Equilibrium Real Options Exercise
Strategies with Multiple Players: The Case of Real Estate Markets

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Abstract
of
Equilibrium Real Options Exercise Strategies with Multiple Players:
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This paper derives a closed form solution for an equilibrium real options strategic exercise model with stochastic revenues and costs for monopoly, duopoly, oligopoly, and competitive markets. To examine the impact of market power on the exercise strategies of option holders, our model also allows one option holder to have a greater production capacity than others. Under a monopolistic environment we find that the optimal option exercise strategy in real estate markets is dramatically opposite to that in a financial (warrant) market, indicating the importance of paying attention to the institutional details of the underlying market when analyzing option exercise strategies. Our model framework can be generalized to the pricing of convertible securities and capital investment decisions involving both stochastic revenues and costs under different types of market structures.
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1. Introduction

The option framework developed in the finance field has been used extensively for analyzing investment decisions related to non-financial assets. For example, Brennan and Schwartz (1985) and Paddock, Siegel and Smith (1988) use an option approach to evaluate natural resource investments and offshore petroleum leases. McDonald and Siegel (1986), Majid and Pindyck (1987), and Ingersoll and Ross (1992) explicitly analyze the impact of option value on capital investment decisions. McDonald and Siegel (1985) and Berger, Ofek and Ewary (1996) address the termination option of an investment project. Childs, Ott and Triantis (1998) use the option valuation framework for analyzing the capital budgeting decisions of interrelated projects. Grenadier (1995, 1997) applies the real options concept to value lease contracts and technological innovations. Schwartz and Zozaya-Gorostiza (2000) design a real options approach for evaluating information technology investments. However, it might be fair to say that, among all the areas embracing the application of the real options concept, real estate markets seem to draw the most attention from researchers. This is probably due to the large size of real estate markets and the availability of empirical data, which make it easier for researchers to empirically test or draw inferences from the real options theories developed in the field.

Titman (1985) and Williams (1991) are the first to apply the real options concept to value real estate developments. Quigg (1993) and Holland, Ott and Riddiough (2000), among others, provide empirical evidence demonstrating that models based on the real options concept can indeed predict property values in real estate markets. However, recognizing that a development decision is not made in isolation and one developer’s decision affects the decisions of others, researchers advance the literature by analyzing option exercise strategies in an equilibrium setting. Williams (1993) first derives symmetric equilibrium exercise strategies for

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1Childs, Riddiough and Triantis (1996) and Williams (1997) extend this concept by studying the value of a re-development option for real assets.
real estate developers.\footnote{Grenadier (2002) also derives a symmetric equilibrium for investment strategies of firms under uncertainty.} However, since this equilibrium assumes that developers will exercise their options simultaneously, it might not be suitable to describe the exercise strategies of certain types of markets. For example, if there are three developers in an office property market and the minimum size of an office building is 20,000 square feet, under a symmetric equilibrium, all the three developers will wait until the market demand reaches a level of around 60,000 square feet for each to build an office with a size of around 20,000 square feet. However, in reality, one developer will start construction when the demand level reaches about 20,000 square feet. Developers might build simultaneously (with a proportional share) if there is a large demand in a short time period that can be shared by them (say a demand for 200 single-family units in one month to be shared by 10 developers). Given this, it might be fair to say that the symmetric equilibrium characterized by Williams (1993) better describes the behavior of single family property markets with a large number of developers (say 10 to 50), but may not be suitable for commercial property markets with fewer developers (say two to nine).

Grenadier (1996) extends Williams’ model to allow for sequential exercise in a duopoly market. However, since most real estate markets (and most markets in which the real options concept has been applied) are not duopoly markets, it is not clear if the result derived for a duopoly market can be generalized to oligopoly markets. Furthermore, there are many important questions that cannot be answered by examining duopoly markets only. For example, will an increase in the number of players in a market affect the exercise decision of the players in the market? Are developers’ exercise strategies the same under different market structures (monopoly, duopoly, oligopoly, and competitive)? More importantly, both Williams’ symmetric equilibrium and Grenadier’s duopoly market equilibrium assume that all developers have an identical production capacity. In reality, some developers might have a higher production capacity than others. For example, in a market with five developable lots, one developer might own three lots and the other two own one each. Will the exercise strategies with this market setting differ from a market where five developers own one lot each (or, at the other extreme case, one developer owns all the five lots)? These questions can...
be better answered by a model that allows for sequential exercises by multiple players.

In Williams’ (1993) and Grenadier’s (1996) strategic-exercise models, the exercise price (the construction cost) is assumed to be fixed over time. While this assumption is a standard one for stock options with a fixed maturity date, it may be problematic for real estate markets. This is true because the maturity of real options is infinite and the optimal exercise time of an option might be the main focus of the model. Simply put, if a developer can build a property with the same cost today or 20 years from today, there must be an incentive for the developer to delay the exercise of the option. Williams (1991) is the first to recognize the importance of construction costs in option exercise decisions and uses the ratio between revenues and costs as the state variable for a developer’s exercise decision. However, to the best of our knowledge, no attempt has been made to include a stochastic exercise price in an equilibrium model that allows for sequential exercises. Indeed, if the level of construction costs is an important factor that drives a developer’s construction decisions, a real options model with only one state variable (revenues) may result in counter-intuitive explanations for the phenomena observed in the real world.

We advance the literature on real options by developing an equilibrium model that incorporates both stochastic demand and construction costs and allows multiple developers in the market to exercise their development options sequentially or simultaneously. We also allow one option holder to have a greater production capacity than others in order to examine the impact of market power on the exercise strategies of the holders. We solve the closed form solution of the model and derive a unique Markovian subgame perfect equilibrium for option exercise strategies in monopoly, duopoly, oligopoly, and competitive markets. This model is developed using real estate markets as the laboratory because the large amount of empirical evidence available in the field allows us to compare our model’s predictions with phenomena observed in the real world. It is also interesting to note that under a monopoly we find that the optimal real-option exercise strategy in

\footnote{Similar types of problems exist with the evaluation of the investment decision for natural resource, mining, petroleum, R&D, and technological innovations using the real options concept.}

\footnote{The construction of the Hearst Castle might serve as an interesting example to demonstrate this concept. When touring the site, the guide told visitors that the Castle was constructed during the 1919 to 1947 period. Most of the construction occurred during the years when the construction costs were low. In this particular example, it was the construction cost that drove the development decision. See www.hearstcastle.org/history/ for a detailed description of the Castle.}
real estate markets is opposite to the optimal warrant exercise strategy described by Constantines (1984). This indicates a need to pay careful attention to the institutional details of a market when developing option exercise strategies. Having said this, we also believe that our model can be revised for the pricing of convertible securities and the analysis of capital investment decisions involving both stochastic revenues and costs under different types of market structures.

The paper is organized as follows. Section 2 explains briefly the real estate market upon which our model is based. Section 3 introduces our model framework. Section 4 solves the equilibrium option values and derives equilibrium exercise strategies for developers. Section 5 presents the implications of our model. Section 6 examines the exercise strategies of a developer under a monopoly while Section 7 reports the exercise strategies of developers when they have different production capacities. Section 8 concludes the paper.

2. A Description of Real Estate Markets

The development decision in a real estate market can be viewed as the exercise of a call option. The exercise price is the construction cost of a building at the time of exercise and the underlying asset is a newly developed property (a building with the underlying land). An owner will develop the property only if the rental level of the property is high enough and/or the construction cost of the property is low enough to justify the exercise of the option. However, since both the rental level and construction cost are a function of the demand and supply levels in the market, one developer’s exercise decision will affect the decision of other developers. The worst scenario is that all developers start construction to capture the same level of demand. All developers will suffer under this circumstance. To avoid the situation where all developers exercise together, developers will have to find an equilibrium set of exercise points so that the profit from the exercise is the same among all developers. With this set of equilibrium points, developers will start to build in sequence whenever the demand level and the construction cost level justify the exercise of an option. To derive this set of equilibrium exercise points, developers will have to estimate the growth rates (and the variances of the growth rates) of future rental levels and construction costs.

The uncertainty about future rents and construction costs seems to be quite high in certain real estate markets. Legg Mason reports the national mean rental growth rates of four property types during the 1990 to 2002 period. The mean

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6Legg Mason compiles their reports from the data supplied by F. W. Dodge. We thank Glenn
growth rates (and the standard deviation) of apartment, office, retail, and industrial properties are 2.07% (2.61%), 1.29% (7.14%), 1.35% (2.68%), and 0.95% (3.91%), respectively. The variance in the rental growth rate is highest for industrial and office properties, where the standard deviation is about 4 to 5.5 times that of the mean growth rate. The variance in the growth rates of the construction costs seems to be comparable to that of the income growth rate. The McGraw Hill Construction Dodge report provides a 20-city average construction cost index during the 1909 to 2002 period. For the entire 94-year period, the mean construction cost growth rate is 4.89% and the standard deviation is 8.39%. (The construction cost growth rate is -19.52% during the 1920 to 1921 period.)

It should be noted that the rental rate and construction cost should be more volatile at the metropolitan level than at the national level (as reported here).

The construction cost consists of two components. The first is a lump-sum initial building cost and the second is the on-going repair, replacement, and renovation cost. After a building is completed, the owner has to incur continuous repair and replacement costs in order to maintain the physical condition of the building. Furthermore, as Grenadier (1996) pointed out, the use of new technologies in new buildings will affect the competitive level of existing buildings. Given this, in order to keep competitive as technologies progress, an owner will have to renovate the building even if its physical condition is well maintained. Consequently, the exercise decisions of developers at later periods affect the on-going maintenance, repair, and innovation costs of existing buildings.

The rental growth rate and construction costs data reported above also indicate that construction costs do not move in tandem with the rental level in the same market. During the 1990 to 2002 period, the correlation coefficients between the construction cost and the rental levels among apartment, office, retail, and industrial proprieties are -7.45%, 0.28%, -8.78%, and -13.40%, respectively. The literature also indicates that both the levels of housing starts and new construc-

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7The information was downloaded from http://dodge.construction.com/Default.asp on March 12, 2003. The index is based on labor and materials costs only. However, in a broad sense, construction costs should also include local building permission fees and environmental study fees. The building permission process could be lengthy and political. The permission cost could be time variant. Including the permission costs into the construction cost will increase the variation in the construction cost greatly.

8The volatility of construction costs has dropped significantly in recent years. During the 1970 to 2002 period, the mean and the standard deviation of the construction cost growth rate are 5.15% and 3.41%, respectively.
tion are inversely related to construction costs (see, for example, Blackley (1999) and Somerville (1999)) and that the level of construction costs negatively affects the investment decisions on commercial properties (see, for example, Holland, Ott and Riddiough (2000)). Capozza, Hendershott, Mach, and Mayer (2002) also indicate that the level of real construction costs plays an important role in dampening real estate cycles. It might be fair to say that both the literature and empirical evidence underscore the importance of having both rental rate and construction cost as decision variables in a real options model for real estate developments.

The production capacities among developers differ. Schwartz and Torous (2003) calculate the Herfindahl ratio using information of the top ten developers of office buildings in 34 metropolitan areas during the 1998 to 2002 period. The mean of the Herfindahl ratio for the 34 metropolitan areas is 0.11. However, the ratios seem to differ dramatically among big cities, with a low of 0.01 in New York City and a high of 0.85 in Indianapolis. Their evidence seems to confirm that developers in real estate markets might have uneven production capacities. Given this, it is also important to explore developers’ exercise strategies when their production capacities differ within a given market.

3. Model Framework

We develop our real option model based on the market characteristics described in Section 2. We start our model by assuming a market with $N$ identical parcels of vacant land owned by $N$ different developers. Each developer has an option to develop the land at any given time in the future. This assumption will be relaxed in Sections 6 and 7, where we allow one developer to hold all the $N$ parcels of land or one developer to hold more parcels of land than other developers. We also assume that this $N$ is not too large so that a developer’s option value is not negligible (and her/his exercise strategies are relevant).

We assume that a new property will begin to produce rents at the time of completion. The rental level will be determined by a downward-sloping inverse demand function. This specification recognizes that an increase in the number of competing properties will reduce the rent levels in the market. The inverse demand function is subjected to continuous demand shocks and can be specified as

$$P(t) = X(t) D[Q(t)] ,$$

where $P(t)$ is the rent inflow at time $t$, $Q(t)$ is the level of property supply at
time $t$, and $D(\cdot)$ is a differentiable function with $D' < 0$.\footnote{To simplify our model presentation, the existing inventory is normalized to zero and $Q(t)$ represents the newly-supplied units to the market.} We define $X(t)$ as a multiplicative demand shock that follows a geometric Brownian motion, or

$$
\frac{dX}{X} = \mu_X dt + \sigma_X dz_1.
$$

The constant $\mu_X$ is the instantaneous conditional expected percentage change in $X$ per unit of time. The constant $\sigma_X$ is the instantaneous conditional standard deviation per unit of time with respect to $z_1$.

The specification of the construction cost (covering both the initial building and on-going replacement costs) needs to reflect the fact that an increase in the number of competing properties will increase construction costs in the market. While we recognize that the initial building cost should be affected by the exercise decision of other developers at the time of exercise, in our continuous time model framework it is not feasible to specify a fixed time period within which the initial building cost increases with multiple developments.\footnote{For example, we cannot say that if two developers build within the same month (or year), then the construction cost will increase. What if the developers build one day after the month (or year)? Given this, we will have to specify the impact of multiple construction activities on the level of construction costs in a continuous fashion.} For this reason, we specify the initial building cost as a series of cash outflows in the future so that the construction decisions of developers can affect one another in a continuous (and diminishing) manner.\footnote{In a way, part of those cash outflows can be viewed as interest and principal payments of the building if the property is financed with mortgages.} Under this specification, when two developers exercise their options within a short time period, the impact on construction costs is greater than if they exercise the options far apart. The on-going replacement cost is also specified as a series of cash outflows in the future and a function of the number of competing properties in the market.

The construction cost outflow at each point will be determined by an upward sloping inverse supply function. The inverse supply function is subjected to continuous supply shocks and can be specified as

$$
C(t) = (1 + k) I(t) S [Q(t)],
$$

where $C(t)$ is the cost outflow at time $t$. $(1+k)$ indicates that the cost outflow is the summation of the initial building cost outflow and the on-going replacement cost outflow. With this specification, we implicitly assume that the on-going
replacement cost is proportional \((k\text{ times})\) to the initial building cost. \(Q(t)\) is the level of property supply at time \(t\), and \(S(\cdot)\) is a differentiable function with \(S' > 0\). We define \(I(t)\) as a multiplicative construction cost shock that also follows a geometric Brownian motion. We specify \(I(t)\) as

\[
\frac{dI}{I} = \mu_I dt + \sigma_I dz_2. \tag{3.4}
\]

The constant \(\mu_I\) is the instantaneous conditional expected percentage change in \(I\) per unit of time. The constant \(\sigma_I\) is the instantaneous conditional standard deviation per unit of time with respect to \(z_2\). Both the developers' initial building costs and on-going replacement costs are stochastic and follow the same construction cost shock.

The instantaneous correlation coefficient \(\rho\) between \(dz_1\) and \(dz_2\) is characterized by

\[
dz_1 dz_2 = \rho dt. \tag{3.5}\]

We assume that construction will take \(\delta\) years to complete. This assumption implies that, when a developer decides to exercise a development option (a decision to build) at time \(\tau\), the property will not begin to receive rents until time \(\tau + \delta\).

We classify the \(N\) developers into a first, second or \(n\)-th player based on the timing of the exercise of the option. When a developer exercises the development option first, we call this developer the first player. The other developers will be called the second, third,..., and \(N\)-th player, respectively. For each of the \(N\) developers, the supply level to the market is normalized to be one unit. The first player will pay the construction cost first and will be able to receive rents at a level that is free of competition. In other words, before other developers enter the market, the first developer will receive a rent inflow at \(P(t) = X(t)D(1)\) and incur a cost outflow at \(C(t) = (1 + k)I(t)S(1)\). However, after the other players exercise their development options, the rental rate received by the first player will be affected. If \(n\) developers have exercised their development options, \(Q(t) = n\), the \(n\) players will receive an oligopoly rent at \(P(t) = X(t)D(n)\) and incur a cost outflow at \(C(t) = (1 + k)I(t)S(n)\).

4. Equilibrium Option Value and Exercise Strategies

To solve the equilibrium exercise strategy of \(N\) developers, we solve the optimal stopping time for the \(N\)-th player first, assuming that all other \(N - 1\) players have
already exercised their development options. We then solve the option value of
the \( N-1 \)-th player, assuming that the \( N \)-th player will exercise her/his option at a
later time. Following the same method, we can define the option value and option
exercise time for all the players.\textsuperscript{12}

4.1. The option value of each player

It is relatively easy to calculate the option value and the optimal exercise time of
the \( N \)-th developer. To analyze the optimal stopping strategy of the \( N \)-th player,
we define

\[ g_N (t, X, I) = e^{-rt}[h_N X(t) - m_N I(t)], \]

where

\[ h_N = \frac{D(N)}{r - \mu_X}e^{-(r-\mu_X)\delta} \quad \text{and} \quad m_N = \frac{(1+k)S(N)}{r - \mu_I}, \]

and \( r \) is a risk-free rate. (We implicitly assume that \( \mu_X < r \) to ensure that the
developers will, on average, exercise their options in finite time.) The term \( h_N X(t) \)
represents the present value of a perpetual rent of \( X(t)D(N) \) to be received by
the property beginning in \( \delta \) years after the exercise of the development option
at time \( t \). The term \( m_N I(t) \) represents the present value of a perpetual cost of
\( (1+k)I(t)S(N) \), which will be incurred right after the exercise of the development
option at time \( t \). Since \( D(N) \) and \( S(N) \) are used to represent the demand and
the supply conditions in the market, it is implicitly assumed that all other \( N-1 \)
players have already exercised the development option. Given this, the value of
the development option of the \( N \)-th player with an initial state \((X, I)\) starting at
time zero is

\[ P_N (X, I) = \sup_{\tau} E^{(X,I)} [g_N (\tau, X_{\tau}, I_{\tau})], \]

where \( \tau \) is the stopping time (or the time for the developer to exercise the option).

Once the exercise time of the \( N \)-th developer is solved, the value of the \( N-1 \)-th
player’s option upon exercise can be calculated as the summation of the following
values.\textsuperscript{13}

\textsuperscript{12}Dutta and Rustichini (1993) use a similar method to solve the optimal stopping time in a
duopoly market.

\textsuperscript{13}The option value of the \( n \)-th player, \( P_n (X, I) \), is calculated in the proof of Proposition 2.
1. A perpetual payment stream $X(t) D(N-1)$ starting at $\delta$, minus a perpetual cost outflow $(1 + k)I(t) S(N-1)$, starting immediately. This payment schedule implies that the $N-1$-th player begins to receive rental income at the $X(t) D(N-1)$ level at the end of the construction period and incurs cost outflows at the $(1 + k)I(t) S(N-1)$ level immediately. The $N-1$-th player will receive this income stream and incur the cost outflows until the $N$-th player exercises her/his development option.

2. A perpetual dividend rate $X(t)[D(N) - D(N-1)]$, starting at $\tau_N + \delta$, where $\tau_N$ is defined as the $N$-th player’s optimal stopping time, minus a perpetual cost outflow payment of $(1 + k)I(t) [S(N) - S(N-1)]$, starting at $\tau_N$.

Similarly, an $n$-th player’s option value can be calculated using a backward induction method. In other words, the option value of the $n$-th player, after exercise, can be calculated as the summation of the following $N-n$ values:

1. A call option that pays a perpetual dividend rate $X(t) D(n)$, starting at $\delta$ and incurs a perpetual cost outflow $(1 + k)I(t) S(n)$, starting immediately.

2. A group of $N-n$ perpetual income streams $X(t)[D(n+1) - D(n)], ..., X(t)[D(N) - D(N-1)]$, starting at $\tau_{n+1} + \delta, ..., \tau_N + \delta$, respectively, minus the corresponding group of $N-n$ perpetual cost outflow payments $(1 + k)I(t) [S(n+1) - S(n)], ..., (1 + k)I(t) [S(N) - S(N-1)]$, starting at $\tau_{n+1}, ..., \tau_N$, respectively. Note that $\tau_i$ for $n + 1 \leq i \leq N$ is the stopping time for the $i$-th developer’s equilibrium exercise point $y_i$ (which will be inductively defined in Proposition 2).

4.2. Equilibrium exercise strategies

After the option value of each player is defined, we are in a position to identify the equilibrium points where developers are indifferent to being the $N$-th (last) developer, the $N-1$-th (the one before the last) developer, or any $n$-th developer. To do this, we first examine the equilibrium points $(y_{N-1}$ and $y_N)$ along the $X/I$ curve at which each developer is indifferent to being the $N-1$-th player or the $N$-th player, given that the remaining $N-2$ players have already exercised their options. Let $\bar{P}_{N-1}(X,I)$ be the $N-1$-th developer’s intrinsic value of the development option. Note that the intrinsic value measures the value of the option if exercised immediately.

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Proposition 1. There exists a unique point \( y_{N-1} \in (0, y_N) \), where \( y_N = \frac{\gamma \cdot m_N}{n_N} \), such that

\[
\bar{P}_{N-1}(X, I) < P_N(X, I), \quad \text{if } \frac{X}{T} < y_{N-1}, \quad (4.4)
\]

\[
\bar{P}_{N-1}(X, I) = P_N(X, I), \quad \text{if } \frac{X}{T} = y_{N-1}, \quad (4.5)
\]

\[
P_{N-1}(X, I) > P_N(X, I), \quad \text{if } y_{N-1} < \frac{X}{T} < y_N, \text{ and} \quad (4.6)
\]

\[
\bar{P}_{N-1}(X, I) = P_N(X, I), \quad \text{if } \frac{X}{T} \geq y_N. \quad (4.7)
\]

Proof. See Appendix.

Proposition 1 indicates that when the initial level of \( \frac{X}{T} \) is less than the trigger value of the \( N-1 \)-th player (\( y_{N-1} \)) the value of the \( N \)-th player’s option is greater than the value of the \( N-1 \)-th player’s option. Under this circumstance, both developers would like to be the \( N \)-th player and no one will exercise the development option first. When the initial level of \( \frac{X}{T} \) is equal to the trigger value of the \( N-1 \)-th player (\( y_{N-1} \)), the value of the \( N-1 \)-th player’s option equals the value of the \( N \)-th player’s option. Under this circumstance, both developers will be indifferent to being the \( N-1 \)-th player or the \( N \)-th player and one of the two developers (by a random process) will exercise the option first. When the initial level of \( \frac{X}{T} \) is between the trigger values \( y_{N-1} \) and \( y_N \), the value of the \( N-1 \)-th player’s option is higher than the value of the \( N \)-th player’s option. Under this circumstance, both developers would prefer to be the \( N-1 \)-th player, but only one of them will be chosen (by a random process). Given this, the \( N \)-th player will only exercise the development option when the level of \( \frac{X}{T} \) first reaches the trigger value \( y_N \). When the initial level of \( \frac{X}{T} \) is equal to or greater than the trigger value \( y_N \), the value of the \( N-1 \)-th player’s option is the same as the value of the \( N \)-th player’s option and both developers will exercise the option simultaneously.

To summarize, the equilibrium exercise strategies of the two developers are based on the two initial levels of \( \frac{X}{T} \), the trigger value of the \( N-1 \)-th player \( y_{N-1} \), and the trigger value of the \( N \)-th player \( y_N \). If the initial level is lower than \( y_{N-1} \), none of the developers should build until \( \frac{X}{T} \) equals \( y_{N-1} \). If the initial level is between \( y_{N-1} \) and \( y_N \), one of the developers will build first (by a random process) and the other will wait until \( \frac{X}{T} \) equals \( y_N \). Under this circumstance, the \( N \)-th player will not exercise the option when \( \frac{X}{T} \) is between \( y_{N-1} \) and \( y_N \). If the initial level of \( \frac{X}{T} \) is larger than \( y_N \), then both the \( N-1 \)-th player and the \( N \)-th player
should exercise the option simultaneously. The next proposition will generalize the result of the $N$-th and $N-1$-th players to all $N$ players.

**Proposition 2.** There exist $N$ points, $y_1 < y_2 < \ldots < y_N = \frac{\gamma}{\gamma-1} \frac{m_N}{h_N}$, such that the first developer will build once $X_I$ reaches $y_1$, the second developer will start to build once $y_2$ is reached, and the $n$-th developer will start to build once $y_n$ is reached. For $n \leq N-1$, $y_n$ must satisfy

$$h_n y_n + \left( m_n y_{n+1}^{-1} - h_n \right) y_{n+1}^{1-\gamma} y_n^{\gamma-1} - m_n = 0. \quad (4.8)$$

**Proof.** See Appendix

Proposition 2 indicates that, in equilibrium, there will be a set of $N$ trigger points $(y_1, y_2, \ldots, y_N)$ for the $N$ developers to exercise their development options. Developers will be indifferent to being the first, second, or last to exercise their development options because the option value to be the first developer is the same as that of all the other developers. Consequently, developers will line up in a sequence and will exercise their development options according to the movement of the level of $X_I$.

Figure 1 illustrates a sample path of the two-dimensional stochastic process $(I(t), X(t))$ that describes the equilibrium strategies of the first two developers in the market.\textsuperscript{14} We divide Figure 1 into three sectors. Sector I includes the area where $X_I$ falls below the line $y_1 = \frac{X_I}{m}$. When $X_I$ falls into this area, the first and second developers will not build. Sector II describes the area where $X_I$ falls below the line $y_2 = \frac{X_I}{m}$ but lies at or above the line $y_1 = \frac{X_I}{m}$. When $X_I$ falls into this sector, one developer has already built while the other developer will wait. Sector III defines the area where $X_I$ lies at or above the line $y_2 = \frac{X_I}{m}$. When $X_I$ falls into this area, both developers have already built.

The most interesting observation we can derive from Figure 1 is that development activities can occur and subside independent of the demand level. Sector III of the figure clearly indicates that both developers will build when the demand level is low. The condition for this to hold is that the construction cost is also low. Consequently, even in a market with a declining demand, constructions could start as long as the construction cost is falling faster than the rental level.\textsuperscript{15} In

\textsuperscript{14}This figure can be made to display all the exercise points of all the $N$ developers. However, the implications derived from the figure with two players will be the same as that derived from a figure with $N$ players.

\textsuperscript{15}Some colleagues suggest that developers might start to construct when the interest rate is low. Since we can treat interest payments as part of the overall construction cost (when a
other words, if the cost of receiving lower rents for a period of time is less than the savings resulting from taking advantage of the current low construction costs, it is rational for a developer to build even with the anticipation that the additional space will result in a lower rental rate after its completion. This implication helps explain the observation described by Grenadier (1996) that developers sometimes start construction when the demand level is falling.\textsuperscript{16} Similarly, sector I of the figure indicates that none of the developers will build even if the demand level is high. This is true because the construction cost of the building is also high. Our implication is consistent with the empirical result provided by Blackley (1999) and Somerville (1999) showing that the level of housing starts or commercial property investment is inversely related to the level of construction cost.

Second, the implication that none of the developers will build even if the demand level is high might be consistent with the observation that developers frequently delay investment decisions when the uncertainty about the future demand is high. In other words, even in a market with a strong demand and high rental level, development might not start if the current construction cost (or the uncertainty about the future demand) is also quite high. This explanation seems to be supported by empirical evidence. Holland, Ott and Riddiough (2000), using commercial real estate data, conduct an empirical test of the explanatory power of competing investment models, finding that the level of construction cost has a negative effect on investment in all types of properties included in their sample.

5. Model Implications

To derive the model’s implications, we need to establish more properties for the set of trigger values \((y_1, ..., y_N)\). The following proposition reports the upper and lower bounds of the trigger value and the expected exercise time for the \(n\)-th unit to be built in a market with \(N\) developers.

**Proposition 3.** In a market with \(N\) developers, the trigger value \(y_n\) for the \(n\)-th unit is

\[
y_n < \frac{\gamma}{\gamma - 1} \frac{m_n}{h_n} < y_{n+1}
\]

\textsuperscript{16}Grenadier (1996) argues that if developers feel a need for preemption, they will build even if the demand level is falling.
or
\[
\frac{\gamma}{\gamma - 1} \frac{m_n - 1}{h_n} < y_n < \frac{\gamma}{\gamma - 1} \frac{m_n}{h_n},
\]
(5.2)

where \( \gamma = \frac{A - (\mu_X - \mu_I) + \sqrt{(A - (\mu_X - \mu_I))^2 + 4(r - \mu_I)A}}{2A} \) and \( A = \frac{1}{2} \sigma_X^2 + \frac{1}{2} \sigma_I^2 - \rho \sigma_X \sigma_I \). The expected exercise time (defined as \( E(\tau_{0,n}) \)) for the \( n \)-th unit to be built, starting at any beginning level \( y_0 \leq y_n \) is
\[
E(\tau_{0,n}) = \frac{\ln \frac{y_n}{y_0}}{\mu_X - \mu_I - \frac{1}{2} (\sigma_X^2 - \sigma_I^2)}.
\]
(5.3)

**Proof.** See Appendix

Let \( \bar{\tau}_N \) be the average expected exercise time of the development options of the \( N \) developers in the market. From equation (5.3), the average expected exercise time of all development options can be specified as
\[
\bar{\tau}_N = \frac{1}{N} \sum_{n=1}^{N} E(\tau_{n-1,n}) = \frac{1}{N} \frac{\ln \frac{y_N}{y_0}}{\mu_X - \mu_I - \frac{1}{2} (\sigma_X^2 - \sigma_I^2)},
\]
(5.4)

where \( E(\tau_{n-1,n}) \) represents the expected exercise time between the \( n - 1 \)-th and \( n \)-th developer. In particular, \( E(\tau_{0,1}) \) represents the expected exercise time for the first developer starting from \( y_0 \) (the initial level in the market). We also assume that both the demand function \( D(.) \) and construction cost function \( S(.) \) have a constant elasticity. Specifically, \( D(x) = x^{-\frac{1}{d}} \), where \( e_d \) is the demand elasticity, and \( S(x) = x^{e_s} \), where \( e_s \) is the construction cost elasticity. To simplify the presentation, we also set \( k \) (on-going replacement costs) = 0. Under this condition, equation (5.4) can be re-written as
\[
\bar{\tau}_N = \frac{1}{N} \left( \frac{1}{e_d} + \frac{1}{e_s} \right) \ln N + \ln \left( \frac{\gamma}{\gamma - 1} \frac{r - \mu_X - \mu_I}{y_0} \right),
\]
(5.5)

Equation (5.5) indicates that \( \bar{\tau}_N \) is a decreasing function of \( N \) and that \( \bar{\tau}_N \rightarrow 0 \) if \( N \rightarrow \infty \). Consequently, as the number of players in a market increases, development activities also increase. Also, when \( N \) is sufficiently large, the developers’ option value will converge to zero. Under this circumstance, each developer’s option exercise strategies will have a negligible effect on those of other developers. Consequently, as \( N \rightarrow \infty \) developers behave like price takers in a competitive market.
The results derived in Proposition 3 allow us to explore three issues further. First, we examine the average exercise time for developers to exercise their development options in relation to the underlying demand and construction cost volatilities (Subsection 5.1). Second, we analyze how demand and construction cost elasticities affect the average exercise time of the developers (Subsection 5.2). Finally, we explore the impact of the number of players on the average exercise time of developers (Subsection 5.3).

5.1. The impact of demand and construction cost volatilities

An option has value because of uncertainty about the future. It is, therefore, interesting to investigate the impacts of demand and construction cost volatilities on the expected exercise times of developers. Our analyses help to explain why development activities could cluster over time for certain areas and why certain areas have more development activities than others.

**Proposition 4.** When demand and construction cost elasticities are constant,

\[
\frac{\partial \bar{\tau}_N}{\partial \sigma_X} = \frac{\sigma_X}{N} \left( \frac{1}{\sigma_d} + \frac{1}{\sigma_e} \right) \ln N + \ln \left( \frac{\gamma}{\gamma-1} \frac{r-\mu_X}{r-\mu_I} y_0^{-1} \right) + \frac{1}{N} \frac{-1}{\gamma(\gamma-1)} \frac{\partial \gamma}{\partial \sigma_X} \mu_X - \frac{1}{2} \sigma_X^2 + \frac{1}{2} \sigma_I^2 > 0. \tag{5.6}
\]

In addition, if (1) \( y_0 \) (the initial \( X \) level) is small enough, (2) \( N \) (the number of players) is large enough, or (3) \( \mu_I \) (the expected growth rate of construction costs) is large enough, then

\[
\frac{\partial \bar{\tau}_N}{\partial \sigma_I} = -\frac{\sigma_I}{N} \left( \frac{1}{\sigma_d} + \frac{1}{\sigma_e} \right) \ln N + \ln \left( \frac{\gamma}{\gamma-1} \frac{r-\mu_X}{r-\mu_I} y_0^{-1} \right) + \frac{1}{N} \frac{-1}{\gamma(\gamma-1)} \frac{\partial \gamma}{\partial \sigma_X} \mu_X - \frac{1}{2} \sigma_X^2 + \frac{1}{2} \sigma_I^2 < 0. \tag{5.7}
\]

**Proof.** See Appendix

Equations (5.6) and (5.7) indicate that \( \frac{\partial \bar{\tau}_N}{\partial \sigma_X} \) is positive and is a decreasing function of \( N, \sigma_d \) and \( \sigma_e \). Given this, Proposition 4 indicates that the higher the demand volatility, the longer will be the expected exercise time between developers.\(^{17}\) This implies that in a market with a high demand volatility it will take a

\(^{17}\)We should mention that Grenadier’s (1996) numerical result shows that the median exercise time between two developers in a duopoly market is shorter when the demand volatility is greater. There are two possibilities for the discrepancy between our results. First, our model analyzes average time while Grenadier analyzes median time. Second, our conclusion is based on analytical solutions while Grenadier’s result is based on numerical solutions.
longer time for developers to exercise their development options. With a higher uncertainty in the demand level, developers will wait until the demand reaches a higher level to begin construction. In other words, the required demand level for developers to start building will be lower for an area (or period) with a lower level of demand volatility. Given this, our analysis predicts that in a period (or in an area) with high demand volatilities, developers in the aggregate will build fewer units when compared to a period (or an area) with low demand volatilities. If we believe that developers in a bull market will have more confidence in the projected future demand, then this might explain why we observe that developers tend to concentrate their building activities in a hot market. We also note that, when \( N \) approaches \( \infty \) (or if we have a competitive market), demand variance has negligible impact on development activities.

Our model’s prediction seems to be supported by two recent empirical studies. Schwartz and Torous (2003) examine office developments in 34 metropolitan areas during the 1998 to 2002 period. They find that a one standard deviation increase in the volatility of the lease rate leads to a 10.9% decrease in the number of new building starts. Bulan, Mayer, and Somerville (2002) examine 1,212 condominium developments in Canada during the 1979 to 1998 period, finding that builders delay developments during times of greater idiosyncratic uncertainty in real estate returns. If we use uncertainty about future lease rates and real estate returns as proxies for uncertainty about future demand, then an increase in demand volatility will decrease the speed of developers in exercising development options in condominium and office markets.

Proposition 4 also indicates that, in most cases (\( y_0 \) is small enough, \( N \) is large enough, or \( \mu_I \) is large enough), \( \frac{\partial \tau_N}{\partial \sigma_I} \) should be negative and an increasing function of \( N \), \( e_d \) and \( e_s \). This implies that the higher the construction cost volatility, the shorter the expected exercise time between two developers. Consequently, in a market with a low construction cost volatility it will take a longer time for both developers to exercise their development options. Given this, an increase in construction cost volatility should accelerate the option exercise decision of developers. Intuitively speaking, holding everything else constant, developers will have more chance to take advantage of the window of opportunity to develop if the future movement of costs is more volatile. Anecdotal evidence tells us that this might be the case. When a city is formulating new and more stringent building codes or environmental protection ordinances (that affect construction costs), we tend to observe more development activity in the city.
To understand why the demand and construction cost volatilities should have an opposite effect on development activities in general, we note that the stochastic process $y = \frac{X}{I}$ has the following form:

$$
\frac{dy}{y} = \mu_y dt + \sigma_y dz
$$

$$
= (\mu_X - \mu_I + \sigma_I^2 - \rho \sigma_X \sigma_I) dt + \sqrt{\sigma_X^2 + \sigma_I^2 - 2 \rho \sigma_X \sigma_I} dz.
$$

(5.8)

Since $y$ is distributed lognormally, the expected continuously compounding growth rate of $y$ is

$$
\mu_y - \frac{1}{2} \sigma_y^2 = \mu_X - \mu_I - \frac{1}{2} \sigma_X^2 + \frac{1}{2} \sigma_I^2.
$$

(5.9)

From equation (5.9), it is clear that a high demand volatility $\sigma_X$ will reduce the growth rate of $y$. A lower growth rate of $y$ means that it will take a longer time for $y$ to reach to an exercise point. Equation (5.9) also demonstrates that, on the opposite side, a high construction cost volatility $\sigma_I$ will increase the growth rate of $y$. This will make it easier for $y$ to reach to an exercise point.

The results derived from Proposition 4 indicate that, in addition to demand and construction cost volatilities, demand and construction cost elasticities are also important determinants of a developer’s exercise decision. We examine this issue next.

### 5.2. The effect of demand and construction cost elasticities

Intuitively speaking, developers are more likely to build when demand elasticity is high. In other words, if developers can increase the demand for space significantly by lowering the rental level slightly then there will be more incentive for developers to build. In the extreme case, when $D(1)$ (the rental level when only one developer builds) and $D(2)$ (the rental level when two developers build) are identical, developers will build simultaneously. Similarly, if developers can lower the construction cost significantly by slightly reducing the construction activities, then there will be a strong incentive for developers not to build simultaneously. In the extreme case, when $S(1)$ (the construction cost when only one developer builds) and $S(2)$ (the construction cost when two developers build) are identical, developers will be indifferent to either building in sequence or building simultaneously. However, if the construction cost level at $S(2)$ is larger than that at $S(1)$, then there will be an incentive for developers to build their units in sequence in order to avoid high construction costs.
Let $e_d$ be the demand elasticity and $e_s$ be the construction cost elasticity. From equation (5.5), we can show that

\[
\frac{\partial \bar{\tau}_N}{\partial e_d} = -\frac{1}{N} \frac{\frac{1}{e_d^2} \ln N}{\mu_X - \mu_I - \frac{1}{2} (\sigma_X^2 - \sigma_I^2)} < 0 \tag{5.10}
\]

and

\[
\frac{\partial \bar{\tau}_N}{\partial e_s} = -\frac{1}{N} \frac{\frac{1}{e_s^2} \ln N}{\mu_X - \mu_I - \frac{1}{2} (\sigma_X^2 - \sigma_I^2)} < 0. \tag{5.11}
\]

Equation (5.10) indicates that, as the demand elasticity increases, the average exercise time between any two developers decreases. In other words, the average exercise time for developers to build one unit of building is shorter when the demand elasticity is larger. This is the case because the developers can reduce the rent slightly to increase the demand level in the face of oversupply. Equation (5.11) indicates that, as the construction cost elasticity increases, the average exercise time between developers also decreases. This makes intuitive sense since the cost for developers to build simultaneously will be about the same as when developers build in sequence.

Equation (5.10) also indicates that the impact of demand elasticity is high when the magnitude of the demand volatility is also high. When developers realize that the cost to fill a vacant space is high, they will be more careful in exercising their development options in the face of uncertainty about the future demand level. Given this, our model predicts that developers will be very careful in exercising development options for special projects such as warehouses and industrial properties. For these types of properties, a slight decrease in the rental rate normally will not increase the demand for space and developers will delay exercising their options when they are uncertain about the future demand. This is probably why the historical vacancy rates of industrial properties (which normally have a lower demand elasticity than office properties) are lower than that of office properties (see Figure 2 of Grenadier (1996)). This is also probably why we rarely observe that the medical complex market is oversupplied. If we believe that the demand elasticity for medical complexes is quite low when compared with other property types in the same market, we would expect that medical complexes

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18 Wang and Zhou (2000) demonstrate that, under certain circumstances, developers will collude with one another on rental rates and will not lower rental rates to increase demand. Under this circumstance, we will see excess vacancy rates in property markets.
will be the one with the least oversupply problem even if there is land available for the construction of such buildings. Similar arguments can be applied to resort hotels. If we believe that the demand elasticity for resort hotels is higher than that of general purpose hotels,\textsuperscript{19} after a long period of hot market conditions followed by a negative demand shock, we would expect to see a more serious oversupply problem in resort hotels than in general purpose hotels. However, an empirical validation of these two testable implications (medical complexes and resort hotels) is required before we can say more about the model’s predictions.

On the other hand, equation (5.11) shows that the impact of construction cost elasticity is high when the magnitude (absolute value) of the construction cost volatility is low. If developers realize that the cost to build units simultaneously is high, they will be more willing to develop their properties in sequence. This is especially true if they are also less certain about the project’s future cost levels. This implies that property types requiring special skilled labor, machinery, or materials will be less likely to experience concentrated developments. This makes intuitive sense if we compare the building activities of office markets in downtown areas and suburban areas of large cities. We know that high-rise, specially designed office buildings (normally in the downtown area) are more likely to require skilled labor, special machinery, and unique materials than low-rise, less-dense office complexes in suburban areas. Given this, it is less likely that developers will concentrate their developments in downtown areas. This speculation has some empirical support. Voith and Crone (1988) report that the average vacancy rate in suburban office markets in 32 large U.S. cities was about 3.1% higher than that in Central Business district (CBD) markets. Similarly, we also suspect that, in an area (such as a small city or remote town) that has difficulty attracting labor on short notice, developers might find it hard to start construction simultaneously.

5.3. An increase in the number of players

In a competitive market, where a developer’s exercise decision does not affect the exercise decisions of other developers, there is no need to study the option exercise strategies of developers. Given this, it is important to understand the impact of the number of players in a market on developers’ exercise strategies.

\textsuperscript{19}It might be easier to attract more people to take vacations by reducing hotel room rates. However, in the aggregate, a lower hotel room rate might not have a big influence on business travellers.
From equation (5.5), we can show that

$$\frac{\partial \bar{\tau}_N}{\partial N} = -\frac{1}{N^2} \left( \frac{1}{e_d} + \frac{1}{e_s} \right) (\ln N - 1) + \ln \left( \frac{\gamma}{\gamma - 1} \frac{r - \mu X}{r - \mu I} y_0^{-1} \right) < 0 \quad (5.12)$$

and

$$\frac{\partial^2 \bar{\tau}_N}{\partial N^2} = \frac{2}{N^3} \left( \frac{1}{e_d} + \frac{1}{e_s} \right) (\ln N - \frac{1}{2}) + \ln \left( \frac{\gamma}{\gamma - 1} \frac{r - \mu X}{r - \mu I} y_0^{-1} \right) > 0. \quad (5.13)$$

Equation (5.12) reports that $\frac{\partial \bar{\tau}_N}{\partial N} < 0$ if $N > 2$. This indicates that the average exercise time of developers decreases as the number of players in a market increases. Furthermore, equation (5.13) reports that $\frac{\partial^2 \bar{\tau}_N}{\partial N^2} > 0$ if $N > 1$. This indicates that the average exercise time of developers decreases at a decreasing rate when the number of players in a market increases. Clearly, the average exercise time of developers in a 3-player market will be shorter than that in a 2-player market. Furthermore, the decrease in the average exercise time will be larger when one more player enters into an existing 2-player market than into an existing 3-player market. When we divide equation (5.12) by equation (5.5), we can see that the ratio of the change in exercise time relative to the average exercise time is slightly lower than $1/N$. This indicates that the decrease in the average exercise time due to the entry of a new player could be significant when $N$ is not too large. Given this, it is clear that the effect of using option exercise strategies in real estate markets could be quite sensitive to the number of players in the market.

For any given $N$, equations (5.12) and (5.13) indicate that both the absolute values of $\frac{\partial \bar{\tau}_N}{\partial N}$ and $\frac{\partial^2 \bar{\tau}_N}{\partial N^2}$ are negatively related to demand and construction cost elasticities ($e_d$ and $e_s$), positively related to demand volatility $\sigma^2_X$, and negatively related to construction cost volatility $\sigma^2_I$ (when $y_0$ is small enough or when $\mu_I$ is large enough). Thus, when demand and construction cost elasticities are small, demand volatility is large, or construction cost volatility is small, the entry of a new developer into a real estate market will cause a meaningful increase in the development activities in the market. In other words, under these conditions, the entry of a new developer will have a larger impact on the exercise strategies of existing developers than under opposite conditions.

To see the interrelationships between the average exercise time, the number of developers, and demand (and construction cost) volatilities, we derive

$$\frac{\partial^2 \bar{\tau}_N}{\partial \sigma_X \partial N} = -\frac{\sigma_X}{N^2} \left( \frac{1}{e_d} + \frac{1}{e_s} \right) (\ln N - 1) + \ln \left( \frac{\gamma}{\gamma - 1} \frac{r - \mu X}{r - \mu I} y_0^{-1} \right) \frac{\left( \mu_X - \mu_I - \frac{1}{2} \sigma^2_X + \frac{1}{2} \sigma^2_I \right)^2}{\left( \mu_X - \mu_I - \frac{1}{2} \sigma^2_X + \frac{1}{2} \sigma^2_I \right)^2}$$

22
Equation (5.14) indicates that, while an increase in demand volatility will increase the average exercise time among developers (see equation (5.6)), the effect is reduced as the number of developers in the market increases. In other words, in a market with more developers, demand volatility has less effect on a developer’s exercise decision. This prediction seems to be supported by empirical evidence. Bulan, Mayer, and Somerville (2002) report that an increase in the number of potential competitors located near a condominium project negates the relationship between the uncertainty of real estate returns and the speed of development.

Similarly, equation (5.15) implies that the effect of construction cost volatility will be reduced as the number of developers in the market increases. The intuition is simple. From equation (5.7), we know that developers are more likely to exercise their options when they have more uncertainty about the level of future construction costs. However, the impact of construction cost volatility on the likelihood of exercising options decreases when there are more developers in the market. This is true because the exercise of more options in a short period by developers will also increase the construction cost to all developers. This will reduce the incentive for developers to exercise the options. Given this, our model predicts that in a period with high cost uncertainty (such as a proposed change in building codes or environmental protection ordinances), the increase in construction activities will be more pronounced in commercial property markets (with fewer developers) than in residential property markets (with more developers).

We also take derivatives with respect to \( N \) using equations (5.10) and (5.11) to see the interrelationships among average exercise time, number of developers, and demand (and construction cost) elasticities. We obtain

\[
\frac{\partial^2 \bar{\tau}_N}{\partial e_d \partial N} = \frac{1}{N^2} \left( \frac{1}{\sigma^2_X} \right) \frac{1}{\sigma^2_I} (\ln N - 1) > 0
\]
\[
\frac{\partial^2 \tau_N}{\partial e_s \partial N} = \frac{1}{N^2} \frac{(\ln N - 1)}{\sigma^2} (\mu_X - \mu_I - \frac{1}{2} (\sigma^2_X - \sigma^2_I)) > 0 \quad (5.17)
\]

if \( N > 2 \).

From equations (5.10) and (5.11), we know that developers tend to slow down construction activities if the demand elasticity (or construction cost elasticity) is low. Equations (5.16) and (5.17) indicate that the impact of demand elasticity (or the construction cost elasticity) on the average exercise time of developers decreases as the number of players in a market increases. Together, these four equations tell us that the impact of elasticities on development activities is weaker when there are more developers in the market. Consequently, an increase in the number of players has a bigger impact for property types with a low demand elasticity than for property types with a high demand elasticity. This seems to be a desirable characteristic for property markets. If the number of players has a larger impact on the development activities of property types with higher demand elasticities, then development activities in those property markets will be very volatile when faced with demand shocks because of the compounding effect.

### 6. Exercise Strategy under Monopoly

While in general a real estate market cannot be characterized as a monopoly, it can be argued that in certain submarkets a developer might hold a monopolistic advantage for a period of time. One example is the development of a very high end residential subdivision or an industrial park. Because of a unique location, view, or amenities, there may not be a suitable substitute in the same market. Under this circumstance, the developer of the subdivision will have a monopolistic advantage in deciding the release schedule (or the phasing) of the lots. The other example is a well developed area where developable sites are limited (such as CBDs in some established cities). In such an area some developers may hold a monopolistic advantage for a short period of time. Finally, this analysis also applies to a situation where a few developers collude with each other in a submarket. Given this, we examine a developer’s option exercise strategy under a monopoly.

We consider a monopolist who can develop \( N \) units sequentially or simultaneously. The developer will maximize the joint value of the \( N \) development
options. To make the comparison between an oligopolistic market and a monopolistic market manageable, we will assume that $nD(n)$ is concave and $nS(n)$ is convex. We will characterize a monopolist’s optimal exercise strategies and compare them with those of $N$ players in an oligopolistic market. To do this, we define $\hat{y}_n$ as the trigger value of the $n$-th unit in a monopolistic market. (Note that $y_n$ is the trigger value of the $n$-th unit in an oligopolistic market.) Let $\hat{\tau}_n$ be the $n$-th exercise time with respect to the trigger value $\hat{y}_n$. The exercise strategy of the developer in a monopolistic market can be derived by solving

\[
\sup_{\hat{\tau}_1 \leq \ldots \leq \hat{\tau}_N} E^{(X,I)} \left[ \sum_{n=1}^{N} e^{-r\hat{\tau}_n} L_n (X_{\hat{\tau}_n}, I_{\hat{\tau}_n}) \right],
\]

where $L_n (X, I) = (nh_n - (n-1)h_{n-1})X - (nm_n - (n-1)m_{n-1})I$.

We take two steps to solve equation (6.1). First, we calculate the expected value of the option $E^{(X,I)} \left[ e^{-r\hat{\tau}_n} L_n (X_{\hat{\tau}_n}, I_{\hat{\tau}_n}) \right]$ for each given trigger value. Second, we select the optimal value of $\hat{y}_n$ such that the sum of the $N$ option values is maximized.

**Proposition 5.** The first trigger value is $\hat{y}_1 = \frac{\gamma}{\gamma - 1} \frac{m_1}{h_1}$. In general, the $i$-th trigger value $\hat{y}_n = \frac{\gamma}{\gamma - 1} \frac{nm_n - (n-1)m_{n-1}}{nh_n - (n-1)h_{n-1}}$ if the demand elasticity $-D(n-1) \frac{1}{\Delta D(n)} \frac{1}{n} > 1$. Otherwise, $\hat{y}_n = \infty$ for $n > 1$. Furthermore, $\hat{y}_n > y_n$.

**Proof.** See Appendix

The concavity of $nD(n)$ and the convexity of $nS(n)$ ensure that $\hat{y}_{n-1} \leq \hat{y}_n$. From Proposition 5, it is clear that if the demand elasticity (as measured by $-D(n-1) \frac{1}{\Delta D(n)} \frac{1}{n}$) is less than one, then the developer will never exercise the second or later development option (or $\hat{y}_n = \infty$ for $n > 1$). This makes intuitive sense. For instance, if the increase of one unit of supply will decrease the revenue of a building by more than half, a monopolist will never construct the second building.

Comparing the magnitude of $\hat{y}_N$ (the trigger value of the $N$-th unit in a monopolistic market) with the magnitude of $y_N = \frac{\gamma}{\gamma - 1} \frac{m_N}{h_N}$ (the trigger value of the $N$-th unit in an oligopolistic market), it is clear that $\hat{y}_N > y_N$. From a reading of Proposition 5 and Proposition 3, it is clear that $\hat{y}_n > y_n$ for $n < N$. Given

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20Note that in an $N$-player market, while developers also maximize the value of their development options, the value of their options must be equal to one another. Otherwise, there will be no equilibrium in an $N$-player market. This constraint is not required for a monopolistic market.
this, we conclude that a monopolist will delay the exercise of development options. Holding everything else constant, at any given period, the units supplied in an oligopolistic market should be no less than that supplied in a monopolistic market. Consequently, the option value is higher for the monopolist.

It is interesting to note that our result is different from the warrant exercise strategy in a monopolistic market. Constantinides (1984) demonstrates that a monopolist exercises more warrants in the early period than warrant holders do in a competitive market, while our result indicates an opposite strategy. This demonstrates that under certain circumstances, option exercise strategies in a financial market and in a real estate market could be different and, therefore, it is important to analyze the institutional details of a market when developing option exercise strategies.

To further analyze the relationship between the \( n-1 \)-th trigger value \( \hat{y}_{n-1} \) and the \( n \)-th trigger value \( \hat{y}_n \), we divide \( \hat{y}_n \) by \( \hat{y}_{n-1} \) and obtain

\[
\frac{\hat{y}_n}{\hat{y}_{n-1}} = \frac{\Delta m_n}{\Delta h_n} \geq 1,
\]

where \( \Delta m_n = nm_n - (n-1)m_{n-1} \) and \( \Delta h_n = nh_n - (n-1)h_{n-1} \). Let \( E(\hat{\tau}_{n-1,n}) \) be the expected time between \( \hat{y}_{n-1} \) and \( \hat{y}_n \), and \( \hat{\tau}_N \) be the average expected exercise time for the monopolist to exercise all the \( N \) development options. Following similar steps to derive \( \bar{\tau}_N \) (the average expected exercise time in an oligopolistic market with \( N \) development options) in equation (5.4), we derive

\[
\hat{\tau}_N = \frac{1}{N} \sum_{n=1}^{N} E(\hat{\tau}_{n-1,n}) = \frac{1}{N} \ln \left( \frac{\gamma^{Nm_N-(N-1)m_{N-1}}}{\mu_X - \mu_I - \frac{\gamma^{N} \sigma_X^2 + \frac{1}{2} \sigma_I^2}} \right).
\]

Note that, from Proposition 5 we know \( \hat{\tau}_N > \bar{\tau}_N \) and \( \hat{\tau}_N \) is finite only when the demand elasticity is larger than one and \( \mu_X - \mu_I - \frac{\gamma}{2} \sigma_X^2 + \frac{1}{2} \sigma_I^2 > 0 \). Otherwise, \( \hat{\tau}_N = \infty \).

Equation (6.3) indicates that the average expected exercise time between trigger values is longer when: 1) the demand elasticity or the construction cost elasticity is smaller, 2) the demand volatility is larger, and 3) the construction cost volatility is lower. If we let \( D(x) = x^{-\frac{1}{e_d}} \) (\( e_d \) is demand elasticity) and \( S(x) = x^{\frac{1}{e_s}} \) (\( e_s \) is construction cost elasticity), from equation (6.3) we immediately have \( \frac{\partial^2 S}{\partial e_d \partial \sigma_X} < 0 \) and \( \frac{\partial^2 S}{\partial e_s \partial \sigma_I} > 0 \). In other words, the impact of volatilities is
higher when demand elasticity or construction cost elasticity is smaller. Furthermore, we can show that $\frac{\partial \hat{\tau}}{\partial N} < 0$ when $e_d > 1$. These implications are similar to those derived in a market with multiple players.

Proposition 5 indicates that the expected exercise time might be the longest for a property type that has a high demand volatility and is also in a monopolistic market. This prediction might be consistent with our casual observation that the sports stadium (or concert hall) market is not very active. It might be fair to say that most sports stadiums (or concert halls) are built only after the owner has already secured players to perform in the stadium (or concert hall).\footnote{Of course, if the stadium or concert hall is built by the government or a non-profit organization, then the consideration might be different.} Given this, in most areas, it might be possible for the developer of a stadium (or a concert hall) to have a monopolistic advantage at least in a short run. If we also believe that the demand volatility for sports stadiums (or concert halls) is also high, then we should observe that the construction of a stadium (or a concert hall) occurs only when developers are very certain about the future demand level. Given this, we should expect that the increase in the number of sports stadiums (or concert halls) will lag behind the increase in the number of spectators. We suggest that this implication is empirically testable.

7. Asymmetric Production Capacity

In Sections 5 and 6, we restricted our analyses to the cases where either each developer can only exercise one development option in an oligopolistic market or all development options are owned by a monopolistic developer. However, there are many mixed cases in between. What if some developers own more developable land than others? Will this change the exercise strategies of all developers?\footnote{To the best of our knowledge, at the time of writing this paper, there is no systematic analysis of issues related to the asymmetric production capacity in a dynamic game-theoretic framework.} In this section, we analyze a simple case to demonstrate this effect. We assume that there are $n + 1$ developers in the market. One of them owns $N - n$ units, where $N - n > 1$ and $N$ is the total number of lots available in the market. The remaining $n$ developers own one lot each. All other assumptions made in the previous sections are retained in this section. Similar to the oligopolistic and monopolistic cases, we define $\tilde{\tau}_N(n)$ as the average expected exercise time of the $N$ development options owned by $n$ small developers and one dominant developer.
Proposition 6. Suppose that there are $n + 1$ developers in the market. One of them is a dominant developer who owns $N - n$ development options ($N - n > 1$). Each of the remaining $n$ small developers owns one development option. In equilibrium, there exists a set of trigger values

$$\tilde{y}_1 < \ldots < \tilde{y}_n < \tilde{y}_{n+1} < \ldots < \tilde{y}_N,$$

such that the first $n$ small developers exercise their options at trigger points $\tilde{y}_1, \ldots, \tilde{y}_n$, while the dominant developer exercises $N - n$ options sequentially at trigger points $\tilde{y}_{n+1}, \ldots, \tilde{y}_N$. Furthermore

$$\bar{\tau}_N < \tilde{\tau}_N(n) < \hat{\tau}_N,$$

where $\tilde{\tau}_N(n)$ is the average expected exercise time for the $N$ development options when developers have uneven production capacities, and $\bar{\tau}_N$ and $\hat{\tau}_N$ are the expected exercise times for the $N$ development options under an oligopolistic market and a monopolistic market, respectively. Keeping $N$ constant, $\tilde{\tau}_N(n)$ decreases as the relative number of small developers $n$ increases.

Proof. See Appendix □

Proposition 6 indicates that in a market with one dominant developer and several small developers, the small developers will exercise their options first, while the dominant one will exercise her/his options last. The intuition behind the result is simple and is based on the result obtained in the monopolistic case (Proposition 5). To show the intuition, we consider a market with seven developable lots and five developers. One dominant developer owns three lots, while the remaining four developers own one lot each.

The basic case is when the dominant developer acts as three individual developers. This will be equivalent to the situation where seven developers own seven developable lots. In this case, the equilibrium result will be the one characterized by Proposition 2, in that there will be seven exercise points and developers are indifferent among these seven exercise points. However, the dominant developer can increase the value of her/his options by selecting the last three exercise points. By doing so, the developer will hold a monopoly position at a later stage of the game. Since the four smaller developers have already exercised their options at the optimal points and they also know that the dominant developer will delay the exercise points under monopoly, it is beneficial to the four small developers to allow the dominant developer to exercise all three options at a later stage of the game. In other words, this strategy will benefit the dominant developer because
of the monopoly position. It will also benefit the four small developers (who have already exercised their options) because they will face competing buildings at a later date.

It should be noted that the dominant developer cannot be allowed to exercise the three options in a monopolistic fashion before all other four developers exercise their options. For example, if the dominant developer exercises the three options first as if she/he holds a monopoly position, the exercise points of these three options will be delayed when compared to the basic case (where one developer holds one option). This will make the remaining four developers worse off than in the basic case. Given this, in equilibrium, the dominant developer will have to exercise her/his options last.

Proposition 6 also indicates that, when the dominant developer holds a larger position (more developable lots), then the average expected exercise time of all developers will be further delayed. This make intuitive sense because large holdings enable the dominant developer to exercise more options under monopoly. In addition, when we count the dominant developer as one developer, the larger the position the dominant developer holds, the fewer the number of smaller developers in the market. When the total number of available lots in a market is fixed, the larger position of the dominant developer’s holdings implies that the total number of players in the market is reduced. Since the total number of players is reduced, developers will, on average, delay exercising their options. On the other hand, when the dominant developer holds only a small position, the market will behave more like an \(N\)-developer market (with an equal production capacity).

This prediction is consistent with the empirical evidence reported by Schwartz and Torous (2003). They find a negative relationship between the \textit{Herfindahl} ratio (calculated from the market share of the top developers in metropolitan areas) and the number of new building starts in the 34 office markets in the U.S. It should be noted that a high \textit{Herfindahl} ratio implies that there are dominant developers in the metropolitan area. The evidence that metropolitan areas with dominant developers tend to have lower building starts is consistent with the model’s predictions in this section.

8. Conclusion

The persistence of excess vacancy and the volatile patterns of housing starts have earned real estate markets the dubious distinction of ranking among the most volatile industries in the U.S. By addressing the interactions among developers’
option exercise strategies, our model is able to explain many development activities that we frequently observe in real estate markets. Our analysis also indicates the importance of paying special attention to the institutional details of a market when developing option exercise strategies.

To the best of our knowledge, our real options model is the first to include two stochastic processes and offer a closed form solution for multiple players in a dynamic game-theoretic equilibrium market setting. The implications of our model to capital investment practices are worth discussing. Our model can be modified to examine the impact of technological uncertainty on corporate strategies and the impact of the market power of a dominant firm on the investment decisions of small firms. In a product market with a low demand elasticity, our model is particularly useful for the timing decision of an investment that involves both stochastic revenues and costs. For example, the production decision of an owner of one oil field (or diamond mine) will affect the total supply of petroleum (or diamonds) in the market, which in turn, will affect the production decisions of the owners of other oil fields (or diamond mines). We might also be able to observe similar strategic exercise patterns from holders of convertible bonds, where both the stock price and the exercise price are stochastic and one holder’s exercise decision affects the value of the underlying securities.
Appendix A

• Proof of Proposition 1:

Proof. In order to prove Proposition 1, we first need to establish three lemmas. Lemma 1A analyzes the optimal time for the N-th (last) developer to exercise her/his option. Lemma 1B and Lemma 1C derive the option values of the N-th player and N − 1-th player, respectively.

Lemma 1A: The optimal stopping time \( \tau_N \) that maximizes

\[
\mathbb{E}^{(X, I)}[g_N(\tau_N, X_{\tau_N}, I_{\tau_N})] = \sup_{\tau} \mathbb{E}^{(X, I)}[g_N(\tau, X_\tau, I_\tau)] = \mathbb{E}^{(X, I)}[g_N(\tau_N, X_{\tau_N}, I_{\tau_N})] \tag{8.1}
\]

is given by

\[
\tau_N = \inf\{t \geq 0, \frac{X(t)}{T(t)} \geq y_N\}, \tag{8.2}
\]

where \( y_N \) (defined as the trigger value of the N-th player) = \( \frac{\gamma}{\gamma - 1} \frac{m_N}{h_N} \),

\[
\gamma = A - (\mu_X - \mu_I) + \sqrt{[A - (\mu_X - \mu_I)]^2 + 4(r - \mu_I)A} \frac{1}{2A}, \tag{8.3}
\]

and \( A = \frac{1}{2} \sigma_X^2 + \frac{1}{2} \sigma_I^2 - \rho \sigma_X \sigma_I \).

To prove Lemma 1A, we note that the value of the development option of the N-th player with an initial state \((X, I)\) can be specified as

\[
P_N(X, I) = \mathbb{E}^{(X, I)}[g_N(\tau, X_\tau, I_\tau)], \tag{8.4}
\]

where \( s \) is the starting time and \( \tau \) is the stopping time. The optimal solution of equation (8.4) requires that \( P_N \) satisfies the condition

\[
\frac{\partial P_N}{\partial s} + \mu_X X \frac{\partial P_N}{\partial X} + \mu_I \frac{\partial P_N}{\partial I} + \sigma_X \sigma_I \rho X I \frac{\partial^2 P_N}{\partial X \partial I} + \frac{1}{2} \sigma_X^2 X^2 \frac{\partial^2 P_N}{\partial X^2} + \frac{1}{2} \sigma_I^2 I^2 \frac{\partial^2 P_N}{\partial I^2} = 0. \tag{8.5}
\]

For simplicity, we set \( s \) to zero. (This implies that the time of valuation is today.) Let \( P_N(s, X, I) = e^{-rs}I f\left(\frac{X}{T}\right) \).\(^{23}\) Since the starting time \( s \) is set to zero, we have

\(^{23}\) Kendall (1995, page 203) provides a verification theorem (theorem 10.18) for the calculation of the optimal stopping time. Since we know that the optimal solution of equation (8.1) must
\[ P_N(X,I) = P_N(0,X,I) = If(y), \] 
where \( y = \frac{X}{T} \). With the aid of equation (8.6), equation (8.5) can be re-written as

\[
-rIf + \mu_X X f' + \mu_I f - \mu_I X f' - \rho \sigma_X \sigma_I \frac{X^2}{T} f''
\]

(8.7)

\[
\frac{1}{2} \sigma_X^2 \frac{X^2}{X} f'' + \frac{1}{2} \sigma_I^2 \frac{X^2}{I} f'' = 0.
\]

Equation (8.7) implies that

\[
(\mu_t - r) f + (\mu_X - \mu_I) y f' + Ay^2 f'' = 0,
\]

(8.8)

where

\[
A = \frac{1}{2} \sigma_X^2 + \frac{1}{2} \sigma_I^2 - \rho \sigma_X \sigma_I.
\]

(8.9)

We also know that equation (8.8) must satisfy the boundary conditions

\[
f(\tilde{y}) = h_N \tilde{y} - m_N,
\]

(8.10)

where \( \tilde{y} > 0 \). (We solve \( \tilde{y} \) explicitly in equation (8.16).) Viewed from time 0, \( \tilde{f}(\tilde{y}) = h_N X - m_N \tilde{I} \) is the value of the option if the developer exercises the option at \( (\tilde{X}, \tilde{I}) \). (This can be derived from equations (4.1) and (4.2).) In addition, if \( X \) hits zero, then the value of the option becomes zero. Thus, we obtain the second boundary condition \( f(0) = 0 \). Given these two boundary conditions, Øksendal (1995, page 199) shows that the general solution of equation (8.8) can be written as

\[
f(y) = C_1 y^{\gamma_1} + C_2 y^{\gamma_2},
\]

(8.11)

where \( C_1 \) and \( C_2 \) are arbitrary constants and

\[
\gamma_i = \frac{A - (\mu_X - \mu_I) \pm \sqrt{[A - (\mu_X - \mu_I)]^2 + 4 (r - \mu_I) A}}{2A},
\]

(8.12)

be unique with respect to \( X \) and \( I \), we try different functional forms of \( X \) and \( I \) to see if we can find a solution. If our selection of the functional form is incorrect, then we will not be able to obtain a solution. However, if the selected functional form is correct, then we should be able to obtain a closed form solution for the problem. In this case, we select \( \frac{X}{I} \) as the functional form for the solution.
where \((i = 1, 2)\) and \(\gamma_2 < 0 < \gamma_1\). Because \(f(y)\) is bounded in the interval \([0, \bar{y}]\), equation (8.11) indicates that, as \(y\) approaches zero, \(C_2\) must also approach zero because \(\gamma_2 < 0\). (This is true because when \(y\) approaches zero, with \(\gamma_2 < 0\), \(f(y)\) will approach infinity if \(C_2 \neq 0\). Since \(f(y)\) is bounded, \(C_2\) must equal zero.)

When \(C_2 = 0\), equation (8.11) can be re-written as

\[
f(y) = C_1 y^{\gamma_1}. \tag{8.13}
\]

Since \(f(\bar{y}) = h_N \bar{y} - m_N\), equation (8.13) can be re-written as

\[
C_1 = \bar{y}^{-\gamma_1} (h_N \bar{y} - m_N). \tag{8.14}
\]

Combining equations (8.13) and (8.14) and replacing \(\gamma_1\) with \(\gamma\), we obtain

\[
f(y) = \bar{y}^{-\gamma} (h_N \bar{y} - m_N) y^{\gamma}. \tag{8.15}
\]

For a given level of \(y\), it is easy to show that the value of \(\bar{y}\) that maximizes \(f(y)\) is

\[
\bar{y} = \frac{\gamma}{\gamma - 1} \frac{m_N}{h_N} = y_N. \tag{8.16}
\]

Equation (8.16) indicates that \(\bar{y}\) is the trigger value that maximizes the option value of the \(N\)-th player. We define this trigger value as \(y_N\). This finishes the proof of Lemma 1A.

Given the trigger value \(y_N = \frac{\gamma}{\gamma - 1} \frac{m_N}{h_N}\), it is straightforward to calculate the value of the \(N\)-th (last) player’s option.

Lemma 1B: The value of the \(N\)-th (last) player’s option can be specified as

\[
P_N(X, I) = h_N \left(\frac{\gamma - 1}{m_N I}\right)^{\gamma - 1} \left(\frac{X}{\gamma}\right)^{\gamma}, \text{ if } \frac{X}{I} < y_N \tag{8.17}
\]

and

\[
P_N(X, I) = h_N X - m_N I, \text{ if } \frac{X}{I} \geq y_N. \tag{8.18}
\]

To prove Lemma 1B, we note that from Lemma 1A

\[
P_N(X, I) = P_N(0, X, I) = I f(y) = I y_N^{\gamma} (h_N y_N - m_N) y^{\gamma}
\]

\[
= h_N \left(\frac{\gamma - 1}{m_N I}\right)^{\gamma - 1} \left(\frac{X}{\gamma}\right)^{\gamma}, \text{ if } \frac{X}{I} < y_N. \tag{8.19}
\]

An immediate exercise of the option leads to \(P_N(X, I) = h_N X - m_N I, \text{ if } \frac{X}{I} \geq y_N.\) This finishes the proof of Lemma 1B.
Lemma 1B is important because in Proposition 2 we analyze the option value of the $N-n$-th player using $P_N(X, I)$ and a backward induction method. At this point, we need to calculate two types of values for the $N-1$-th player’s option in order to characterize the equilibrium of our model. The first is the intrinsic value of the development option $ar{P}_{N-1}(X, I)$. The second is the value of the development option, which measures the value of the option when exercised at the optimal time and is denoted by $P_{N-1}(X, I)$.

Lemma 1C: The intrinsic value of the $N-1$-th player’s option is

$$\bar{P}_{N-1}(X, I) = h_{N-1}X + I^{1-\gamma}X^\gamma (\Delta h_N y_N - \Delta m_N) y_N^{-\gamma} - m_{N-1}I$$

(8.20)

if $X < y_N$, and the value of the $N-1$-th player’s option is

$$P_{N-1}(X, I) = \bar{P}_{N-1}(X, I) = h_N X - m_N I$$

if $X \geq y_N$, (8.21)

where $\Delta h_N = \frac{D(N)-D(N-1)}{r-\mu_X} e^{-(r-\mu_X)\delta}$ and $\Delta m_N = \frac{S(N)-S(N-1)}{r-\mu_I}$.

To prove Lemma 1C, we note that the value of the perpetual payment stream $X(t)D(N-1)$, starting at $\delta$, minus the perpetual payment stream of $I(t)S(N-1)$, starting at $\tau_N$, can be calculated as

$$L_1 = h_{N-1}X - m_{N-1}I,$$

(8.22)

where $h_{N-1} = \frac{D(N-1)}{r-\mu_X} e^{-(r-\mu_X)\delta}$ and $m_{N-1} = \frac{S(N-1)}{r-\mu_I}$.

We define $L_2$ as the value of the perpetual payment stream $X(t)[D(N)-D(N-1)]$, starting at $\tau_N + \delta$, minus the perpetual payment stream of $I(t)[S(N)-S(N-1)]$, starting at $\tau_N$. We know that when $X < y_N$, $L_2$ must satisfy the condition

$$\frac{\partial L_2}{\partial s} + \mu_X X \frac{\partial L_2}{\partial X} + \mu_I I \frac{\partial L_2}{\partial I} + \sigma_X \sigma_I \rho X I \frac{\partial^2 L_2}{\partial X \partial I}$$

$$+ \frac{1}{2} \sigma_X^2 X^2 \frac{\partial^2 L_2}{\partial X^2} + \frac{1}{2} \sigma_I^2 I^2 \frac{\partial^2 L_2}{\partial I^2} = 0,$$

(8.23)

subject to the boundary condition that

$$L_2(X, I) = \Delta h_N X - \Delta m_N I$$

at $X = y_N = \frac{\gamma}{\gamma - 1} \frac{m_N}{h_N}$. (8.25)

Following the same approach for solving the option value of the $N$-th player, we have

$$L_2 = IC_1y^\gamma,$$

where $C_1 = y_N^{-\gamma} [\Delta h_N y_N - \Delta m_N]$. (8.26)
Consequently,
\[ L_2 = Iy_N^\gamma [\triangle h_Ny_N - \triangle m_N] y^\gamma, \quad \text{if } \frac{X}{I} < y_N, \quad (8.27) \]
and
\[ L_2 = \triangle h_NX - \triangle m_NI, \quad \text{if } \frac{X}{I} \geq y_N. \quad (8.28) \]
Combining equations (8.22) and (8.27), we have
\[ \bar{P}_{N-1}(X,I) = h_{N-1}X - m_{N-1}I + Iy_N^{-\gamma} [\triangle h_Ny_N - \triangle m_N] y^\gamma. \quad (8.29) \]
This finishes the proof of Lemma 1C.

We are now in a position to prove Proposition 1. Given the option value of the \( N-1 \)-th player (specified in equations (8.20) and (8.21)) and the option value of the \( N \)-th player (specified in equations (8.17) and (8.18)), we obtain
\[ h_{N-1}X - m_{N-1}I + Iy_N^{-\gamma} [\triangle h_Ny_N - \triangle m_N] y^\gamma = h_N^\gamma \left( \frac{\gamma - 1}{m_NI} \right)^{\gamma-1} \left( \frac{X}{I} \right)^\gamma. \quad (8.30) \]
Dividing both sides of equation (8.30) by \( I \), we have
\[ h_{N-1}y_{N-1} - m_{N-1} + [m_{N-1} - h_{N-1}y_N] y_{N-1}^{-\gamma} y_N^{-\gamma} = 0. \quad (8.31) \]
Let
\[ \chi(y) = h_{N-1}y - m_{N-1} + [m_{N-1} - h_{N-1}y_N] y_N^{-\gamma} y_1^{-\gamma}. \quad (8.32) \]
We know that \( \chi(y_N) \) and \( \lim_{y \to 0} \chi(y) \) are negative. In addition
\[ \chi''(y) = \gamma(\gamma - 1) [m_{N-1} - h_{N-1}y_N] y_N^{-\gamma} y_1^{-2} < 0. \quad (8.33) \]
Given this, there must be a unique \( y_{N-1} \in (0, y_N) \) such that \( \chi(y_{N-1}) = 0 \).

- Proof of Proposition 2:

**Proof.** To prove this proposition, we will have to establish Lemma 2A first, which calculates the option value of the \( n \)-th player.

Lemma 2A: Let \( y_{n+1} \) be inductively defined by equation
\[ h_{n+1}y_{n+1} - m_{n+1} + [m_{n+1} - h_{n+1}y_{n+2}] y_{n+1}^{-\gamma} y_{n+2}^{-\gamma} = 0. \quad (8.34) \]
If \( \frac{X}{I} < y_{n+1} \), then the intrinsic value of the \( n \)-th player’s option is

\[
\bar{P}_n(X, I) = h_n X + I^{1-\gamma} X^\gamma \sum_{i=n}^{N-1} (\Delta h_{i+1} y_{i+1} - \Delta m_{i+1}) y_{i+1}^{\gamma} - m_n I
\]  

(8.35)

and the option value of the \( n+1 \)-th player is

\[
P_{n+1}(X, I) = I^{1-\gamma} X^\gamma \left[ h_{n+1} y_{n+1} - m_{n+1} + \sum_{i=n+1}^{N-1} \left[ \Delta h_{i+1} y_{i+1} - \Delta m_{i+1} \right] y_{i+1}^{\gamma} \right].
\]  

(8.36)

If \( \frac{X}{I} \geq y_{n+1} \), then the intrinsic value and the value of the \( n+1 \)-th player’s option are

\[
\bar{P}_{n+1}(X, I) = \bar{P}_n(X, I) = P_{n+1}(X, I),
\]

(8.37)

where

\[
\Delta h_{i+1} = \frac{D(i+1) - D(i)}{r - \mu_X} e^{-(r-\mu_X)\delta}
\]

(8.38)

and

\[
\Delta m_{i+1} = \frac{S(i+1) - S(i)}{r - \mu_I}.
\]

(8.39)

To prove Lemma 2A, we follow the same steps as those used to prove Proposition 1. In other words, after the first \( n - 1 \) players exercise their options, the value of the \( n \)-th player’s option \( P_n(X, I) \) if exercised immediately can be decomposed into \( N - n + 1 \) components. For simplicity, we denote them by \( L_n, ..., L_N \), respectively.

The value of the perpetual payment stream of \( X(t) \) \( D(n) \), starting at \( \delta \), minus the perpetual payment stream of \( I(t) \) \( S(n) \), starting at \( \tau_N \), can be calculated as

\[
L_n = h_n X - m_n I.
\]

(8.40)

We define \( L_{n+i} \) (where \( i = 1, ..., N - n \)) as the value of the perpetual payment stream \( X(t) \) \( [D(n+i) - D(n+i-1)] \), starting at \( \tau_{n+i} + \delta \), minus the perpetual payment stream \( I(t) \) \( [S(n+i) - S(n+i-1)] \) starting at \( \tau_{n+i} \). Following the same steps as in equations (8.26) to (8.27), we have

\[
L_{n+i} = I y_{n+i}^{\gamma} [\Delta h_{n+i} y_{n+i} - \Delta m_{n+i}] y_{n+i}^{\gamma}, \text{ if } \frac{X}{I} < y_{n+1}.
\]

(8.41)
Thus,
\[ \bar{P}_n(X, I) = h_n X - m_n I + I^{1-\gamma} X^{\gamma} \sum_{i=n}^{N-1} [\Delta h_{i+1} y_{i+1} - \Delta m_{i+1}] y_{i+1}^{-\gamma}, \]
(8.42)
if \( X \leq y_{n+1} \). Similarly, in the case \( X > y_{n+1} \), the \( n + 1 \)-th player will not exercise her/his option until \( y_{n+1} \) is reached. We denote this value as \( P_{n+1}(X, I) \).

\[ P_{n+1}(X, I) = I^{1-\gamma} X^{\gamma} \left[ h_{n+1} y_{n+1} - m_{n+1} + \sum_{i=n+1}^{N-1} [\Delta h_{i+1} y_{i+1} - \Delta m_{i+1}] y_{i+1}^{-\gamma} \right]. \]
(8.43)

In the case \( X \geq y_{n+1} \), following equation (8.34) inductively, we know that the \( n \)-th player and the \( n+1 \)-th player will always exercise their options simultaneously. Thus
\[ P_n(X, I) = \bar{P}_{n+1}(X, I) = P_{n+1}(X, I). \]
(8.44)
Note that \( \bar{P}_{n+1}(X, I) \) is the intrinsic value of the option if exercised immediately. This finishes the proof of Lemma 2A.

Now we are in a position to prove Proposition 2. Suppose that the last \( N - n \) points, \( y_{n+1}, ..., y_N \), have already been identified by induction, all we need to do is to show that there exists a unique point \( y_n \in (0, y_{n+1}) \) such that
\[ \bar{P}_n(X, I) = P_{n+1}(X, I). \]
(8.45)
Combining equations (8.42) and (8.43), we have
\[ h_n y - m_n + [m_n - h_n y_{n+1}] y_{n+1}^{\gamma} y_{n+1}^{-\gamma} = 0. \]
(8.46)
The uniqueness of the solution of equation (8.46) follows from the same arguments in Proposition 1.

- Proof of Proposition 3:

**Proof.** Note that by solving equation (4.8) we have
\[ y_{n+1} = \frac{\gamma^{1-\gamma} y_n^{-\gamma-1}}{\gamma y_n^{\gamma-1} y_n^{-1} - 1} \frac{m_n}{h_n} > \frac{\gamma m_n}{\gamma - 1 h_n}, \]
(8.47)
where \( y_n < y_{n+1} \). Similarly, since
\[ h_n + \gamma \left( m_n y_{n+1}^{1-\gamma} - h_n \right) y_{n+1}^{\gamma-1} y_n^{-\gamma} > 0, \]
(8.48)
combining equation (4.8) with equation (8.48), we have

\[ y_n < \frac{\gamma}{\gamma - 1} \frac{m_n}{h_n}. \]  

(8.49)

From Øksendal (1995, page 121), we know that the expected exercise time of an option starting at any beginning level \( y_0 \) is

\[ E(\tau_{0,n}) = \ln \frac{\frac{y_n}{y_0}}{\mu_X - \mu_I - \frac{1}{2} (\sigma_X^2 - \sigma_I^2)} \]  

(8.50)

if \( \mu_X - \mu_I - \frac{1}{2} (\sigma_X^2 - \sigma_I^2) > 0 \). Otherwise, \( E(\tau_n) = \infty \). \( \blacksquare \)

• Proof of Proposition 4:

**Proof.** Equations (5.6) and (5.7) come directly from equation (5.5). To prove the results, we need to show that \( \frac{\partial \gamma}{\partial \sigma_X} < 0 \) and \( \frac{\partial \gamma}{\partial \sigma_I} < 0 \). It is easy to check (from equations (8.9) to (8.12)) that given \( \rho \),

\[ \gamma = \frac{A - (\mu_X - \mu_I) + \phi}{2A} > 1, \]  

(8.51)

where

\[ \phi = \sqrt{[A - (\mu_X - \mu_I)]^2 + 4(r - \mu_I)A} \]

is a decreasing function of \( \sigma_X \) and \( \sigma_I \). Thus

\[ \frac{\partial \gamma}{\partial \sigma_X} < 0 \text{ and } \frac{\partial \gamma}{\partial \sigma_I} < 0. \]  

(8.52)

There are two opposite forces in equation (5.7). First, an increase in \( \sigma_I \) allows \( X_I \) a greater chance of hitting the trigger value \( y_N \) because an increase in \( \sigma_I \) gives \( I \) a greater chance of hitting a low bound (a property of the Geometric Brownian motion). This effect is captured by the negative sign of the first term in equation (5.7). Second, an increase in \( \sigma_I \) results in a higher trigger value \( y_N \), which is captured by the positive sign of the second term in equation (5.7). It is clear that the sign of equation (5.7) should be negative if \( y_0 \) is small enough or the growth rate of the construction cost is large enough. The effect of \( N \) is obvious. \( \blacksquare \)

• Proof of Proposition 5:
Proof. Suppose that a monopolist chooses a set of trigger values \( \hat{y}_1 \leq \hat{y}_2 \leq \ldots \leq \hat{y}_N \) to maximize her/his objective function \( \sum_{n=1}^{N} e^{-r\tau_n} L_n(X_{\tau_n}, I_{\tau_n}) \). Equations (6.1) and (8.27) indicate that

\[
\sum_{n=1}^{N} e^{-r\tau_n} L_n(X_{\tau_n}, I_{\tau_n}) = \sum_{n=1}^{N} \hat{y}_n^{-\gamma} \left[ \hat{\Delta} h_n \hat{y}_n - \hat{\Delta} m_n \right] I^{1-\gamma} X^{\gamma},
\]  

where \( \hat{\Delta} h_n = \frac{nD(n)-(n-1)D(n-1)}{r-\mu_X} e^{-(r-\mu)\delta} \) and \( \hat{\Delta} m_n = \frac{nS(n)-(n-1)S(n-1)}{r-\mu_I} \). We first solve \( \hat{y}_1 \), taking \( \hat{y}_2, \ldots, \hat{y}_N \) as given. Under this circumstance, we have

\[
\sup_{\hat{\tau}_1} E^{(X,I)} \left[ \sum_{n=1}^{N} e^{-r\tau_n} L_n(X_{\tau_n}, I_{\tau_n}) \right] = \left[ h_1 \hat{y}_1^{-1-\gamma} - m_1 \hat{y}_1^{-\gamma} + \sum_{n=1}^{N} \hat{y}_n^{-\gamma} \left( \hat{\Delta} h_n \hat{y}_n - \hat{\Delta} m_n \right) \right] I^{1-\gamma} X^{\gamma},
\]

where

\[
\hat{y}_1 = \frac{\gamma}{\gamma - 1} \frac{m_1}{h_1}.
\]

It should be noted that \( \hat{y}_1 \) is independent of \( \hat{y}_2, \ldots, \hat{y}_N \). Similarly, solving other trigger values \( \hat{y}_n (n > 1) \) can be reduced to solving

\[
\max_{\hat{y}_n} \hat{y}_n^{-\gamma} \left( \hat{\Delta} h_n \hat{y}_n - \hat{\Delta} m_n \right) I^{1-\gamma} X^{\gamma}.
\]

Combining equations (8.54) and (8.56), we have

\[
\hat{y}_n = \frac{\gamma}{\gamma - 1} \frac{\hat{\Delta} m_n}{\hat{\Delta} h_n}, \text{ if } nh_n \text{ is increasing}
\]

and \( \hat{y}_n = \infty \), if \( nh_n \) is decreasing.

• Proof of Proposition 6:

Proof. Our proof consists of four stages. In the first stage, we characterize the exercise strategies of a market with one dominant developer (who owns two development options) and one small developer (who owns one development option). In the second stage, we characterize the exercise strategies of a market with one dominant developer (who owns two development options) and \( n \) small developers (who each owns one development option). In the third stage, we expand the
market to include a dominant developer (who owns more than two development options) and \( n \) small developers (who each owns one development option). The last stage derives the main result of the proposition.

**Stage One:**

Suppose that there are two developers in the market. One is a dominant developer and the other is a small developer. The small developer has only one unit to build, while the dominant developer has two units to build (which can be exercised either simultaneously or sequentially). For notational simplicity and without loss of generality, in the proof we will take \( S(n) \) as a constant and thus write \( m_n = m \). We will prove this case in two steps. In the first step, we will show that the dominant developer will be the last player and exercises one of her/his development options last. In the second step, we will show that, in equilibrium, the small developer will exercise her/his option earlier than the dominant developer. This means that the dominant developer will exercise two options after the small developer exercises her/his option.

We define the small developer’s option value and intrinsic value (when she/he acts as the \( i \)-th player) as \( P_s(X, I) \) and \( \bar{P}_s(X, I) \), respectively. We also define the dominant developer’s option value and intrinsic value (when she/he acts as the \( i \)-th player) as \( P_d(X, I) \) and \( \bar{P}_d(X, I) \), respectively. The small developer has three alternatives when exercising her/his option.

First, the small developer is the first player and exercises the option at an initial level \( y \). Under this circumstance, the dominant developer will act as both the second and third developers (and will exercise the two options with monopolistic power). We have

\[
\bar{P}_1(X, I) = I \left[ h_1 y - m + \Delta h_2 \tilde{y}_2^{1-\gamma} y + \Delta h_3 \tilde{y}_3^{1-\gamma} y^{\gamma} \right], 
\]

where \( \tilde{y}_2 = \frac{\gamma}{\gamma-1} \frac{m}{h_2} \) and \( \tilde{y}_3 = \frac{\gamma}{\gamma-1} \frac{m}{2h_3-h_2} \), which are obtained by following the same approach in Proposition 5. It is easy to check that \( y_2 \leq \tilde{y}_2 \leq \hat{y}_2 \) and \( y_3 \leq \tilde{y}_3 \leq \hat{y}_3 \).

Second, the small developer is the last (third) player. The dominant developer will act as both the first and second developers. Under this circumstance, the trigger value for the small developer to exercise the option is \( y_3 = \frac{\gamma}{\gamma-1} \frac{m}{h_3} \). Thus

\[
P_3(X, I) = I y_3^{1-\gamma} [h_3 y_3 - m] y^{\gamma}.
\]

Third, the small developer is the second player. In this case, the dominant player will act as both the first and last players. To calculate \( P_2(X, I) \), we need to find a trigger value \( y_2^* \) first. To do this, it is logical to assume that the dominant
developer has already exercised one option (and has one more option left). The two developers will decide who should be the second player (or the last player). If the small developer builds first, the intrinsic option value is

$$\bar{P}_2^s(X, I) = I[h_2y - m + \triangle h_3\tilde{y}_3^{1-\gamma}y^\gamma].$$  \hspace{1cm} (8.60)$$

Let $y_2'$ be the point that makes $\bar{P}_2^s(X, I)$ and $P_3^s(X, I)$ equal. Since $\tilde{y}_3 > y_3$, it is easy to show that $y_2' < y_2$. We next show that $y_2' \leq y_2' \leq \gamma-1 \frac{m}{h_2}$. To see this, we note that if the dominant developer exercises her/his second option at the trigger point $y_2'$, then the value of the option is

$$V_2 = P_1^d(X, I) + P_2^d(X, I) = Ih_1y +$$

$$I y_2'^{-\gamma}[(2h_2 - h_1)y'_3 - m]y^\gamma + 2I y_3^{1-\gamma} \triangle h_3y^\gamma. \hspace{1cm} (8.61)$$

If the small developer exercises the option at the trigger point $y_2'$, then the value of the option of the dominant developer is

$$V_3 = P_1^d(X, I) + P_3^d(X, I) = Ih_1y +$$

$$I y_3^{1-\gamma} \triangle h_3y^\gamma + I \tilde{y}_3^\gamma [(2h_2 - h_1)\tilde{y}_3 - m] y^\gamma. \hspace{1cm} (8.62)$$

It is straightforward to check that

$$\frac{V_2 - V_3}{I y^\gamma} = \frac{P_2^d(X, I) - P_3^d(X, I)}{I y^\gamma} =$$

$$\tilde{y}_3^{-\gamma} [(3h_3 - 2h_2)\tilde{y}_3 - m] - \tilde{y}_3^{-\gamma} [(3h_3 - 2h_2)\tilde{y}_3 - m] < 0. \hspace{1cm} (8.63)$$

Given this, the dominant developer has no incentive to be the second player (or to exercise the option before or at the trigger point $y_2'$). The small developer will exercise the option at the point where $V_2 = V_3$ is reached or at the point $\gamma-1 \frac{m}{h_2}$ (where the maximum value of the option is obtained), whichever is smaller. Thus, if the dominant developer has already exercised one option, then the possible exercise sequence is that the small developer exercises the second option, followed by the dominant developer. Under this circumstance, we have

$$P_2^s(X, I) = I \left[(h_2 y_2^s - m)y_2^s - \gamma y^\gamma + \tilde{y}_3^{1-\gamma} \triangle h_3 y^\gamma \right]. \hspace{1cm} (8.64)$$

Given that the last option is always exercised by the dominant developer in equilibrium, our next step is to find out who will act as the first player.
Note that, at any point $y$, both the dominant developer and the small developer will compare option values (to be the first or the second player) when making exercise decisions. Let $\Delta_s$ and $\Delta_d$ be the option value differences (to be the first player or the second player) for the small developer and the dominant developer, respectively. Under this circumstance, we have

$$\Delta_s = \bar{P}^s_1 - P^s = I \left[ h_1 y - m + \bar{y}_2^{1-\gamma} \triangle h_2 y^{\gamma} - (h_2 y_2^* - m) y_2^{*-\gamma} y^{\gamma} \right]$$

and

$$\Delta_d = \bar{P}^d_1 - P^d = I \left[ h_1 y - m + y_2^{*1-\gamma} \triangle h_2 y^{\gamma} - (h_2 \bar{y}_2 - m) \bar{y}_2^{*-\gamma} y^{\gamma} \right].$$

When the initial level $y$ is small, both $\Delta_s$ and $\Delta_d$ are negative. In equilibrium, the developer whose option value difference becomes positive first will exercise the option. Since

$$\Delta_s - \Delta_d = I \left[ (2h_2 - h_1) \bar{y}_2 - m \right] \bar{y}_2^{*-\gamma} y^{\gamma} - I \left[ (2h_2 - h_1) y_2^* - m \right] y_2^{*-\gamma} y^{\gamma} \geq 0,$$

the small developer will exercise her/his option first. The exercise point $\bar{y}_1$ can be solved by setting $\Delta_d = 0$. This concludes our proof for the first stage, where we have one small player (with one development option) and one dominant player (with two development options).

**Stage Two:**

We now start with the case in which there are $n$ small developers and one dominant developer who owns two development options. Again, our proof is divided into two steps. First, we show that, given that the dominant developer has already exercised the first option, then it is best for the dominant developer to be the last player when exercising the remaining option. Second, we show that the first strategy is inferior to the one where the dominant developer exercises the two options at the end of the game.

Suppose that the dominant developer has already exercised one option as the first player. Let $V_i = P^d_1(X, I) + P^d_i(X, I)$ be the dominant developer's option value when the second option is exercised as the $i$-th player. $V_{i+1}$ is similarly defined. We then have

$$\frac{V_i - V_{i+1}}{I y^{\gamma}} = y_i^{1-\gamma} \triangle h_i < 0$$

for $i = 2, \ldots, n$. In case $i = n + 1$, following the same procedure as in Stage One, we obtain an equation that is similar to equation (8.63), or

$$\frac{V_{n+1} - V_{n+2}}{I y^{\gamma}} =$$

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\[ y_{n+2}^\gamma [(3h_{n+2} - 2h_{n+1}) y_{n+2} - m] - \tilde{y}_{n+2}^\gamma [(3h_{n+2} - 2h_{n+1}) \tilde{y}_{n+2} - m] < 0. \] (8.68)

Note that, in equation (8.68),
\[ y_{n+2} = \frac{\gamma}{\gamma - 1} \frac{m}{h_{n+2}} < \frac{\gamma}{\gamma - 1} \frac{m}{2h_{n+2} - h_{n+1}} = \tilde{y}_{n+2}. \]

Thus, if the dominant developer has already exercised one option, she/he will exercise the remaining option as the last player. The trigger point for the last option is \( \tilde{y}_{n+2} = \frac{\gamma}{\gamma - 1} \frac{m}{2h_{n+2} - h_{n+1}}. \)

We now calculate the value of the first option when the dominant developer chooses to exercise her/his first option as the \( i \)-th player, assuming that the dominant developer will exercise the second option as the last player. Under this condition, we have \( P_d^i(X, I) = P_d^{i+1}(X, I) \) for \( i = 1, \ldots, n - 1 \) and
\[ \frac{P_d^n(X, I) - P_d^{n+1}(X, I)}{\gamma} = y_{n+1}^*(h_{n+1} y_{n+1}^* - m) - \tilde{y}_{n+1}^*(h_{n+1} \tilde{y}_{n+1} - m), \] (8.69)

where \( y_{n+1}^* \) is the trigger point of the \( n + 1 \)-th small developer if the dominant developer exercises his/her first option as the \( i \)-th player \( (i \leq n) \). Following the same procedure as in Stage One (where \( n = 1 \)), we obtain \( y_{n+1}^* \leq \tilde{y}_{n+1} = \frac{\gamma}{\gamma - 1} \frac{m}{h_{n+1}} \), which implies that \( P_d^n(X, I) \leq P_d^{n+1}(X, I) \). In other words, in equilibrium, the dominant developer will not exercise her/his first option until all small developers have exercised their options.

**Stage Three:**

The result for a more general case where the dominant developer owns \( w \) \((w > 2)\) options can be shown by induction. Following the same logic as in the case \( w = 2 \), we first examine the scenario where the dominant developer has already exercised one option as the \( i \)-th player, where \( 1 \leq i \leq n + 1 \). Under this condition, the game after the dominant developer has exercised one option is reduced to the case in which there are \( n - i + 1 \) small developers and one dominant developer who owns \( w - 1 \) development options. By assumption, it is optimal for the dominant developer to exercise her/his remaining \( w - 1 \) options after all small developers have exercised their options. We next show that it is also optimal for the dominant developer to exercise her/his first option as the \( n + 1 \)-th player, given that the remaining \( w - 1 \) options are exercised as the last \( w - 1 \) players.

To show this, we note that, as in the case of \( w = 2 \), \( P_d^i(X, I) = P_d^{i+1}(X, I) \) for \( i = 1, \ldots, n - 1 \) and
\[ \frac{P_d^n(X, I) - P_d^{n+1}(X, I)}{\gamma} = y_{n+1}^*(h_{n+1} y_{n+1}^* - m) - \tilde{y}_{n+1}^*(h_{n+1} \tilde{y}_{n+1} - m). \] (8.70)
In equation (8.70), \( y_{n+1}^* \) represents the \( n + 1 \)-th small player’s optimal trigger point when the dominant developer exercises one option as the first player and then exercises the remaining \( w - 1 \) options as the last \( w - 1 \) players. Similar to the case in Stage One, we can show that \( y_{n+1}^* \leq \tilde{y}_{n+1} \), which implies that \( P_n^d(X, I) \leq P_{n+1}^d(X, I) \).

We have shown that, in equilibrium, the dominant developer exercises her/his \( N - n \) options as the last \( N - n \) players. Given this, the equilibrium trigger points \( (\tilde{y}_1, ..., \tilde{y}_n, \tilde{y}_{n+1}, ..., \tilde{y}_N) \) can be determined in two steps. For trigger points \( (\tilde{y}_{n+1}, ..., \tilde{y}_N) \), we can apply the result in Proposition 5 to obtain

\[
\tilde{y}_{n+1} = \frac{\gamma}{\gamma - 1} \frac{m}{h_{n+1}}, ..., \tilde{y}_N = \frac{\gamma}{\gamma - 1} \frac{m}{(N-n)h_N - (N-n-1)h_{N-1}}. 
\] (8.71)

However, solving trigger points \( (\tilde{y}_1, ..., \tilde{y}_n) \) or \( \tilde{y}_i \) for \( 1 \leq i \leq n \) is more involved. First, we solve \( \tilde{y}_n \). To do this we are actually characterizing a game equilibrium in which there are one small developer and one dominant developer who owns \( N - n \) options, with \( n - 1 \) units of existing inventory outstanding. In this game, let \( y^*_j \) be the trigger point for the small developer to act as the \( j \)-th player, at which

\[
P^d_j(X, I) - P^d_{j+1}(X, I) = 0
\] (8.72)

for \( j = n, ..., N - 1 \), and \( y_N^* = y_N = \frac{\gamma}{\gamma - 1} \frac{m}{h_N} \). Clearly, \( \tilde{y}_n = y_n^* \). To determine \( y_n^* \), we note that equation (8.72) implies that

\[
h_j y_j^* - m + y_{j+1}^{*\gamma} \Delta h_{j+1}y_{j+1}^{*\gamma} - \tilde{y}_{j+1}^{r\gamma}(h_{j+1}\tilde{y}_{j+1} - m)y_{j+1}^{r\gamma} = 0.
\] (8.73)

Thus, \( y_n^* \) (and hence \( \tilde{y}_n \)) can be solved by backward induction. Once \( \tilde{y}_n \) is solved by equation (8.73), it is straightforward to check that \( \tilde{y}_i \) for \( 1 \leq i \leq n - 1 \) is determined by

\[
h_i \tilde{y}_i - m + [m - h_i \tilde{y}_{i+1}] \tilde{y}_i^{r\gamma} \tilde{y}_{i+1}^{r\gamma} = 0.
\] (8.74)

**Stage Four:**

From equations (8.50) and (8.71), we obtain

\[
\tilde{\tau}_N(n) = \frac{1}{N} \ln \frac{\tilde{y}_N}{\tilde{y}_N} = \frac{1}{N} \ln \left( \frac{1}{N} \frac{\gamma}{\gamma - 1} \frac{m}{(N-n)h_N - (N-n-1)h_{N-1}} \right), \] (8.75)

where \( \tilde{\tau}_N(n) \) is the expected exercise time for the \( N \) development options in a market where developers have uneven production capacities. It is clear from an examination of equation (8.75) that \( \tilde{\tau}_N(n) \) is a decreasing function of \( n \). The fact that \( y_N < \tilde{y}_N < \tilde{y}_N \) implies that \( \tilde{\tau}_N < \tilde{\tau}_N(n) < \tilde{\tau}_N \).
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