Comparing the Accuracy of the Minimum-Variance Grid Method to Multiple Regression in Appraised Value Estimates

Tsong-Yue Lai* and Ko Wang**

The adjustment-grid method and the multiple-regression method are the two most frequently used techniques in the sales comparison approach. This paper demonstrates that although both techniques provide unbiased estimators, the minimum-variance grid estimator should result in a smaller standard deviation than the multiple-regression estimator. A technique is also derived to estimate the confidence interval or to perform hypothesis tests for the minimum-variance grid estimator.

The adjustment-grid method is probably the most frequently used approach to estimating the market value of single-family houses.\(^1\) In light of the wide applications of the technique, it is puzzling as to why the academic community has placed so little attention on this subject. Indeed, considering the importance of unbiased and accurate property value estimates to real estate lending and investment practices, a scientific investigation into the appraisal methodology seems long overdue.

Colwell, Cannady and Wu (1983) are the first to provide an analytical foundation for the adjustment-grid method. However, their analysis does not provide a scientific method for the selection of comparable weights.\(^2\) Recently, Vandell (1991) advances the literature by formalizing the adjustment-grid method using a minimum-variance criterion for comparable weights selection.\(^3\) With Vandell’s contribution, the adjustment-grid method is being

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1 Colwell et al. (1983, p. 12) indicate that the adjustment grid method is required for most appraisals of single-family houses for lending purposes.

2 Colwell et al. (1983, p. 23) indicate that “the choice of a weighting scheme may be a matter of judgement tempered by experience.”

transformed from an art to a scientific method. However, there are at least two more issues that need to be addressed before the adjustment-grid method can be viewed as complete.

First, as suggested by both Colwell et al. (1983) and Vandell (1991), the hedonic-price coefficients estimated from a hedonic pricing equation can be used as the adjustment factors for the adjustment-grid method. Under this paradigm, the application of the adjustment-grid method requires the use of the multiple-regression method first. In this case, it seems natural to use the multiple-regression method to estimate property values because hedonic-price coefficients are readily available. Indeed, because a multiple-regression estimator is unbiased and is readily available once the hedonic-price coefficients are obtained, it is not clear why the adjustment-grid method should be used. This is probably why Vandell (1991, p. 236) specifically suggests that a comparison between the adjustment-grid method and the multiple-regression method is warranted.

Second, because the distribution properties of the adjustment-grid estimator are unknown at the present time, the adjustment-grid method does not allow construction of a confidence interval or performance of hypothesis tests of an estimated property value. In the real world, it is not uncommon for lenders or investors to be interested in a range of estimated property values. They might also want to know whether a particular value (say, the offering price) is statistically different from the true property value. The lack of ability to perform such analyses renders the adjustment-grid method less valuable than the multiple-regression method.

This paper examines these two issues. Section two discusses the model framework. Section three compares the prediction errors of the minimum-variance grid estimator and multiple-regression estimator. Section four derives a technique that estimates the confidence interval and performs hypothesis tests of the minimum-variance grid estimator. The last section is the conclusion.

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4 It is common knowledge that, due to data limitations (see Colwell 1983, p. 20), the multiple regression method has been a less popular technique than the adjustment-grid method. However, when the adjustment factors used in the adjustment-grid method are the hedonic-price coefficients derived from a multiple-regression equation, the use of the adjustment-grid method is also subject to the same data constraints.

5 Colwell et al. (1983) argue that the adjustment-grid is superior to the multiple-regression method if an omitted variable problem exists. However, they do not give guidelines as to which technique should be used if the classic OLS assumptions hold.
Model Framework

**OLS and Adjustment-Grid Estimator**

Assume that the value of a subject property can be determined by a hedonic pricing equation with a vector of property attributes. or:

\[ V = X\beta + \epsilon, \]  \hspace{1cm} (1)

where the \( n \times 1 \) vectors \( \epsilon \) and \( V \) are the error term and observed sales price vectors. The \( n \times k \) matrix \( X \) represents the property attributes of the comparable properties (\( n \) is the number of observed property sales and \( k \) is the number of property attributes). \( \beta \) is a \( k \times 1 \) vector representing the true but unobservable hedonic coefficients of property attributes. The error term vector \( \epsilon \) is assumed to be multivariate normal and homoskedastic with zero mean and standard deviation \( \sigma \) (i.e., \( E(\epsilon \epsilon') = \sigma^2 I_n \), where \( I_n \) is the \( n \times n \) identity matrix and \( E(.) \) is an expectation operator).

Given the hedonic pricing equation, the true subject property value, \( V_s \), can be specified as

\[ V_s = X_s \beta + \epsilon_s, \]  \hspace{1cm} (2)

where \( X_s \) is an \( 1 \times k \) vector representing the property attributes of a subject property. \( \epsilon_s \) is the difference between the true property value (\( V_s \)) and the value predicted by \( X_s \beta \). \( \epsilon_s \) is assumed to be normally distributed with zero mean and the same standard deviation \( \sigma \).

The unknown vector \( \beta \) in equation (1) can be estimated from the \( n \) observed property sales. If the classic assumptions of the ordinary least squares (OLS) model are satisfied, the OLS estimator of \( \beta \) (or \( \hat{\beta} \)) is equal to \( (X'X)^{-1}X'V \).

In this circumstance, the prediction of the subject property (or the multiple-regression estimator) can be specified as

\[ \hat{V}_{ols} = X_s \hat{\beta}, \]  \hspace{1cm} (3)

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\(^{\text{6}}\) A notation in a boldface refers to either a vector or a matrix. "'" denotes the transpose vector or matrix.
When applying the adjustment-grid method, both Colwell et al. (1983) and Vandell (1991) advocate that the hedonic price coefficients (\( \hat{\beta} \)) estimated from a hedonic pricing equation can be used as the adjustment factors of the property attributes. Following their methodology, the adjustment-grid estimator (\( \hat{V}_{ols} \)) can be specified as

\[
\hat{V}_{ols} = \sum_{i=1}^{\kappa} \omega_i (X_s - X_{ci}) \hat{\beta} + \omega' V_c = X_s \hat{\beta} + \omega'(V_c - X_c \hat{\beta}) = X_s \hat{\beta} + \omega'(X_c (\beta - \hat{\beta}) + \epsilon_c) = \hat{V}_{ols} + \omega' u_c.
\]  

where the \( \kappa \times 1 \) vector \( V_c \) represents the observed comparable sales prices, the \( \kappa \times k \) matrix \( X_c \) represents the property attributes of the \( \kappa \) comparables and the \( 1 \times \kappa \) vector \( \omega' \) is the weights of the comparable (\( \omega_i \) is the weight on the \( i \)th comparable, \( i = 1, 2, \ldots, \kappa \)). The sum of these weights must be equal to unity. The \( \kappa \times 1 \) vector \( u_c \) (i.e., \( V_c - X_c \hat{\beta} \)) is the prediction error of the comparable based on the regression method, and \( \kappa \times 1 \) vector \( \epsilon_c \) is the difference between the observed values \( V_c \) and \( X_c \hat{\beta} \). Equation (4) shows that the estimated property value using the adjustment-grid method is just the sum of the multiple-regression estimator (\( \hat{V}_{ols} \)) and the weighted average of the prediction errors (\( V_c - X_c \hat{\beta} \)) of the comparable properties.

The Prediction Error of the OLS Estimator

When a hedonic pricing equation (specifically, the OLS method) is used to estimate the subject property value, the prediction error of the multiple-regression estimator is determined by

\[
u_{ols} = V_s - \hat{V}_{ols}.
\]  

Substituting equation (2) into equation (5), the expectation of \( u_{ols} \) can be specified as

\[
E(u_{ols}) = E(\epsilon_c) + X_s[\beta - E(\hat{\beta})] = 0.
\]
Because $E(\varepsilon) = 0$ and because $\hat{\beta}$ is an unbiased estimator of $\beta$, the multiple-regression estimator must be unbiased. The variance of the prediction error of a multiple-regression estimator ($u_{ols}$) can be specified as\footnote{See Johnston (1984, p. 199) for details.}

\begin{equation}
\sigma_{ols}^2 = \sigma^2[X_s(X'X)^{-1}X'_s + 1].
\end{equation}

\section*{The Prediction Error of the Adjustment Grid Estimator}

When an adjustment-grid method is used to estimate the property value, the prediction error of the adjustment-grid estimator is the difference between equations (2) and (4), or,

\begin{equation}
u_G = V_s - \hat{V}_G = V_s - \hat{V}_{ols} - \omega'u_c
= u_{ols} - \omega'u_c.\end{equation}

Equation (8) illustrates the relationship between the prediction errors of the adjustment-grid method and the multiple-regression method. That is, the prediction error resulting from the adjustment-grid method must be equal to the prediction error of the subject property less the weighted average of the prediction errors of comparable properties. The expectation of $u_G$ can be specified as

\begin{equation}
E(u_G) = E(u_{ols}) - \omega'E(u_c)
= 0.
\end{equation}

Because the expectations of $u_{ols}$ and $u_c$ are both zero, equation (9) shows that the adjustment-grid estimator is also unbiased.

Define $\varepsilon_i$ to be the $i^{th}$ element of the $\varepsilon$ vector. If the covariance of $\varepsilon_i$ and $\varepsilon_{i'}$ (i.e., $\text{Cov}(\varepsilon_i, \varepsilon_{i'})$) equals zero for all comparable $i$, the variance of the prediction error of an adjustment-grid estimator ($u_G$) is\footnote{Note that, because $E(\hat{\beta}) = \beta$, $u_G = E(u_G) = X_s(E(\hat{\beta}) - \hat{\beta}) + \varepsilon_i - \omega'|X_s(E(\hat{\beta}) - \hat{\beta}) + \varepsilon_i$.}

\begin{equation}
\sigma_G^2 = \sigma_{ols}^2 + \omega^2X_c(X'X)^{-1}X'_c + I_{ss}])\omega - 2\omega'X_c(X'X)^{-1}X'_c.\end{equation}
Equation (10) implies that a necessary condition for $\sigma^2_G < \sigma^2_{\text{opt}}$ is a positive $\omega'X_c(X'X)^{-1}X'_c$. Since $\omega'X_c(X'X)^{-1}X'_c \sigma^2 = \text{Cov}(\omega'X_c \hat{\beta}, X'_c \hat{\beta})$, the condition for a positive $\omega'X_c(X'X)^{-1}X'_c$ is that $\omega'X_c \hat{\beta}$ and $X'_c \hat{\beta}$ are positively correlated. To minimize the variance of the prediction error of an adjustment-grid estimator (from equation (10)), the weighted average of the property attributes of the comparables used for the adjustment-grid method must be as close as possible to the property attributes of the subject property. In other words, the correlation between the weighted average predictions of the comparable properties and the prediction of the subject property should be as high as possible. This finding underscores the importance of a correct comparable selection process (such as the minimum-variance approach advanced by Vandell (1991)) in the application of the adjustment-grid method.

A Comparison

The adjustment-grid method advocated by Vandell (1991) selects the weight of each comparable based on a minimum-variance criterion. In other words, given a fixed number of comparables, the estimator derived from the minimum-variance technique must be the one with the least variance among all possible selections. To minimize $\sigma^2_G$ under the constraint $\omega'e_\kappa = 1$, the first order condition of equation (10) must satisfy

$$
\sigma^2[(X_c(X'X)^{-1}X'_c + I_\kappa)\omega - X_c(X'X)^{-1}X'_c] - \lambda e_\kappa = 0, \tag{11}
$$

where $2\lambda$ is the Lagrange multiplier of the constraint $\omega'e_\kappa = 1$, and $e_\kappa$ is the $\kappa \times 1$ vector of one on each element.

Solving for $\omega$, the optimal solution $\omega^*$ can be specified as

$$
\omega^* = A^{-1}B + \frac{1 - e'_\kappa A^{-1}B}{e'_\kappa A^{-1}e_\kappa} A^{-1}e_\kappa. \tag{12}
$$

and $\lambda$ is determined by

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9 The second condition automatically holds because $X_c(X'X)^{-1}X'_c + I_\kappa$ is a positive definite matrix and equation (10) is a quadratic equation of $\omega$. 
\[ \lambda = \sigma^2 \frac{1 - e'_e A^{-1} B}{e'_e A^{-1} e_e}, \]  \hspace{1cm} (13) 

where,

\[ A = X_c(X'X)^{-1}X'_c + I_k, \]
\[ B = X_c(X'X)^{-1}X'_s. \]

It should be noted that \( A^{-1}B \) in equation (12) represents the optimal weights of equation (10) when the unity constraint is relaxed. With equation (12) and the assumptions of \( \epsilon \) identified in equation (1), the following two propositions are derived.

**Proposition One:** If \( \beta \) is estimated using an OLS method and if \( \epsilon \) is assumed to be an identically and independently distributed normal random variable, then the minimum-variance weights \( \omega^* \) derived from the adjustment-grid method can be decided by the observable property attributes \( X, X_c, X_s \). In addition, the variance of the prediction error of the minimum-variance grid estimator \( \sigma^2_G \) can also be determined by \( X, X_c, X_s \) and \( \sigma \).

To see this, \( \omega^* \) is substituted for \( \omega \) in equation (10). The variance of the prediction error under the minimum-variance-grid method becomes

\[ \sigma^2_G = \sigma^2_{ols} + \sigma^2 \left[ \frac{(1 - e'_e A^{-1} B)^2}{e'_e A^{-1} e_e} - B'A^{-1}B \right] 
- \sigma^2 \left[ 1 + X_s(X'X)^{-1}X'_s \right] \frac{(1 - e'_e A^{-1} B)^2}{e'_e A^{-1} e_e} - B'A^{-1}B \right]. \]  \hspace{1cm} (14) 

It should be stressed that this proposition holds if and only if the error terms satisfy the *i.i.d.* assumption. If this assumption is violated, the estimated covariance matrix of \( \epsilon \) will be incorporated into equation (12) because \( \hat{\beta} \) will be estimated from a generalized least squares (GLS) equation. Under these circumstances, the minimum-variance weights depend on both the observable property attributes, \( X, X_c, X_s \), and the estimated covariance matrix of the error terms. It should also be noted that this method does not require the estimation of parameters (see Vandell (1991, Table 4) for the calculation of the optimal weights).
Proposition Two: A necessary and sufficient condition for $\sigma_G^2 < \sigma_{ols}^2$ is
\[(1 - e_{x'}^TA^{-1}B)^2 < (e_{x'}^TA^{-1}e_x)(B'A^{-1}B)\].

From equation (14), it is clear that $\sigma_G^2 < \sigma_{ols}^2$ if and only if the second term, $[(1 - e_{x'}^TA^{-1}B)^2((e_{x'}^TA^{-1}e_x)] - (B'A^{-1}B)$ of equation (14) is negative. This condition is equivalent to $(1 - e_{x'}^TA^{-1}B)^2 < (e_{x'}^TA^{-1}e_x)(B'A^{-1}B)$. Because $A^{-1}$ is a positive definite matrix, $(1 - e_{x'}^TA^{-1}B)^2 < (e_{x'}^TA^{-1}e_x)(B'A^{-1}B)$ can be rewritten as $1 - e_{x'}^TA^{-1}B < [(e_{x'}^TA^{-1}e_x)(B'A^{-1}B)]^{0.5}$. Since $A^{-1}B$ represents the optimal weights of the comparables (see equation (10)) when the unity constraint is relaxed, a sufficient condition for $\sigma_G^2$ to be less than $\sigma_{ols}^2$ is that the sum of the components of $A^{-1}B$ is greater than one (i.e., $1 < e_{x'}^TA^{-1}B$).

It is difficult to specify the conditions under which the inequality $(1 - e_{x'}^TA^{-1}B)^2 < (e_{x'}^TA^{-1}e_x)(B'A^{-1}B)$ holds. Thus, it is equally difficult to conclude which estimator (minimum-variance grid estimator or the multiple-regression estimator) will have a lower variance for its prediction error. Nevertheless, to gain more insight on the impact of the term $e_{x'}^TA^{-1}B$, the values of $e_{x'}^TA^{-1}B$ are estimated using different types of matrices $A$ and $B$. In every estimation the value of $(1 - e_{x'}^TA^{-1}B)^2$ is always less than $(e_{x'}^TA^{-1}e_x)(B'A^{-1}B)$.

In other words, although the conditions are not specified under which $(1 - e_{x'}^TA^{-1}B)^2 < (e_{x'}^TA^{-1}e_x)(B'A^{-1}B)$, it appears that the variance of the prediction error of the minimum-variance grid estimator is lower than that of a multiple-regression estimator.

To substantiate the observations that the variance of the prediction error of a minimum-variance grid estimator may be lower than that of a multiple-regression estimator, the impact of the number of comparable properties on the variance of the prediction error is analyzed. As suggested by Vandell (1991) and Gau, Lai and Wang (1992), the variance of a minimum-variance grid estimator decreases (or stays the same) as the number of comparables used to estimate the property value increases. Also, given a set of property attributes (i.e., $X_s$), the variance of a multiple-regression estimator will be a constant $\sigma_{ols}^2 = \sigma^2(X_s(X_s'X)^{-1}X_s' + 1)$. Thus, it is clear that an increase in

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10 See Johnston (1984, p. 110) for a discussion of this proposition. A detailed proof is also available from the authors upon request.

11 An alternative way to think about this issue is to imagine that there is a house with no characteristics. Under this scenario (and restricting the intercept of the regression to be zero), the fitted value of the house must be zero. When this house is included in the weighting scheme and as long as all comparables receive positive weights, appraisers will be better off using the adjustment-grid method than using the fitted value alone.
the number of comparables makes the minimum-variance grid method the preferred technique to the multiple regression method.

As shown in equations (4), (6) and (8), the prediction error of the adjustment-grid estimator consists of two components: (1) the estimation error of \( \beta \) (the deviation between the estimated \( \hat{\beta} \) and the true \( \beta \)); and (2) the error terms of \( \epsilon, \epsilon_s \) and \( \epsilon_c \) in equations (1), (2) and (4), respectively. If there are \( k \) property attributes and \( \kappa \) \((\geq k + 1)\) comparables, a mathematical programming model can be used to solve for the minimum \( \omega' I_\kappa \omega \) subject to the \( \kappa + 1 \) linear equations \( \omega' X_c = X_s \) and \( \omega' e_c = 1 \). As a result, there exists an optimal replicated weight vector, \( \omega^* \), such that the property attributes of the subject property can be duplicated by a linear combination of the property attributes of the comparable properties. If the property attributes of the subject property are duplicated, the estimation error of \( \hat{\beta} \) can be completely eliminated. Given this, the prediction error of the adjustment-grid estimator will only be caused by the error terms, and the prediction error of a minimum-variance grid estimator will be solely decided by \( \sigma^2 [1 + \omega^* I_\kappa \omega^*] \).

Because the optimal replicated weights \( \omega^* \) may not be equal to the minimum-variance weight \( \omega^* \) derived from the first-order condition (equation (11)), the variance of the estimator using the replicated weights must be greater than that of the minimum-variance weight. Derived from equations (6) and (10) is

\[
\sigma^2 \leq \sigma^2 [1 + \omega^* I_\kappa \omega^*].
\]  

(15)

As the number of comparables increases, the magnitude of \( \omega^* I_\kappa \omega^* \) decreases.\(^\text{12}\) When the number of comparables is sufficiently large, the magnitude of \( \omega^* I_\kappa \omega^* \) must be less than the positive constant \([X_s' (X' X)^{-1} X_s']\).\(^\text{13}\) Thus, the variance of a multiple-regression estimator

\(^\text{12}\) Given the fact that only a finite number of observations are available and that observations used in the regression may not be used as comparables in the grid, the optimal allocation of the observations is an important issue. An increase in the number of comparables will decrease the variance of the grid method (see Gau et al. (1992, p. 114). On the other hand, however, an increase in the number of comparables will also result in a decrease in the number of observations in a regression. This will, in turn, increase the estimated variance of the residual (the \( \epsilon \)) and reduce the accuracy of the estimated adjustment coefficients (the \( \beta \)).

\(^\text{13}\) To see this, assume that there exists a constant \( d \) and the weight of each comparable is bounded by \( d/\kappa \) (where \( d \) is a positive finite number and \( \kappa \) represents the number of comparable properties). In this situation, the magnitude of \( \omega^* I_\kappa \omega^* \) will be bounded by \( (d^2/\kappa^2) \) or \( d^2/\kappa \). Consequently, \( \omega^* I_\kappa \omega^* \) decreases as \( \kappa \) increases. When \( \kappa \) approaches infinity, \( \omega^* I_\kappa \omega^* \) should approach zero.
Table 1: Matrix of $(X'X)^{-1}$ and property attributes of sampled properties.

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<th></th>
<th>BDRMS</th>
<th>BATHS</th>
<th>SQ.FT.</th>
<th>LOTSZ</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel A: Calculation of the $(X'X)^{-1} \times 10^6$ matrix.(^a)</td>
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<td>.05</td>
<td>.01</td>
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Panel B: The property attributes $(X_c)$ matrix.\(^b\)

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\(^a\) $(X'X)^{-1}$ matrix is the variance-covariance of $\hat{\beta}$ (obtained from Vandell) divided by $s^2$ ($s = 10,746$).

\(^b\) Obtained from Vandell (1991, Table 2).

(equals $\sigma^2[1 + X_c(X'X)^{-1}X'_c]$) is likely to be greater than that of a minimum-variance grid estimator.

A Numerical Example

These propositions can be demonstrated easily using a numerical example. To make the comparison more meaningful, data reported in Tables 2 and 3 of Vandell's (1991) article are used to conduct the analysis.\(^{14}\) Panels A and B of Table 1 report the calculations of the $(X'X)^{-1}$ matrix, which is the variance-covariance matrix of $\hat{\beta}$ divided by $s^2$ ($s^2$ is the unbiased estimator of $\sigma^2$). Both the $s^2$ and the $X_c$ matrix are obtained from Vandell's Table 2.

\(^{14}\) The matrix reported in Panel A of Table 1 is the one reported in the corrective note published in the Journal of the American Real Estate and Urban Economics Association, Volume 20, No. 1, 1992, p. 151.
Table 2 ■ Calculation of the $A$ matrix, $B$ vector and the optimal weights of the comparables.

<table>
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<td>Panel A: Calculation of the $A$ matrix.&lt;sup&gt;a&lt;/sup&gt;</td>
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<td>Panel B: Calculation of the $B'$ vector.&lt;sup&gt;b&lt;/sup&gt;</td>
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<td>Panel C: Calculation of the optimal weights $\omega^*$ of comparables&lt;sup&gt;c&lt;/sup&gt; (%)</td>
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<td>13</td>
<td>4</td>
<td>14</td>
<td>9</td>
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</table>

<sup>a</sup> $A = X'(X'X)^{-1}X'_c + I_x$, where $I_x$ is an identity matrix.

<sup>b</sup> $B = X'_c(X'X)^{-1}X'_r$, where $B$ is a column vector.

<sup>c</sup> $\omega^* = A^{-1}B + \frac{1 - e'e'A^{-1}B}{e'e'A^{-1}e_x}A^{-1}e_x$

Panels A and B of Table 2 report the calculations of the $A$ matrix and $B$ vector, respectively. The optimal weights of the comparables can be easily calculated from equation (12). It should be noted that the optimal comparable weights derived and reported in Panel C of Table 2 are identical to those reported by Vandell (1991). Thus, Proposition One demonstrates that the minimum-variance weight $\omega^*$ derived from the adjustment-grid method can be decided only by the observable property attributes $X$, $X_c$ and $X_r$ when $\beta$ is estimated using the OLS method and if $\epsilon$ is assumed to be i.i.d. with the normality property.

Table 3 compares the variances of the minimum-variance grid estimator and the multiple-regression estimator. The variance of the minimum-variance grid estimator ($\sigma_G^2$) is the sum of the variance of the prediction error of a multiple-regression estimator ($\sigma_{ols}^2$) and the term $\sigma^2 \{[(1 - e'e'A^{-1}B)^2 / (e'_x A^{-1} e_x)] - (B'A^{-1}B)\}$. Thus, in order for $\sigma_G^2$ to be greater than $\sigma_{ols}^2$, the
Table 3  The variances of the prediction errors of the minimum-variance grid and multiple-regression estimators.

<table>
<thead>
<tr>
<th></th>
<th>Description</th>
<th>Variance</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>(B' A^{-1} B)</td>
<td>5505.39%</td>
</tr>
<tr>
<td>2</td>
<td>(\frac{[(1 - e'A^{-1}B)^2/(e_A^{-1}e_s)]}{X_s} + 1)</td>
<td>.14%</td>
</tr>
<tr>
<td>3</td>
<td>(\frac{[(1 - e'A^{-1}B)^2/(e_A^{-1}e_s)]}{X_s} - (B' A^{-1} B))</td>
<td>-5505.25%</td>
</tr>
<tr>
<td>4</td>
<td>(X_s (X'X)^{-1} X_s + 1)</td>
<td>5618.48%</td>
</tr>
<tr>
<td>5</td>
<td>(\sigma^2)</td>
<td>115,476,516</td>
</tr>
<tr>
<td>6</td>
<td>(\sigma_{\text{ols}})</td>
<td>80,548</td>
</tr>
<tr>
<td>7</td>
<td>(\sigma_G)</td>
<td>11,434</td>
</tr>
</tbody>
</table>

\(^a\) The \(A\) matrix and \(B\) vector are obtained from Table 2.

\(^b\) It should be noted that \(\sigma_{\text{ols}}^2 = \sigma^2 [X_s (X'X)^{-1} X_s + 1]\).

\(^c\) In this particular application, the variance of the prediction error of a
point estimator is \((\sigma_{\text{ols}}^2 = \sigma^2 [X_s (X'X)^{-1} X_s + 1])\). The standard deviation of the prediction error of the mean multiple-regression estimator
\((\sigma_{\text{ols}}^2 = \sigma^2 [X_s (X'X)^{-1} X_s])^{0.5}\) is 79,828.

\(^d\) \(\sigma_{\text{ols}}^2 = \sigma_{\text{ols}}^2 + \sigma^2 [((1 - e'A^{-1}B)^2/(e_A^{-1}e_s)] - (B' A^{-1} B)]\). When \(\sigma_{\text{ols}}^2\) is replaced by \(\sigma_{\text{ols}}^2\) (specified as \(\sigma^2 [X_s (X'X)^{-1} X_s]\)), \(\sigma_G\) equals 3,907.

The term \(B' A^{-1} B\) must be positive and sufficiently large to offset the impact of the term \(\frac{[(1 - e'A^{-1}B)^2/(e_A^{-1}e_s)]}{X_s} + 1\). The first two lines of Table 3 report the values of the term \(B' A^{-1} B\) and the term \(\frac{[(1 - e'A^{-1}B)^2/(e_A^{-1}e_s)]}{X_s} - (B' A^{-1} B)\) (55.05 and .001, respectively). The difference in these two terms is 55.05 (see line 3 of Table 3).

It is also known that \(\sigma_{\text{ols}}^2 = \sigma^2 [X_s (X'X)^{-1} X_s + 1]\). Line 4 of Table 3 reports that \(X_s (X'X)^{-1} X_s + 1 = 56.18\), or alternatively, \(X_s (X'X)^{-1} X_s = 55.18\). It should be noted that the estimation error of \(X_s \hat{\beta}\) is determined by the magnitude of the constant term \(X_s (X'X)^{-1} X_s\). In this particular example, because the number of comparables is quite large (10 comparables), 99.8% (or 55.05\(\sigma^2\) out of the 55.18\(\sigma^2\)) of the estimation error of \(X_s \hat{\beta}\) can be eliminated by the use of the minimum-variance grid estimator. As a result, the standard deviation of the prediction error of the minimum-variance grid estimator \((\sigma_G = 11.423)\) is much smaller than that of the multiple-regression estimator \((\sigma_{\text{ols}} = 80.524)\).\(^{15}\) However, it should be noted that this comparison holds only when the assumptions of OLS conditions are satisfied and when a

\(^{15}\) The variance of the prediction error of a mean estimator, \(\sigma_{\text{ols}}^2\), can be specified as \(\sigma^2 [X_s (X'X)^{-1} X_s]\). (See equations (22) and (23) for a detailed explanation of these two types of estimators.) When \(\sigma_{\text{ols}}^2\) is used, \(\sigma_G = 3,907\) and \(\sigma_{\text{ols}} = 79,828\). It is also important to note the standard deviation of 3,907 is identical to that reported by Vandell (1991).
"typical" set of parameter values is used. Under a different set of parameters or when OLS conditions are not satisfied, the numerical result might change.

**Confidence Interval and Hypothesis Testing**

**The Prediction of a Point Estimator**

Given the OLS assumptions and from equation (8), \( u_G \) is a linear combination of normal variables \( V_s \) and \( \hat{V}_G \) and, therefore, must also be normally distributed. Thus, obtained with the help of equations (9) and (14)\(^{16} \) is

\[
\frac{u_G}{\sigma \sqrt{1 + X_s (X'X)^{-1} X'_s + \frac{(1 - e'_s A^{-1} B)^2}{e'_s A^{-1} e_s} - B'A^{-1} B}} \sim N(0,1), \tag{16}
\]

where \( \omega^* \) (the minimum-variance comparable weights using the adjustment-grid method) is determined by equation (12). Replacing the unknown \( \sigma \) with its estimator \( s = \left[ u'u / (n - k) \right]^{0.5} \) (where \( u = V - X\hat{B} \)) produces

\[
\frac{V_s - \hat{V}_G}{s \sqrt{1 + X_s (X'X)^{-1} X'_s + \frac{(1 - e'_s A^{-1} B)^2}{e'_s A^{-1} e_s} - B'A^{-1} B}} \sim t(n - k), \tag{17}
\]

where \( t(n - k) \) is the Student’s \( t \) distribution with \( n - k \) degrees of freedom.\(^{17} \) From equations (16) and (17), the following proposition is derived.

**Proposition Three:** Under the assumptions of homoskedasticity and normality of \( \epsilon \) in equation (1), the prediction error of the property value based on the minimum-variance grid method is normally distributed with mean zero and variance given by equation (14). If \( s^2 \) is an unbiased estimator of the \( \sigma^2 \) used in equation (16), then the \( V_s - \hat{V}_G \) divided by its estimated standard deviation will be \( t \)-distributed with \( n - k \) degrees of freedom.

\(^{16} \) The notation "\( \sim \)" means "is distributed." \( N(0,1) \) represents the standard normal distribution.

\(^{17} \) It should be noted that comparables sharpen the value estimates indicating that they add information. However, given that only a finite number of observations are available, the degrees of freedom \( n - k \) in equation (17) decrease as the number of comparables used in the grid method increases. Also see footnote 12 for a discussion of this issue.
Equation (17) can be used for hypothesis testing. That is, given an estimated \( \hat{V}_G \), an appraisal value can be tested to see if it is significantly different from the true value of the property. In other words, the null hypothesis can be rejected, \( H_0: V_s = V_o \), at the 100\(\alpha\) percent level of significance if

\[
\left| \frac{V_0 - \hat{V}_G}{s \sqrt{1 + X_s(X'X)^{-1}X'_s + \frac{(1 - e'_s A^{-1}B)^2}{e'_s A^{-1}e_s} - B'A^{-1}B}} \right| > t_{\alpha/2}(n - k), \tag{18}
\]

where, \( t_{\alpha/2}(n - k) \) is the Student's \( t \)-Statistic with \( n - k \) degrees of freedom and a probability of \( 1 - \alpha/2 \).

Since the values of all the terms in equation (17) are known, the 100(1 - \(\alpha\)) percent confidence interval for \( V_s \) can be derived as

\[
X_s \hat{\beta} + \omega^*(V_c - X_c \hat{\beta})
\]

\[
\pm t_{\alpha/2}s \sqrt{1 + X_s(X'X)^{-1}X'_s + \frac{(1 + e'_s A^{-1}B)^2}{e'_s A^{-1}e_s} - B'A^{-1}B}.	ag{19}
\]

The result from equations (18) and (19) can be summarized into the following proposition.

**Proposition Four:** Under the assumptions of homoskedasticity and normality of \( \mathbf{e} \) in equation (1), the null hypothesis, \( H_0: V_s = V_o \), can be rejected at the 100\(\alpha\) percent level if equation (18) holds. In addition, the 100(1 - \(\alpha\)) percent confidence interval of the true property value based on the minimum-variance grid method can be determined by equation (19).

**The Prediction of the Mean Value of an Estimator**

Johnston (1984, p. 44) indicates that it is also important to find the confidence interval (or perform a hypothesis test) of the mean value of a point estimator. In this case, the mean value \( E(V_s) \) is equal to \( X_s \hat{\beta} \). Following the same procedure used in previous sections, the prediction error (\( \tilde{u}_G \)) of the \( E(V_s) \) is

\[
u_G = E(V_s) - \hat{V}_G = X_s(\beta - \hat{\beta}) - \omega'[X_s \hat{\beta} + e_c - X_s \hat{\beta}]. \tag{20}\]

The expectation of \( \tilde{u}_G \) is
\[ E(\bar{u}_G) = X_c(\beta - E(\hat{\beta})) - \omega'[X_c(\beta - E(\hat{\beta})) - E(e_c)] \]
\[ = 0. \quad (21) \]

The variance of the prediction error \( \bar{u}_G \) is

\[ \sigma^2_G = \sigma^2 [X_c(X'X)^{-1}X_c' + \omega'(X_c(X'X)^{-1}X_c' + I)\omega - 2\omega'X_c(X'X)^{-1}X_c'] \]
\[ = \sigma^2 [X_c(X'X)^{-1}X_c' + \omega'A\omega - 2\omega'B]. \quad (22) \]

From an examination of equations (10) and (22), it is clear that the optimal solution \( \omega^* \) in equation (12) is also the optimal (the minimum-variance) solution for equation (22). Therefore,

\[ \sigma^2_G = \sigma^2 \left[ X_c(X'X)^{-1}X_c' + \frac{(1 - e_c'A^{-1}B)^2}{e_c'A^{-1}e_c} - B'A^{-1}B \right]. \quad (23) \]

An implication from equations (22) and (23) is that if the number of comparables is greater than the number of adjustment factors, then the error term from the estimated \( \hat{\beta} \) could be completely eliminated. That is, there exists an optimal replicated weight vector, \( \omega^* \), which minimizes \( \omega^*'I_x\omega^* \) subject to \( \omega^*'X_c = X_c \) and \( \omega^*'e_\kappa = 1 \). Given this, \( \sigma^2_G \) in equation (23) must satisfy

\[ \sigma^2_G \leq \sigma^2[\omega^*'I_x\omega^*]. \quad (24) \]

Equation (24) demonstrates that when \( \kappa \) approaches infinity, there exists an optimal replicated weight vector, \( \omega^* \), such that \( \omega^*'I_x\omega^* \) approaches zero. Consequently, the prediction error of the mean of the true property value also approaches zero.\(^{18}\)

Since equation (20) is a linear combination of normal variables, \( \bar{u}_G \) must be normally distributed with mean zero and variance given by equation (23). Given this, the null hypothesis is rejected, \( H_0: E(V_s) = V_0 \), at the 100\( \alpha \) percent level of significance if

\(^{18}\) This concept is consistent with the law of large numbers.
\[
\left| \frac{\bar{V}_o - \bar{V}_o^*}{s \sqrt{X_s^t(X'X)^{-1}X'_o + \frac{(1 - e'_e A^{-1}B)^2}{e'_e A^{-1}e_e} - B'A^{-1}B}} \right| > t_{\alpha/2}(n - k). \tag{25}
\]

A 100(1 - \(\alpha\)) percent confidence interval for predicting the mean value of \(V_s\) under the minimum-variance grid method, therefore, is determined by

\[
X_s \hat{\beta} + \omega^*(V_c, X_s, \hat{\beta})
\]

\[
\pm t_{\alpha/2} s \sqrt{X_s^t(X'X)^{-1}X'_s + \frac{(1 - e'_e A^{-1}B)^2}{e'_e A^{-1}e_e} - B'A^{-1}B}. \tag{26}
\]

Equations (25) and (26) can be summarized in the following proposition.

**Proposition Five:** Under the assumptions of homoskedasticity and normality of \(\varepsilon\) in equation (1), the null hypothesis, \(H_0: F(V_s) = V_o\), can be rejected at the 100\(\alpha\) percent level if equation (25) holds. In addition, the 100(1 - \(\alpha\)) percent confidence interval of the expected true subject property value, based on the minimum-variance grid method, can be determined by equation (26).

**A Numerical Example**

The same numerical example that was used in the previous section can be used to demonstrate Propositions Four and Five. To estimate the confidence interval of an adjustment-grid estimator, the value of the subject property needs to be estimated. Given the optimal weights \(\omega^*\) reported in Panel C of Table 2, and based on equation (4), the estimated property value is 233,532.

Line 7 of Table 3 indicates that \(\sigma_o\) equals 11,434. From equation (19), the 95% confidence interval of \(V_s\) ranges from 211,121 to 255,943. The confidence level of the mean value, \(E(V_s)\), of the subject property, \(\bar{\sigma}_o\), equals 3,907 (see Table 3). Given equation (26), the 95% confidence interval of the \(F(V_s)\) ranges from 225,874 to 241,190.

It should be noted that the confidence intervals resulting from the minimum-variance grid method are much tighter than those resulting from the multiple-regression method. Given the \(X_s\) and \(\hat{\beta}\) reported in Vandell (1991, Tables 2 and 3), the estimated property value using the multiple-regression method is 296,522. Line 6 of Table 3 reports that the standard deviation (\(\sigma_{o_e}\)) of the prediction error of the point estimator and standard deviation (\(\sigma_{o_e}\)) of the
prediction error of the mean value are 80,548 and 79,828, respectively. Consequently, the 95% confidence interval of a point estimator $V_s$ ranges from 138,648 to 454,396. The 95% confidence interval of the mean $V_s$ ranges from 140,059 to 452,985.

In this particular example, the variance of the minimum-variance grid method is much smaller than that of the multiple-regression estimator. This is true because the standard error of the hedonic pricing equation is very high ($s = 10,746$) and because the use of the minimum-variance grid method eliminates most of the estimation error of $\hat{X},\hat{\beta}$ (see the section titled "A Comparison" for a discussion of this issue). In addition, it is interesting to note that the multiple-regression estimator (296,522) is much greater than the minimum-variance grid estimator (233,532).

To investigate whether the estimator derived from the multiple-regression method is significantly different from the estimator derived from the minimum-variance grid method, a hypothesis test was performed using equation (18). The difference between these two estimators is statistically significant ($t$-Statistic = 5.5).19 In other words, at the 99% confidence level, the hypothesis that the multiple regression method produces a property estimator similar to that of the minimum-variance grid method is rejected.

It should be noted that the results of the numerical example are obtained using a linear hedonic specification and assuming homoskedasticity and normality of $\epsilon$. However, it is well-known that the distribution of residuals are typically heteroskedastic (see, for example, Randolph 1988) and the residuals are typically spatially correlated (see, for example, Dubin 1988). In addition, there is evidence to indicate that a nonlinear specification is preferred when compared to a linear specification (see, for example, Halvorsen and Palmquist 1981). In the presence of heteroskedastic and spatially correlated residuals, instead of using the OLS procedure, the GLS method should be used for the regression equation. When the GLS procedure or a non-linear specification are used, the estimated variance of the residuals and the estimated variances of the adjustment coefficients will be smaller than those derived using the OLS method.20 This paper did not examine whether

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19 The $t$-Statistic equals 16.1 when the variance of the prediction error of the mean estimator is used (see equation (25)).

20 In other words, these results should be interpreted merely as an example of the relative magnitude of the standard deviations and confidence intervals under a typical set of parameter values and the assumption that OLS conditions are satisfied.
the conclusions still hold when OLS conditions are not satisfied or when a different set of parameters are used. This topic is recommended for future research.

Conclusion

This paper examines the statistical properties of the minimum-variance grid estimator and the multiple-regression estimator. Although both estimators are unbiased, the variance of the prediction error of a minimum-variance grid estimator should be lower than that of a multiple-regression estimator under realistic scenarios (i.e., the number of comparables is sufficiently large). This implies that the minimum-variance grid method advanced by Vandell (1991) should be the preferred approach when compared to the multiple-regression method.

It should also be noted that, when compared to the multiple regression method, the traditional adjustment-grid method lacks the ability to estimate the confidence interval (or to perform a hypothesis test) of the true property value. This paper advances the literature by deriving a technique to perform those analyses for the minimum-variance grid method. With the aid of this technique, the minimum-variance grid method can be viewed as being more complete than before.

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References


