearing, captures the following essence of competition: prices are competitive if firms cannot favorably change them. A change is favorable when a firm cutting a price can make profits, expecting that other firms may exit only if it is in their interest to do so, and stay otherwise. When prices are immune from price cuts, the price system achieves incentive efficiency:

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\(^1\)For an interpretation of lotteries as sunspots or rationing, see PT (1984a,b) and Kehoe, Levine and Prescott (2002), or Gale (1996), respectively.

\(^2\)Lotteries do not play any fundamental role in the analysis: as the paper shows, dropping them yields payoff-equivalent equilibrium outcomes. An equivalent market formulation would instead allow individuals only to buy slots at mechanisms subject to a budget constraint.
at equilibrium, prices are such that all agents participate the constrained efficient mechanism, so that a first welfare theorem holds. Moreover, an equilibrium always exists. With two types the equilibrium is also outcome-unique and coincides with the separating mechanism most preferred by the high-quality types among the constrained optimal allocations giving at least the RS contract utility to the low-quality types. With more than two types, an equilibrium exists under Monotonicity and Sorting, mild generalizations of conditions used in the related literature. Constrained optimality generally allows for cross-subsidies and, with type-dependent utilities, for randomizations, so our equilibrium outcome is not always the RS outcome.

Related literature

There is no agreed-upon notion of competition or decentralization in economies with adverse selection. RS introduced a notion of competition among contract designers and obtained the disconcerting result that equilibrium may not exist and may not be constrained efficient. Later, C. Wilson (1977), Riley (1979) and Hellwig (1987) did propose versions of the notion of competition that regained existence. Most closely related to this paper, Miyazaki (1977) (see also Crocker and Snow (1985)) using C. Wilson’s notion of equilibrium studied a labor market economy with adverse selection (with only two types) where firms offer mechanisms. The equilibrium then coincides in allocation with ours. From this point of view, our contribution can be seen as a general competitive equilibrium translation of Miyazaki’s setting, and our notion of equilibrium a competitive version of Wilson’s. Crucially, in those papers individuals can choose only among the few mechanisms offered by the firms, and the profit-maximizing firms only offer resource feasible mechanisms. A price system is absent. Instead, in our market system individuals, and not just firms, can a priori choose among all possible mechanisms, resource feasible or not. The price system bears the burden of making the markets function, that is, of matching firms and individuals, and guarantees feasibility.

Gale (1992, 1996) also explored a notion of competitive market for contracts and proposed some stability-based refinement of beliefs to pin down the equilibrium, delivering coherence of incentives and competition, i.e., existence — see also Dubey and Geanakoplos (2002) and Zame (2007). However,

\[\text{Competition à la RS necessarily destroys cross-subsidies even when they would Pareto improve upon a separating contract. This takes the form of nonexistence when the pooling allocation, allowing cross-subsidies, dominates the separating contract.} \]
none of these market systems and associated refinements guarantees constrained optimality. Instead, our market system achieves incentive efficiency at competitive prices for two reasons: first, it prices mechanisms, not just contracts; second, it puts conditions on prices not via small action set perturbations, but by requiring that prices at inactive mechanisms cannot be favorably changed —a selection criterion over the set of (Walrasian) equilibria.\footnote{A market that prices only contracts, and not mechanisms, may not support incentive efficient outcomes even if prices cannot be favorably changed (see Section 10). In a related note (Citanna and Siconolfi (2011)), we show that pricing mechanisms, but using stability-based refinements à la Gale and Dubey and Geanakoplos, may not result in incentive efficient equilibrium outcomes either.}

Finally, following the PT work Bisin and Gottardi (2006) used a notion of Lindahl equilibrium to decentralize constrained efficient allocations. Nevertheless, their price system is coupled with restrictions on market participation, a fundamental ingredient for their optimality result (see Rustichini and Siconolfi, 2008). It also requires a distribution of consumption rights tailored to the actual distribution of types in the population. These conditions clash with the goal of decentralization à la R. Wilson (1987), that asks for the system to be universal —i.e., type distribution independent. In our construction, individuals only face prices for slots at various mechanisms, and no other institutional constraint limiting their choices. The price system is specified and works without requiring prior knowledge of the type distribution in the population. Of course, the cost we pay for a less informationally demanding mechanism is its computational complexity.

The paper is organized as follows. In Section 2 we introduce our insurance economies. In Section 3 we define the allocation problem as an assignment of individuals to mechanisms, and introduce other standard preliminary notions and the two main assumptions. In Section 4 we describe our market for mechanisms. Section 5 defines the notion of equilibrium we use, and states our main result, Theorem 2. Section 6 provides examples of applications beyond insurance economies. Section 7 describes preliminary properties derived from optimization, market clearing and rational expectations. Sections 8 and 9 provide the argument proving Theorem 2, discussing in turn incentive efficiency and existence of equilibrium. Section 10 discusses alternative market structures. Proofs are found in the Appendix.
2 The economies

We look at a simple, large economy with private information: there is only one physical consumption good, and a continuum of individuals with unobservable type \( s \in S \), a finite set. We denote by \( \pi_s \) the fraction of the type \( s \) agents in the population, with \( \pi_s > 0 \) and \( \sum_{s \in S} \pi_s = 1 \). Individuals have type-invariant, uncertain endowments, subject to finitely many idiosyncratic shocks \( \omega \in \Omega \). The individual endowment is \( e_\omega \) with \( e_\omega > e_{\omega'} > 0 \) for any \( \omega, \omega' \in \Omega \) with \( \omega > \omega' \).

A type \( s \in S \) is identified by a probability vector over \( \Omega \), \((\pi(\omega|s))_{\omega \in \Omega}\), and a cardinality index \( v_s : \mathbb{R}_+ \rightarrow \mathbb{R} \), a continuous, strictly increasing and strictly concave function. Thus, the utility to a type-\( s \) individual generated by a net trade \( z \equiv (z_\omega)_{\omega \in \Omega} \geq -e \equiv -(e_\omega)_{\omega \in \Omega} \) is

\[
u(z, s) \equiv \mathbb{E}_s [v_s(z + e)],
\]

where \( \mathbb{E}_s \) denoted the expectation operator given \( s \). Types are private information.

We order types by the expected value of their endowment \( \mathbb{E}_s(e) \) which we assume without essential loss of generality to be strictly increasing in \( s \). In line with many insurance models we interpret type \( s = 1 \) as the ‘bad’ type. First-order stochastic dominance and therefore the monotone likelihood ratio property imply this condition on \( \mathbb{E}_s(e) \), allowing us to also interpret the bad type as the high-risk type.

A special case of our setup is represented by what we call *Wilson economies* (Wilson, 1977). These are economies with type invariant cardinality indexes, \( v_s = v \) for all \( s \); moreover, \( \Omega = 2 \) (states are ‘High’, \( s = 2 \), or ‘Low’, \( s = 1 \)). The probability of the High state for type \( s \) is strictly increasing in \( s \). Wilson economies are themselves a generalization of *RS economies*, where \( S = 2 \).

3 The allocation problem

The classical point of view sees the allocation problem as an assignment of net trades (i.e., insurance levels) to individuals. Because of incentive compatibility due to adverse selection, it is instead convenient to think of the allocation problem here as consisting in assigning agents to mechanisms (see also PT, as well as Gale (1992)).
Mechanisms

A (direct) mechanism is a menu of possibly random insurance contracts, giving rise to a probability distribution over state-contingent net trades or consumption. Individual trades are assumed to be fully verifiable and enforceable—i.e., contracts are exclusive.

More precisely, a mechanism $\zeta$ is a collection of $S$ (Borel) probability measures, or contracts, $\zeta_s$, over $K \subseteq \mathbb{R}^\Omega$, a compact set of net trades, $z \geq -e$. The set of mechanisms is $Z$, a convex and (weak*) compact set.

Individuals rank probability distributions over net trades by using expected utility. Thus, the utility for type $s$ of a probability measure $\zeta_{s'}$ is:

$$u(\zeta_{s'}, s) = \int_K u(z, s) d\zeta_{s'} = \sum_{s'} \left[ \int_K v_s(z_\omega + e_\omega) d\zeta_{s'} \right] \pi(\omega|s),$$

A type-$s$ individual assigned to mechanism $\zeta$ can hide her type and select the probability measure that she likes the most. Thus, without loss of generality we can restrict attention to (or define the set of mechanisms as) the set $X$ of $S$-tuple of incentive compatible probability measures over net trades, that is,

$$X = \{ \zeta \in Z : u(\zeta_s, s) \geq u(\zeta_{s'}, s), \text{ all } s, s' \in S \},$$

and denote by $U_s(\zeta)$ the utility that a type-$s$ individual attaches to a mechanism $\zeta$. It is $U_s(\zeta) \equiv u(\zeta_s, s)$. As the maps $U_s(\zeta)$ are linear in $\zeta$, the set of incentive compatible mechanisms $X$ is convex (and since is closed, it is weak*-compact), and its size is to be further specified below after feasibility is introduced.

To use a unified notation we denote type $s$ expected net trade generated by mechanism $\zeta$ by

$$\mathbb{E}_s(\zeta) = \sum_{s'} \left[ \int_K z_{s'} d\zeta_s \right] \pi(\omega|s).$$

A mechanism $\zeta$ is pooling if $\zeta_s = \zeta_{s'}$ for all $s, s'$, and it is elementary if $\zeta = \delta_z$ where $\delta$ is the Dirac function and $z$ a collection of net trades $z_s \in K, s \in S$.

Parts of the analysis below use the following two assumptions placed on individual contracts:

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5 As noted among others by PT (1984b) and Kehoe, Levine, Prescott (2002), randomizations can ease the incentive compatibility constraints when the cardinality index is type dependent.
Monotonicity Let $\zeta \in \Delta(K)$ be such that $U_s(\zeta) \geq v_1(e_1)$ and $E_s(\zeta) \leq 0$. Then, $E_s(\zeta) \leq E_s(\zeta) \leq 0$ for all $\sigma > s$.\footnote{Monotonicity is generally stated as a property of individual contracts, that is, $E_s(z) \leq E_s(z)$, for all $z \in K$, $\sigma > s$, (see, e.g., Guerrieri, Shimer and Wright, 2010). Our version is weaker and can be further weakened by replacing as lower bound the contract $(E_1(e) - e_0)\omega \in Q$ with the mechanism $\zeta^{RS}$ that generalizes the separating RS outcome, defined in Section 7.}

Sorting Let $\zeta, \zeta' \in \Delta(K)$ be such that $U_s(\zeta) \geq U_s(\zeta')$, while $U_{s+1}(\zeta) \leq U_{s+1}(\zeta')$. Then,

$$U_\sigma(\zeta) \leq U_\sigma(\zeta') \text{ all } \sigma > s,$$

$$U_{\sigma'}(\zeta) \geq U_{\sigma'}(\zeta') \text{ all } \sigma' \leq s.$$ 

Versions of Monotonicity and Sorting are widely used in the contract literature, even in recent equilibrium analyses of competition with contracts (see, e.g., Guerrieri et al. (2010)). Wilson economies satisfy Monotonicity and Sorting, as it is easily proved. We emphasize that these assumptions do not have bite when $S = 2$: Monotonicity holds immediately as $U_1(\zeta) \geq U_1(\zeta^{RS})$ and $E_1(\zeta) \leq 0$ imply that $\zeta = \zeta^{RS}$; since $\zeta^{RS}$ provides full insurance and $E_s(e)$ is already assumed to be increasing with $s$, the claim follows; Sorting, instead, is vacuous. Hence, when $S = 2$ and utilities are type-dependent, the usual ‘single-crossing’ condition may be violated, but our analysis goes through. More generally, for $S > 2$ an economy may fail ‘single-crossing’, while still satisfying Sorting.

Many results below are proved without using Monotonicity or Sorting. Hereafter, unless we explicitly state them, it must be understood that these assumptions are not needed.

Lotteries, feasibility and incentive efficiency

As in the PT tradition, the assignment problem can be recast in terms of lotteries. As we will show below, though, lotteries do not play here any fundamental role, and are used mainly for convenience of exposition, and for ease of comparison with the literature. Let $\Delta(X)$ be the set of lotteries, i.e., (Borel) probability measures, over $X$. The utility from a lottery $\nu_s \in \Delta(X)$ is

$$U_s(\nu_s) = \int_X U_s(\zeta) d\nu_s.$$ 

Given the linearity of the map $U_s(\nu_s)$, hereafter we write $U_s\nu_s$. Since $U_s\delta_s = U_s(\zeta)$, we feel free to switch from $U_s\delta_s$ to $U_s(\zeta)$ in notation whenever more
convenient. We also denote by $\text{supp}(\nu_s)$ the support of lottery $\nu_s$. Even if $\nu_s = \delta_\zeta$ is degenerate, it may allow randomizations within mechanism $\zeta$.

As the economy is large, by the appropriate version of the Law of Large Numbers the set of feasible lotteries or allocations is

$$Y = \{\nu \in \Delta(X)^S : \sum_s \pi_s \int_X E_s(\zeta) d\nu_s \leq 0\}$$

and $\bar{Y}$ is the set of allocations satisfying exact feasibility, i.e., feasibility with equality. Feasible allocations are ex-post incentive compatible because their support is contained in $X$. Feasibility of allocations itself implies budget balance in the aggregate only: given a feasible allocation $\nu$, it is still possible for a mechanism $\zeta \in \text{supp}(\nu)$ to consume more resources than those belonging to its participants, i.e., $\sum_s \pi_s E_s(\zeta) > 0$. A mechanism $\zeta$ is (exactly) feasible if $\delta_\zeta \in Y$ ($\bar{Y}$).

As mentioned earlier, to make the analysis interesting we assume that the set $X$ (and hence $Z$) is rich. It contains all feasible and incentive compatible elementary mechanisms as well as $\bar{z} = (\bar{z}_s)_{s \in S}$, a pooling elementary mechanism satisfying $\bar{z} \gg 0$, $\bar{z}_{\omega,s} + e_{\omega}$ is $\omega$-invariant, and $\bar{z}_s \geq z$, for all $z \in K$. As preferences display risk aversion and strict monotonicity, it is

$$U_s(\bar{z}) > U_s(0) \quad \text{for all } s, \text{ all } \nu_s \in \Delta(X) \setminus \{\delta_\bar{z}\}. $$

Thus, in particular, individual preferences are nonsatiated in the feasible set.

An allocation $\nu$ is individually rational if $U_s \nu_s \geq U_s(0)$, all $s$; it is (ex-ante) incentive compatible if

$$U_s \nu_s \geq U_s \nu_{s'}, \text{ all } s, s' \in S; \quad \text{(IC)}$$

it is a Constrained Pareto Optimal (hereafter CPO) if it is feasible, individually rational, incentive compatible and there is no other feasible, incentive compatible allocation $\tilde{\nu}$ such that

$$U_s \tilde{\nu}_s \geq U_s \nu_s, \text{ all } s \in S,$$

with one strict inequality; it is a strict CPO if the previous inequalities are strict for all $s \in S$. As preferences are not satiated within the feasible set, strict CPO are elements of $\bar{Y}$. 
We now describe a market system for assigning agents to mechanisms. We allow agents to buy slots at mechanisms in $X$, given prices $p(\zeta)$ for an individual slot at mechanism $\zeta$. Such prices $p(\zeta)$ can be positive or negative. One can think of $p(\zeta)$ as the fee paid by an individual to participate mechanism $\zeta$; if this number is negative, the agent receives money to enter the mechanism. Individuals’ behavior is restricted by a budget constraint.

Simultaneously, slots at mechanisms, or simply mechanisms, are offered by a large, but finite number of profit maximizing firms/intermediaries. Offering one unit of mechanism $\zeta \in X$ yields $p(\zeta)$ revenues to the firm.

The market matches agents and firms into mechanisms via prices. A more Walrasian view of this matching market can be equivalently used: prices direct demand for and supply of slots at each mechanism.

We will show that such a (matching) market can function well, in that in a suitable, natural equilibrium state, agents and firms are matched via the price system into mechanisms in the right numbers, and the equilibrium is incentive efficient. One may wonder if pricing only elementary mechanisms or, as in most of the literature, just individual contracts would not be enough for our purposes. We will address this modeling issue in Section 10 after having presented all necessary tools.

Public randomization devices are not needed to coordinate the agents’ decisions in this market. However, once again for completeness and comparison with the literature, we allow agents access to lotteries $\mu \in \Delta(X)$ over mechanisms, and thus consider the consumption space to be $\Delta(X)$, as opposed to just $X$. Then, prices $p$ are assumed to be linear in the objects of trade, that is, in lotteries, and are identified with real-valued, continuous functions with domain $X$.

Firms and the aggregate production set

Firms guarantee that there will be resources to consume at each mechanism, and verify and enforce the agents’ resulting net trades at a mechanism. In order to make profit-maximizing choices, firms need to know both the market price as well as the cost of running mechanisms. The latter is determined by the proportion of types choosing each mechanism, that is, by the demand. Since demand and supply are formed simultaneously, firms need to form beliefs $\beta(\zeta)$ about demand in each mechanism $\zeta$. As customary, we assume
that beliefs are common across firms.\textsuperscript{7} The expected cost of having type \( s \) in mechanism \( \zeta \) is \( \mathbb{E}_s(\zeta) \). Thus, denoting by \( \beta(s; \zeta) \) the believed fraction of type \( s \) in mechanism \( \zeta \), the expected cost of running such a mechanism is

\[
\mathbb{E}_\beta(\zeta) = \sum_s \mathbb{E}_s(\zeta) \beta(s; \zeta).
\]

For later interpretation it is worth stressing that each offered mechanism is large, as it may potentially serve a large number of individuals, and yet small as it may be offered by many firms. These many firms may either each offer multiple mechanisms, or each a different mechanism. As usual in competitive analysis, it suffices to describe the aggregate production set, i.e., the sum of each individual firm’s production set. Following PT, the aggregate production set is the set of positive, finite (Borel) measures over \( X \), \( \mathcal{M}_+(X) \), and displays constant returns to scale. This is to say that firms running mechanism \( \zeta \) can supply any positive number of slots at such mechanism independently of what they do at other mechanisms. An aggregate supply is a positive measure \( \nu^* \in \mathcal{M}_+(X) \), and the total cost of offering a measure \( \nu^* \) of mechanisms is \( \int_X \mathbb{E}_\beta(\zeta) d\nu^*(\zeta) \).

**Zero profits, budgets and prices**

Since firms are profit maximizers and the aggregate technology, \( \mathcal{M}_+(X) \), displays constant returns to scale, profits must be zero. Otherwise, the firm(s) operating mechanism \( \zeta \) would realize unbounded profits by supplying infinite slots at such a mechanism. For this reason, individual property rights over mechanisms do not need to be specified. Thus, at given \( p \) the individual budget set is simply identified with lotteries \( \mu \) satisfying

\[
p \mu = \int_X p(\zeta) d\mu \leq 0.
\]

The zero-profit condition restricts prices to satisfy

\[
p(\zeta) \leq \mathbb{E}_\beta(\zeta) \text{ for all } \zeta.
\]

This break-even or zero-profit condition is common to related competition models (see, e.g., Gale (1992), Dubey and Geanakoplos (2002), and Zame (2007)). Without loss of generality, hereafter we identify the price domain with

\textsuperscript{7}Alternatively, commonality of beliefs can be introduced as an equilibrium condition. We also restrict \( \beta \) to be measurable as a function from \( X \) to \( \Delta(S) \).
\[ P = \{ p \in \mathbb{C}(X, \mathbb{R}) : p(\zeta) = \mathbb{E}_\beta(\zeta) \text{ all } \zeta, \text{ for some belief system } \beta \}. \]

Equivalently, \( p \in P \) if and only if \( p(\zeta) \in [p_m(\zeta), p_M(\zeta)] \) for \( p_m(\zeta) = \min_s \mathbb{E}_s(\zeta) \) and \( p_M(\zeta) = \max_s \mathbb{E}_s(\zeta) \). As \( p \in P \), the cost of no trade (the elementary mechanism \( z_s = 0, \text{ all } s \)) is zero so that individuals can (and will) make individually rational choices. As all individuals face identical budget constraints, they can and will buy (ex-ante) incentive compatible lotteries. Moreover, for all \( p \), not necessarily continuous, but satisfying \( p(\zeta) \in [p_m(\zeta), p_M(\zeta)] \), it is \( p(\tilde{z}) > 0 \) (with \( \tilde{z} \) defined above) so that preferences are not satiated in the budget set.

## 5 Equilibrium

We introduce a notion of competitive equilibrium to study the market allocation outcomes. The natural definition of competitive equilibrium must include a consistent collection of prices, allocations, and beliefs via three standard requirements: optimization, market clearing and rational expectations. These requirements are common to existing definitions in the literature (as in Gale (1992), Dubey and Geanakoplos (2002), and Zame (2007)), though our price system differs in view of what is tradeable. In economies without adverse selection, those are the only criteria needed for defining Walrasian equilibria. With adverse selection the set of equilibria so defined can be quite large, as we explain later on. To restrict the equilibrium set, the existing literature adds as well a refinement or selection criterion based on some notion of stability or forward induction.\(^8\) Instead, as a second departure from the literature, we use a refinement close in spirit to C. Wilson’s (1977) notion of ‘anticipatory equilibrium’.

In order to make this notion precise we need to introduce and explain three ingredients: a profit function, expressing unit profits of mechanisms as a function of price-allocation pairs; a market indicator, making precise the idea of shutting down the mechanisms that lose money; price cuts, formalizing the notion that prices for some mechanisms are reduced.

**Profit function, market indicator, and price cuts**

\(^8\) The RS analysis of competitive screening can also be seen as a refinement imposing too strong restrictions on inactive mechanism beliefs – hence the nonexistence.
The profit function $\Pi$ measures the unit profits of any mechanism for a given allocation price pair $(\mu, p) \in \Delta(X)^S \times \mathbb{P}$. The allocation $\mu$ defines a set of active mechanisms (the set $\text{supp}(\sum_{s'} \pi_{s'} \mu_{s'})$) and a set of inactive mechanisms (the set $X \setminus \text{supp}(\sum_{s'} \pi_{s'} \mu_{s'})$). The proportion of type $s$ in active mechanism $\zeta$ is equal to the number of individuals of type $s$ in $\zeta$ divided by the total number of individuals in $\zeta$. Formally, this is $\frac{\pi_s d\mu_s(\zeta)}{\sum_{s'} \pi_{s'} d\mu_{s'}(\zeta)}$. Hence, the profit of active mechanism $\zeta \in \text{supp}(\sum_{s'} \pi_{s'} \mu_{s'})$ is

$$\Pi(\zeta; (\mu, p)) = p(\zeta) - \sum_s E_s(\zeta) \frac{\pi_s d\mu_s(\zeta)}{\sum_{s'} \pi_{s'} d\mu_{s'}(\zeta)}.$$ 

For inactive mechanisms $\zeta \in X \setminus \text{supp}(\sum_{s'} \pi_{s'} \mu_{s'})$ (a set of $\sum_{s'} \pi_{s'} \mu_{s'}$-zero measure), we set

$$\Pi(\zeta; (\mu, p)) = 0.$$ 

Next, a market indicator is a measurable map $\eta : X \to \{0, 1\}$ where $\eta(\zeta) = 0$ means that mechanism $\zeta$ is shut down, while $\eta(\zeta) = 1$ means that $\zeta$ is tradeable, and $X_\eta$ is the set of tradeable mechanisms. Given a price system $p$ and a market indicator $\eta$, individuals choose lotteries over tradeable mechanisms, that is they solve

$$\max_{\mu \in \Delta(X_\eta)} U_s \mu \text{ subject to } p \mu \leq 0.$$ 

Let $M_\eta(p, \eta) \subset \Delta(X_\eta)$ be the set of optimal solutions, possibly empty, and $M(p, \eta) = \times_s M_\eta(p, \eta)$.

An effective price cut against price system $p$ is $(p', \eta)$ a price - market indicator pair, with $p' < p$, a bounded measurable function such that:

1. $p'(\zeta) < p(\zeta) \implies \eta(\zeta) = 1$;
2. (viability) for some $\mu \in M(p', \eta)$, $\eta(\zeta) = 1 \implies \Pi(\zeta; (\mu, p')) \geq 0$ with strict inequality if $p'(\zeta) < p(\zeta)$;
3. (maximality) for all $\eta' \geq \eta$, $\eta'(\zeta) - \eta(\zeta) = 1 \implies \Pi(\zeta; (\mu', p')) < 0$, for some $\mu' \in M(p', \eta')$.

Each probability measure $\nu_s$ is absolutely continuous with respect to the measure $\sum_s \pi_s \nu_s$. Thus, the Radon-Nikodym derivative of $\pi_s \nu_s$ with respect to $\sum_s \pi_s \nu_s$ is well defined.

The Radon-Nikodym derivative of $\pi_s \mu_s$ with respect to $\sum_s \pi_s' \mu_{s'}$ is unique only $\sum_s \pi_s \nu_s$-a.e.. Hence, for inactive mechanisms it can be chosen arbitrarily so that $\Pi(\zeta; (\mu, p)) = 0$. 

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We stress that, given a price \( p_0 \), there may be several functions \( \eta \) that make \((p', \eta)\) an effective price cut. Viability and maximality can be slightly modified without affecting any of the results.\(^{11}\) They play a fundamental role in restricting price cuts—hence, indirectly, beliefs and therefore prices—at equilibrium.

**Equilibrium**

We are now ready to define our equilibrium notion.

**Definition 1** An equilibrium is an array \((\nu, \beta, \nu^*, p)\) of an allocation \(\nu\), beliefs \(\beta\), an aggregate supply of the firms \(\nu^*\), and a price system \(p \in \mathbb{P}\) such that:

- **(O) (Optimization)** given \(p\), \(\nu_s \in \arg\max_{\mu \in \Delta(X)} U_s \mu\) s.t. \(p_\mu \leq 0\), all \(s \in S\);
  
given \(p\) and \(\beta\), \(\nu^* \in \arg\max_{\mu \in \mathcal{M}^+}{\int}_X [p(\zeta) - \mathbb{E}_\beta(\zeta)]d\mu;\)

- **(M) (Market clearing)** \(\sum_s \pi_s \nu_s(B) = \nu^*(B)\) for all \(B \in \mathcal{B}(X)\);

- **(C) (Belief consistency)** \((\nu, \beta)\) are compatible, or

\[
\pi_s \nu_s(B) = \int_B \beta(s; \zeta) d\left(\sum_{s'} \pi_s \nu_{s'}\right), \quad \text{all } s \in S, B \in \mathcal{B}(X);
\]

- **(NPC) (no price cuts)** the price system \(p\) is immune from effective price cuts: there is no effective price cut against \(p\).

The first three equilibrium requirements are standard. Requirement (O) is the usual behavioral restriction on individuals and firms. Notwithstanding the large number of firms, the constant returns to scale feature of the aggregate production set \(\mathcal{M}^+\) and their common beliefs allow for collapsing the firms’ choice problems into the “aggregate” profit maximization problem. Requirement (M), or ‘demand equals supply’, highlights the allocation problem as an assignment problem, as mentioned above. Requirement (C) is a rational expectations condition. It says that the proportions of types in any subset of mechanisms are ultimately determined at equilibrium by the optimal individual choices. That is, whenever \(\nu^*(B) > 0\)—and therefore via

\(^{11}\)For example, we can replace ‘for some \(\mu' \in M(p', \eta')\)’ in condition (3) with ‘for all \(\mu' \in M(p', \eta')\)’; or impose even stricter conditions regarding mechanism shut-down, say, by asking that a mechanism can be shut down if, upon reopening, it loses money and makes the undercut mechanism not lose money.
(M) \( \sum_s \pi_s \nu_s(B) > 0 \) – beliefs are uniquely determined by \( \nu^* \) and they are consistent with Bayes’ rule:

\[
\beta(s; B) = \frac{\int_B \beta(s; \zeta) d\nu^*(\zeta)}{\nu^*(B)} = \frac{\pi_s \nu_s(B)}{\sum_{s'} \pi_{s'} \nu_{s'}(B)}.
\]

However, (C) does not pin down beliefs for inactive mechanisms, that is, for \( \zeta \) outside sets of \( \nu^* \) zero measure. Importantly, \( \nu \) may not be (and typically will not be) absolutely continuous with respect to the Lebesgue measure. Thus, beliefs may be disconnected from allocations in large sets of the mechanisms space.

Requirement (NPC) roughly states that at equilibrium it is not possible to make positive profits by cutting prices of inactive mechanisms while expecting that other firms react by shutting down mechanisms that lose money as a result. This is the condition echoing C. Wilson’s (1977) notion of ‘anticipatory equilibrium’. As Wilson (1977), we think of our equilibrium as a stationary point of an otherwise unspecified dynamic process.

To gain insight into (NPC), it is useful to revisit the meaning of price taking behavior for a standard competitive firm. A firm takes prices as given at equilibrium because it has no incentive to change them. Why? By increasing prices the firm loses all of its customers, as they can get the same product from the competitors at a lower price. By decreasing prices, the firm is pricing below the exogenously given marginal cost, thereby decreasing its profits. In our equilibrium the price of an active mechanism also is equal to its marginal (and average) cost, which is nevertheless endogenously determined by the demand. Moreover, for inactive mechanisms, this cost is determined by beliefs unrelated to the statistics of market participation. A firm operating an inactive mechanism may thus want to challenge (or not agree with) the market belief and decrease the price of this mechanism. As we allow other firms to react to the price cut, it is pointless to let these firms reduce the prices of other mechanisms as these possibilities are already included in the notion of price cut. Hence, the only possible competitive reaction to a price cut is to shut down loss-yielding mechanisms. The extent to which mechanisms can be shut down is limited only by the firms’ interest to do so (a free exit rationale), via the notion of maximal shut-down.

We could have modified our definition of effective price cut in two opposite directions. At one extreme, we could consider unrestricted price cuts where no firm leaves the market. This may leave the equilibrium set empty exactly
for the same reason that the RS equilibrium may fail to exist: such price cuts can be used to recreate RS-style cream-skimming. At the opposite extreme, we could instead limit attention to price cuts that are consistent with nonnegative profits for all mechanisms, substantially reducing the firms’ ability to compete with each other. We would still obtaining existence, but potentially support lots of inefficient outcomes.

The main result of the paper is stated as follows.

**Theorem 2** An equilibrium allocation is incentive efficient. Under Monotonicity and Sorting, an equilibrium exists.

As shown and discussed below (see Lemmas 2 and 7), existence of equilibrium in this market does not hinge upon agents trading lotteries. Instead, the RS nonexistence is bypassed here because equilibria are parametrized by beliefs—as in, e.g., Gale (1992)—and beliefs are disciplined by condition (NPC). In fact, lotteries expand the agents’ trading opportunities, and make price cuts potentially more attractive, hence existence of equilibrium potentially more problematic.

The proof of Theorem 2 occupies much of the rest of the paper. Before entering the details, to address the robustness of our construction and of the theorem it may help going over possible further applications of the main model where the result still applies.

### 6 Further applications

Our framework and results extend beyond insurance to similar setups of large adverse selection economies with essentially only one physical good; this is the case, say, of Spence’s (1973) labor market signaling model, or of Akerlof’s (1976) ‘rat race’ analysis (as modeled for instance by Miyazaki (1977)). We illustrate this via two examples.

**Labor market signaling**

Consider the simplest such economy, where education does not affect workers’ productivity. Here $S$ represents the number of worker types, $\pi_s$ the fraction of the type $s$ workers in the population. Type $s$ affects a worker’s utility as well as his productivity, $\theta_s$. Workers can supply inelastically one

\[\text{In fact, as it is obvious, the proofs in the Appendix show that equilibria which are immune only from these restricted price cuts always exist.}\]
unit of labor yielding $\theta_s$ units of output. In addition, (prior to working) they can invest, without loss of generality, at most one unit of time in education. A ‘net trade’ is $z = (z_1, z_2)$, where $z_1 \geq 0$ is the salary received by the worker, and $z_2 \geq 0$ the level of education required. The worker’s utility is $u(z, s) = v_s(z_1, 1 - z_2)$, and $v_s$ is strictly increasing in its arguments. Moreover, it is assumed that $\theta_s$ is increasing in $s$.

In these economies, firms and workers match on the basis of wage-education mechanisms, and of their prices. The set $X$ of mechanisms $\zeta$ is described as in Section 3 and so are utilities $U_s(\zeta)$. Once the map $\mathbb{E}_s(z)$ is replaced by $r_s(z) = z_1 - \theta_s$, and $\mathbb{E}_s(\zeta)$ by $r_s(\zeta) = \int r_s(z) d\zeta_s$, every definition carries verbatim: lotteries over $X$, feasibility, incentive efficiency, the price system and firms. For instance, feasibility is $\sum_s \pi_s \int_X r_s(\zeta) dv_s \leq 0$; the cost of running mechanism $\zeta$ is $r_\beta(\zeta) = \sum_s r_s(\zeta) \beta(s; \zeta)$; $X^0$ is defined with $r_s(\zeta) \leq 0$, all $s$, and the generalized RS mechanism has $r_1(\zeta^{RS}) = 0$. ‘Full insurance’ obviously coincides with the full information type-$s$ optimal allocation, solving

$$\max_{\zeta \in \Delta(X)} U_s(\zeta) \quad \text{s.t.} \quad r_s(\zeta) \leq 0,$$

here implying $z_2 = 0$ (zero education, or no signaling). Monotonicity is immediate from $\theta_s$ increasing in $s$, and Sorting is implied by the commonly adopted assumption of single crossing — though the latter is not necessary. Our Theorem 2 then goes through, and we obtain that, whenever constrained efficient, cross-subsidies arise at equilibrium in the market, in contrast with zero-subsidy separation typically obtained with other trading systems or equilibrium refinements.

The rat race

Here type $s$ affects a worker’s utility as well as his productivity, $f_s(\ell)$, where $\ell$ denotes the amount of work (or speed) provided. Workers choose labor supply as leisure affects their utility function. A ‘net trade’ is $z = (z_1, z_2)$, where $z_1 \geq 0$ is the salary received by the worker, and $z_2 = \ell \geq 0$ the amount of labor offered. For these economies mechanisms represent menus of labor contracts, which are priced by the market system. The worker’s utility again is $u(z, s) = v_s(z_1, 1 - z_2)$, assumed to satisfy standard conditions. The set $X$ and $U_s(\zeta)$, $\zeta \in X$, are described in the exact same way as in Section 3. The map $\mathbb{E}_s(z)$ is replaced now by $r_s(z) = z_1 - f_s(z_2)$. ‘Full insurance’ again coincides with the full information type-$s$ optimal allocation. Monotonicity is implied by $f_s(\ell)$ increasing in $s$, for all $\ell \in [0, 1]$, and Sorting by the commonly adopted assumption of single crossing. Again, Theorem 2 holds.
7 Preliminary properties

Feasibility of equilibrium
At equilibrium firms make zero profits, and (ZP) holds with equality. This implies that equilibrium allocations are resource feasible. To see this, let \((\nu, \beta, \nu^*, p)\) be an equilibrium; then since \(\nu_s\) is budget feasible for all \(s\)

\[
\int p(\sum_s \pi_s d\nu_s) \leq 0.
\]

Thus, using (ZP), (M) and (C), we obtain

\[
0 \geq \sum_s \pi_s \int p(\zeta)d\nu_s = \int_X E_{\beta}(\zeta)d\nu^* = \sum_s \pi_s \int_X E_s(\zeta)d\nu_s,
\]

i.e., feasibility. Moreover, as \(p \in \mathbb{P}\), preferences are not satiated in the budget set and each \(\nu_s\) satisfies the budget constraint with equality, so that the expression above is an equality. Thus, equilibrium allocations belong to \(\hat{Y}\).

Generalized RS mechanisms
A generalized Rothschild and Stiglitz (or RS) mechanism, \(\zeta^{RS}\), is the solution to the vector maximization problem

\[
\max_{\zeta \in X^0} (U_s(\zeta))_{s \in S}, \quad (RS)
\]

where

\[
X^0 = \{\zeta \in X : E_s(\zeta) \leq 0, \text{ all } s\}.
\]

The set of generalized RS mechanisms \(X^{RS}\) may contain many mechanisms, though all of them are payoff equivalent by construction. For RS economies, the set \(X^{RS}\) is a singleton coinciding with the RS separating allocation. For general economies, the set \(X^{RS}\) is characterized by the following lemma whose proof is left to the reader.

Lemma 1 (generalized RS mechanism) \(X^{RS}\) is a nonempty subset of \(X^0\). Moreover, \(\zeta^{RS}_{1} = \delta_{z^{RS}_{1}}\) with \(z^{RS}_{\omega,1} + e_\omega = E_1(e)\), for all \(\omega\), and for all \(\zeta^{RS} \in X^{RS}\).

Only the second statement of Lemma 1 needs a short explanation. It stems from a simple generalization of the RS argument and relies on two
observations. First, the pooling allocation assigning $\zeta_{RS}^{1}$ to all $s$ is an element of $X^{0}$ as $E_{s}(\zeta_{RS}^{1}) = E_{1}(e) - E_{s}(e) \leq 0$, all $s$. Second, $\zeta_{RS}^{1}$ is the optimum at type 1’s odds.

Since $\zeta_{RS}^{1} \in X^{0}$, any price system in $P$ gives nonpositive value to these mechanisms. Mechanisms $\zeta_{RS}^{1}$ serve as an outside option for each type as any type is always able to buy them in the market.

Existence and indeterminacy of weak equilibrium allocations

We call weak equilibrium an array $(\nu, \beta, \nu^{*}, p)$ satisfying all equilibrium conditions, but (NPC). As in related work by Gale (1992), weak equilibria exist, but they may be severely indeterminate. Here, we give a precise characterization of such phenomenon.

If $\mu$ is a weak equilibrium allocation it is an element of the set

$$\Delta' = \{ \mu \in \Delta(X)^{S} | U_{s} \mu_{s} \geq U_{s}(\zeta_{RS}), \zeta_{RS} \in X^{RS}, \text{all } s \}.$$  

In the next lemma we show that all mechanisms in $\Delta' \cap \tilde{Y}$ are weak equilibrium allocations, and that the payoff indeterminacy of weak equilibrium is entirely exhausted by such mechanisms. Therefore, if RS mechanisms are CPO, weak equilibria coincide with $X^{RS}$ in the allocation space. When this is the case, RS equilibria exist and equilibrium and weak equilibrium coincide. However, if RS equilibria do not exist, weak equilibria are indeterminate.

A tight characterization of the weak equilibrium set necessitates the introduction of two ancillary notions. First, a mechanism $\pi_{s} \in X$ is equivalent to an allocation $\nu$ if

$$U_{s}(\nu) = U_{s}(\pi_{s}) \text{ and } \int E_{s}(\zeta) d\nu_{s} = \int E_{s}(\pi_{s}) d\pi_{s}, \text{ all } s.$$  

In the Appendix we argue that given $\nu$ the standard operation of compounding defines an equivalent mechanism.

Second, a price $p \in P$ supports mechanism $\zeta$ if there exists $\beta$ such that $(\nu, \beta, \nu^{*}, p)$ with $\nu^{*} = \nu_{s} = \delta_{\zeta}$, all $s$, is a weak equilibrium; $P(\zeta) \subset P$ is the set of prices that support $\zeta$. Hence, if $P(\zeta) \neq \emptyset$, mechanism $\zeta$ is a weak equilibrium allocation.

**Lemma 2** Any mechanism $\zeta^{*} \in \Delta' \cap \tilde{Y}$ is a weak equilibrium allocation. In particular, there exists a collection of scalars $(\lambda_{s})_{s \in S} \gg 0$ such that the price function $\bar{p}$ defined as

$$\bar{p}(\zeta) = \max \left\{ \max_{s \in S} \frac{U_{s}(\zeta) - U_{s}(\zeta^{*})}{\lambda_{s}}, p_{n}(\zeta) \right\}$$  

18
is an element of $P(\zeta^*)$. Moreover, if $\nu$ is a weak equilibrium allocation, then there exists an equivalent mechanism $E\nu$ with $P(E\nu) \neq \emptyset$.

Hence, a price system $p$ that evaluates each mechanism $\zeta$ at utility value relative to a reference point $\zeta^* \in \Delta'$ and for the type that values $\zeta$ the most delivers a weak equilibrium.

It is worth making two comments about Lemma 2. First, lotteries do not play here the role of mixed strategies in the strategic analysis of Dasgupta and Maskin (1986); that is, randomizations are not used here to restore existence. Indeed, by the first part of Lemma 2, weak equilibria exist even if we restrict the individual choice set simply to mechanisms in $X$ with $p(\zeta) \leq 0$. By the last part of Lemma 2, every weak equilibrium allocation is payoff equivalent to a degenerate weak equilibrium allocation. Thus, lotteries play no essential role in the analysis of weak equilibrium, and therefore of equilibrium. Consider Wilson economies and recall that their CPO allocations are elementary mechanisms. Therefore, for such economies the set of elementary mechanisms in $\Delta'$ is never empty. Then Lemma 2 implies that weak equilibria exist even if the individual choice set is restricted to elementary mechanisms. Existence of weak equilibria is not restored by having allowed for random mechanisms either.

In fact, weak equilibria exist and are indeterminate because beliefs (for inactive mechanisms) are treated as unrestricted endogenous variables. As we will see below, condition (NPC) in Definition 1 implicitly restricts beliefs, without compromising existence –or incentive efficiency. Instead, to compare, treat beliefs as in RS: drop condition (NPC) and instead, at a competitive equilibrium, let the price of any mechanism be determined by the odds of the types that strictly prefer it to the candidate equilibrium allocation –a restriction on beliefs. With such a treatment of beliefs, a competitive equilibrium may not exist. Indeed, consider a RS economy where the pooling allocation most preferred by the low-risk type $z_p; 2$ strictly Pareto dominates the separating allocation $z^{RS}$. As known, these economies fail to have RS equilibria. By a very similar argument, these economies fail to have a competitive equilibrium with the RS treatment of beliefs. To see this, recall that single-crossing implies that for every mechanism $\zeta \in \Delta' \cap Y$ with $\zeta \neq \delta z^{RS}$ there exists a cream-skimming mechanism $\zeta'$, i.e., such that $U_2(\zeta') > U_2(\zeta)$, $U_1(\zeta') < U_1(\zeta)$, and $E_2(\zeta') < 0$. Then, by the RS treatment of beliefs, it is $p(\zeta') = E_2(\zeta') < 0$. As $\zeta'$ is budget feasible, the low-risk individuals cannot choose optimally $\zeta$, that fails therefore to be an equilibrium. Thus, we are
left considering $z^{RS}$ as the only possible equilibrium candidate. However, $U_s(z^{RS}) < U_s(\bar{z}^{RS})$, all $s$, and $\sum \pi_s E_s(\bar{z}^{RS}) = 0$ imply that $p(z^{RS}) = 0$. Again, since $\bar{z}^{RS}$ is budget feasible individuals cannot choose optimally $z^{RS}$. Therefore, with the RS treatment of beliefs the competitive equilibrium set is potentially empty.

8 Incentive efficiency of equilibrium

With the preliminary properties stated, we can now show that an equilibrium must be a strict CPO, establishing the first half of Theorem 2.

Lemma 3 If $\zeta^*$ is an equilibrium allocation, then it is a strict CPO.

Why? A strictly inefficient mechanism can only be sustained as a weak equilibrium by prices for strictly Pareto superior mechanisms —including a strictly feasible one such mechanism— that are too high, that is, positive. Competition across mechanisms (through price cuts, i.e., condition (NPC)) insures that such a situation is not stable, as firms can favorably change prices—from positive to zero—to increase their profits on the strictly feasible, Pareto superior mechanism while every type will flock to this mechanism at zero price.

However, competition is not strong enough to destroy equilibrium allocations that are strict CPO, but not CPO. Indeed, because of the nature of the incentive compatibility constraints, strict monotonicity of preferences does not suffice to make strict CPO and CPO allocations equivalent. The argument in the absence of incentive constraints goes as follows: if $\zeta$ is a strict, but not a weak Pareto optimum, there is a feasible $\zeta'$ such that $U_s(\zeta') \geq U_s(\zeta)$ for all $s$, with strict inequality for at least one $\bar{s}$; then, by continuity, $U_{\bar{s}}(\zeta'') > U_{\bar{s}}(\zeta)$ for some $\zeta''$ with $E_{\bar{s}}(\zeta'') < E_{\bar{s}}(\zeta')$ arbitrarily close to $\zeta''$; the gap $E_{\bar{s}}(\zeta' - \zeta'') > 0$ can then be redistributed among the other individuals $s \neq \bar{s}$ so to find a feasible allocation $\zeta$ that makes everybody better off, a contradiction to $\zeta$ being a strict Pareto optimum. However, in adverse selection economies this argument breaks down because $\zeta$ may fail to be incentive compatible. It is known that if the cardinality indexes are type invariant, $v_s = v$ for all $s$, as in Wilson economies, then strict CPO and CPO allocations coincide. Interestingly, the argument exploits the $s$ invariance of $v_s$ reducing lotteries over $\zeta$ and $\zeta'$ to payoff-equivalent elementary mechanisms that use less resources. At that point, and only then, it exploits
strict monotonicity of preferences. It is an open question to find minimal conditions under which strict CPO and CPO allocations coincide.

9 Existence of equilibrium

If \( X^{RS} \) is contained in the strict CPO set, equilibrium price systems always exist and the equilibrium allocation is payoff unique. This is shown in the next lemma, which does not require either Sorting or Monotonicity.

Lemma 4 If \( X^{RS} \) is contained in the strict CPO set, then \( X^{RS} \) is contained in the set of equilibrium allocations. If \( X^{RS} \subset CPO \), then the equilibrium set coincides with \( X^{RS} \).

This is because if a mechanism \( \zeta \) is a strict CPO, then \( \zeta \in \hat{Y} \). Therefore, mechanisms in \( X^{RS} \) have \( E_s(\zeta^{RS}) = 0 \) for all \( s \), and they cannot be shut down after an effective price cut, due to the maximality requirement on \( \eta \) (condition (3)). If the price cut were effective, there should be an allocation Pareto dominating \( \zeta^{RS} \) and strictly feasible, as it has zero cost and generates positive profits. Such an allocation cannot exist because nothing strictly feasible can Pareto dominate a strict CPO. The second part of Lemma 4 is a consequence of the first, of the fact that equilibrium is a subset of weak equilibrium, and of Lemma 2.

S-CPO

Existence of equilibrium is generally proved by showing that a specific strict CPO is an equilibrium allocation. Consider the following recursive family of programming problems indexed by \( s = 2, \ldots, S \). Let \( V_1(1) \equiv U_1(\zeta^{RS}) \). For \( s \geq 2 \), the \( s \)-th problem is appended to \( s - 1 \) reservation values \( V_\sigma(\sigma) \), \( \sigma \leq s - 1 \), and is defined as

\[
V_s(s) \equiv \max_{\zeta \in \mathcal{X}} U_s(\zeta) \\
U_\sigma(\zeta) \geq V_\sigma(\sigma), \quad \text{all } \sigma < s, \\
\sum_{\sigma \leq s} \pi_\sigma E_\sigma(\zeta) \leq 0.
\]

Let \( \zeta(s) \) be (one of) the optimal solution(s) and let \( V(s) = (V_\sigma(s))_{\sigma=1}^s \equiv (U_\sigma(\zeta(s)))_{\sigma=1}^s \) be the utilities assigned to the \( \sigma \leq s \) types in the \( s \)-th programming problem. For each type \( \sigma < s \) the value of the \( \sigma \)-th programming problem for such a type provides the reservation utility for \( s \)-th programming problem. In the Appendix we show several key properties of the \( s \)-th programming problems, summarized in the following statement.
Lemma 5 Under Monotonicity and Sorting, for each $s$:

(i) there exists a solution $\zeta(s)$ to the $s$-th problem;
(ii) any solution $\zeta(s)$ satisfies $\sum_{s' \leq \sigma} \pi_{s'} E_{s'}(\zeta(s)) \geq 0$ for all $\sigma \leq s$, with equality for $\sigma = s$;
(iii) $\zeta(S) \in \Delta' \cap \bar{Y}$;

Property (ii) is key, and it states that any $s$-th problem is characterized by a specific cross-subsidization pattern. We call it the ‘top-down’ property of cross-subsidies. It says that after partitioning types into two groups, those lower than and higher than an arbitrarily given type, respectively, the lower-type group never subsidizes the higher-type group.

The next lemma characterizes the welfare properties of mechanisms $\zeta(S)$—without using Sorting or Monotonicity.

Lemma 6 (i) any $\zeta(S)$ is a strict CPO, and there is a $\zeta(S)$ in the CPO set;
(ii) if $\zeta^{RS}$ is a strict CPO, then $\zeta^{RS}$ is an optimal solution to the $S$-th problem;
(iii) when $S = 2$, all $\zeta(S)$ are in the CPO set;
(iv) if $v_s = v$ all $s$, then $\zeta(S)$ is unique, is a CPO, and $\zeta(S) = z(S)$.

Let $X(S) \subset X$ be the set of $\zeta(S)$ which are CPOs. Property (i) justifies calling $\zeta(S) \in X(S)$ a $S$-CPO, the CPO preferred by the highest type in the set of mechanisms satisfying $U_s(\zeta) \geq V_s(s)$, all $s < S$. Properties (iii) and (iv) identify two situations in which Pareto optimality and uniqueness, respectively, of $\zeta(S)$ are satisfied. By construction, when $S = 2$, $\zeta(2)$ is the mechanism most preferred by the low-risk type ($s = 2$) within the set of mechanisms providing the high-risk type ($s = 1$) a utility no less than $U_1(\zeta^{RS}) = v_1(\mathbb{E}_1(e))$. Therefore, if $X^{RS} \subset CPO$, $\zeta(2)$ is a separating RS allocation, otherwise at $\zeta(2)$ low-risk individuals subsidize high-risk individuals. Under Property (iv), $\zeta(S)$ is also elementary, i.e., only $\omega$-dependent. If $v_s = v$, all $s$, and $S = 2$, mechanism $\zeta(2)$ is nothing else than the equilibrium allocation of Miyazaki (1977). This cannot come as a surprise since Miyazaki’s equilibrium is Wilson’s applied to a world where firms supply (elementary) mechanisms rather than contracts (or net trades).

Existence and uniqueness
The second half of our main result, existence of equilibrium, follows now from this lemma.
Lemma 7  Under Monotonicity and Sorting, every $\zeta(S)$ is an equilibrium allocation for every economy.

Lemma 7 not only establishes that price cut immunity is not too demanding a belief restriction, but also provides a way to compute an equilibrium via the sequence of $s$-th programming problems. Since only one mechanism is active at this equilibrium, transfers across types, if any, occur within the active mechanism. Prices (of mechanisms) not only signal to agents which mechanism they should select, guaranteeing feasibility (budget balancedness) in the process, but also reflect such transfers in that the price paid by a contributing (receiving) type is zero, when it should have been negative (positive) at his odds. By Lemma 6.ii, $\zeta^{RS}$ is a $\zeta(S)$ mechanism whenever it is a strict CPO, thereby reconciling Lemma 7 with Lemma 4.

Why does an equilibrium exist—in contrast with the nonexistence of RS equilibria discussed in Section 7? The strength of (NPC) is balanced by two properties of effective price cuts. The first is maximality. The second is a somewhat hidden property ensuing from viability: if $(p', \eta')$ is an effective price cut against $p$, the allocation $\mu$ making $\eta$ viable is feasible—and in $\Delta'$. It must be so because since $\mu$ is budget feasible—in fact, its price is zero—it generates nonpositive revenues; and since it must yield positive profits, it must have a negative cost. As the total cost of $\mu$ is just the total net resources used by $\mu$, $\mu$ is therefore feasible.

To outline the role of these two properties in establishing existence, we illustrate the argument for economies where $S = 2$, $\zeta^{RS}$ is not a strict CPO and where RS equilibria fail to exist. To simplify the discussion we assume that all inequalities involved in the discussion are strict.

Pick $\zeta(2)$ and remember that since $\zeta^{RS}$ is not incentive efficient, $E_2(\zeta(2)) < 0$. Since $\zeta(2)$ is both feasible and welfare superior to $\zeta^{RS}$, by Lemma 2 $\zeta(2)$ is a weak equilibrium allocation. Among the many possible supporting prices in $P(\zeta(2))$, we pick a price $p \in P(\zeta(2))$ that satisfies the following property: if $U_s(\zeta) \leq U_s(\zeta(S))$, all $s$, and $0 \in [p_m(\zeta), p_M(\zeta)]$, it is $p(\zeta) = 0$.

Suppose by contradiction that there exists a price cut $(p', \eta)$ against $p$ with $\mu \in M(p', \eta)$, the allocation that makes $\eta$ viable. There are two possible cases: either low-risk individuals prefer $\mu$ to $\zeta(2)$, or the opposite.

The first case contradicts the definition of $\zeta(2)$: nothing feasible and in $\Delta'$ can be preferred by $s = 2$ to $\zeta(2)$. Thus, the second key property of (NPC) shuts down cream-skimming of low-risk individuals.
As for the second case, consider the pooling mechanism \( \bar{\zeta} = (\zeta_2(2), \zeta_2(2)) \). Since by construction \( U_s(\bar{\zeta}) \leq U_s(\zeta(2)) \), all \( s \), and \( 0 \leq E_1(\bar{\zeta}) \), it is \( p(\bar{\zeta}) = 0 \), and therefore \( \mu_1(\bar{\zeta}) = 0 \), all \( \mu_1 \in M_1(p) \), while \( \bar{\zeta} \in M_2(p) \). Since \( E_2(\bar{\zeta}) = E_2(\zeta(2)) < 0 \), it must be that \( \bar{\zeta} \) is shut down, contradicting the maximality of \( \eta \). Mechanism \( \bar{\zeta} \) acts as a ‘sterilizer’ of price cuts: thus, the first key property prevents the closing of pooling mechanisms that sterilize potential price cuts.

In contrast, \( \zeta(2) \) fails to be a RS equilibrium (or a competitive equilibrium with RS treatment of beliefs) because it can be killed by the nonfeasible mechanism \( \zeta' \) attracting low-risk individuals only. Such a mechanism is profitable in RS because high-risk individuals keep participating the \( \zeta(2) \) mechanism, that loses therefore money. As in Wilson (1977), this is just impossible with (NPC): if \( \zeta' \) is part of a price cut \( (p', \eta) \), \( \zeta(2) \) would need to be shut down at \( \eta \), by maximality, making therefore also \( \zeta' \) nonprofitable—because non-feasible. Equilibrium then restricts implicitly beliefs at \( \zeta' \) so that the price of \( \zeta' \) is high enough to discourage low-risk agents from participating such mechanism.

What happens when \( S > 2 \), and what is the role played by Monotonicity and Sorting then?

Basically, Monotonicity and Sorting not only guarantee existence of \( \zeta(S) \) in the general case (Lemma 5.i), but also shape other properties of this mechanism and of the sterilizer, \( \bar{\zeta} \): Sorting guarantees that there is agreement of preferences in the tails of \( S \), those who end up preferring the sterilizer to the price cut. Monotonicity ensures that these agents contribute resources at the sterilizer, therefore making \( \bar{\zeta} \) profitable if reopened.

For \( S > 2 \), we do not know if the equilibrium set coincides with \( X(S) \). Instead, when \( S = 2 \) a stronger result holds: we can identify the equilibrium set with \( X(S) \), so the equilibrium is payoff unique. This is because any incentive efficient allocation other than \( \zeta(2) \) can be undercut by shutting down mechanisms that are preferred, relatively to \( \zeta(2) \), by the lowest type.

**Proposition 1** When \( S = 2 \), \( \zeta \) is an equilibrium allocation only if \( \zeta = \zeta(2) \) for some \( \zeta(2) \in X(2) \).

Together with Lemma 6.ii, Proposition 1 also shows that when \( S = 2 \) the equilibrium must be a CPO.
10 Alternative market structures

We have established that a Walrasian market for mechanisms can decentralize incentive efficient allocations if prices are immune from price cuts. As mentioned earlier, our result depends on the market pricing not only contracts, but also mechanisms; and not only elementary, but also random mechanisms. We conclude our analysis by briefly explaining why this is the case.

Let us call $C$ the market structure where trade is limited to just (lotteries over) individual contracts, while $E$ denotes the structure where trade is limited to just (lotteries over) elementary mechanisms. Evidently, market structures $C$ and $E$ differ from our market structure in that they restrict the set of tradeable mechanisms to a subset of $X$.

Since in all market structures individuals can trade lotteries over the available mechanisms or contracts, the sets of feasible and incentive compatible allocations are equivalent in the three structures that therefore have identical CPO allocations. Equivalent here means that there exist isomorphisms across these allocation sets preserving payoffs as well as resource consumption.

**Proposition 2** The sets of feasible and incentive compatible allocations are equivalent in the three structures.

Notwithstanding this equivalence, the three market structures have different equilibrium outcomes, and $C$ and $E$ may fail to decentralize incentive efficient allocations. By the zero profit condition, at equilibrium $E$ does not allow transfers across mechanisms, failing therefore to achieve optimality when the latter calls for random transfers. Market structure $C$ suffers from the same problem, but in addition does not allow subsidies across types at equilibrium. To be as clear as possible we illustrate these failures in economies where $S = 2$ and mechanisms $\zeta^{RS}$ are not CPO.

To study inefficiency of equilibrium with $C$ structures, consider a RS economy without a RS equilibrium. Recall that in these economies CPO allocations are elementary mechanisms, and the feasible and pooling allocation most preferred by the low-risk type, $\bar{z}_{p,2}$, strictly Pareto dominates the separating allocation $z^{RS}$. Efficiency excludes the use of lotteries at equilibrium. Also, $C$ markets cannot decentralize separating contracts $z_s$, $s = 1, 2$, requiring cross subsidization, say $E_1(z_1) > 0 > E_2(z_2)$. Otherwise, it would be $p(z_1) = 0$, to allow type-1 individuals to buy $z_1$, and then $E_1(z_1) > 0$ implies that firms offering such a contract make losses. This excludes $\zeta(2)$.
as an equilibrium outcome. In fact, the pooling CPO mechanism $\tilde{z}^P$ is then the only CPO mechanism potentially decentralizable as a competitive equilibrium, but it fails to be so for two reasons. First, with smooth preferences, since $\pi(\omega|1) \neq \pi(\omega|2)$, it is $\tilde{z}^P \neq \tilde{z}^{p,2}$ and, by definition of $\tilde{z}^{p,2}$, it is also $U_2(\tilde{z}^{p,2}) > U_2(\tilde{z}^p)$. Second, by single-crossing, there exists a contract $z' \in K$ such that $E_2(z') < 0$, $U_2(\tilde{z}^{p,2}) > U_2(z') > U_2(\tilde{z}^p)$, and $U_1(z') < U_1(\tilde{z}^p)$. Then, $\tilde{z}^P$ cannot be an equilibrium outcome as $z'$ can be used as the basis for a price cut against any $p \in P(\tilde{z}^p)$. The previous discussion can be easily turned into a formal argument, left to the reader, proving the next lemma.

**Proposition 3** The equilibrium allocation of such an economy with CPO markets is not a CPO.

We then come to $E$ structures, which suffer from the same inefficiency problem, but for different issues. Consider an economy with type-dependent utilities such that the S-CPO mechanism $\zeta(2)$ satisfies:

i) $E_1(\zeta(2)) > 0 > E_2(\zeta(2))$; and

ii) $\zeta(2)$ is essential, that is, $(U_2(z'), E_2(z))$ is not constant $\zeta(2)$-a.e..

By Proposition 1, $\zeta(2)$ is the unique equilibrium allocation of our market structure. However, for such an economy the equilibrium allocations with $E$ structures are inefficient. Here, the argument is complicated and we just sketch its two driving forces. First, recall that CPO allocations provide full insurance to some types. Thus, if equilibrium allocations were efficient, type-1 individuals would be fully insured. Then, any mechanism in the support of their optimal lottery would provide the same full insurance consumption and cost zero. However, by condition (i), equilibrium efficiency calls for transfers from type-2 individuals, and imposes no losses on each mechanism. We show that this implies that the equilibrium of $E$ is essentially degenerate: it is randomizing among feasible and payoff-equivalent mechanisms. Second, it follows that for the economy satisfying (i) and (ii), $E$ markets can decentralize at most CPO elementary mechanisms. Such mechanisms however must provide type-2 individuals with no less utility than $\tilde{z}(2)$, the optimal solution to the $S$-problem when $X$ is replaced by the set of elementary mechanisms. Hence, $E$ markets can decentralize at most $\tilde{z}(2)$ which by the assumptions (i) and (ii) is not a CPO.

**Proposition 4** Let $S = 2$. Let $\tilde{v} \in \Delta'$ be a strict CPO such that its mean, $\tilde{\zeta} = E\tilde{v}$, has $\tilde{\zeta}_2$ essential, provides full insurance to type 1, and $E_1(\tilde{\zeta}) > 0$. Then, $\tilde{v}$ is not price supportable in $E$.  


11 Appendix

Auxiliary definitions and lemmas. First we settle topological issues. We consider \( C(K; \mathbb{R}) \) endowed with the sup-norm topology. Since \( K \subset \mathbb{R}^\Omega \) is compact, the norm dual of \( C(K; \mathbb{R}) \) endowed with the sup-norm topology is \( M(K) \), the space of finite Borel measures over \( K \) (see, e.g., Aliprantis and Border (1999), Theorem 13.12). Then, we consider the (normed) dual pair \( < C(K; \mathbb{R}), M(K) > \) and endow \( M(K) \) with the weak* topology. Also, \( \Delta(K) \subset M(K) \) is weak* compact (see, e.g., Aliprantis and Border (1999), Theorem 14.11), and so is the closed set \( X \subset \Delta(K)^S \). When considering \( C(X; \mathbb{R}) \), \( X \) is endowed with the weak* topology induced by \( < C(K; \mathbb{R}), M(K) > \), while \( C(X; \mathbb{R}) \) is endowed with the sup-norm topology. The norm dual of \( C(X; \mathbb{R}) \) is therefore \( M(X) \), and \( \Delta(X) \subset M(X) \) is weak* compact (where the weak* is induced this time by the dual pair \( < C(X; \mathbb{R}), M(X) > \)).

Equivalent mechanisms (defined in Section 7) are well defined for every allocation via the notion of compound, or mean lottery. For any lottery \( \nu \in \Delta(X) \), the mean of \( \nu \) is the mechanism \( E \nu \in X \) satisfying

\[
\int f dE \nu = \int \int f d\zeta d\nu, \quad \text{for all } f \in C(K; \mathbb{R}).
\]

which is well defined and weakly continuous in \( C(K; \mathbb{R}) \), by the Riesz Representation Theorem for linear functionals over \( C(K; \mathbb{R}) \) (see Aliprantis and Border (1999), Theorem 13.12). Since \( E \nu \in X \), it is \( E \nu = (E \nu_s)_{s \in S} \), with \( E \nu_s \in \Delta(K) \). Intuitively, mechanism \( E \nu \) compounds lottery \( \nu \) with mechanisms \( \zeta \) in \( \text{supp}(\nu) \). By linearity of payoffs and resources, we have

\[
U_s(\nu) = \int u(\zeta, s) d\nu = \left( \int u(z, s) d\zeta_s \right) d\nu = \int u(z, s) dE \nu = U_s(E \nu),
\]

as well as

\[
\int \left( \sum_s \pi_s E_s(\zeta) \right) d\nu_s = \left[ \int \sum_s \pi_s E_s(z) d\zeta_s \right] d\nu_s = \left[ \sum_s \pi_s E_s(z) dE \nu_s \right],
\]

so compounding preserves resources and payoffs. In the sequel, whenever convenient and without loss of generality we identify allocations with mechanisms.

We list the relevant properties of strict CPO and incentive compatible allocations as auxiliary lemmas whose proof is left to the reader.
The first auxiliary result states that resources are exhausted and some types are fully insured at a strict CPO allocation. Recall that a CPO allocation is a strict CPO.

** Auxiliary Lemma 1** Let \( \zeta \) be a strict CPO, then \( \zeta \in \bar{Y} \). Moreover, \( \zeta \) provides full insurance for some type \( s \).

The next auxiliary lemma is concerned with incentive constraints that do not bind, under Sorting.

** Auxiliary Lemma 2** Let \( \zeta \) and \( \zeta' \) be elements of \( \Delta(K) \). Under Sorting, if \( U_{s+1}(\zeta') > U_{s+1}(\zeta) \) and \( U_s(\zeta) \geq U_s(\zeta') \) for some \( s \), then \( U_j(\zeta') > U_j(\zeta) \) for all \( j \geq s + 1 \); similarly, if \( U_{s+1}(\zeta') \geq U_{s+1}(\zeta) \) and \( U_s(\zeta) > U_s(\zeta') \) for some \( s \), then \( U_j(\zeta) > U_j(\zeta') \) for all \( j \leq s \).

In the analysis we also repeatedly take advantage of the order structure of incentive compatible allocations, with Sorting. For any two mechanisms \( \zeta \) and \( \zeta' \) in \( X \), and for a subset \( S' \) of \( S \), let \( \zeta \vee_{S'} \zeta' \) be the mechanism defined as

\[
(\zeta \vee_{S'} \zeta')_s = \begin{cases} 
\zeta'_s & \text{if } s \in S', \\
\zeta_s & \text{otherwise}.
\end{cases}
\]

** Auxiliary Lemma 3** Let \( S' = \{s : s_* \leq s < s^*\} \) for \( s_* \leq s^* \leq S \), and \( \zeta, \zeta' \in X \) be such that \( U_{s^*}(\zeta) \geq U_{s^*}(\zeta') \) and \( U_{s_*}(\zeta) \geq U_{s_*}(\zeta') \), while \( U_{s_*-1}(\zeta') \geq U_{s_*-1}(\zeta) \) and \( U_{s_*+1}(\zeta') \geq U_{s_*+1}(\zeta) \). Then, under Sorting, \( \zeta \vee_{S'} \zeta' \in X \).

To prove Lemma 7, we will need to focus on a particular supporting price which we call \(*\)-supporting. A price \( q \) is \(*\)-supporting for \( \zeta^* \in \Delta' \cap \bar{Y} \) if \( q \in \mathbb{P}(\zeta^*) \) and

\[
q(\zeta) = \min[0, p_M(\zeta)], \text{ for } \zeta \in Z_-(\zeta^*),
\]

where

\[
Z_-(\zeta^*) = \{\zeta : U_s(\zeta) \leq U_s(\zeta^*) \text{ for all } s, \text{ and } p_m(\zeta) \leq 0\}.
\]

For weak equilibria, \(*\)-supporting prices always exist.

** Auxiliary Lemma 4** Let \( \zeta^* \in \Delta' \cap \bar{Y} \). Then, there exists a \(*\)-supporting price of \( \zeta^* \).
We now give a simpler characterization for price cuts.

**Auxiliary Lemma 5** Suppose that \((p', \eta)\) is an effective price cut against \(p\).

Then there is an effective price cut \((p'', \eta')\) with \(p''(\zeta) = p(\zeta')\) for all, but one \(\zeta\), say \(\zeta'\), and \(p''(\zeta') = 0\), \(\zeta' \in M_s(p'', \eta)\), all \(s\), and \(\sum_s \pi_s E_s(\zeta') < 0\).

As a consequence of the lemma, hereafter we denote an effective price cut against \(p\) with the pair \((p', \eta)\). It is understood that \(\zeta'\) is the only mechanism whose price has been cut (to zero) as in the statement above and that \(\zeta'\) is an optimal choice at \((p', \eta)\) for all \(s\).

**Preliminary properties. Proof of Lemma 2:** The following is an obvious but important preliminary observation.

\((\diamondsuit)\) For given \(s\), let \(\nu'\) be the solution to the type-\(s\) agent’s optimization problem at prices \(p'\), and let \(\nu\) be the solution at \(p\). Then, if \(p \leq p'\), \(U_s \nu \geq U_s \nu'\). (Therefore, if \(\nu\) is budget feasible at \(p'\), \(\nu = \nu'\).)

[Indeed, since \(p \leq p'\), \(\int pd\nu' \leq \int p' d\nu' = 0 = \int pd\nu\). Hence, \(\nu'\) is budget feasible at \(p\), but \(\nu\) is chosen, whence the claim, by revealed preference, proving \((\diamondsuit)\).]

We are now ready to prove the proposition. We just need to show that \(\mathbb{P}(\zeta^*) \neq \emptyset\). We first construct prices \(\bar{p}_s \in \mathbb{P}\) such that \(\zeta^* \in M_s(p_s)\), and then show that \(\bar{p} = \max_s \bar{p}_s\) supports \(\zeta^*\).

We claim that \(\zeta^{RS} \in X^{RS}\) is an optimal solution to the individual programming problem at \(p_M\) for all types \(s\). First, \(p_M(\zeta^{RS}) = 0\) for any \(\zeta^{RS} \in X^{RS}\), by Lemma 1. Second, by the convexity of the map \(\max_s [\int \mathbb{E}_s z d\zeta]\) and Jensen’s inequality:

\[ p_M \nu = \int p_M(\zeta) d\nu = \int \max_s \int \mathbb{E}_s z d\zeta d\nu \geq \max_s \int \mathbb{E}_s z d\mathbb{E} \nu = p_M \delta_{\nu}\nu. \]

Therefore, without loss of generality, for each \(s\), the optimal solution to the individual programming problem at \(p_M\) is a mechanism \(\tilde{\zeta}(s) \in X\) with \(p_M(\tilde{\zeta}(s)) = 0\). However, \(p_M(\tilde{\zeta}(s)) = 0\) implies \(\mathbb{E}_{\nu'}(\tilde{\zeta}(s)) \leq 0\) for all \(s'\). Therefore, \(\tilde{\zeta}(s) \in X^0\) and, by definition, \(\tilde{\zeta}(s) = \zeta^{RS}\).
Take any $s$; from Luenberger (1969, Theorem 1, p. 249, combined with Theorem 2, p. 178), the first order conditions for optimality guarantee that there exists $\lambda_s > 0$ such that

$$U_s(\zeta) - U_s(\zeta^{RS}) \leq \lambda_s p_M(\zeta)$$

for all $\zeta \in X$ and, since $U_s(\zeta^*) \geq U_s(\zeta^{RS})$, it is

$$U_s(\zeta) - U_s(\zeta^*) \leq \lambda_s p_M(\zeta)$$

for all $\zeta \in X$. (1)

Define the price functional $\bar{p}_s$ as

$$\bar{p}_s(\zeta) = \max \left\{ \frac{U_s(\zeta) - U_s(\zeta^*)}{\lambda_s}, p_m(\zeta) \right\}.$$ 

Then, $\bar{p}_s$ satisfies three key properties. First, $\bar{p}_s$ is the maximum of two continuous functions and it is therefore continuous. Second, $\bar{p}_s$ is belief-based or, equivalently, $\bar{p}_s \in \mathbb{P}$, since it satisfies the following inequalities:

$$p_m(\zeta) \leq \bar{p}_s(\zeta) \leq p_M(\zeta),$$

where the first inequality holds by construction, while we used (1) for the last inequality.

Third, it is $\zeta^* \in M_s(\bar{p}_s)$. Indeed, it is $\zeta^* \in M_s\left(\frac{U_s(\zeta) - U_s(\zeta^*)}{\lambda_s}\right)$. The contract $\zeta^*$ is budget feasible at $\bar{p}_s$, that is, $\bar{p}_s(\zeta^*) = 0$; and $\bar{p}_s > \frac{U_s(\zeta) - U_s(\zeta^*)}{\lambda_s}$. Thus, the claim follows by (1).

Define $\bar{p}$ as in the statement of the proposition. Then $\bar{p}(\zeta) = \max_s \bar{p}_s(\zeta)$, and it is therefore continuous. It is $\bar{p} \geq \bar{p}_s$, all $s$, $\bar{p}(\zeta^*) = 0$ and thus, by (1), $\zeta^* \in M_s(\bar{p})$, all $s$. Finally since $p_m \leq \bar{p} \leq p_M$, it is $\bar{p} \in \mathbb{P}$. Hence, there exists beliefs $\beta$ such that $\bar{p}(\zeta) = \mathbb{E}_\beta(\zeta)$, all $\zeta$, and since $\bar{p}(\zeta^*) = 0$ and $\zeta^* \in \bar{Y}$, $\beta$ can be chosen to satisfy (C). Finally, as $\zeta^*$ is profit-maximizing at $(\bar{p}, \beta)$, (M) is also satisfied. Therefore, $\bar{p}$ supports $\zeta^*$.

Suppose now that $\mu$ is a nondegenerate weak equilibrium allocation. Then, it is $U_s(\mu) \geq U_s(\zeta^{RS})$, and $\mu \in \Delta' \cap \bar{Y}$. Therefore, the mean of $\mu$, $\mathbb{E}(\mu)$, is payoff equivalent and it is also an element of $\Delta' \cap \bar{Y}$. Importantly, it is degenerate, as desired.

**Incentive efficiency. Proof of Lemma 3:** The proof of this lemma is based on the following observation:
∀ζ ∈ Y and not a strict CPO, ∃ ζ’ ∈ Y such that U_s(ζ’) > U_s(ζ) all s, and
Σ_s π_s E(s)(ζ’) < 0.

[To see this, observe that since ζ is not a strict CPO there exists ζ” ∈ Y such that U_s(ζ”) > U_s(ζ), all s. Let ζ’ be the mechanism assigning probability α to ζ” and (1 − α) to the pooling mechanism z = e. Now ζ’ ∈ Y, and
Σ_s π_s E_s(ζ’) < Σ_s π_s E_s(ζ) ≤ 0, for all α. Moreover, for α close enough to 1, U_s(ζ’) > U_s(ζ), all s, proving ⟨1⟩.

Now let ζ* be an equilibrium allocation with p ∈ P(ζ*), and suppose otherwise, ζ* is not a strict CPO. Because it is an equilibrium, ζ* ∈ Y, so let ζ’ be the contract defined in ⟨1⟩. First, observe that p(ζ’) > 0, otherwise we have an immediate contradiction with U_s(ζ*) < U_s(ζ’) all s. We construct an effective price cut (ζ’, η) against p, with η(ζ) = 1, for all ζ ∈ X. We claim that at (p’, η), ζ’ is an optimal solution to the individual programming problem for all s. For suppose not. Pick an s and let μ_s be an optimal solution with U_s μ_s > U_s(ζ’) and p’ μ_s = 0. Then, by the optimality conditions and by the fact that p’(ζ’) = 0, it is μ_s(ζ’) = 0. Therefore, μ_s is also budget feasible at p (not just at p’), while U_s μ_s > U_s(ζ*), violating the definition of ζ*, a contradiction.

Then, by construction it is ζ’ ∈ Μ_s(p’, η), all s. Since η = 1 everywhere, (p’, η) obviously satisfies condition (1) in the definition of effective price cut, and it satisfies (3) if it satisfies (2) –viability. To show viability, Π(ζ; ζ’, p’) = 0 for all ζ ≠ ζ’. Next, p’(ζ’) = 0 and ζ’ optimal for all s imply that Π(ζ’; (ζ’, p’)) = 0 − Σ_s π_s E_s(ζ’) > 0 by the strict feasibility of ζ’ —from ⟨1⟩. Hence, (p’, η) satisfies (1)-(3) and is an effective price cut. This contradicts the fact that ζ* is an equilibrium allocation.

Existence of equilibrium. Proof of Lemma 4: Since XRS is contained in the set of strict CPO and, by definition, E_s(ζRS) ≤ 0 for all s ∈ S, it is, by Auxiliary Lemma 1, E_s(ζRS) = 0 for all s ∈ S and all ζRS ∈ XRS. First we show that if ζRS is a strict CPO, then it is an equilibrium allocation; and then that if XRS ⊂ CPO, any mechanism ζ’ ∉ XRS cannot be an equilibrium allocation.

Fix a ζRS ∈ XRS and let p ∈ P(ζRS) be its supporting price, which exists by Lemma 2. By contradiction, suppose that there exists an effective price cut (ζ’, η). By Auxiliary Lemma 5, Σ_s π_s E_s(ζ’) < 0. Hence, by Auxiliary Lemma 1, ζ’ is not a strict CPO and therefore U_s(ζ’) < U_s(ζRS) for some s ∈ S. Thus, p(ζRS) = p(ζRS) = 0 implies that η(ζRS) = 0, or ζ’ would
not be optimal at \((p', \eta)\) for such \(s\). Define \(\eta' > \eta\) as \(\eta'((\zeta)) = \eta(\zeta)\) for all \(\zeta \neq \zeta^RS\), while \(\eta'(\zeta^RS) = 1\). Then, either \(U_s(\zeta^RS) > U_s(\zeta')\) or \(U_s(\zeta^RS) \leq U_s(\zeta')\). Since \(p'(\zeta^RS) = 0\), in the first case \(M_s(p', \eta') = \delta_{\zeta'}\), while in the second \(M_s(p', \eta) \subset M_s(p', \eta')\). Therefore, \(M(p', \eta')\) is nonempty. Then, since \(E_s(\zeta^RS) = 0\) for all \(s \in S\) and the set \(\{s' : U_{s'}(\zeta') < U_{s'}(\zeta^RS)\}\) is nonempty, it is \(\Pi(\zeta^RS; (v', s')) = 0\) for all \(v' \in M(p', \eta')\), violating the required maximality of \(\eta\), a contradiction that ends the proof.

Now suppose, again by contradiction that \(\zeta' \notin X^{RS}\), but \(\zeta'\) is an equilibrium allocation for some \(p \in P\). Since \(\zeta^RS\) is a CPO, by Auxiliary Lemma 1, \(E_s(\zeta^RS) = 0\), for all \(s\) and \(\zeta^RS \in X^{RS}\). Since \(\zeta'\) is an equilibrium allocation, \(U_s(\zeta') \geq U_s(\zeta^RS)\) for all \(s\). Therefore, \(U_s(\zeta') = U_s(\zeta^RS)\) for all \(s\). Then, \(\zeta' \notin X^{RS}\) means that \(E_{\sigma}(\zeta') > 0\), for some \(\sigma \in S\). Let \(\sigma^*\) be one of them. Let \(\zeta'' = \zeta' \vee_{\sigma^*} \zeta^RS\). The allocation \(\zeta''\) is incentive compatible because, for all \(s \neq \sigma^*,\) \(U_s(\zeta'') = U_s(\zeta') \geq U_s(\zeta^RS) = U_s(\zeta_{\sigma^*})\), while \(U_{\sigma^*}(\zeta'') = U_{\sigma^*}(\zeta^RS) \geq U_{\sigma^*}(\zeta_{\sigma^*})\) for all \(s\). However, \(\sum_s \pi_s E_s(\zeta'') < \sum_s \pi_s E_s(\zeta') = \sum_s \pi_s E_s(\zeta_{\sigma^*}) = 0\). The latter contradicts \(X^{RS} \subset CPO\) by Auxiliary Lemma 1, thereby concluding the argument.

**Proof of Lemma 5**

Assuming a solution exists, we first prove (ii); then (iii) and (i).

(ii) Let \(\zeta(s)\) solving the \(s\)-th problem be given. First, we show that 
\[
\sum_{s' \leq s} \pi_{s'} E_{s'}(\zeta(s)) = 0.
\]
For suppose not, and 
\[
\sum_{s' \leq s} \pi_{s'} E_{s'}(\zeta(s)) < 0.
\]
Let \(\bar{z}\) be the pooling contract dominating any mechanism in \(X\). For some \(\varepsilon > 0\), the mean \(E_{\nu_{\varepsilon}}\) of the lottery assigning \(\bar{z}\) with probability \(\varepsilon\) and \(\zeta_{s'}(s)\) with probability \((1 - \varepsilon)\) to \(s' \leq s\) has 
\[
\sum_{s' \leq s} \pi_{s'} E_{s'}(E_{\nu_{\varepsilon}}) < 0,
\]
\(E_{\nu_{\varepsilon}} \in X\) and 
\(U_{s'}(E_{\nu_{\varepsilon}}) > U_{s'}(\zeta(s))\) all \(s' \leq s\), thereby contradicting \(\zeta(s)\) optimal in the \(s\)-th problem.

Again by contradiction, suppose now that 
\[
\sum_{s' = \sigma + 1}^s \pi_{s'} E_{s'}(\zeta(s)) > 0
\]
for some \(\sigma < s\). Since 
\[
\sum_{s' \leq s} \pi_{s'} E_{s'}(\zeta(s)) = 0,
\]
it is 
\[
\sum_{s' \leq \sigma} \pi_{s'} E_{s'}(\zeta(s)) < 0.
\]
By the definition of \(\zeta(s)\) it is 
\(U_{s'}(\zeta(s)) \geq V_{s'}(s')\), all \(s' \leq \sigma\). Then, \(\zeta(s)\) belongs to the constraint set of the \(\sigma\)-problem and 
\(U_{\sigma}(\zeta(s)) \geq V_{\sigma}(\sigma)\), but 
\[
\sum_{s' \leq \sigma} \pi_{s'} E_{s'}(\zeta(s)) < 0,
\]
contradicting the first part of the claim.

(iii) We are going to show by induction that a solution \(\zeta(s)\) to the \(s\)-th problem exists and 
\(V_{s}(\sigma) \geq U_{s}(\zeta(s)^RS)\) for all \(\sigma \leq s\), which together with (ii) implies (i) and (iii) for \(s = S\).

For \(s = 2\), \(\zeta^RS\) is obviously an element of the constraint set, which is then nonempty; continuity of the objective and weak*-compactness of the constraint set are obvious, so a solution \(\zeta(2)\) exists and 
\(V_{\sigma}(\sigma) \geq U_{\sigma}(\zeta(2)^RS)\) for all \(\sigma \leq 2\). Suppose that the statement is true for \(s - 1 > 1\). Since
by assumption $\zeta(s - 1)$ exists, let $S' = \{\sigma : \sigma < s\}$ and consider the two allocations $\tilde{\zeta}^1(s) = (\tilde{\zeta} \vee s', \zeta(s - 1))$, with $\tilde{\zeta}' = \zeta_{s-1}(s - 1)$, all $s'$, and $\zeta^2(s) = (\zeta^{RS} \vee s', \zeta(s - 1))$. Clearly, $U_\sigma(\zeta^1_\sigma(s)) \geq V_\sigma(\sigma)$ all $\sigma < s$ and $\kappa$. By part (ii) and $\zeta^1_\sigma(s) = \zeta_\sigma(s - 1)$ all $\sigma < s$, it is $\sum_{\sigma < s} \pi_\sigma E_\sigma(\zeta^1_\sigma(s)) = 0$, $\kappa = 1, 2$. Furthermore, by Auxiliary Lemma 3, $\tilde{\zeta}^1(s) \in X$. We show that $\sum_{\sigma \leq s} \pi_\sigma E_\sigma(\zeta^1(s)) \leq 0$. By (ii), $E_{s-1}(\zeta(s - 1)) \leq 0$, and by the inductive assumption $U_{s-1}(\zeta_{s-1}(s)) = V_{s-1}(s - 1) \geq U_{s-1}(\zeta^{RS})$. Thus, Monotonicity implies $E_s(\tilde{\zeta}^1(s - 1)) \leq 0$. Hence, it is $\sum_{\sigma \leq s} \pi_\sigma E_\sigma(\zeta^1(s)) \leq 0$ as claimed.

Instead, since $E_s(\zeta^{RS}) \leq 0$, it is $\sum_{\sigma \leq s} \pi_\sigma E_\sigma(\zeta^2(s)) \leq 0$.

Now there are two possibilities, either $U_s(\tilde{\zeta}^1(s)) = U_s(\zeta_{s-1}(s - 1)) \geq U_s(\zeta^{RS})$, or vice versa. If the inequality holds true, as $\tilde{\zeta}^1(s)$ is in the constraint set for the $s$-th problem, and as before continuity and weak*-compactness are obvious, a solution $\zeta(s)$ exists, or (i); clearly, $V_s(s) = U_s(\zeta(s)) \geq U_s(\zeta^1(s)) \geq U_s(\zeta^{RS})$, establishing (iii). Otherwise, notice that since by the inductive assumption $U_\sigma(\zeta(s - 1)) \geq V_\sigma(\sigma) \geq U_\sigma(\zeta^{RS})$, all $\sigma < s$, and since $U_s(\zeta^1_s) > U_s(\zeta_\sigma(s - 1))$, by Auxiliary Lemma 3, $\tilde{\zeta}^2(s) \in X$, once again concluding the argument. ■

**Proof of Lemma 6**

The proof of (i) is trivial, and therefore omitted.

(ii) Suppose otherwise and pick any $\zeta(S)$, solution to the the $S$-th problem. Thus, it must be $U_s(\zeta(S)) \geq U_s(\zeta^{RS})$ for all $s$, by Lemma 5.iii, and $U_S(\zeta(S)) \geq U_S(\zeta^{RS})$. Let $s_* = \max\{s : U_s(\zeta^{RS}) = U_s(\zeta(S))\}$. Since $\zeta^{RS}$ is a strict CPO such an $s_*$ exists and by the contradiction assumption $s_* < S$. Let $S' = \{s : s > s_*\}$. By Auxiliary Lemma 3, the mechanism $\zeta'(\zeta^{RS} \vee s', \zeta(S))$ is in $X$. By the definition of $\zeta^{RS}$ and by Lemma 5.ii, $\zeta' \in Y$. Moreover, for $s > s_*$ and $\sigma \leq s_*$, it is

$$U_s(\zeta') = U_s(\zeta(S)) > U_s(\zeta^{RS}) > U_s(\zeta^{RS}_\sigma) = U_s(\zeta'_\sigma).$$

Consider the mechanism $\zeta^\mu$, $\mu \in (0, 1)$ defined as follows:

- for $s \in S'$, $\zeta^\mu_s$ assigns $-e$ with probability $\mu$, and $\zeta'_s$ with probability $(1 - \mu);
- for $s \notin S'$, $\zeta^\mu_s = \zeta'_s = \zeta^{RS}_s$.

For each $\mu \in (0, 1)$, it is $\sum_s \pi_\sigma E_\sigma(\zeta^\mu_s) < 0$. For $\mu$ arbitrarily close to 0, by the inequality above it is $\zeta^\mu \in X$, and $U_s(\zeta^\mu) \geq U_s(\zeta^{RS})$ for all $s$. Thus, $\zeta^\mu$ is a strict CPO, contradicting Auxiliary Lemma 1. ■

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(iii) Since any $\zeta(S)$ is a strict CPO, we just have to rule out that $U_1(\zeta(S)) > U_1(\zeta'(S))$ for $\zeta(S)$ and $\zeta'(S)$, two distinct $S$-CPOs. Argue by contradiction. By Auxiliary Lemma 1, and the assumption that $E_s(e)$ is increasing in $s$, type 1 must always be fully insured; then, it is $\zeta_1(S) = \hat{z}_1 > \hat{z}_1' = \zeta_1'(S)$. Now consider the mean $E\nu_{1/2}$ of the lottery assigning probability 1/2 to $\zeta(S)$ and 1/2 to $\zeta'(S)$. Obviously, $E\nu_{1/2}$ is also a solution, but type 1 is not fully insured, a contradiction.

(iv) Given a mechanism $\zeta \in X$, we define the $\omega$-certainty equivalent $\hat{x} = (\hat{x}_s(\omega))_{s \in S} \in \mathbb{R}^S_+$ as the solution to $\int v(z_\omega + e_\omega) d\zeta_{\omega,s} = v(\hat{x}_s(\omega))$, where $\zeta_{\omega,s}$ is the marginal distribution of $\zeta_s$ given $\omega$; again, we let $\hat{z}_s$ be the corresponding net trade, i.e., $\hat{z}_s = \hat{x}_s(\omega) - e_\omega$, all $\omega$.

Suppose not and let $\zeta(S), \zeta'(S)$ be two solutions. As in part (iii), mechanism $E\nu_{1/2}$ corresponding to the lottery $\nu_{1/2}$ assigning both $\zeta(S)$ and $\zeta'(S)$ with probability 1/2 is a solution as well. The $\omega$-certainty equivalent of $E\nu_{1/2}$, $\hat{z}$, has $\sum \pi_s E_s(\hat{z}) < 0$, while by construction $U_s(\hat{z}_s') = U_s(\zeta'(S))$ all $s$ and $s'$. Thus, $\hat{z}$ also is a solution to the $S$-problem, but $\sum \pi_s E_s(\hat{z}) < 0$, contradicting Auxiliary Lemma 1.

Proof of Lemma 7: We show that there cannot be any effective price cut against $p$, where $p$ is a $\star$-supporting price for $\zeta(S)$, which exists by Auxiliary Lemma 4. The argument is by contradiction and is divided into steps. Suppose otherwise, and let $(\zeta', \eta)$ be an effective price cut. By Auxiliary Lemma 5, it is $p' = 0 \leq p(\zeta'), \sum \pi_s E_s(\zeta') < 0$, and $U_s(\zeta') \geq U_s(\zeta(S))$ for some $s$. The first step shows that the price cut must attract the highest type.

**Step 1** If $(\zeta', \eta)$ is a price cut against $p$, then $U_S(\zeta') \geq U_S(\zeta(S))$.

Proof. Suppose otherwise and let $s^* - 1 < S$ be the highest type satisfying $U_{s^* - 1}(\zeta') \geq U_{s^* - 1}(\zeta(S))$. Then, $U_{s^*}(\zeta(S)) > U_{s^*}(\zeta')$, while $U_{s^* - 1}(\zeta(S)) \leq U_{s^* - 1}(\zeta')$. Let $\tilde{\zeta}$ be the pooling mechanism $\tilde{\zeta} = \zeta_s(\zeta)(S)$ for all $s$, and let for $S' = \{s : s < s^*\}$. Notice that, by Auxiliary Lemma 3, mechanism $\tilde{\zeta} = \zeta(S) \vee_{S'} \tilde{\zeta}$ satisfies $\tilde{\zeta} \in X$. By construction, $U_s(\tilde{\zeta}) \leq U_s(\zeta(S))$ for all $s$.

Now consider the mean $E\nu_\varepsilon$ of the lottery $\nu_\varepsilon$ assigning $\tilde{\zeta}$ with probability $\varepsilon$ and the pooling mechanism $z = -e$ with probability $(1 - \varepsilon)$. For $\varepsilon$ close enough to one, it is $U_s(E\nu_\varepsilon) > U_s(\zeta)$ all $s \geq s^*$, while $U_{s^* - 1}(E\nu_\varepsilon) < U_{s^* - 1}(\zeta')$. Then, by Auxiliary Lemma 2, $U_s(\zeta') \geq U_s(\zeta_{s^*}(S)) > U_s(E\nu_\varepsilon)$ for all $s < s^*$. Furthermore, by construction of $E\nu_\varepsilon$, both $U_s(E\nu_\varepsilon) < U_s(\zeta(S))$ all $s$ and $E_S(E\nu_\varepsilon) < E_S(\zeta(S)) \leq 0$. 

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Since $p$ is a $\ast$-supporting price, it is $p'(E\nu_\varepsilon) = p(E\nu_\varepsilon) = \min[0, p_M(E\nu_\varepsilon)] \leq 0$. Since $U_s(E\nu_\varepsilon) > U_s(\zeta')$ for some $s \geq s^*$ (e.g., $S$), it must be that $\eta(E\nu_\varepsilon) = 0$.

Define $\eta' > \eta$ as $\eta'(\zeta) = 1$ if either $\zeta$ is in the weak*-closure of the set $\eta^{-1}(1)$ or if $\zeta = E\nu_\varepsilon$, while $\eta'(\zeta) = 0$ otherwise. The set $X_{\eta'}$ is therefore weak*-compact so that $M(p, \eta')$ is nonempty. If $U_s\mu_s > U_s(\zeta')$, for $\mu_s \in M_s(p, \eta')$, $p'(\zeta') = 0$ implies that $M_s(p', \eta') = M_s(p, \eta')$. If instead $U_s\mu_s \leq U_s(\zeta')$, for $\mu_s \in M_s(p, \eta')$, $p'(\zeta') = 0$ implies that $\zeta' \in M_s(p', \eta')$. Therefore, $M(p', \eta')$ is nonempty.

If $\min[0, p_M(E\nu_\varepsilon)] = 0$, then $p'(E\nu_\varepsilon) = 0$ and $U_s(E\nu_\varepsilon) < U_s(\zeta')$ imply that $E\nu_\varepsilon \not\in \text{supp}(\nu_s)$, for all $\nu_s \in M_s(p', \eta')$ and for all $s < s^*$. By Lemma 5.ii, $\sum_{s \geq s^*} \pi_s E\nu_\varepsilon(\zeta) \leq 0$. Then,

$$\Pi(E\nu_\varepsilon; \tilde{p}', p') = 0 - \varepsilon \sum_{s \geq s^*} \frac{\pi_s}{\sum_{s' \geq \sigma} \pi_{s'}} E\nu_\varepsilon(\zeta) + (1 - \varepsilon) \sum_{s \geq s^*} \frac{\pi_s}{\sum_{s' \geq \sigma} \pi_{s'}} E\nu_\varepsilon(e) \geq 0,$$

contradicting the maximality of $\eta$. If $\min[0, p_M(E\nu_\varepsilon)] = p_M(E\nu_\varepsilon) < 0$, then for each $\nu \in \times_s M_s(p', \eta')$, it is $\nu_s(E\nu_\varepsilon) > 0$ for $s \geq s^*$, and therefore:

$$\Pi(E\nu_\varepsilon; \tilde{p}', p') = p_M(E\nu_\varepsilon) - \sum_s \frac{\pi_s \nu_s(E\nu_\varepsilon)}{\sum_{s'} \pi_{s'} \nu_{s'}(E\nu_\varepsilon)} E\nu_\varepsilon(E\nu_\varepsilon) \geq 0,$$

concluding the argument.\qed

The first step ends the proof when $S = 2$. Indeed, it must be $\zeta' \in \Delta'$, and then $U_S(\zeta') \geq U_S(\zeta(S))$ and $U_1(\zeta') \geq U_1(\zeta^{RS})$ imply necessarily that $\zeta'$ is also a Pareto optimal $S$-CPO. However, as in Lemma 6.iii, it cannot be $U_1(\zeta(S)) \neq U_1(\zeta')$, so $\zeta'$ cannot exist.

The second step shows that there must be a type $\hat{\sigma}$ whose ‘reservation utility’ is guaranteed by $\zeta(S)$, and at the price cut is making less than his ‘reservation utility’. Formally, let

$$\hat{S} = \{s : U_s(\zeta') < V_s(s)\}.$$

**Step 2** Let $(\zeta', \eta)$ be an effective price cut against $p$. Then, there exists $\hat{s} \in \hat{S}$ such that $V_{\hat{s}}(S) = V_{\hat{s}}(\hat{s})$.

**Proof.** First, since by Step 1 $U_S(\zeta') \geq U_S(\zeta(S))$, it must be that $\hat{S} \neq \emptyset$, otherwise $\zeta'$ is an element of the constraint set of the $S$-problem, delivers a value greater than or equal to $V_S(S)$, but $\sum_s \pi_s E\nu_\varepsilon(\zeta') < 0$, a contradiction.
to Lemma 5.ii. Now suppose otherwise: \( V_s(S) > V_\hat{s}(s) \) for all \( s \in \hat{S} \) —since by the definition of the \( S \)-problem \( V_{\hat{s}}(S) \geq V_s(s) \) for all \( s \). Consider the mean \( E\nu_\varepsilon \) of the lottery \( \nu_\varepsilon \) assigning \( \zeta^* \) with probability \( \varepsilon > 0 \) and \( \zeta(S) \) with probability \( (1 - \varepsilon) \). For all \( \varepsilon \in (0, 1) \), it is \( E\nu_\varepsilon \in X \), and \( \sum_s \pi_s E_s(E\nu_\varepsilon) < 0 \). Also, \( U_s(E\nu_\varepsilon) \geq V_s(s) \) for all \( s \not\in \hat{S} \). For \( \varepsilon \) close enough to 0, it is as well \( U_s(E\nu_\varepsilon) \geq V_\hat{s}(s) \) for all \( s \in \hat{S} \). Hence, there exists \( \varepsilon' \) so that mechanism \( E\nu_{\varepsilon'} \) is an element of the constraint set of the \( S \)-th problem, delivers a value greater than or equal to \( V_\hat{s}(S) \), but \( \sum_s \pi_s E_s(E\nu_{\varepsilon'}) < 0 \), a contradiction to Lemma 5.ii. Thus, \( V_\hat{s}(\hat{s}) = V_\hat{s}(S) \) for some \( \hat{s} \in \hat{S} \). \( \square \)

When \( \sigma < S \), the optimal solutions \( \zeta(\sigma) \) to the \( \sigma \)-problems pin down only the first \( \sigma \) components \( \zeta_\sigma(\sigma) \), \( s \leq \sigma \), while they just restrict the remaining \( S - \sigma \) components in order to guarantee \( \zeta(\sigma) \in X \). Hereafter, we identify the last \( S - \sigma \) components of \( \zeta(\sigma) \) as \( \zeta_\sigma(\sigma) = \zeta_\sigma(\sigma) \), for all \( s \geq \sigma \).

In the next step we show that at \( \zeta(S) \) the highest type is not transferring resources below \( \hat{\sigma} \); in fact, this type is using the same aggregate resources as \( \hat{\sigma} \) would use.

**Step 3** If \( \sigma \) is such that \( V_\sigma(S) = V_\sigma(\sigma) \), then:

i) \( \sum_{s \leq \sigma} \pi_s E_s(\zeta(S)) = \sum_{s \leq \sigma} \pi_s E_s(\zeta(\sigma)) = 0 \); and

ii) \( \zeta(S) \) is an optimal solution to the \( \sigma \)-problem.

**Proof.** i) By Lemma 5.ii, \( \sum_{s \leq \sigma} \pi_s E_s(\zeta(S)) \geq \sum_{s \leq \sigma} \pi_s E_s(\zeta(\sigma)) = 0 \). By contradiction, suppose that \( \sum_{s \leq \sigma} \pi_s E_s(\zeta(S)) \geq \sum_{s \leq \sigma} \pi_s E_s(\zeta(\sigma)) \). Consider the allocation \( \zeta^* = \zeta(S) \cap \{ s : s \leq \sigma \} \). By Auxiliary Lemma 3, if \( U_{\sigma + 1}(\zeta(S)) \geq U_{\sigma + 1}(\zeta(\sigma)) \) and \( U_\sigma(\zeta(S)) \leq U_\sigma(\zeta(\sigma)) \), then \( \zeta^* \in X \). Since \( V_\sigma(\sigma) = V_\sigma(S) \), it is

\[
U_\sigma(\zeta(\sigma)) = V_\sigma(\sigma) = V_\sigma(S) \geq U_\sigma(\zeta(S)).
\]

On the other hand, \( E_\sigma(\zeta(\sigma)) \leq 0 \) by Lemma 5.ii and \( U_\sigma(\zeta(\sigma)) \geq U_\sigma(\zeta^RS) \) by Lemma 5.iii, therefore \( E_{\sigma + 1}(\zeta_\sigma(\sigma)) \leq 0 \) by Monotonicity. Hence, by revealed preferences,

\[
U_{\sigma + 1}(\zeta(S)) = V_{\sigma + 1}(S) \geq U_{\sigma + 1}(\zeta(\sigma)) = U_{\sigma + 1}(\zeta_\sigma(\sigma)).
\]

Therefore \( \zeta^* \in X \) and \( U_s(\zeta^*_s) \geq V_s(s) \) for all \( s \). However,

\[
\sum_s \pi_s E_s(\zeta^*) = \sum_s \pi_s [E(s) - E_s(\zeta(S))] = \sum_{s \leq \sigma} \pi_s [E_s(\zeta(\sigma)) - E_s(\zeta(S))] < 0.
\]
Thus, $\zeta^*$ is an element of the constraint set of the $S$-problem, delivers a value $V_\sigma(S)$, but $\sum_s \pi_s B_s(\zeta^*) < 0$, a contradiction to Lemma 5.ii. Thus, i) holds true.

ii) The allocation $(\zeta_s(S))_{s=1}^S$ is an element of the constraint set of the $\sigma$-problem, delivers a value $V_\sigma(\sigma)$ to the $\sigma$-th type, and by i) it is $\sum_{s \leq \sigma} \pi_s B_s(\zeta_s(S)) = 0$. It is therefore an optimal solution to the $\sigma$-problem.\]

**Step 4** $\zeta(S)$ is an equilibrium allocation.

**Proof:** Let $\hat{\sigma}$ be the highest integer in $\hat{S}$ such that $V_\hat{\sigma}(\hat{\sigma}) = V_\hat{\sigma}(S)$ (such an integer exists by Step 2).

If $U_s(\zeta') \geq U_s(\zeta(S))$ for some $s < \hat{\sigma}$, let $s_\ast$ be the highest such integer.

If $U_s(\zeta') < U_s(\zeta(S))$ for all $s < \hat{\sigma}$, set $s_\ast = 0$.

For $\hat{\zeta}$ denoting the pooling mechanism $\hat{\zeta}_s = \zeta_{s_\ast + 1}(S)$ for all $s$, and $S' = \{s : s > s_\ast\}$, let $\zeta'' = \hat{\zeta} \lor S' \zeta(S).$ By Auxiliary Lemma 3, $\zeta'' \in X$. For $\tilde{\zeta}$ denoting the pooling mechanism $\tilde{\zeta}_s = \zeta_\hat{\sigma}(S)$ for all $s$, and for $S'' = \{s : s > \hat{\sigma}\}$, let $\tilde{\zeta} = \zeta'' \lor S'' \zeta$. Once again by Auxiliary Lemma 3, $\tilde{\zeta} \in X$ and moreover $U_s(\tilde{\zeta}) \leq U_s(\zeta(S))$ for all $s$, and by Step 3, $\mathbb{E}_s(\tilde{\zeta}) \leq 0$ for some $s$.

Next, observe that by definition of $s_\ast$, $\zeta(S)$, and $\zeta'$, it is

$$U_{s_\ast}(\zeta_{s_\ast}) \geq U_{s_\ast}(\zeta(S)) \geq U_{s_\ast}(\zeta_{s_\ast + 1}(S)),$$

while

$$U_{s_\ast + 1}(\zeta'_{s_\ast}) \leq U_{s_\ast + 1}(\zeta'_{s_\ast + 1}) \leq U_{s_\ast + 1}(\zeta_{s_\ast + 1}(S)).$$

Then, $U_s(\zeta') \geq U_s(\zeta)$ all $s \leq s_\ast$, by Sorting. We also have that $U_s(\zeta') < U_s(\tilde{\zeta})$ all $s$ with $s_\ast < s < s^*$, for $s^* > \hat{\sigma}$ denoting the lowest integer above $\hat{\sigma}$ such that $U_{s^*}(\zeta') \geq U_{s^*}(\tilde{\zeta}) = U_{s^*}(\zeta_\hat{\sigma}(S))$. Then, by Sorting again we must have $U_s(\zeta') \geq U_s(\tilde{\zeta})$ for all $s \geq s^*$.

The next table summarizes the construction of $\tilde{\zeta}$. The first row specifies allocations, the second matches allocation and intervals of types, and the third utility changes:

<table>
<thead>
<tr>
<th>$\zeta$</th>
<th>$\zeta_{s_\ast + 1}(S)$</th>
<th>$\zeta_\hat{\sigma}(S)$</th>
<th>$\zeta_{s^*}(S)$</th>
<th>$\zeta_{s^*}(S)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S$</td>
<td>$[1, s_\ast]$</td>
<td>$[s_\ast + 1, \hat{\sigma}]$</td>
<td>$[\hat{\sigma}, s^* - 1]$</td>
<td>$[s^*, S]$</td>
</tr>
<tr>
<td>$U_s(\zeta') - U_s(\zeta)$</td>
<td>$\geq 0$</td>
<td>$&lt; 0$</td>
<td>$&lt; 0$</td>
<td>$\geq 0$</td>
</tr>
</tbody>
</table>

Now consider the mean $\mathbb{E}_{\nu_\varepsilon}$ of the lottery $\nu_\varepsilon$, assigning $\tilde{\zeta}$ with probability $\varepsilon \in (0, 1)$, and the pooling mechanism $-e$ with probability $(1 - \varepsilon)$. By the convexity of $X$, $\mathbb{E}_{\nu_\varepsilon} \in X$ and, by construction, $U_s(\mathbb{E}_{\nu_\varepsilon}) < U_s(\zeta(S))$.
for all $s$. Then, since $\mathbb{E}_s(\mathbb{E}\nu_e) < 0$, for some $s$, it is $p(\mathbb{E}\nu_e) = p'(\mathbb{E}\nu_e) = \min\{0, p_M(\mathbb{E}\nu_e)\} \leq 0$, as $p$ is a *-supporting price. However, for $\varepsilon$ close enough to 1, $U_s(\mathbb{E}\nu_e) > U_s(\zeta')$, all $s$ such that $s_* < s < s^*$. Therefore, for such $\varepsilon$ it must be $\eta(\mathbb{E}\nu_e) = 0$ at the price cut.

Define $\eta' > \eta$ as $\eta'(\zeta) = 1$ if either $\zeta$ is in the weak*-closure of the set $\eta^{-1}(1)$ or if $\zeta = \mathbb{E}\nu_e$, while $\eta'(\zeta) = 0$ otherwise. As in Step 1, $M(p', \eta')$ is nonempty.

If $\min\{0, p_M(\mathbb{E}\nu_e)\} = 0$, then since $U_s(\mathbb{E}\nu_e) > U_s(\zeta')$ for all $s$ such that $s_* < s < s^*$, while $U_s(\mathbb{E}\nu_e) < U_s(\zeta')$ for all $s \geq s^*$ and $s \leq s_*$, it is $M_s(p', \eta') = \delta_{\mathbb{E}\nu_e}$ for $s_* < s < s^*$, while for $\nu_s \in M_s(p', \eta')$, it is $\mathbb{E}\nu_e \notin \text{supp}(\nu_s)$, all $s \geq s^*$ and $s \leq s_*$. Hence, the expression for $\Pi(\mathbb{E}\nu_e; (\tilde{p}', p'))$ is

$$0 - \varepsilon \sum_{s_* < s < s^*} \sum_{s_* < s' < s^*} \pi_{s,s'} \mathbb{E}_s(\tilde{\zeta}) - (1 - \varepsilon) \sum_{s_* < s < s^*} \sum_{s_* < s' < s^*} \pi_{s,s'} \mathbb{E}_s(-\varepsilon).$$

Now,

$$\sum_{s_* < s < s^*} \pi_s \mathbb{E}_s(\tilde{\zeta}) = \sum_{s_* < s \leq \hat{s}} \pi_s \mathbb{E}_s(\zeta(S)) + \sum_{\hat{s} < s < s^*} \pi_s \mathbb{E}_s(\zeta(\hat{s})(S)) \leq 0.$$

By Step 3 and Lemma 5.ii, it is both $\sum_{s_* < s \leq \hat{s}} \pi_s \mathbb{E}_s(\zeta(S)) \leq 0$ and $\mathbb{E}_\hat{s}(\zeta(\hat{s})(S)) \leq 0$. By Lemma 5.iii, it is $U_\hat{s}(\zeta(S)) \geq U_\hat{s}(\zeta(\hat{s})^R_S)$. Therefore, Monotonicity implies $\mathbb{E}_s(\zeta(\hat{s})(S)) \leq 0$ for all $s \geq \hat{s}$. Thus, $\sum_{\hat{s} < s \leq s^*} \pi_s \mathbb{E}_s(\zeta(\hat{s})(S)) \leq 0$. It follows that $\Pi(\mathbb{E}\nu_e; (\tilde{p}', p')) > 0$, contradicting the maximality of $\eta$.

If $\min\{0, p_M(\mathbb{E}\nu_e)\} = p_M(\mathbb{E}\nu_e) < 0$, then for each $\nu \in \times_s M_s(p', \eta')$, it is $\mathbb{E}_s(\mathbb{E}\nu_e) > 0$ for $s_* < s < s^*$, and therefore

$$\Pi(\mathbb{E}\nu_e; (\tilde{p}', p')) = p_M(\mathbb{E}\nu_e) - \sum_s \frac{\pi_s \mathbb{E}_s(\mathbb{E}\nu_e)}{\pi_s \mathbb{E}_s(\mathbb{E}\nu_e)} \mathbb{E}_s(\mathbb{E}\nu_e) \geq 0,$$

concluding the argument. ■

**Proof of Proposition 1:** We break the proof into three steps.

**Step 1** If $\mathbb{E}_S(\zeta(S)) < 0$, then $U_s(\zeta(S)) > U_s(\zeta^R_S)$ for all $s$.

**Proof:** We claim that if $\mathbb{E}_S(\zeta(S)) < 0$, then $U_1(\zeta(S)) > U_1(\zeta^R_S)$, then $\zeta(S) \neq \zeta^R_S$. This is so because by Lemma 6.i and Auxiliary Lemma 1 type 1 must be fully insured at $\zeta(S)$ and, by Lemma 1, also at $\zeta^R_S$, while $\mathbb{E}_1(\zeta(S)) = \frac{p_{\mathbb{E}_1}}{p_1} \mathbb{E}_S(\zeta(S)) > \mathbb{E}_1(\zeta^R_S) = 0$. Now, because $\zeta^R_S$ is feasible for the $S$-problem, it must be $U_S(\zeta(S)) \geq U_S(\zeta^R_S)$. If $U_S(\zeta(S)) = U_S(\zeta^R_S)$, then $\zeta^R_S \in X(S)$, and we contradict Lemma 6.ii, by the same argument. ■

The next step simplifies the optimal lotteries chosen for given prices $p \in \mathbb{P}$. 38
Step 2 For each $p \in \mathbb{P}$, each $s \in S$ and each $\mu \in M_s(p)$, there exists $\nu_\alpha \in M_s(p)$ such that $\text{supp}(\nu_\alpha) \subset \text{supp}(\mu)$ and $\text{supp}(\nu_\alpha)$ contains at most two points.

Proof: Let $\nu_s \in M_s(p)$, and $V_s = U_s \nu_s$. From Luenberger (1969), the FOCs are given by

$$U_s(\zeta) - V_s = \lambda_s p(\zeta), \nu_s\text{-a.e.}$$

$$U_s(\zeta) - V_s \leq \lambda_s p(\zeta), \text{all } \zeta \in X,$$

for some $\lambda_s > 0$, by local nonsatiation. Let $\Sigma_+ = \{ \zeta : p(\zeta) > 0 \}$ and $\Sigma_- = \{ \zeta : p(\zeta) < 0 \}$. Since $p \nu_s = 0$, it must be that $\nu_s(\Sigma_+)$ = 0 if and only if $\nu_s(\Sigma_-) = 0$. Thus, either $\nu_s(\Sigma_+) = \nu_s(\Sigma_-) = 0$ or both $\nu_s(\Sigma_+)$ > 0, and $\nu_s(\Sigma_-) > 0$. In both cases, we can find $\zeta_1, \zeta_2 \in \text{supp}(\mu)$ (in the first case, with $\zeta_1 = \zeta_2$) such that $U_s(\zeta_1) - V_s = \lambda_s p(\zeta_1)$, $\kappa = 1, 2$. Thus, we can construct a lottery $\nu_\alpha$ by putting probability $\alpha$ on $\zeta_1$ and $(1 - \alpha)$ on $\zeta_2$ such that $U_s \nu_\alpha = V_s$ and $p \nu_\alpha = 0$.\]

Step 3 There exists an effective price cut against $p$, all $p \in \mathbb{P}(\zeta)$, all $\zeta \neq \zeta(S)$.

Assume that for any $\zeta(S)$ it is $\zeta(S) \notin X^{RS}$, otherwise Lemma 4 concludes the argument.

Suppose that $\zeta$ is an equilibrium allocation and by contradiction that $\zeta \neq \zeta(S)$ for any $\zeta(S)$. Let $p \in \mathbb{P}(\zeta)$. Since $p(\zeta^{RS}) \leq 0$, it is $\zeta \in \Delta'$. Since $\zeta$ is feasible, it is $U_S(\zeta(S)) > U_S(\zeta)$ – otherwise we’d contradict the definition of $\zeta(S)$.

Since $\zeta(S) \notin X^{RS}$, we have $\mathbb{E}_S(\zeta(S)) < 0$, and then by Step 1, $U_S(\zeta(S)) > U_S(\zeta^{RS})$ all $s$.

Next, let $\zeta'$ be the mechanism assigning $\zeta(S)$ with probability $1 - \varepsilon$ and $-\varepsilon$ with probability $\varepsilon$. Then, $\sum_s \pi_s \mathbb{E}_s(\zeta') < 0$ for all $\varepsilon \in (0, 1)$, while by continuity for $\varepsilon$ small enough we have that $U_S(\zeta') > U_S(\zeta)$ and $U_1(\zeta') > U_1(\zeta^{RS})$. Therefore, it must be that $p(\zeta') > 0$.

We now construct a market indicator $\eta$ so that $(\zeta', \eta)$ is an effective price cut.

Let $\hat{Z}$ be the set of contracts satisfying: for each $\zeta \in \hat{Z} \subset X \setminus \{ \zeta' \}$, there exists a lottery $\mu_1$ with $\#\text{supp}(\mu_1) \leq 2$ and $\mu_1(\zeta) > 0$, such that three conditions hold:
i. \( U_1(\zeta') < U_1\mu_1 \);

ii. \( p'\mu_1 \leq 0 \);

iii. \( \mathbb{E}_1(\zeta) > p'(\zeta) \).

It is easy to show that the set \( \hat{Z} \) is open. Now set \( \eta(\zeta) = 0 \) if and only if \( \zeta \in \hat{Z} \). Hence, the set \( X_{\eta} = X \setminus \hat{Z} \) is weak*-compact. Therefore, the set \( M(p, \eta) \) is nonempty. Thus, there are two possibilities: either \( U_s\mu_s > U_s(\zeta') \), for some \( \mu_s \in M_s(p, \eta) \); or \( U_s\mu_s \leq U_s(\zeta') \), for some \( \mu_s \in M_s(p, \eta) \). It is now easy to show that if the latter holds true, then \( \zeta' \in M_s(p', \eta) \); while if the former, then \( M_s(p', \eta) = M_s(p, \eta) \). Hence, the set \( M(p', \eta) \) is nonempty. Therefore, to show that \( (\zeta', \eta) \) is an effective price cut against \( (\zeta, p) \) it suffices to prove that:

a) at \((p', \eta)\), \( \zeta' \) is an optimal solution for all \( s \);

b) \( \eta \) is maximal, i.e., it satisfies (3).

For type \( S \) claim (a) is obvious. For type \( s = 1 \), suppose by contradiction that \( U_1\mu_1 > U_1(\zeta') \) for \( \mu_1 \in M_1(p', \eta) \). We show that this implies that \( \text{supp}(\mu_1) \cap \hat{Z} \neq \emptyset \), contradicting the definition of the market indicator \( \eta \).

By the definition of \( \zeta' \) and the properties of \( \zeta^{RS} \), it is \( U_1(\zeta') \geq U_1(\zeta^{RS}) \), and therefore \( U_1\mu_1 > U_1(\zeta^{RS}) \). The latter implies \( \int \mathbb{E}_1(\zeta)d\mu_1 > 0 \) for all \( \mu_1 \in M_1(p', \eta) \). However, by Step 2, for each \( \mu_1 \in M_1(p', \eta) \) there exists a utility-equivalent lottery \( \mu'_1 \in M_1(p', \eta) \) with \( \text{supp}(\mu'_1) \subset \text{supp}(\mu_1) \) and with \( \text{supp}(\mu'_1) = \{\zeta_1, \zeta_2\} \). Hence, \( p'\mu'_1 \leq 0 \) implies

\[
\sum_{\kappa=1,2} \mu'_1(\zeta_{\kappa})[\mathbb{E}_1(\zeta_{\kappa}) - p'(\zeta_{\kappa})] \geq \sum_{\kappa=1,2} \mu'_1(\zeta_{\kappa})[\mathbb{E}_1(\zeta_{\kappa})] > 0.
\]

Therefore, Condition (iii) above is satisfied as at least one point \( \zeta_{\kappa} \) must have expected net trade for \( s = 1 \) higher than its price. Condition (ii) is obviously satisfied by \( \mu'_1 \), and Condition (i) is also satisfied by assumption. Then, it is \( \zeta_{\kappa} \in \hat{Z} \) for some \( \kappa \), contradicting \( \eta(\zeta_{\kappa}) = 0 \).

Finally in order to show that \( \eta \) is maximal, pick \( \zeta'' \in \hat{Z} \) (if \( \hat{Z} = \emptyset \), we are done), and define \( \eta' \) as \( \eta'(\zeta) = \eta(\zeta) \) for all \( \zeta \neq \zeta'' \), while \( \eta(\zeta'') = 1 \). Notice that the set \( X_{\eta'} = X_\eta \cup \{\zeta''\} \) is weak*-compact. Hence, \( M(p, \eta') \) is nonempty and by the argument above so is \( M(p', \eta') \). Therefore, \( \nu'_S \in M_S(p', \eta') \) if and only if \( \nu'_S = \delta_{\zeta'} \). As for \( s = 1 \), by the definition of \( \hat{Z} \), Condition 1 holds true for some \( \mu_1 \) with \( \mu_1(\zeta'') > 0 \). Therefore, \( U_1\nu'_1 > U_1(\zeta') \) for all \( \nu'_1 \in M_1(p', \eta') \), so that by Step 2 \( \text{supp}(\nu'_1) \cap \hat{Z} \neq \emptyset \) for all such \( \nu'_1 \). Equivalently, \( \nu'_1(\zeta'') > 0 \) at
all optimal solutions. However, since \( \mathbb{E} \left( \zeta'' \right) > p(\zeta'') \), it is \( \Pi(\zeta''; (\nu', p')) < 0 \) for all \( \nu' \in M(p', \eta') \), concluding the argument.

On pricing random mechanisms. Proof of Proposition 2: First we argue that this is true for \( C \) and \( E \), then we complete the argument. The feasible and incentive compatible allocations sets are, respectively, \( Y_C \setminus IC_C \), i.e.,

\[
\left\{ \mu \in \Delta(K)^S : \sum_s \pi_s \int_K (\sum \pi(\omega|s)z_{\omega,s})d\mu_s \leq 0, U_s\mu_s \geq U_s\mu_{s'}, \text{ all } s, s' \right\},
\]

and, denoting with \( X^E \) the set of incentive compatible and elementary mechanisms, \( Y^E \setminus IC^E \), i.e.,

\[
\left\{ \mu \in \Delta(X^E)^S : \sum_s \pi_s \int_K \mathbb{E}_s(z)d\mu_s \leq 0, U_s\mu_s \geq U_s\mu_{s'}, \text{ all } s, s' \right\}.
\]

Since an element of \( K \) can be identified with a pooling elementary mechanism in \( X^E \), \( Y^C \setminus IC^C \) is equivalent to a subset of \( Y^E \setminus IC^E \). For the other direction, recall that by definition of \( X \) and \( X^E \) for \( z = (z_s)_{s \in S} \in X^E \), it is \( U_s(z) = \mathbb{E}_s(v_s(z_s + e)) \) as well as \( \mathbb{E}_s(z) = \sum \pi(\omega|s)z_{\omega,s} \). For any probability measure \( \nu \) over \( X^E \), let \( \nu(s) \) be the probability measure over \( K \) defined as \( \nu(s)(B) = \int_{B \times \{s \neq s'\} \times K} d\nu \), for all \( B \in \mathcal{B}(K) \). Then \( U_s\mu_s = \int_{X^E} \mathbb{E}_s(z)d\mu_s = \int_K \mathbb{E}_s[v_s(z_s + e)]d\mu_s(s) \). Therefore, \( (\mu_s(s))_{s \in S} \in \Delta(K)^S \) is payoff equivalent to \( \mu \in \Delta(X^C)^S \) and it consumes the same resources. As already discussed at the beginning of this Appendix, the assumptions on individual preferences imply that, by compounding, an element of \( \Delta(\Delta(K)^S) \) is payoff (and resource) equivalent to an element of \( \Delta(K)^S \), establishing the equivalence of the three sets.

Proof of Proposition 4: We denote \( \tilde{z}_1 \) as \( z_1^* \), a full insurance net trade. Suppose by contradiction that \( \tilde{\nu} \) is price supportable in \( \mathcal{E} \), with \( p \) the supporting price and \( \tilde{\nu}_s \in M_s(p) \). Since \( p\tilde{\nu}_1 = 0 \), by the optimality conditions and the definition of \( z_1^* \), it must be that \( p(z) = 0 \), \( \tilde{\nu}_1 \)-a.e.. Also, it must be that \( \tilde{\nu}_1 \) is absolutely continuous with respect to \( \tilde{\nu}_2 \). Otherwise, if \( \tilde{\nu}_1(B) > 0 \),
but $\tilde{\nu}_2(B) = 0$ for some $B \in \mathcal{B}(X)$, then

$$\int_B \Pi(z; (\tilde{\nu}, p)) d\left(\sum_s \pi_s \tilde{\nu}_s\right) = \int_B \Pi(z; (\tilde{\nu}, p)) d(\pi_1 \tilde{\nu}_1)$$

$$= - \int_B \mathbb{E}_1(z) \pi_1 d\tilde{\nu}_1 = - \mathbb{E}_1(z^*_1) \pi_1 \tilde{\nu}_1(B) < 0.$$  

The latter implies that $\tilde{\nu}_2(B) > 0$ for all $B$ with $\tilde{\nu}_1(B) > 0$.

We claim that for some $B \subset \text{supp}(\tilde{\nu}_1)$ with $\sum_s \pi_s \tilde{\nu}_s(B) > 0$,

$$\int_B \Pi(z; (\tilde{\nu}, p)) d\left(\sum_s \pi_s \tilde{\nu}_s\right) < 0,$$

violating the zero profit condition (ZP) for some $z \in \text{supp}(\tilde{\nu})$.

The condition $p(z) = 0$, $\tilde{\nu}_1$-a.e., and the optimality conditions imply that $U_2(z) = U_2 \tilde{\nu}_2$, $\tilde{\nu}_1$-a.e.. However, $z \in \text{supp}(\tilde{\nu}_1)$ is incentive compatible and $z_1 = z^*_1$, $\tilde{\nu}_1$-a.e.. Thus, if it were $\sum_s \pi_s \mathbb{E}_s(z) < 0$ in a subset of positive $\tilde{\nu}_1$ and therefore $\tilde{\nu}_2$ measure, $\tilde{\nu}$ would not be a strict CPO, a contradiction. Then, $\sum_s \pi_s \mathbb{E}_s(z) \geq 0$, $\tilde{\nu}_1$-a.e.. Therefore, if the claim were not true and by contradiction $\Pi(z; (\tilde{\nu}, p)) \geq 0$, $\tilde{\nu}_1$-a.e., then $\tilde{\nu}_2(B) \geq \tilde{\nu}_1(B)$ for all $B \subset \text{supp}(\tilde{\nu}_1)$ with $\tilde{\nu}_1(B) > 0$. As a consequence, $\tilde{\nu}_2(\text{supp}(\tilde{\nu}_1)) = 1$ and therefore $\tilde{\nu}_2(B) = \tilde{\nu}_1(B)$ for all $B \in \mathcal{B}(X)$, i.e., $\tilde{\nu}_1 = \tilde{\nu}_2$. It follows that $(U_2(z), \mathbb{E}_2(z))$ is constant $\tilde{\nu}_2$-a.e., hence $\tilde{\zeta}_2$-a.e., contradicting $\tilde{\zeta}_2$ essential. This establishes the claim, and therefore the statement. ■

References


