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Substandard models of finite set theory

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A survey of the isomorphic submodels of V_ω, the set of hereditarily finite sets. In the usual language of set theory, V_ω has $2^{\aleph_0}$ isomorphic submodels. But other set-theoretic languages give different systems of submodels. For example, the language of adjunction allows only countably many isomorphic submodels of V_ω.

1 Introduction

This is an area where two different usages of the word "standard" collide: in set theory and in models of arithmetic.[1)] The starting point is V_ω, the set of hereditarily finite sets. Might this be designated *the* standard model of finite set theory? I have no need of an answer to that question here, so I don't dwell on it.

All other models considered here are *submodels* of V_ω, in the model-theorist's sense,[2)] and are standard in the rather strong sense that they are isomorphic to V_ω, although the exact meaning of *isomorphic* will have to be made precise. And there's the rub: what language are we talking? Both *submodel* and *isomorphic* are defined relative to a language.

Let S be a constant, function, or relation symbol, or a set of such symbols, and let $\mathfrak{L}(S)$ be the first order language of equality augmented with S.

In principle, for any element of V_ω, function from V_ω^n to V_ω, or relation on V_ω^n, or set of such, there is a corresponding language with appropriate symbol(s). Assuming, that is, we have enough symbols. But in all cases I shall consider, the symbol(s) S will have a specified natural or conventional interpretation S^{V_ω} on V_ω, which will usually be denoted simply S.

As the most important example, the symbol $\in$ is given its conventional interpretation as the restriction to V_ω of the membership relation, in what is commonly called *the language of set theory*. This actually has two versions: $\mathfrak{L}(\in)$ and $\mathfrak{L}(0\ \in)$ where 0 is the empty set.

There are other set-theoretic languages as well, notably $\mathfrak{L}(;)$, the language of *adjunction*: $[x; y] =_{\mathrm{df}} x \cup \{y\}$ (see [7]).

For many purposes, the languages $\mathfrak{L}(\in)$ and $\mathfrak{L}(;)$ are equivalent, just as 0 can be left in or out without any change in expressive power. Definitional expansions or circumlocutions can always be found.

But the systems of isomorphic submodels for these languages have very different properties. For example, we shall see that $\langle V_\omega, \in\rangle$ has $2^{\aleph_0}$ isomorphic submodels (Theorem 4.8) but $\langle V_\omega, ;\rangle$ has only countably many (Theorem 3.2).

Instead of talking in terms of submodels, I shall use an equivalent formulation in terms of isomorphic embeddings of V_ω into itself.

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1) There is also the nonstandard analyst's usage, along (of course) with the underlying natural language usage.
2) E.g. Chang and Keisler [1].

Definition 1.1 Let X, Y be subsets of V_ω. Then $f : X \overset{S}{\hookrightarrow} Y$ will mean that f is an isomorphic injection of $\langle X, S\rangle$ into $\langle Y, S\rangle$ in the language $\mathfrak{L}(S)$, where S is given (the restrictions to X and Y of) its natural or conventional interpretation on V_ω.

Here, *isomorphic injection* means that f is one-to-one and is an isomorphism between $\langle X, S\rangle$ and $\langle f\text{"}X, S\rangle$. I shall sometimes refer to $f : X \overset{S}{\hookrightarrow} Y$ as an *S-embedding*, and if it is a bijection as an *S-isomorphism*.

In particular, $f : X \overset{0}{\hookrightarrow} Y$ means that f is one-to-one and $f(0) = 0$, and $f : X \hookrightarrow Y$ (so that the language is that of equality only) means simply that f is one-to-one. On the other hand, we need equality in the language even to express the notions of one-to-one and S-embedding.

Now I make precise how embeddings correspond to submodels.

Definition 1.2 Let $X, Y \subseteq V_\omega$. X is an S-submodel of Y means: $\langle X, S\rangle$ is an isomorphic submodel of $\langle Y, S\rangle$ (in the language $\mathfrak{L}(S)$).

Isomorphic here means that there exists an S-isomorphism between X and Y. Proposition 1.3 says that the S-submodels of V_ω are just the images of S-embeddings of V_ω:

Proposition 1.3 *If $f : V_\omega \overset{S}{\hookrightarrow} V_\omega$, then $f\text{"}V_\omega$ is an S-submodel of V_ω (a* substandard *model). Conversely, suppose $M \subseteq V_\omega$ is an S-submodel of V_ω. Then there is an S-isomorphism $f : V_\omega \longrightarrow M$ which also acts as an S-embedding $f : V_\omega \overset{S}{\hookrightarrow} V_\omega$ whose image is M.*

Definitions such as the above can clearly be generalized to other domains beyond V_ω, and many of the basic results on embeddings in this paper apply in a wider setting. Some of them (e.g. Theorem 3.2) also apply to models of ZF. But I shall stick inside V_ω for simplicity, both of presentation and of assumptions, except for some further remarks on this topic at the end of Section 3.

Throughout, an important tool will be the arithmetic of sets [5]. This is introduced, along with some other basic notions and prerequisites, in Section 2.

Section 3 deals with the simpler case of $\mathfrak{L}(;)$, before we turn to $\mathfrak{L}(\in)$ in Section 4. This will give rise in Section 5 to a topological description of the $\in$-embeddings as a closed subset of a space homeomorphic to the Baire space.

After brief sections discussing two other examples of set-theoretic languages, in Section 8 I give some basic general results on the effects of changing languages and apply them to various set-theoretic languages. Section 9 takes a glance at embeddings for arithmetical languages.

2 Preliminaries

If R is a relation symbol, the property of being an R-embedding can be broken down:

Definition 2.1 Let R be a relation and $f : X \longrightarrow Y$.

(i) f *preserves* R if and only if $(\forall x_1 \cdots x_n \in X)\ (R(x_1, \dots, x_n) \Rightarrow R(f(x_1), \dots, f(x_n))$.

(ii) f *preserves* $\neg R$ if and only if $(\forall x_1 \cdots x_n \in X)\ (R(f(x_1), \dots, f(x_n)) \Rightarrow R(x_1, \dots, x_n))$.

We could treat $\neg R$ as a new relation symbol for the complement of R. Thus

$$f : X \overset{R}{\hookrightarrow} Y \text{ if and only if } f \text{ is one-to-one and } f \text{ preserves both } R \text{ and } \neg R.$$

In the case where R is the membership relation $\in$, the clause "f is one-to-one" can be dropped because, by extensionality:

Lemma 2.2 *If f preserves both $\in$ and $\notin$, then f is one-to-one.*

Proof. If $x \neq y$, say $(z \in x \wedge z \notin y)$, then $(f(z) \in f(x) \wedge f(z) \notin f(y))$. □

In fact if $f : V_\omega \overset{R}{\hookrightarrow} V_\omega$, then f preserves any relation built from R by Boolean combination. These observations will be extended in Section 8.

A function $f : V_\omega \longrightarrow V_\omega$ is defined $\in$-*inductively* when $f(x)$ is defined in terms of $f\text{"}x$. f is defined ;-*inductively* if $f([x; y])$ is defined in terms of $f(x)$ and $f(y)$, and $f(0)$ is given.

We are not concerned with theories here, therefore nor with provability. In the background, of course, is a metatheory, consisting of enough set theory to define V_ω and establish its properties, including the definitions $V_0 = 0$, $V_{n+1} = PV_n$, $V_\omega = \bigcup_{n\in\omega} V_n$, where P is the power set operator.

I next introduce some other languages. They fall into two groups. First, some other *set-theoretic* languages, based on elementary set operators and relations. They are mostly weaker in expressive power than $\in$ and adjunction. They include $\subset$, which will be the subject of Section 7, and $\cup$ and $\cap$, for which see Section 8.

Following Flavio Previale [8], I use $x < y$ to denote $x \in \mathrm{TC}(y)$, where $\mathrm{TC}(y)$ is the transitive closure of y. I take a look at $<$-embeddings in Section 6.

Notice that $\in$, $<$ and $\subset$ are all extensions of the usual order relation $<$ on ω to V_ω. The corresponding embeddings turn out to differ.

The second group of languages to consider are *arithmetical.* In Section 9 I shall look briefly at embeddings for languages based on ordinal addition and multiplication of sets [5]. These operations, originally due to Alfred Tarski and Dana Scott respectively, generalize the usual ordinal operations to all sets, although here we are restricting them to V_ω. Even before we consider arithmetical languages in Section 9, addition and multiplication will be used to characterize and provide examples of embeddings: in Section 3 addition will allow a simple characterization of the ;-embeddings. For now I give some definitions and basic properties of addition and multiplication, together with two auxiliary notions $\trianglelefteq$ and λ_a from [5] which I shall call upon.

Definition 2.3 Define, $\in$-inductively in the first two cases:

$$a + b = a \cup \{a + x \mid x \in b\}.$$
$$a \cdot b = \{a \cdot q + r \mid q \in b \wedge r \in a\}.$$
$$\lambda_a(b) = \{a + x \mid x \in b\}.$$
$$a \trianglelefteq b \Leftrightarrow \exists x(a + x = b).$$

Here are some results on arithmetic of sets from [5].[3)]

Theorem 2.4

(i) *Addition and multiplication on V_ω are associative, noncommutative operations that left-distribute, i.e., $a \cdot (b + c) = a \cdot b + a \cdot c$, and left-cancel, i.e., $a + b = a + c$ implies $b = c$, and similarly for $\cdot$ if $a \neq 0$.*

(ii) *Each non-empty $a \in V_\omega$ can be uniquely written as a sum $a_1 + \cdots + a_n$, where the a_i are additively irreducible, i.e., $a_i \neq 0$ and $a_i = x + y$ implies $x = 0$ or $y = 0$.*

(iii) *$a + c = b + d$ implies $a \trianglelefteq b$ or $b \trianglelefteq a$.*

(iv) *$\mathrm{TC}(a) \cap \lambda_a(b) = 0$. Thus $a + b$ is the disjoint union of a and $\lambda_a(b)$.*

I shall have occasion to quote other results from [5] about the algebraic properties of $+$ and $\cdot$.

$\triangleleft$ provides a fourth extension to V_ω of the order relation on ω.

In general, questions about embeddings of V_ω can be regarded as analogues of the same questions asked about embeddings of ω. Thus, it is easy to establish that ω has only countably many s-embeddings, where s is the successor function, but that ω has $2^{\aleph_0}$ $\in$-embeddings, i.e., order-preserving embeddings. So these properties of ω transfer to V_ω, if we consider adjunction as an analogue of successor. But not all properties of embeddings on ω transfer in this way, for instance ω has only countably many $+$-embeddings, whereas V_ω has $2^{\aleph_0}$ (Section 9).

3 Adjunction

The ;-embeddings $f : V_\omega \overset{;}{\hookrightarrow} V_\omega$ can be simply characterized using addition of sets.

Definition 3.1 For $a \in V_\omega$, define $a_+ : V_\omega \longrightarrow V_\omega$ by $a_+(x) = a + x$.

Theorem 3.2 *Suppose $f : V_\omega \longrightarrow V_\omega$. Then*

$$f : V_\omega \overset{;}{\hookrightarrow} V_\omega \ \textit{ iff } \ (\exists a \in V_\omega)\,(f = a_+).$$

[3)] In [5] I erroneously stated that some of these results were provable from $\mathsf{ZF}\backslash\{\mathrm{Inf}\}$. In fact they also need the axiom of transitive containment: see [3]. In the present paper I am concerned in any case with truth in V_ω, not provability.

P r o o f. $a_+ : V_\omega \overset{;}{\hookrightarrow} V_\omega$ follows from two simple results in [5]. [4] Namely, left cancellation of addition says that a_+ is one-to-one. And $a + [x; y] = [(a + x); (a + y)]$ tells us that $a_+([x; y]) = [a_+(x); a_+(y)]$. (In [6] this latter idea is used as an ;-inductive definition of addition on V_ω.).

Conversely, suppose $f : V_\omega \overset{;}{\hookrightarrow} V_\omega$. Let $a = f(0)$. Then $f(x) = a + x$ can be shown by a simple ;-induction on x. □

Corollary 3.3

(i) *The set of ;-embeddings of* V_ω *into itself is countable.*

(ii) *The only* $f : V_\omega \overset{0\,;}{\hookrightarrow} V_\omega$ *is the identity function* 0_+.

Statement (ii) shows a difference between the languages $\mathfrak{L}(;)$ and $\mathfrak{L}(0\,;)$.

Proposition 3.4 *If* $f : V_\omega \overset{;}{\hookrightarrow} V_\omega$*, then* $f : V_\omega \overset{\in}{\hookrightarrow} V_\omega$.

P r o o f. This has a simple direct proof because $x \in y$ if and only if $[y; x] = y$. But bearing in mind 3.2 we can cite [5, 3.4] which says that $a + x \in a + y \leftrightarrow x \in y$. □

In 8.5, a more general proposition will be given. In Section 4 we'll see that the converse of 3.4 is false. However it's true for isomorphisms, since the identity function is the only $\in$-automorphism of V_ω, and hence (by 3.4) its only ;-automorphism, because an $\in$-minimal set moved by $f : V_\omega \overset{\in}{\hookrightarrow} V_\omega$ cannot be in the image of f.

Similarly, for any $\langle M, E\rangle$ (not necessarily a submodel of V_ω) which is isomorphic to $\langle V_\omega, \in\rangle$, the isomorphism (call it F) is unique. If we wish to add 0 to the language, there is a (unique) suitable candidate for 0^M. And F induces an adjunction $;^M$ which is also unique and is definable in $\langle M, E\rangle$ in such a way[5] that

– F is also an isomorphism in $\mathfrak{L}(0\,;)$, and

– it can be expressed as $t^M \longmapsto t^{V_\omega}$ for each term t of $\mathfrak{L}(0;)$.[6]

Generalizing the models-of-arithmetic sense of "standard", say that $\langle M, S^M\rangle$ is S-standard if and only if it is isomorphic to $\langle V_\omega, S\rangle$. Then the above remarks show that a model is $\in$-standard if and only if it is ;-standard, or more precisely that each of these types of model can be converted into the other by the obvious definitional expansions. Furthermore, our results about $\in$-submodels of V_ω apply to the $\in$-submodels of any $\in$-standard model.

On the other hand, by restricting attention to submodels of V_ω at the start, we have simplified our ontological commitments and therefore some definitions and arguments.

4 $\in$-embeddings

Arithmetic of sets provides some further natural examples of $\in$-embeddings:

Proposition 4.1 *Let* $a, b \in V_\omega$ *and define* $f(x) = a + b \cdot x$*. Then* $f : V_\omega \overset{\in}{\hookrightarrow} V_\omega$ *if and only if* $0 \in b$.[7]

P r o o f. a_+ is dealt with in Section 3, and [5, 4.13] says that $x \longmapsto b \cdot x$ is an $\in$-embedding if and only if $0 \in b$. □

In particular, when $0 \in b \neq 1$ this f is an $\in$-embedding which is not an ;-embedding. In fact we shall see that there are $2^{\aleph_0}$ $\in$-embeddings. First we shall provide in 4.2 and 4.4 a characterization of $\in$-embeddings.

Lemma 4.2 *Let* x *be transitive and* $f : x \hookrightarrow y$*. Then*

$$f \text{ preserves } \in \text{ iff } (\forall u \in x)(f(u) \supseteq f\text{"}u).$$

P r o o f. $f\text{"}u \subseteq f(u)$ is equivalent to $(\forall v \in u)\,(f(v) \in f(u))$. □

Definition 4.3 If f is a function, define $f_-(u) =_{\mathrm{df}} f(u) \setminus f\text{"}u$.

(This will be of interest when $f(u) \supseteq f\text{"}u$ so that $f(u)$ is the disjoint union of $f\text{"}u$ and $f_-(u)$.)

4) [5, 3.4(ii) and 3.2(iii)].

5) This is put differently in Section 8.

6) See [6] for a syntactic equivalent.

7) Narciso Garcia [2] gives what amounts to part of this result, viz. that if $0 \in b$, then f preserves $\in$.

Lemma 4.4 *Let x be transitive and $f : x \hookrightarrow y$. Then*

$$f \text{ preserves } \notin \text{ iff } (\forall u \in x)\,(f_-(u) \cap f\text{"}x = 0).$$

Proof. Suppose that f preserves $\notin$ but the right side is false: for some $u \in x$ there is an element w in $f_-(u) \cap f\text{"}x$, say $w = f(v)$, $v \in x$. Then $f(v) \in f(u)$ and hence $v \in u$. It follows that $w \in f\text{"}u$, contradicting $w \in f_-(u)$.

Now suppose the right side true: for $u, v \in x$, $f(v) \notin f_-(u)$. Let u, v be such that $f(v) \in f(u)$. I shall show that $v \in u$. Since $f(v) \notin f_-(u)$, it follows that $f(v) \in f\text{"}u$: say $f(v) = f(w)$ with $w \in u$. Since x is transitive, $w \in x$. Since f is one-to-one, $v = w$, and so $v \in u$. □

Moving towards a construction for $\in$-embeddings, I first note a simple fact:

Lemma 4.5 *Let x be transitive. Then $[x; y]$ is transitive if and only if $y \subseteq x$.*

Definition 4.6 y is an *external subset* of x if $y \subseteq x$ and $y \notin x$.

The next lemma is a tool for building $\in$-embeddings.

Lemma 4.7 *Suppose $f : x \overset{\in}{\hookrightarrow} V_\omega$, where x is a transitive subset of V_ω. Let y be an external subset of x and let z be any element of V_ω such that*

(a) $z \cap f\text{"}x = 0$, *and* (b) $f\text{"}y \cup z \notin \bigcup_{u \in x} f_-(u)$.

Then $f \cup \langle y, f\text{"}y \cup z\rangle : [x; y] \overset{\in}{\hookrightarrow} V_\omega$, i.e., we can extend f to an $\in$-embedding with domain $[x; y]$ by assigning $f_-(y) = z$ and $f(y) = f\text{"}y \cup z$.

Proof. To show that the extended f is still one-to-one, suppose that $f(y) = f(u)$ for some $u \in x$. Then, by assumption (a), $f\text{"}y = f(y) \cap f\text{"}x = f(u) \cap f\text{"}x = f\text{"}u$, where the last equality uses 4.2 and 4.4 applied to the original f. But u and y are both subsets of x and f is one-to-one on x, so $f\text{"}y = f\text{"}u$ implies $y = u$, contradicting the assumption that y is an external subset of x.

The extended f preserves $\in$ because the original does and $f(y) \supseteq f\text{"}y$. It remains to verify that the extended f preserves $\notin$, i.e., $(\forall u \in [x; y])(f_-(u) \cap f\text{"}[x; y] = 0)$. For $u \in x$ this follows from 4.2 and our assumption (b) which guarantees that $f(y) \notin f_-(u)$. For $u = y$ use (a) and the fact that $f(y) \notin f_-(y)$ since $f_-(y) \subseteq f(y)$. □

Theorem 4.8 *If $f : x_0 \overset{\in}{\hookrightarrow} V_\omega$, where $x_0 \in V_\omega$ is transitive, there are $2^{\aleph_0}$ $\in$-embeddings $f^* : V_\omega \overset{\in}{\hookrightarrow} V_\omega$ such that $f^* \supset f$.*

Proof. In [6, Section 3] there is defined an enumeration $\{s_n \mid n \in \omega\}$ of V_ω such that $s_0 = 0$ and for every n, $s_{n+1} \subseteq \{s_0, \dots, s_n\}$, and hence (see 4.5) each $\{s_0, \dots, s_n\}$ is transitive. Another way to build an enumeration with the same property is by listing, in any order, all elements of $V_{n+1} \setminus V_n$ before moving on to $V_{n+2} \setminus V_{n+1}$.

Now given $f : x_0 \overset{\in}{\hookrightarrow} V_\omega$ delete all elements of x_0 from such an enumeration to obtain an ordering $\{s_n \mid n \in \omega\}$ of $V_\omega \setminus x_0$ such that, if $x_n = x_0 \cup \{s_0, \dots, s_{n-1}\}$, then each x_n is transitive and s_n is an external subset of x_n. Suppose we have already extended f to $f : x_n \overset{\in}{\hookrightarrow} V_\omega$. We can apply 4.7 with $x = x_n$ and $y = s_n$. Since x is finite, so are $f\text{"}x$ and $\bigcup_{u \in x} f_-(u)$, and therefore there are infinitely many $z \in V_\omega$ satisfying (a) and (b) of 4.7. (For example z could be $\{v\}$, where v is any element of V_ω with rank greater than the rank of $f\text{"}x$.) Hence there are infinitely many extensions of f to $f : x_{n+1} \overset{\in}{\hookrightarrow} V_\omega$. Thus the set of extensions of f to $\in$-embeddings with domains x_n form an infinitely branching tree, and the union of any infinite branch of this tree is an $\in$-embedding from V_ω to V_ω. □

In particular, there are $2^{\aleph_0}$ $f : V_\omega \overset{0\in}{\hookrightarrow} V_\omega$, and for any $a \in V_\omega$ there are $2^{\aleph_0}$ $\in$-embeddings $f : V_\omega \overset{\in}{\hookrightarrow} V_\omega$ such that $f(0) = a$.

Theorem 4.9 *If $f : \omega \longrightarrow \omega$ is strictly increasing, then f has $2^{\aleph_0}$ extensions to $\in$-embeddings $f^* : V_\omega \overset{\in}{\hookrightarrow} V_\omega$ such that $f^* \supset f$.*

Proof. In the same vein as 4.8 we can fix an enumeration of $V_\omega \setminus \omega$ and check that there are infinitely many z satisfying conditions (a) and (b) of 4.7 when $x = \omega \cup a$ and a is finite. For example z could be $\{\{n\}\}$ for any sufficiently large $n \in \omega$. □

5 An analogue of the Baire space

The proof of 4.8 has a topological content which will be expressed by generalizing the Baire space $\mathcal{N}$ of functions from ω to ω as follows.

Let $\mathcal{M}$ be the set of all functions from V_ω to V_ω. The role played by finite *sequences* vis-à-vis the Baire space is here played by finite functions *with transitive domain*: let V_ω^* be the set of hereditarily finite functions with transitive domains, so $V_\omega^* \subset V_\omega$. We assign to $\mathcal{M}$ the topology whose basis consists of all sets of form $\mathcal{M}_s = \{f \in \mathcal{M} \mid s \subset f\}$ for $s \in V_\omega^*$: this is a basis for a topology because it is closed under finite intersections.

Adapting well-known approaches to $\mathcal{N}$ (e.g. [4]), define a *quasi-tree* T to be a subset of V_ω^* closed under subsets, i.e., $x \subseteq y \in T$ implies $x \in T$. T is *pruned* if every $s \in T$ has a proper extension in T. A *bough* of a quasi-tree T is a function $f \in \mathcal{M}$ such that $(\forall x \in V_\omega)\,(x \text{ transitive} \to f \restriction x \in T)$, and $[T]$ is the set of all boughs of T. Then the map $T \longmapsto [T]$ is a bijection between pruned quasi-trees and closed subsets of $\mathcal{M}$, and the proof of 4.8 shows:

Theorem 5.1 *The set of* $\in$*-embeddings from* V_ω *to* V_ω *is a closed subset of* $\mathcal{M}$.

In fact that proof shows that it's a perfect subset.

Theorem 5.2 $\mathcal{M}$ *is homeomorphic to the Baire space* $\mathcal{N}$.

Proof. I call on a classical theorem:

Theorem (Alexandrov-Urysohn, [4, I.7.7]) $\mathcal{N}$ *is the unique, up to homeomorphism, non-empty Polish zero-dimensional space for which all compact subsets have empty interior.*

The verification that $\mathcal{M}$ satisfies these conditions is a straightforward adaptation of the same verification for $\mathcal{N}$. For a complete metric on $\mathcal{M}$, define, when $f \neq g$, $d(f,g) = 1/2^\varrho$, where ϱ is the least rank of x such that $f(x) \neq g(x)$.

The basis for $\mathcal{M}$ given above is a countable basis of clopen sets, and $\mathcal{M}$ is easily seen to be Hausdorff, so that $\mathcal{M}$ is zero-dimensional. To verify that compact subsets of $\mathcal{M}$ have empty interiors, it suffices to show that if T is pruned, then $[T]$ is compact if and only if T is finitely splitting, i.e., for every $s \in T$ with domain x and every subset y of x, there are in T at most finitely many extensions of s to domain $[x; y]$. □

Next I draw out further the topological content of the construction in 4.8.

Definition 5.3 Suppose $g : x \longrightarrow y$, where x is transitive. Define $g_+ : x \longrightarrow V$ by $g_+(u) = g_+``u \cup g(u)$.

This is an $\in$-inductive definition, and immediately g_+ satisfies the condition of Lemma 4.2 for preserving $\in$:

Lemma 5.4 *Suppose* g *has transitive domain. Then* g_+ *preserves* $\in$.

For $a \in V_\omega$, let a also denote the constant function (and element of $\mathcal{M}$) $x \longmapsto a$. Then the reader may verify that a_+ as defined here agrees with a_+ as defined earlier in 3.1. Also if $g : V_\omega \longrightarrow V_\omega$, then $g_+ : V_\omega \longrightarrow V_\omega$.

Lemma 5.5 *Let* x *be transitive and* $g : x \longrightarrow y$. *Suppose that*

$$(1) \qquad (\forall u \in x)\,(g(u) \cap g_+``x = 0).$$

Then $g_+ : x \overset{\in}{\hookrightarrow} g_+``x$ *and* $g = (g_+)_-$.

Proof. By Lemmas 5.4 and 2.2, we only need to verify that g_+ preserves $\notin$. But (1) says that $g(u) = (g_+)_-(u)$, and so by Lemma 4.4 g_+ preserves $\notin$. □

Furthermore, if h is an extension of g (with both domains transitive), then h_+ extends g_+. So summing up:

Proposition 5.6 *The function* $f \longmapsto f_-$ *is a homeomorphism between, on the one hand, the set of all* $f : V_\omega \overset{\in}{\hookrightarrow} V_\omega$, *and, on the other hand, the set of functions* $g \in \mathcal{M}$ *with the property* (1). *Its inverse is* $g \longmapsto g_+$.

It should be noted, however, that when all functions from V_ω to V_ω are considered, $g \longmapsto g_+$ is not one-to-one.

6 $<$-embeddings

Proposition 6.1 *Let $x, y \subseteq V_\omega$ with x transitive. If $f : x \overset{;}{\hookrightarrow} y$, then $f : x \overset{<}{\hookrightarrow} y$.*

Proof. Using 3.2, which applies to any $f : x \longrightarrow V_\omega$ with x transitive, I show that a_+ is a $<$-embedding. In [5, 3.6] it is shown that

$$z < a + y \;\Leftrightarrow\; z < a \;\vee\; (\exists v < y)(z = a + v). \tag{2}$$

So $a + x < a + y$ if and only if $(\exists v < y)(a + x = a + v)$, because $a + x \not< a$ ([5, 3.3]). But $a + x = a + v$ if and only if $x = v$ (i.e., $a_+ : V_\omega \hookrightarrow V_\omega$), so $a + x < a + y$ if and only if $x < y$. □

Proposition 6.2 *If f preserves $\in$ and the domain of f is transitive, then f preserves $<$.*

Proof. If $u < v \in x$, say $u \in u_1 \in \cdots \in u_n \in v$, then $f(u) \in f(u_1) \in \cdots \in f(u_n) \in f(v)$. □

I shall need some other, related results about products from [5]:

Lemma 6.3 *Let $b \neq 0$.*

(i) *$b \cdot u = b \cdot v$ implies $u = v$.* ([5, 4.8])

(ii) $\mathrm{TC}(b \cdot x) = \{b \cdot q + r \mid q < x \wedge r < b\}$. ([5, 4.15])

(iii) *If $r < b$, $s < b$, and $b \cdot x + r = b \cdot y + s$, then $x = y$ and $r = s$.* ([5, 4.7])

Now 6.1 can be generalized:

Proposition 6.4 *If $f(x) = a + b \cdot x$ with $b \neq 0$, then $f : V_\omega \overset{<}{\hookrightarrow} V_\omega$.*

Proof. 6.1 deals with a_+, 6.3(i) says that $x \longmapsto b \cdot x$ is one-to-one, and 6.3(ii) implies that if $u < v$ and $b \neq 0$, then $b \cdot u < b \cdot v$. So it remains to show that $u < v$ assuming $b \cdot u < b \cdot v$. By 6.3(ii), if $b \cdot u < b \cdot v$, then $b \cdot u = b \cdot q + r$ for some $q < v$ and $r < b$. By 6.3(iii), $u = q$. □

Proposition 6.2 can't be strengthened to an implication between $\in$-embeddings and $<$-embeddings:

Example 6.5 Neither of $f : V_\omega \overset{\in}{\hookrightarrow} V_\omega$ and $f : V_\omega \overset{<}{\hookrightarrow} V_\omega$ implies the other.

Proof. Suppose $b \neq 0$ and $0 \notin b$. By 4.1 and 6.4, $x \longmapsto b \cdot x$ is a $<$-embedding which is not an $\in$-embedding. I next give an example of an embedding f such that $f : V_\omega \overset{\in}{\hookrightarrow} V_\omega$ but not $f : V_\omega \overset{<}{\hookrightarrow} V_\omega$. Recall that $V_3 = \{0, 1, \{1\}, 2\}$. In the notation of Lemma 4.7, let $x = \{0, 1, \{1\}\}$ and let $f : x \overset{\in}{\hookrightarrow} V_\omega$ be the identity embedding, with $y = 2$ and $z = \{\{\{1\}\}\}$. Verifying that $z \cap f\text{"}x = 0$ and $f_-(u) = 0$ for $u \in x$, 4.7 says that we can extend f to $f : V_3 \overset{\in}{\hookrightarrow} V_\omega$ with $f(2) = \{0, 1, \{\{1\}\}\}$. Since $f(\{1\}) < f(2)$ but $\{1\} \not< 2$, f does not preserve $\not<$. Use 4.8 to extend f to domain V_ω. □

Lemma 6.6 *Let x be transitive and $f : x \hookrightarrow y$. Then*

$$f \textit{ preserves } < \textit{ iff } (\forall u \in x)\,(\mathrm{TC}(f(u)) \supseteq f\text{"}\mathrm{TC}(u)).$$

Definition 6.7 If f is a function, define $\hat{f}(u) =_{\mathrm{df}} \mathrm{TC}(f(u)) \setminus f\text{"}\mathrm{TC}(u)$.

Lemma 6.8 *Let $f : x \hookrightarrow y$. Then*

$$f \textit{ preserves } \not< \textit{ iff } (\forall u \in x)\,(\hat{f}(u) \cap f\text{"}x = 0).$$

Proof. Suppose that f preserves $\not<$ but for some u and v in x, $f(v) \in \hat{f}(u)$. Then, since $\hat{f}(u) \subseteq \mathrm{TC}(f(u))$, it follows that $f(v) < f(u)$, hence $v < u$ so $f(v) \in f\text{"}\mathrm{TC}(u)$, contradicting the definition of $\hat{f}$.

Conversely, assume the right side true and $f(v) < f(u)$. Since $f(v) \notin \hat{f}(u)$ we must have $f(v) \in f\text{"}\mathrm{TC}(u)$. As in the proof of 4.4, because f is one-to-one we can deduce that $v \in \mathrm{TC}(u)$, i.e., $v < u$. □

7 $\subset$-embeddings

The subset relation is another generalization of the order relation on ω to V_ω. The definitions of addition and multiplication quite easily yield:

Proposition 7.1 *If $f(x) = a + b \cdot x$ and $b \neq 0$, then $f : V_\omega \overset{\subset}{\hookrightarrow} V_\omega$.*[8)]

So $f : V_\omega \overset{;}{\hookrightarrow} V_\omega$ implies $f : V_\omega \overset{\subset}{\hookrightarrow} V_\omega$, but once again there is only a limited logical connection between $\in$-embeddings and $\subset$-embeddings:

Proposition 7.2 *If $f : x \overset{\in}{\hookrightarrow} y$ and x is transitive, then f preserves $\not\subset$.*

P r o o f. If $f(u) \subseteq f(v)$ and $z \in u$, then $f(z) \in f(u)$, so $f(z) \in f(v)$, so $z \in v$. □

Example 7.3 Neither of $f : V_\omega \overset{\in}{\hookrightarrow} V_\omega$ and $f : V_\omega \overset{\subset}{\hookrightarrow} V_\omega$ implies the other.

P r o o f. By 4.1 and 7.1 the function $x \longmapsto a + b \cdot x$, $0 \notin b \neq 0$ of Example 6.5 is a $\subset$-embedding which is not an $\in$-embedding. And here is an $\in$-embedding which is not a $\subset$-embedding. Using Lemma 4.7, extend the identity embedding $f : 3 \overset{\in}{\hookrightarrow} V_\omega$ to $f : V_3 \overset{\in}{\hookrightarrow} V_\omega$ by assigning $f(\{1\}) = \{1, \{\{1\}\}\}$, i.e. in the notation of 4.7, $z = \{\{\{1\}\}\}$. f does not preserve $\subset$ because $f(\{1\}) \not\subset f(2)$. Use 4.8 to extend f to domain V_ω. □

Another example of a $\subset$-embedding is $\lambda_a : V_\omega \overset{\subset}{\hookrightarrow} V_\omega$ (see Definition 2.3), which is neither an $\in$-embedding nor a $<$-embedding unless $a = 0$.

Theorem 8.9 will show (rather more than) that V_ω has $2^{\aleph_0}$ $\subset$-embeddings.

8 Comparing languages

The main job of this section will be to establish some basic results linking S-embeddings for different choices of S, and apply them to some set-theoretic languages. But I begin by filling in some details about preservation, which was defined in Definition 2.1 only for relations. What about preserving a function?

Suppose $f \in \mathcal{M}$, and suppose G is an n-ary function symbol with a given interpretation on V_ω. Let $\Gamma(G)$ denote the graph of G. Set-theoretically G^{V_ω} and $\Gamma(G)^{V_\omega}$ are the same subset of V_ω^{n+1}, but linguistically $\Gamma(G)$ is a relation symbol instead of a function symbol.

Say that f *preserves* G if and only if f preserves $\Gamma(G)$, i.e., f and G commute:

$$f(G(x_1, \ldots, x_n)) = G(f(x_1), \ldots, f(x_n)).$$

So $f : V_\omega \overset{G}{\hookrightarrow} V_\omega$ if and only if f preserves G and f is one-to-one. Also say that f *preserves* $\neg G$ if and only if f preserves $\neg\Gamma(G)$, i.e., if and only if $G(f(x_1), \ldots, f(x_n)) = f(y)$ implies that $G(x_1, \ldots, x_n) = y$.

This is sometimes redundant:

Proposition 8.1 *If $f : V_\omega \overset{G}{\hookrightarrow} V_\omega$, then f preserves $\neg G$.*

P r o o f. If $G(f(x_1), \ldots, f(x_n)) = f(y)$, then, since f preserves G, $f(G(x_1, \ldots, x_n)) = f(y)$. But f is one-to-one. □

Proposition 8.2 *If f preserves G and f preserves $\neg G$, then $f \restriction (G"V_\omega^n)$ is one-to-one.*

P r o o f. Suppose $f(G(x_1, \ldots, x_n)) = f(y)$. Since f preserves G, it follows that $G(f(x_1), \ldots, f(x_n)) = f(y)$. Since f preserves $\neg G$, $G(x_1, \ldots, x_n) = y$. □

Corollary 8.3 *Suppose f preserves adjunction. Then*

$$f \text{ preserves } \neg; \text{ iff } f \text{ is one-to-one iff } f : V_\omega \overset{;}{\hookrightarrow} V_\omega.$$

P r o o f. The special case of $f(0)$ left out of 8.2 is easily dealt with. □

8) Once again, Garcia [2] stated in effect part of this, namely that f preserves $\subset$.

Next I make precise some arguments, already used implicitly in special cases, along the lines that if f is an S-embedding and T is another language whose symbols[9)] are simply defined in $\mathfrak{L}(S)$, then f is a T-embedding.

Definition 8.4 Suppose $\mathfrak{L}(S)$ and $\mathfrak{L}(T)$ are languages. I shall say that T is *QF-definable from S* if

(i) for every relation symbol R in T, $R^{\langle V_\omega, T\rangle}$ is definable in $\langle V_\omega, S\rangle$ by a quantifier-free formula of $\mathfrak{L}(S)$,

(ii) for every function symbol F in T, the graph of $F^{\langle V_\omega, T\rangle}$ is definable in $\langle V_\omega, S\rangle$ by a quantifier-free formula of $\mathfrak{L}(S)$, and

(iii) for every constant symbol C of T, $C^{\langle V_\omega, T\rangle}$ is named in $\langle V_\omega, S\rangle$ by a closed term of $\mathfrak{L}(S)$.

Proposition 8.5 *If T is QF-definable from S, then every S-embedding is a T-embedding.*

Proof. If R is a relation symbol in T defined by the quantifier-free φ, and f is an S-embedding, then (cf. the remark after Lemma 2.2):

$$\begin{aligned} \langle V_\omega, T\rangle \vDash R(\vec{x}) \quad &\Leftrightarrow \quad \langle V_\omega, S\rangle \vDash \varphi(\vec{x}) \\ &\Leftrightarrow \quad \langle V_\omega, S\rangle \vDash \varphi(f(\vec{x})) \\ &\Leftrightarrow \quad \langle V_\omega, T\rangle \vDash R(f(\vec{x})). \end{aligned}$$

Function and constant symbols are dealt with similarly. □

One corollary is Proposition 3.4, because $x \in y$ is definable by the formula $[y; x] = x$. 8.5 was also implicitly used earlier (e.g. 7.2) when I rather casually identified $<$-embeddings with $\leq$-embeddings and $\subset$-embeddings with $\subseteq$-embeddings. Here is another corollary:

Proposition 8.6 *Every $\cup$-embedding is a $\subset$-embedding. Likewise, every $\cap$-embedding is a $\subset$-embedding.*

Proof. $x \subset y$ if and only if $x \cup y = y \wedge x \neq y$. Similarly for $\cap$. □

It is straightforward to verify that if $b \neq 0$, then $x \longmapsto a + b \cdot x \ : \ V_\omega \overset{\cup\cap}{\hookrightarrow} V_\omega$, in fact that $x \longmapsto a + b \cdot x$ also preserves the *unary* operators $\bigcup$ and $\bigcap$.

I use $\{\}$ to denote the singleton function $x \longmapsto \{x\}$.

Proposition 8.7 *The only $f : V_\omega \longrightarrow V_\omega$ which is both a $\cup$-embedding and a $\{\}$-embedding is the identity.*

Proof. Suppose $f : V_\omega \overset{\cup\{\}}{\hookrightarrow} V_\omega$. By 8.5 f is an ;-embedding, since $[x; y] = x \cup \{y\}$. So by 3.2 f is a_+ for some a. But it is easy to see that a_+ is not a $\{\}$-embedding unless $a = 0$. □

So you can't hope to preserve everything in a non-trivial embedding, just as we saw earlier that there are no $f : V_\omega \overset{0\,;}{\hookrightarrow} V_\omega$ other than the identity. But some combinations of symbols are more fruitful.

Proposition 8.8 *If $f : V_\omega \overset{\subset\{\}}{\hookrightarrow} V_\omega$, then $f : V_\omega \overset{0\in}{\hookrightarrow} V_\omega$.*

Proof. $x \in y$ if and only if $\{x\} \subseteq y$. To show that $f(0) = 0$, suppose $f(0) = a$ and pick any $b \in V_\omega$. Since $0 \subset \{b\}$, it follows that $a \subset f(\{b\}) = \{f(b)\}$ which implies that $a = 0$. □

In contrast to 8.7, there are many $f : V_\omega \overset{\subset\{\}}{\hookrightarrow} V_\omega$:

Theorem 8.9 *There are $2^{\aleph_0}$ $f : V_\omega \overset{\subset\{\}}{\hookrightarrow} V_\omega$.*

Proof. Modify the construction in the proof of 4.8. Let $\{s_0 = 0, s_1, s_2, \dots\}$ be an enumeration of V_ω such that, as before, $s_n \subseteq x_n = \{s_0, \dots, s_{n-1}\}$, but now with the following additional property: if $u \subset v$, then u occurs before v in the enumeration. This is easy to arrange.

9) Or rather the interpretations of the symbols of T on V_ω.

Start with $f(0) = 0$ and suppose, inductively on $n > 0$, that we have constructed $f : x_n \longrightarrow V_\omega$ satisfying the following six conditions for each $k < n$:

(a) $f_-(s_k) \cap f\text{"}x_k = 0$,
(b) $f(s_k) \notin \bigcup_{i<k} f_-(s_i)$,
(c) $f(s_k) \supseteq f\text{"}(s_k)$,
(d) $f_-(s_k) \supset f_-(s_j)$ for each non-empty $s_j \subset s_k$,
(e) no element of $f_-(s_k)$ is a singleton, and
(f) if s_k is a singleton, then $f_-(s_k) = 0$.

The first two conditions are essentially (a) and (b) of Lemma 4.7, and allow us to prove, by an induction on $k < n$ using 4.7, that $f : x_n \overset{\in}{\hookrightarrow} V_\omega$. It follows by Proposition 7.2 that f preserves $\not\subset$.

If $s_j \subset s_k$ and s_j is non-empty, then $j < k$ and from (c) and (d) at stage k we see that $f(s_j) \subset f(s_k)$. And since f is one-to-one, $0 = f(0) \subset f(s_k)$ for $k > 0$. So f also preserves $\subset$. Thus $f : x_n \overset{\subset}{\hookrightarrow} V_\omega$.

Since x_n is finite it is not closed under the singleton operator so it wouldn't be correct to call it a structure for $\mathfrak{L}(\{\})$. This means that f can't (at this stage) be described as a $\{\}$-embedding, but we have the next best thing:

(†) For any $k < n$, if s_k is a singleton, then $s_k = \{s_i\}$ for some $i < k$, and by (c) and (f) at stage k we have $f(s_k) = \{f(s_i)\}$.

Now I show how to extend f to domain x_{n+1} satisfying (a) – (f) with $k = n$, by assigning a value to $f(s_n)$.

When s_n is a singleton, say $s_n = \{s_i\}$, $i < n$, then to satisfy (c) and (f) we are obliged to make $f_-(s_n) = 0$ and $f(s_n) = \{f(s_i)\}$. This means that $f(s_n)$ is not an element of $f_-(s_j)$ for any $j < n$ since we assured (e) at stage j. Thus (b) holds for the extended function. The other conditions are satisfied trivially.

When s_n is not a singleton, I claim that there are infinitely many choices for $f(s_n)$ satisfying (a) – (f) with $k = n$. Specifically, let

$$z = \bigcup\{f_-(s_j) \mid s_j \subset s_n\} \cup \{\{v, v+1\}\},$$

where v is any element of V_ω of larger rank than $f\text{"}x_n$. My claim is that $f(s_n) = z \cup f\text{"}s_n$ will do.

Applying Lemma 4.4 to $f : x_n \overset{\in}{\hookrightarrow} V_\omega$, we see that

(‡) for any $i < n$ and $s_j \subset s_n$, $f(s_i) \notin f_-(s_j)$.

Hence in particular $z \cap f\text{"}s_n = 0$, i.e., $f_-(s_n) = z$. With this established, (‡) also tells us that (a) holds with $k = n$. Conditions (b) – (f) are now easily verified.

We now have a tree of partial embeddings $f : x_n \overset{\in\subset}{\hookrightarrow} V_\omega$ which is infinitely branching at non-singleton stages, and each satisfies (†), so the union of any branch gives an embedding $f^* : V_\omega \overset{\subset\{\}}{\hookrightarrow} V_\omega$. □

Definition 8.10 Say that T is $\forall$-*definable* from S if and only if the three clauses of Definition 8.4 hold with "quantifier-free" replaced by "universal". Likewise for $\exists$-*definable* use "existential".

Note that we are not using bounded quantifiers. The next Proposition is really just a form of the preservation of universal formulæ by submodels in basic model theory:

Proposition 8.11

(i) *If T is $\forall$-definable from S and $f : V_\omega \overset{S}{\hookrightarrow} V_\omega$, then f preserves $\neg T$.*

(ii) *If T is $\exists$-definable from S and $f : V_\omega \overset{S}{\hookrightarrow} V_\omega$, then f preserves T.*

Proof. Suppose the relation $R(\vec{x})$ is defined by $\forall \vec{z}\, \varphi(\vec{z}, \vec{x})$, where φ is a quantifier-free formula of $\mathfrak{L}(S)$. Then

$$\begin{aligned} \langle V_\omega, R\rangle \vDash R(f(\vec{x})) \quad &\Leftrightarrow \quad \langle V_\omega, S\rangle \vDash \forall \vec{z} \varphi(\vec{z}, f(\vec{x})) \\ &\Rightarrow \quad \text{for all } \vec{z} \in V_\omega,\ \langle V_\omega, S\rangle \vDash \varphi(f(\vec{z}), f(\vec{x})) \\ &\Leftrightarrow \quad \langle V_\omega, S\rangle \vDash \forall\, \vec{z} \varphi(\vec{z}, \vec{x}) \\ &\Leftrightarrow \quad \langle V_\omega, R\rangle \vDash R(\vec{x}). \end{aligned}$$

Similarly for functions and constants and for (ii). □

A corollary of Proposition 8.11 is Proposition 7.2. There will be another in the next section. I conclude this section by applying its results to TC-embeddings.

Proposition 8.12

(i) *If* $f : V_\omega \overset{\in\mathrm{TC}}{\hookrightarrow} V_\omega$, *then* $f : V_\omega \overset{<}{\hookrightarrow} V_\omega$.

(ii) *If* $f : V_\omega \overset{\in<}{\hookrightarrow} V_\omega$, *then f preserves* $\neg\mathrm{TC}$.

Proof. (i) $<$ is QF-definable from $\in$ and TC, and (ii) $y = \mathrm{TC}(x)$ if and only if $\forall z(z < x \Leftrightarrow z \in y)$. □

Proposition 8.13

(i) a_+ *is a* TC-*embedding if and only if a is transitive.*

(ii) $x \longmapsto b \cdot x$ *is a* TC-*embedding if and only if b is transitive and* $b \neq 0$.

Proof.

(i) In [5] it is shown that $\mathrm{TC}(a + x) = \mathrm{TC}(a) \cup \lambda_a(\mathrm{TC}(x))$ (cf. (2) above), and this union is a disjoint union by virtue of 2.4(iv). But $a + \mathrm{TC}(x) = a \cup \lambda_a(\mathrm{TC}(x))$, so the two are equal if and only if $a = \mathrm{TC}(a)$.

(ii) Compare 6.3(ii) with $b \cdot \mathrm{TC}(x) = \{b \cdot q + r \mid q < x \wedge r \in b\}$. □

This furnishes examples (when a or b is not transitive) to show that not all $\in$- and $<$-embeddings are TC-embeddings.

As an example of a function which is a TC-embedding but not an $\in$-embedding nor a $<$-embedding, consider $f(x) = \mathrm{TC}(x) + x$. To show that this f preserves TC, verify that $\mathrm{TC}(f(x)) = f(\mathrm{TC}(x)) = \mathrm{TC}(x) + \mathrm{TC}(x)$. To verify that f is one-to-one, suppose

(3) $$\mathrm{TC}(x) + x = \mathrm{TC}(y) + y.$$

By 2.4(iii), for some r, $\mathrm{TC}(x) + r = \mathrm{TC}(y)$ (or vice versa). Then by left cancellation $r + y = x$, and so $\mathrm{TC}(r+y)+r = \mathrm{TC}(y)$. It follows that $\lambda_r(\mathrm{TC}(y)) \subseteq \mathrm{TC}(y)$ ((2) again) and hence (since y is finite) that $r = 0$. Hence $\mathrm{TC}(x) = \mathrm{TC}(y)$, and by left cancellation from (3), $x = y$.

9 Arithmetical embeddings

What embeddings preserve $+$, $\cdot$ or $\lhd$ (see Definition 2.3)? Here I mention just a few elementary facts. It follows from [5, 4.9] that, in the present notation:

Proposition 9.1 *Let* $0 \neq b \in V_\omega$. *Then the function* $x \longmapsto b \cdot x$ *is a* $+$-*embedding.*

With help from a related result, [5, 4.3(iv)], we can show that $x \longmapsto b \cdot x$ is also a λ-embedding, where λ is the *binary* lift function $\langle x, y\rangle \longmapsto \lambda_x(y)$.

Since $\lhd$ is existentially defined from $+$, we have this corollary of 8.11:

Proposition 9.2 *Any* $+$-*embedding preserves* $\lhd$.

On ω, the only $+$-embeddings are those given by 9.1. But V_ω admits many more $+$-embeddings, in fact by 2.4(ii) it has $2^{\aleph_0}$ $+$-*automorphisms*: for let AI be the set of additively irreducible elements of V_ω. Then any permutation π of AI induces a $+$-embedding of V_ω onto itself by $a_1 + \cdots + a_n \longmapsto \pi(a_1) + \cdots + \pi(a_n)$.

Do other results about arithmetical embeddings on ω fail on V_ω? For instance, is there a non-trivial $f \in \mathcal{M}$ which is both a $+$-embedding and a $\cdot$-embedding? ω has $2^{\aleph_0}$ $\cdot$-automorphisms obtained by permuting the primes. What about V_ω?

For an example of a $\lhd$-embedding which is not a $+$-embedding, consider a_+ with $a \neq 0$. For an example of a $+$-embedding which is not a $\lhd$-embedding, note that (by 2.4(ii) again) any $+$-embedding is determined by its action on AI, and define a $+$-embedding f by $f(1) = \{1\}$, $f(\{1\}) = \{1\} + 1$, and $f(a) = a$ for every other element a of AI, checking that f is one-to-one. But f does not preserve $\neg\lhd$ because $f(1) \lhd f(\{1\})$. This example also shows that you can't improve 8.11(ii) by demanding that $f : V_\omega \overset{T}{\hookrightarrow} V_\omega$.

References

[1] C. C. Chang and H. J. Keisler, Model Theory (North-Holland, 1973).
[2] N. Garcia, Operating on the universe. Arch. Math. Logic **27**, 61 – 68 (1988).
[3] R. Kaye and T. L. Wong, On interpretations of arithmetic and set theory. Notre Dame J. Formal Logic **48**, 497 – 510 (2007).
[4] A. S. Kechris, Classical Descriptive Set Theory (Springer-Verlag, 1995).
[5] L. Kirby, Addition and multiplication of sets. Math. Logic Quarterly **53**, 52 – 65 (2007).
[6] L. Kirby, A hierarchy of hereditarily finite sets. Arch. Math. Logic **47**, 143 – 157 (2008).
[7] L. Kirby, Finitary set theory. Notre Dame J. Formal Logic **50**, 227 – 244 (2009).
[8] F. Previale, Induction and foundation in the theory of the hereditarily finite sets. Arch. Math. Logic **33**, 213 – 241 (1994).