Arch. Math. Logic (2008) 47:143–157
DOI 10.1007/s00153-008-0073-7

Mathematical Logic

A hierarchy of hereditarily finite sets

Laurence Kirby

Received: 3 January 2005 / Revised: 7 June 2006 / Published online: 31 May 2008

Abstract This article defines a hierarchy on the hereditarily finite sets which reflects the way sets are built up from the empty set by repeated adjunction, the addition to an already existing set of a single new element drawn from the already existing sets. The structure of the lowest levels of this hierarchy is examined, and some results are obtained about the cardinalities of levels of the hierarchy.

Keywords Finite set theory · Adjunction · Adduction · Hierarchy

Mathematics Subject Classification (2000) 03E20

1 Introduction

It is well known that the hereditarily finite sets are generated from the empty set by repeated use of the *adjunction* (or *adduction*) operator

$$\langle x, y\rangle \mapsto x \cup \{y\}.$$

Givant and Tarski [1] proposed a form of induction based on this operator to characterize the hereditarily finite sets. Their induction was a stronger version of an induction introduced half a century earlier by Tarski [8].

In this paper I use the adjunction operator in another way, to define a hierarchy on the hereditarily finite sets that differs from the usual cumulative hierarchy. This hierarchy naturally mirrors the process of generating the hereditarily finite sets using

L. Kirby (✉)
Department of Mathematics, Baruch College, City University of New York,
1 Bernard Baruch Way, New York, NY 10010, USA
e-mail: laurence_kirby@baruch.cuny.edu

adjunction, and I shall argue that it is essentially finitist in nature. I study the early levels of the adjunctive hierarchy in ways that are not feasible for the cumulative hierarchy because the latter grows too fast.

2 Basics

We work in a language $\mathcal{L}(0\,;)$ for set theory which differs from the usual language $\mathcal{L}(0\,\in)$. Instead of the membership relation $\in$ we employ as basic operation *adjunction* or *adduction*, a binary operation denoted by $[x;\, y]$. So $\mathcal{L}(0\,;)$ has a binary function symbol for adjunction as well as a constant symbol 0. The intended interpretation of $[a;\, p]$ is $a \cup \{p\}$.

Informally I shall frequently drop the outermost brackets where no ambiguity results, and use the notation: $a;\, p, q = [a;\, p];\, q$. Thus

$$a;\, p_1, \ldots, p_n = [\ldots[[a;\, p_1];\, p_2];\, \ldots];\, p_n.$$

The symbol 0 will be used in two ways: as the constant symbol in the formal language, and for the empty set which is the standard interpretation of this symbol. Likewise the semi-colon does duty both as the function symbol and as its interpretation in sets. So $0;\, p_1, p_2, \ldots, p_n = \{p_1, p_2, \ldots, p_n\}$. Also $p \in a$ is defined to mean $a;\, p = a$. So in fact $\mathcal{L}(0\,\in)$ and $\mathcal{L}(0\,;)$ are equivalent: each of the symbols is definable in terms of the other in a common extension $\mathcal{L}(0 \in ;\,)$.

As a background theory, assume a familiar set theory: ZF will do. As W. W. Tait [7] emphasizes, when we describe a finitist operator in general terms, we are doing so from an infinitist standpoint.

Following the usual development of set theory, the finite von Neumann ordinals $0, 1, 2, 3, \ldots$ are defined by iterating the operator $n \mapsto [n;\, n] = n + 1$. The successor operation for numbers has become the *diagonalization* of the adjunction operator.

Kirby [2] gives details of the connection between the adjunction operator and the operations of ordinal arithmetic. And [3] uses adjunction as a basis for axiomatizing finite set theory and generating the primitive recursive set functions.

V_ω is the set of *hereditarily finite sets*, defined as usual by means of the cumulative hierarchy

$$V_0 = 0,\;\; V_{n+1} = PV_n,\;\; V_\omega = \bigcup\{V_n \mid n \in \omega\},$$

where Px is the power set of x.

In the iterative conception of set theory, the hierarchy of the V_n is the way of building up the hereditarily finite sets starting from 0. But if we examine this process of generation in detail, especially taking into account considerations of feasibility, it becomes evident that it does not conform with Ockham's razor. The number 5, for example, doesn't occur until V_6, by which time we have also obtained $2^{65536} - 1$ other sets, including many complicated structures. This number is far more than the number of protons in the universe of physics, indeed in 2^{65270} such universes. This seems a grossly inefficient way to generate sets, in as much as long before we get to small, simple, useful sets such as 6 we have generated more sets than we can ever possibly use or name in practice. Of course this is a version of a familiar complaint about exponentiation which, especially along with iteration of functions, results in unfeasible growth too soon and too fast.

In the language $\mathcal{L}(0\,;)$ we can define a natural, slower-growing hierarchy for the hereditarily finite sets, the *adjunctive hierarchy*:

$$A_0 = [0; 0] = \{0\} = 1, \quad A_{n+1} = A_n \cup \{\, [x; y] \mid x, y \in A_n\}.$$

(The subscripts n and the names A_n are here in the metalanguage).

It is intuitively clear that this hierarchy generates all hereditarily finite sets: $V_\omega = \bigcup\{A_n \mid n \in \omega\}$. And it conforms well with the following picture of the hereditarily finite sets given by Flavio Previale ([5, p. 214]):

> Hered. finite sets are exactly what can be obtained by starting from the empty set and then iterating the operation of adding, to a set already obtained (by the procedure itself), an element taken in its turn among the sets already obtained.

Note that $5 \in A_5$, and we shall see that the cardinality of A_5 is 11680, so we have a more reasonable amount of baggage along with the number 5. The situation has not improved sufficiently, however, to be called feasible. A_6 has become unwieldy at 135717904 elements.

How fast does the cardinality of A_n grow? Clearly an upper bound for $|A_{n+1}|$ is $|A_n| + |A_n|^2$, so that the growth of the size of A_n, as a function of n, is double-exponential (i.e., of order 2^{2^n}), rather than iterated exponential as is the case for V_n. A heuristic argument suggests that this upper bound is not a bad approximation for the size of A_n since one would expect relatively few repetitions among the new sets $[x; y]$ that are added at stage $n + 1$. This argument is somewhat borne out by the results of Sect. 4.

The cumulative hierarchy extends, of course, to the infinite sets by taking unions at limit stages, but the present hierarchy does not have a natural transfinite extension: we can define $A_\omega = \bigcup_{n\in\omega} A_n = V_\omega$, but it is easy to verify that:

Proposition 2.1 $V_\omega \cup \{\, [x; y] \mid x, y \in V_\omega\} = V_\omega$.

So the generative process stalls, and adjunction alone is not enough to generate the transfinite sets, even with $\bigcup$ as an additional generator. The adjunctive hierarchy is an essentially finitist notion. However, adjunction itself works fine as a basic notion (instead of the membership relation) for infinitary set theory; see [2].

Some basic properties of the A_n can be proved by induction on n:

Lemma 2.2

(i) *For each* $n \in \omega,\ n \in A_n$.
(ii) $x \in A_n \rightarrow |x| \le n$.
(iii) $x \in A_n \rightarrow x - \{y\} \in A_n$, *for any* y.

(*Notation*: $|x|$ is the cardinality of x. In this paper, the symbol '$-$' will be used for the set theoretic difference operation, and only occasionally for arithmetical subtraction in clearly arithmetical contexts such as within the subscript of A_{n-1}).

Proof of Lemma 2.2 (iii) $0 - \{y\} = 0$, so (iii) is true for $n = 0$. Suppose true for n, and let $x \in A_{n+1}$: say $x = a; p$ with $a, p \in A_n$. If $p = y$ then $x - \{y\} = a - \{y\}$ which is in A_n by the induction hypothesis. If $p \neq y$ and $y \in a$, then $x - \{y\} = (a - \{y\}); p$ and, again using the induction hypothesis, this is in A_{n+1}. If $p \neq y$ and $y \notin a$, then $x - \{y\} = x \in A_{n+1}$.

The first part of the next Lemma strengthens (iii) of the previous Lemma. J. C. Shepherdson ([6, Sect. 2.4]) uses a version of the property stated in the first part of the next Lemma as a condition for absoluteness of cardinals in models of GB set theory. A. R. D. Mathias [4] uses the name *supertransitive* for the conjunction of properties (i) and (iii).

Lemma 2.3

(i) $x \in A_n \rightarrow Px \subseteq A_n$.
(ii) *If* $x \in A_{n+1}$ *and* $y \in x$ *then* $y \in A_n$.
(iii) A_n *is transitive.*

Proof (i) Inductively, assume (i) proven for n. Suppose $x \in A_{n+1}$ and $y \subseteq x$. We need to show that $y \in A_{n+1}$. Say $x = a; p$ with $a, p \in A_n$. By Lemma 2.2(iii), $a - \{p\} \in A_n$. Since $y - \{p\} \subseteq a - \{p\}$, by the induction hypothesis $y - \{p\} \in A_n$. Now y is equal to either $y - \{p\}$ or $[(y - \{p\}); p]$, depending on whether $p \notin y$ or $p \in y$. In either case, $y \in A_{n+1}$.

(ii) The case $n = 0$ is easy to verify since $A_1 = \{0, \{0\}\} = \{0, 1\} = 2$. Inductively, suppose $y \in x = a; p$ with $a, p \in A_{n+1}$. Then either $y \in a$ in which case the inductive hypothesis supplies $y \in A_n$, or $y = p$ so $y \in A_{n+1}$.

(iii) follows from (ii).

Each closed term of $\mathcal{L}(0\,;)$ represents a hereditarily finite set. But a given non-empty hereditarily finite set has more than one such representation. We shall need to consider when two closed terms represent the same set. For closed terms s, t of $\mathcal{L}(0\,;)$, define $s \equiv t \Leftrightarrow s^{V_\omega} = t^{V_\omega}$, where t^{V_ω} is the interpretation of the closed term t in V_ω (considered as a structure for $\mathcal{L}(0\,;)$). Equivalently, $s \equiv t \Leftrightarrow V_\omega \models s = t$. There will be a syntactical equivalent for this concept in Theorem 2.4 below.

Let PS_{00} be the theory consisting of $0; 0 \neq 0$ together with the universal closures of the following axioms:

$$x; y, y = x; y. \tag{1}$$

$$x; y, z = x; z, y. \tag{2}$$

PS_{00} is true in V_ω. The name PS_{00} is to suggest a very restricted form of Peano set theory, following Flavio Previale [5]. In [3], two other basic axioms will be added to PS_{00} along with a schema for the first-order version of the Givant-Tarski induction to get the full theory PS. Meanwhile, for Theorem 2.4 we shall only have need of this weaker theory.

We also need a hierarchy on the set $\mathcal{T}$ of closed terms of $\mathcal{L}(0\,;)$, corresponding to the A_n hierarchy, defined by

$$\mathcal{T}_0 = \{0\}, \ \mathcal{T}_{n+1} = \mathcal{T}_n \cup \{\, [s; t] \mid s, t \in \mathcal{T}_n\}.$$

So $a \in A_n \leftrightarrow$ for some $t \in \mathcal{T}_n, \ t^{V_\omega} = a$.

Part (i) of the next Theorem is a weak completeness result. It shows in particular that the relation $s \equiv t$ is primitive recursive. Part (ii) is a refinement of Lemma 2.2(iii) needed for the proof:

Theorem 2.4

(i) *For closed terms* s, t, $s \equiv t$ *if and only if* $PS_{00} \vdash s = t$.
(ii) *Let* $n \geq 1$. *If* $t \in \mathcal{T}_n$ *and* $p \in t^{V_\omega}$ *then there exist* $t_0 \in \mathcal{T}_n$ *and* $s \in \mathcal{T}_{n-1}$ *such that* $PS_{00} \vdash [t_0; s] = t$, $s^{V_\omega} = p$, *and* $p \notin t_0^{V_\omega}$.

Proof We prove both parts together by a joint induction. As inductive hypothesis assume that (i) is proven for $s, t \in \mathcal{T}_n$, and (ii) is proven for n. The case $n = 1$ is straightforward and uses $0; 0 \neq 0$. In the proof, the symbol '$\vdash$' will refer to provability in PS_{00}.

First we prove (ii) with n replaced by $n + 1$. So assume $t \in \mathcal{T}_{n+1}$ and $p \in t^{V_\omega}$. Say $t = u; v$ with $u, v \in \mathcal{T}_n$. We need $t_0 \in \mathcal{T}_{n+1}$ and $s \in \mathcal{T}_n$ satisfying the conditions in (ii). If $p = v^{V_\omega}$ and $p \notin u^{V_\omega}$, then we can take $t_0 = u$, $s = v$. If $p = v^{V_\omega}$ and $p \in u^{V_\omega}$, then by the inductive hypothesis for (ii) applied to u, take $t_0 \in \mathcal{T}_n$, $s \in \mathcal{T}_{n-1}$ such that $\vdash t_0; s = u$, $s^{V_\omega} = p$, and $p \notin t_0^{V_\omega}$. By the inductive hypothesis for (i), $\vdash s = v$, and hence, using axiom (1),

$$\vdash t_0; s = t_0; s, s = t_0; s, v = u; v = t.$$

If $p \neq v^{V_\omega}$, then $p \in u^{V_\omega}$ and as before use the inductive hypothesis for (ii) to obtain $u_0 \in \mathcal{T}_n$ and $s \in \mathcal{T}_{n-1}$ such that $\vdash u_0; s = u$, $s^{V_\omega} = p$, and $p \notin u_0^{V_\omega}$. Then

$$\vdash u; v = u_0; s, v = u_0; v, s = [u_0; v]; s \qquad \text{(using axiom (2))}$$

and $p \notin [u_0; v]^{V_\omega}$, so we can take t_0 to be $u_0; v$.

To complete the inductive step, we prove, given $s, t \in \mathcal{T}_{n+1}$ with $s^{V_\omega} = t^{V_\omega}$, that $\vdash s = t$. Say $s = q; r$ and $t = u; v$ with $q, r, u, v \in \mathcal{T}_n$.

Case 1 $r \equiv v$. Working in V_ω, we have $q; r = u; r$ and hence one of three subcases must occur: either $V_\omega \models q = u$, or $V_\omega \models q; r = u \wedge q; r \neq q$, or $V_\omega \models u; r = q \wedge u; r \neq u$. In the first subcase, by the inductive hypothesis for (i), $\vdash q = u$ and hence $\vdash s = q; r = u; r = t$. In the second subcase, use the inductive hypothesis for (ii) to obtain $s \in \mathcal{T}_{n-1}$ and $u_0 \in \mathcal{T}_n$ with $\vdash u_0; s = u$, $s^{V_\omega} = r^{V_\omega}$, and $r^{V_\omega} \notin u_0^{V_\omega}$. Now $q^{V_\omega} = u_0^{V_\omega} = u^{V_\omega} - \{r^{V_\omega}\}$. By the inductive hypothesis for (i), $\vdash s = r$ and $\vdash q = u_0$. Hence $\vdash q; r = u$. Using axiom (1), $\vdash u; r = q; r, r = q; r$. The third subcase is symmetrical with the second.

Case 2 $r \not\equiv v$. We have $r^{V_\omega} \in u^{V_\omega}$ and $v^{V_\omega} \in q^{V_\omega}$. Using the inductive hypothesis for (ii), obtain $u_0 \in \mathcal{T}_n$ and $r_0 \in \mathcal{T}_{n-1}$ such that $\vdash u = u_0; r_0$ and $r_0^{V_\omega} = r^{V_\omega}$. By the inductive hypothesis for (i), $\vdash r_0 = r$. In the same way obtain $q_0 \in \mathcal{T}_n$ and $v_0 \in \mathcal{T}_{n-1}$ such that $\vdash q = q_0; v_0 \wedge v_0 = v$. Now

$$q_0^{V_\omega} = q^{V_\omega} - \{v^{V_\omega}\} = u^{V_\omega} - \{r^{V_\omega}\} = u_0^{V_\omega},$$

and by the inductive hypothesis for (i), $\vdash q_0 = u_0$. Hence

$$\vdash q; r = q; r_0 = q_0; v_0, r_0 = q_0; r_0, v_0 = u_0; r_0, v_0 = u; v_0 = u; v.$$

3 The small adjunctive classes

In this section we shall explore the classes A_n for $n \leq 5$ through some visualizations.

The idea of adjunction, and the associated hierarchy, give rise to a natural primitive recursive algorithm for generating a list $s_0, s_1, \ldots s_n, \ldots$ of all the hereditarily finite sets. The idea is to start with $s_0 = 0$ and $A_0 = \{0\}$. Given $A_n = \{s_0, \ldots, s_{|A_n|-1}\}$, generate A_{n+1} by adding in all sets of form $[x; y]$ with x and y in A_n, ignoring repetitions. Here are more details. It is convenient to work with terms of $\mathcal{L}(0\,;)$ rather than sets, so let $t_0 = 0$ and $\mathcal{T}_0 = \{0\}$. Given $\mathcal{T}_n = \{t_0, \ldots, t_{|\mathcal{T}_n|-1}\}$, generate $\mathcal{T}_{n+1}$ as follows. Consider in turn each of the terms of form $[t_i; t_j]$ with $i, j < |\mathcal{T}_n|$, ordering these terms by the lexicographic order on $\langle i, j\rangle$. For each term check whether it is equivalent to (i.e. represents the same set as, see Section 2) any of the terms already in the list, and add it to the end of the list if not.

The resulting list is the *adjunctive-lexicographic ordering* of V_ω, and each set in V_ω is uniquely represented in the list by its *canonical term*. An implementation of this algorithm using *Mathematica* software was used to produce the visualizations and empirical calculations of this section and the next. Without computer assistance one can start the process:

$$A_1 = 0; 0, 1 = 2$$
$$A_2 = 2; [0; 1], 2$$
$$A_3 = 2; [0; 1], 2, [0; [0; 1]], [0; 2], [1; [0; 1]], [1; 2], [[0; 1]; [0; 1]], [[0; 1]; 2], [2; [0; 1]], 3$$

or in traditional setbuilder notation:

$$A_1 = \{0, 1\} = 2$$
$$A_2 = \{0, 1, \{1\}, 2\}$$
$$A_3 = \{0, 1, \{1\}, 2, \{\{1\}\}, \{2\}, \{0, \{1\}\}, \{0, 2\}, \{1, \{1\}\}, \{1, 2\}, \{0, 1, \{1\}\}, 3\}.$$
$$|A_1| = 2,\ |A_2| = 4,\ |A_3| = 12.$$

It will be proved that $|A_4| = 112,\ |A_5| = 11680,\ |A_6| = 135717904.$

Before proceeding to some visualizations based on the adjunctive-lexicographic ordering of V_ω, a note about pictures. Stephen Wolfram's book, *A New Kind of Science* [9], demonstrates that the behaviour of complex finite systems with sizes in the order of hundreds to hundreds of thousands of units is often best perceived visually. The hierarchy of this paper gives an opportunity to study the generation of the hereditarily finite sets on a visualizable scale.

Figure 2 is a three-dimensional bar chart showing the generation of A_4 from A_3, using the canonical terms $s_0, s_1, s_2, \ldots$ It is a visualization of the "multiplication table" for the adjunction operation: for $0 \leq i, j \leq 11$, if $[s_i; s_j] = s_k$, then the height of the bar at position (i, j) is k. Thus the tallest bar, corresponding to the last element to be added to A_4 in the adjunctive-lexicographic ordering, has height 111 and represents the set $s_{111} = [s_{11}; s_{11}] = [3; 3] = 4$. Figure 2 also indicates the first occurrences of 1, 2 and 3, corresponding to their canonical terms. The empty set itself is not represented in this and the other visualizations.

Figure 1 similarly shows A_5 generated from the 112 elements of A_4. The tallest bar, with height 11679, represents the set 5. A_4 occupies the front 12×12 square, flattened compared with Fig. 2 because the vertical scale is compressed.

A different kind of visualization is obtained by giving each canonical term a visual form. I explain the process by an example: Fig. 3 shows how the visualization of $s_9 = [[0; 1]; [1; 1]]$ is constructed from visualizations of [0; 1] and [1; 1]. Figure 4 then illustrates the construction of a visualization of A_3 by gluing together visualizations of its elements, aligning them at their central semi-colons.

Figure 5 shows A_4 in the same way.

Figure 6 shows a different aspect of A_4, and Fig. 7 extends the view as far as s_{311}, from yet another viewpoint. In these two figures, the terms are aligned at their left ends rather than at their central semi-colons as in Figs. 4 and 5. Since Fig. 6 is viewed from the rear, the representation of the last set in A_4, viz. $4 = [3; 3]$, is clearly visible.

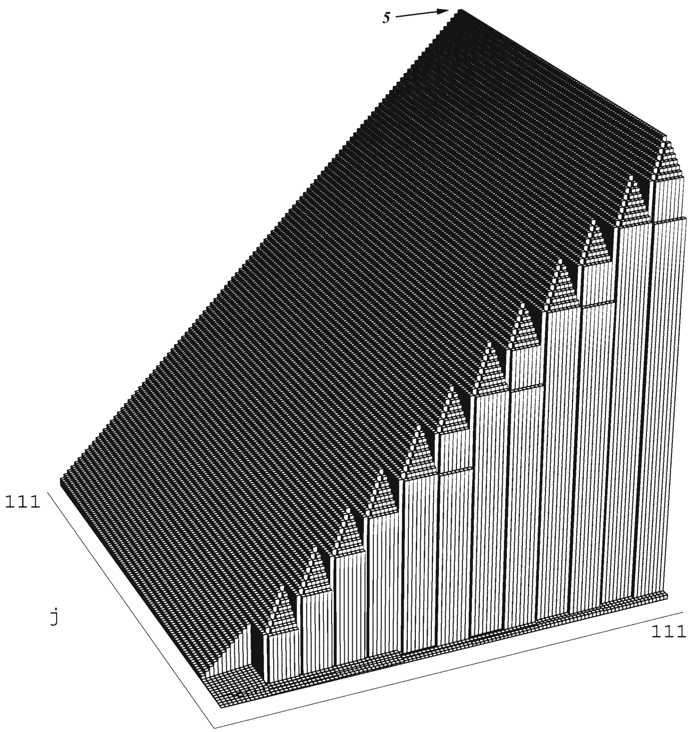


Fig. 1 A_5

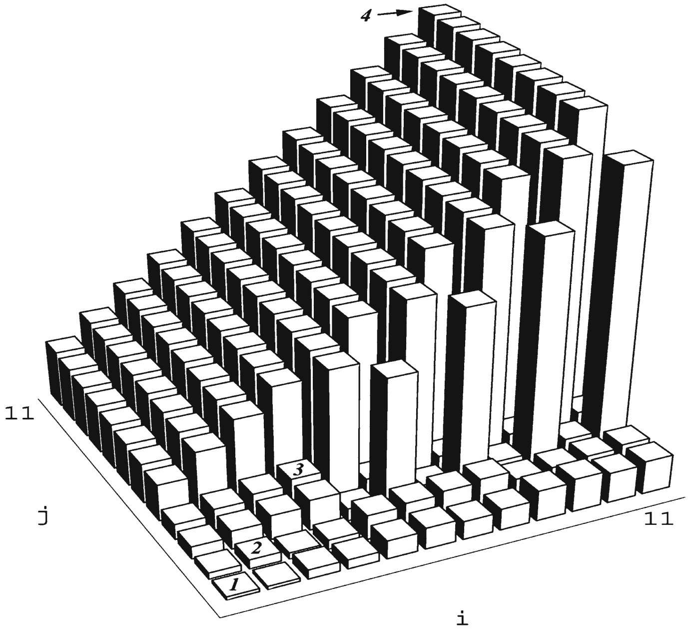
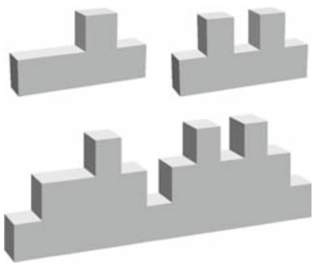


Fig. 2 A_4

Fig. 3 $s_9 = [s_2; s_3]$ (*bottom*) is built up from $s_2 = [0; [0; 0]] = \{1\}$ (*top left*) and $s_3 = [[0; 0]; [0; 0]] = 2$ (*top right*)



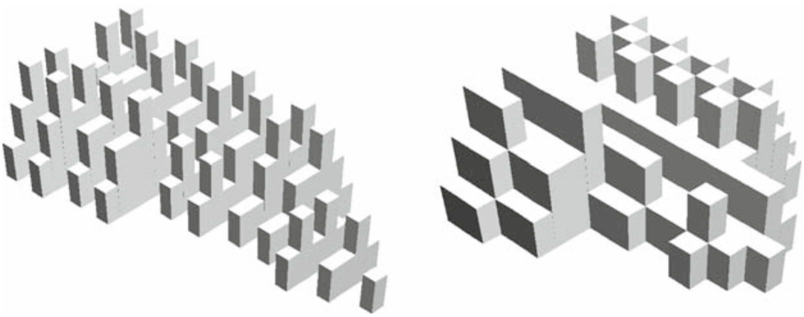


Fig. 4 A_3. *Left*, the sets s_1 through s_{11}, from *right* to *left*, centre aligned. *Right*, the same sets are glued together

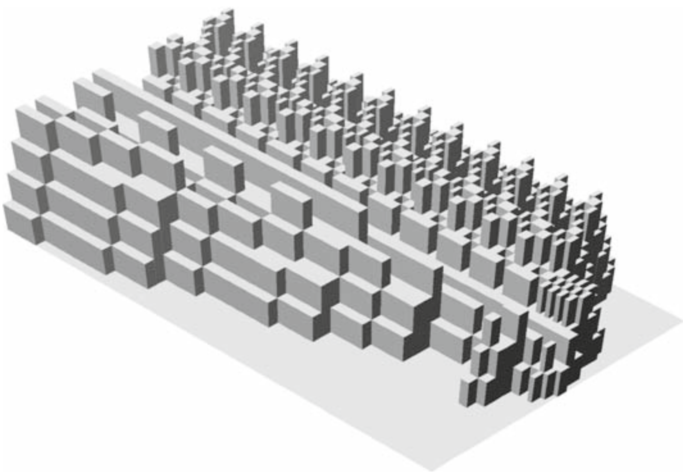


Fig. 5 A_4, centre aligned

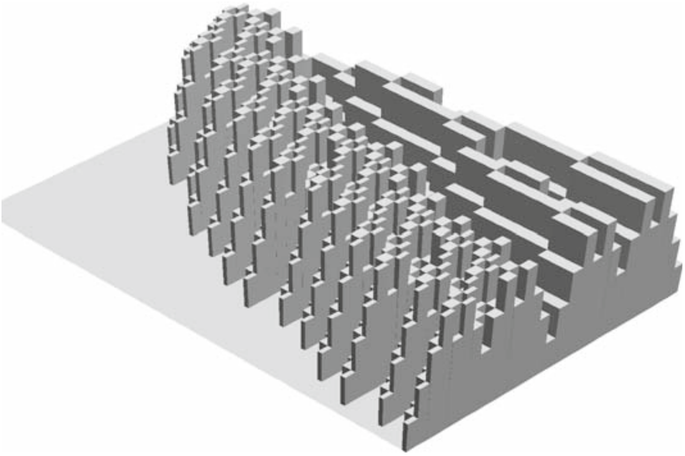


Fig. 6 A_4, left aligned

The *nest depth* of a term of $\mathcal{L}(0\,;)$ is the nest depth of its brackets, for example $s_9 = [[0; [0; 0]]; [[0; 0]; [0; 0]]]$ has nest depth 3. So the heights of the visualizations in Figs. 3, 4, 5, 6 and 7 are equal to the nest depths of the terms represented. With this in mind, the following characterization of the classes A_n, easily proved by induction on n, is visible in the figures:

Proposition 3.1 *For any set $a \in V_\omega$, $a \in A_n$ if and only if the canonical term for a has nest depth $\leq n$.*

Density plots, wherein the increasing heights in Figs. 3, 4, 5, 6 and 7 are rendered instead by darker shades, give another kind of visualization: Figs. 8 and 9.

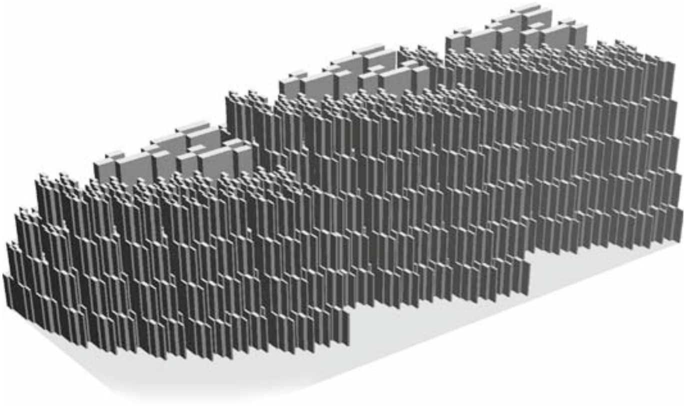


Fig. 7 s_1–s_{311}, left aligned. The first range is A_4 (cf. Fig. 6), and the first 200 sets of A_5 follow

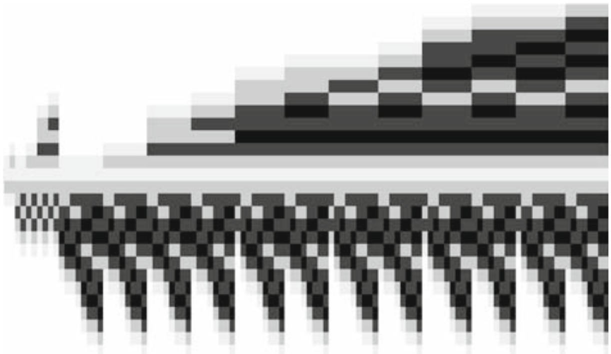


Fig. 8 Density plot of A_4, corresponding to Fig. 5

4 Subclasses and cardinalities of classes of the hierarchy

A_6 is too big to be feasibly visualized by the methods of Sect. 3. A little of its structure can be discerned by investigating some subclasses of the A_n, in the following results which are valid for larger classes as well.

Definition

(i) $x \Gamma y = x \cup y \cup \{[u; v] \mid u \in x \wedge v \in y\}$.
(ii) $A_{m,n} = A_m \Gamma A_n$.

Thus $A_{Max(m,n)} \subseteq A_{m,n} \subseteq A_{Max(m,n)+1}$ and $A_{n,n} = A_{n+1}$. Figure 10 shows a bar chart for $A_{4,3}$.

Lemma 4.1

(i) *For* $n \geq 1$, $A_{n,0} = A_n$.
(ii) *For* $n \geq 2$, $A_{n,1} = A_n$.

Fig. 9 Three density plots of A_5. *Right*, centrally aligned. *Middle*, left aligned. *Left*, using representations of sets derived from the traditional setbuilder notation

Proof (i) $A_{1,0} = A_1$ because $0; 0 = 1; 0 = 1$. Inductively, suppose $A_{n,0} = A_n$ and show $A_{n+1,0} = A_{n+1}$. Let $u \in A_{n+1}$. Then $u = x; y$ for some $x, y \in A_n$. Hence $u; 0 = x; y, 0 = [x; 0]; y$ by Axiom (2). By the inductive hypothesis, $x; 0 \in A_n$ and hence $u; 0 \in A_{n+1}$.

(ii) Once one has verified that $A_{2,1} = A_2$, the inductive proof is like that for (i).

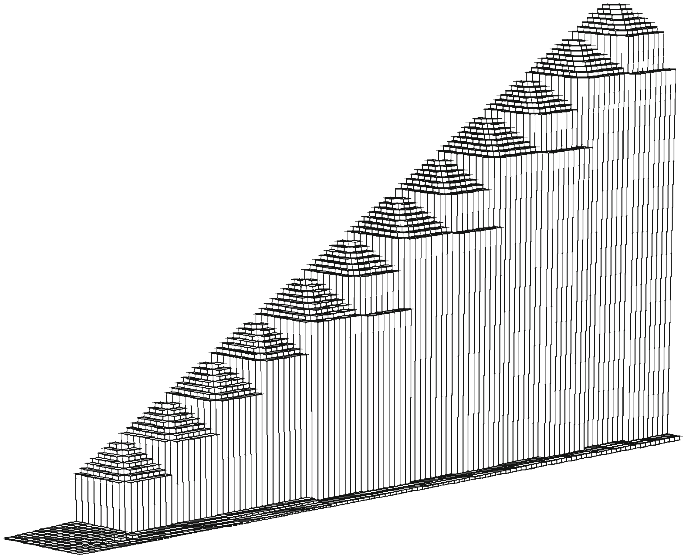


Fig. 10 $A_{4,3}$ (Compare Fig. 1)

Proposition 4.2 *If* $n > 0$, $x; y = x'; y'$, $x \in A_n$, $x' \in A_n$, $y \notin A_{n-1}$, *and* $y' \notin A_{n-1}$, *then* $x = x'$ *and* $y = y'$.

Proof $x; y = x'; y' \to y' \in x \vee y' = y$. But Lemma 2.3(ii) tells us that $y' \notin x$, hence $y' = y$. Since symmetrically $y \notin x'$, it follows that $x = x'$.

Corollary 4.3

(i) *For* $n \geq 1$, $\left|A_{n+1} - A_{n,n-1}\right| = |A_n| \cdot |A_n - A_{n-1}|$.

(ii) *For* $0 \leq m < n - 1$, $\left|A_{m+1,n} - A_{m,n}\right| = |A_{m+1} - A_m| \cdot |A_n - A_{n-1}|$.

Proof (i) First note that if $u = x; y$ with $x \in A_n$ and $y \in A_n - A_{n-1}$, then u is not an element of $A_{n,n-1}$. For suppose it were: then for some $v \in A_n$ and $w \in A_{n-1}$, $u = x; y = v; w$. This implies that $y = w$ or $y \in v$. The former contradicts the suppositions that $y \notin A_{n-1}$ and $w \in A_{n-1}$, and the latter is impossible by Lemma 2.3(ii). It follows that $A_{n+1} - A_{n,n-1} = A_{n,n} - A_{n,n-1}$ consists exactly of all sets of form $x; y$ with $x \in A_n$ and $y \in A_n - A_{n-1}$. Proposition 4.2 says that these are all distinct for distinct pairs $\langle x, y\rangle$.

(ii) If $u \in A_{m+1,n} - A_{m,n}$, say $u = [x; y]$ with $x \in A_{m+1}$ and $y \in A_n$, then of course $x \notin A_m$. Also $y \notin A_{n-1}$: for suppose $y \in A_{n-1}$. Then $u \in A_{m+1,n-1} \subseteq A_{n-1,n-1} = A_n \subseteq A_{m,n}$. The proof concludes like that of (i).

A manifestation of 4.2 and 4.3(i) with $n = 4$ can be discerned by examining Figs. 1 and 10.

Lemma 4.4 *Let* $n \geq k \geq 1$. *If* $x \in A_n$ *then* x *has at most* $k - 1$ *elements which are not in* A_{n-k}.

Proof by induction on k that the Lemma holds for all $n \geq k$. The case $k = 1$ is Lemma 2.3(ii). If true for k and $x \in A_n$ with $n \geq k + 1$, say $x = y; z$ with y, z in A_{n-1}. By inductive hypothesis at most $k - 1$ elements of y are not in $A_{n-(k-1)}$ (and *a fortiori* not in A_{n-k}) and the only additional candidate for an element of x not in A_{n-k} is z.

Proposition 4.5 *Let $n \geq 2$.*

(i) $a \in A_{n,n-1} - A_{n,n-2}$ *if and only if there exist $u \in A_{n-1}$, and distinct elements v and w of $A_{n-1} - A_{n-2}$, such that $a = u; v, w$.*
(ii) *If $u; v, w = u'; v', w'$ with $u \in A_{n-1}$, and $v \neq w$ with $v, w \in A_{n-1} - A_{n-2}$, and likewise for u', v', w', then $u = u'$ and $\{v, w\} = \{v', w'\}$.*

Proof (i) Let $a \in A_{n,n-1} - A_{n,n-2}$, say $a = x; w$ with $x \in A_n$ and $w \in A_{n-1} - A_{n-2}$. In turn let $x = u; v$ with u and v in A_{n-1}. So $a = u; v, w$ and one direction of the proof is complete if we can show that $v \notin A_{n-2}$. But suppose v were in A_{n-2}: then $a = [u; w]; v$ would be in $A_{n,n-2}$ because $[u; w] \in A_n$.
Conversely, given $a = u; v, w$ as specified in (i), then $a \in A_{n,n-1}$ because $u; v \in A_n$. We need to show that $a \notin A_{n,n-2}$. But suppose it were: $a = u; v, w = x; y$ with $x \in A_n$, $y \in A_{n-2}$. Since v and w cannot equal y, they must both be in x, contradicting Lemma 4.4 with $k = 2$.
(ii) follows from the fact that none of v, w, v', w' can be in u nor in u' by Lemma 2.3(ii).

Corollary 4.6 *For $n \geq 3$,* $\left|A_{n,n-1} - A_{n,n-2}\right| = |A_{n-1}| \cdot \binom{|A_{n-1} - A_{n-2}|}{2}$.

Proposition 4.7 *Let $n \geq 3$.*

(i) $a \in A_{n,n-2} - A_{n,n-3}$ *if and only if there exist $u \in A_{n-2}$, and distinct elements v, w, x such that $v, w \in A_{n-2} - A_{n-3}$ and $x \in A_{n-1} - A_{n-3}$, such that $a = u; v, w, x$.*
(ii) *If $u; v, w, x = u'; v', w', x'$ with u, v, w, x as in the statement of (i), and likewise for u', v', w', x', then $u = u'$ and $\{v, w, x\} = \{v', w', x'\}$.*

Proof (i) For "only if", suppose $a \in A_{n,n-2} - A_{n,n-3}$. Write $a = b; w$ with $b \in A_n$ and $w \in A_{n-2} - A_{n-3}$, and in turn write $b = c; x$ with c and x in A_{n-1}, and $c = u; v$ with u and v in A_{n-2}. Thus $a = u; v, w, x$ and it remains to show that v, w, x are distinct and none of them are in A_{n-3}. For distinctness, if (say) $w = x$ then $a = u; v, w$ would be in A_n. And suppose that (say) $v \in A_{n-3}$: then $a = [u; w, x]; v$ would be in $A_{n,n-3}$ since $[u; w, x] \in A_n$.
For "if", suppose $a = u; v, w, x$ as given. It is straightforward that $a \in A_{n,n-2}$. If $a \in A_{n,n-3}$, say $a = u; v, w, x = y; z$ with $y \in A_n$ and $z \in A_{n-3}$, then v, w and x cannot equal z so they must all be in y, contradicting Lemma 4.4 with $k = 3$.
(ii) follows from the fact that none of v, w, x, v', w', x' can be in u or u'.

Corollary 4.8 *For $n \geq 3$,* $\left|A_{n,n-2} - A_{n,n-3}\right|$

$$= |A_{n-2}| \cdot \left(\binom{|A_{n-2} - A_{n-3}|}{2} \cdot |A_{n-1} - A_{n-2}| + \binom{|A_{n-2} - A_{n-3}|}{3}\right).$$

This corollary is obtained by considering two cases in the characterization of $A_{n,n-2}$ given in Proposition 4.7, depending upon whether x is in $A_{n-1} - A_{n-2}$ or in $A_{n-2} - A_{n-3}$.

The same methods can be used to obtain:

Proposition 4.9 *Let* $n \geq 4$.

(i) $a \in A_{n,n-3} - A_{n,n-4}$ *if and only if there exist* $u \in A_{n-3}$, *and distinct elements* v, w, x, y *such that* $v, w \in A_{n-3} - A_{n-4}$, $x \in A_{n-2} - A_{n-4}$, *and* $y \in A_{n-1} - A_{n-4}$, *such that* $a = u; v, w, x, y$.

(ii) *If* $u; v, w, x, y = u'; v', w', x', y'$ *with* u, v, w, x, y *as in the statement of* (i), *and likewise for* u', v', w', x', y' *then* $u = u'$ *and* $\{v, w, x, y\} = \{v', w', x', y'\}$.

Corollary 4.10 *For* $n \geq 4$, $\left|A_{n,n-3} - A_{n,n-4}\right|$

$$= |A_{n-3}| \cdot \left\{ \binom{|A_{n-3} - A_{n-4}|}{2} \cdot \left[|A_{n-2} - A_{n-3}| \cdot |A_{n-1} - A_{n-2}| + \binom{|A_{n-2} - A_{n-3}|}{2} \right] + \binom{|A_{n-3} - A_{n-4}|}{3} \cdot |A_{n-2} - A_{n-3}| + \binom{|A_{n-3} - A_{n-4}|}{4} \right\}.$$

Using these results one can fill in the numbers in Figs. 11 and 12. In these figures the lines represent inclusions between classes, and the number adjacent to a line is the number of elements of the class on the right which are not in the class on the left. All the numbers as far as $A_{5,2}$ have been confirmed empirically. The only number not obtained directly as a case of one of the above results is the size of $A_6 - A_{4,5}$. By

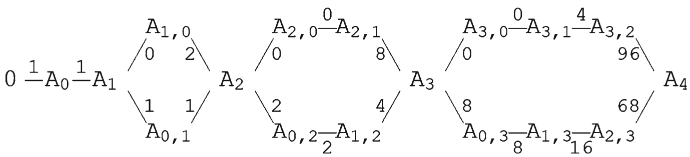


Fig. 11 Inclusions between the small classes and subclasses

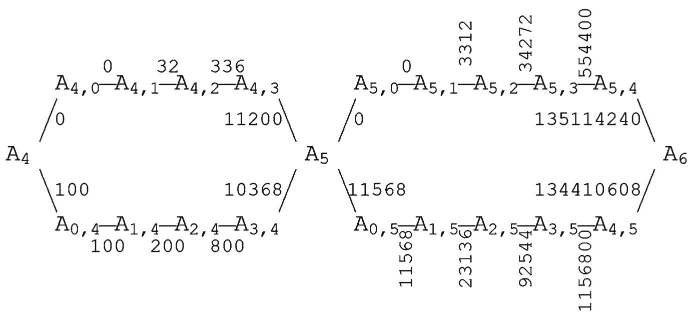


Fig. 12 Continuation of Fig. 11

summing the upper numbers we obtain $|A_6| = 135717904$, and from this the value indicated for $A_6 - A_{4,5}$ is deduced.

References

1. Givant, S., Tarski, A.: Peano arithmetic and the Zermelo-like theory of sets with finite ranks. Not. Am. Math. Soc. 77T-E51:A-437 (1977)
2. Kirby, L.: Addition and multiplication of sets. Math. Log. Q. **53**(1), 52–65 (2007)
3. Kirby, L.: Finitary set theory. Notre Dame J. Formal Log. (to appear)
4. Mathias, A.R.D.: Slim models of Zermelo set theory. J. Symb. Log. **66**(2), 487–496 (2001)
5. Previale, F.: Induction and foundation in the theory of the hereditarily finite sets. Arch. Math. Log. **33**, 213–241 (1994)
6. Shepherdson, J.C.: Inner models for set theory—Part I. J. Symb. Log. **16**(3), 161–190 (1951)
7. Tait, W.W.: Finitism. J. Philos. **78**(9), 524–546 (1981)
8. Tarski, A.: Sur les ensembles finis. Fundam. Math. **6**, 45–95 (1924)
9. Wolfram, S.: A New Kind of Science. Wolfram Media, Champaign (2002)