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Ordinal operations on graph representations of sets

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Any set x is uniquely specified by the graph of the membership relation on the set obtained by adjoining x to the transitive closure of x. Thus any operation on sets can be looked at as an operation on these graphs. We look at the operations of ordinal arithmetic of sets in this light. This turns out to be simplest for a modified ordinal arithmetic based on the Zermelo ordinals, instead of the usual von Neumann ordinals. In this arithmetic, addition of sets corresponds to concatenating graphs, and multiplication corresponds to replacing each edge of a graph by a copy of another graph. Characterizations for the von Neumann case are also given.

1 Introduction

Let $\mathrm{TC}(x)$ denote the transitive closure of the set x. Any set x can be represented by the directed graph $G(x)$ whose nodes are the elements of $\mathrm{TC}(x) \cup \{x\}$ and which has an edge $a \to b$ just when $b \in a$.

If ν_0 is a node of a directed graph G, by a *path* from ν_0, we shall mean a sequence of nodes $(\nu_0, \nu_1, \nu_2, \cdots)$ such that G contains an edge $\nu_i \to \nu_{i+1}$ for each i.

In Aczel's language from [1], the graph $G(x)$ is an example of an *accessible pointed graph*: a directed graph G with a unique *source* or initial node $P = P_G$ (the *point*) such that every other node of G is on a finite path from P. A *picture* of a set x is an accessible pointed graph G which has an assignment of a set to each node such that the elements of the set assigned to each node ν are the sets assigned to the *children* of ν (i.e., nodes μ such that $\nu \to \mu$ is an edge), and x is assigned to the point. So $G(x)$, with each node assigned to itself, can be called the canonical picture of x (up to isomorphism, and in a sense which can be made precise, cf. Theorem 1.3 below).

Unlike Aczel, we am concerned here with *well-founded* sets, so we can simplify Aczel's framework. In this paper, *all sets are well-founded*, i.e., we have the Foundation Axiom for sets. Hence any accessible pointed graph which is a picture of a set is well-founded as a graph (in Aczel's sense, i.e., it has no infinite paths). For hereditarily finite sets, this is equivalent to saying that that the graph is *acyclic*. And using Mostowski's Collapsing Lemma [1]:

Theorem 1.1 *Any well-founded accessible pointed graph is a picture of a unique set.*

Any operation on sets, then, corresponds to an operation on graphs, in particular the operations of ordinal arithmetic, extended to all sets (cf. Section 2 below). We shall see, however, that the correspondence is particularly simple when we replace the usual von Neumann ordinal arithmetic by one that, in the finite case, extends the arithmetic of the Zermelo ordinals (cf. Section 2). The Zermelo sum corresponds to concatenation of graphs (Theorem 3.1). The Zermelo product corresponds to the operation of replacing each edge of a graph by a copy of another graph (Theorem 4.1).

We shall also see how von Neumann addition can be expressed as *weighting* each node of a graph by another graph (Theorem 3.3). Von Neumann multiplication is also characterized in Section 4, and the characterizations are recast in terms of the canonical graphs $G(x)$.

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In the rest of this section, we shall cover some basics of pictures of sets and introduce some notation. Section 2 will review the ordinal arithmetic of sets and introduce the Zermelo arithmetic. Section 3 will give the graph characterizations for addition, and Section 4 for multiplication. Section 5 will offer brief remarks on the relationship of the current work to earlier generalizations of arithmetic to sets and graphs. Finally, Section 6 will generalize some of the results on addition to provide a glimpse of the structure of the $\in$-embeddings of sets. We studied these in [9]; for our present purposes we can characterize them by saying that the corresponding graph operations are graph monomorphisms. The Zermelo sum will be seen to have a special role among these, providing in a sense the smallest (Theorem 6.3).

By $x < y$, we shall mean that x is an element of $\mathrm{TC}(y)$. If G is a directed graph and ν is a node of G then ν_G is the set of children of ν. $G\nu$ is the the subgraph of G induced by the set of all nodes which are on paths from ν. Some basic facts about the graphs $G(x)$:

Proposition 1.2

(i) *If $x < y$ then $x_{G(y)} = x$, and $G(y)x = G(x)$.*

(ii) *If x is hereditarily finite, the rank of x (in the cumulative hierarchy) is equal to the length of the longest path in $G(x)$.*[1]

(iii) *The number of nodes of $G(x)$ is* $|\mathrm{TC}(x)| + 1$.

A directed graph G is *extensional* iff distinct nodes have distinct sets of children. When an accessible pointed graph G is well-founded and extensional, it has a unique *sink* or terminal node, accessible via paths from every other node, which we shall call O or O_G and which is of course assigned the empty set 0 when G is a picture of a set.

Any well-founded graph can be *extensionalized*, or thinned down to an extensional accessible pointed graph, by an inductive process. At each stage consider the nodes all of whose children have already been considered. Identify nodes that have identical sets of children. The identifying can be seen as taking a quotient of the graph. By $\in$-induction, any set has a unique (up to isomorphism) extensional picture:

Theorem 1.3 *The extensionalization of any picture of x is isomorphic to $G(x)$.*

The process of extensionalization can also, in a sense, be reversed. If G is a picture of x then we "unfold" G into a rooted tree[2] T_G as follows: the nodes of T_G are paths $(P_G, \nu_1, \dots, \nu_i)$ in G, with the root being the trivial path (P_G), and T_G has edges $(P_G, \nu_1, \dots, \nu_i) \to (P_G, \nu_1, \dots, \nu_i, \nu_{i+1})$. T_G is still a picture of x when we assign to each path the set which was assigned in G to the path's final node. G is extensional just when T_G is what is called an *identity tree*: a rooted tree whose only automorphism fixing the root is the identity. Meir, Moon and Mycielski [10] defined the natural one-to-one correspondence between hereditarily finite sets and identity trees, in our terms $x \mapsto T_{G(x)}$, and obtained asymptotic results about frequencies of certain properties of identity trees.

2 Von Neumann arithmetic and Zermelo arithmetic

Our ordinals are the usual von Neumann ordinals. In [6], we studied ordinal addition and multiplication of sets (due respectively to Alfred Tarski and Dana Scott) which generalize to all sets the operations of ordinal arithmetic:

$$x + y = x \cup \{x + r \mid r \in y\}$$
$$\text{and} \quad x \cdot y = \{x \cdot q + r \mid q \in y \wedge r \in x\}.$$

But the graph viewpoint will bring to the fore a parallel arithmetic of sets which we shall call the Zermelo arithmetic because it generalizes in the same way the arithmetic of the finite *Zermelo ordinals*, which are defined by $0_z = 0$, $(n+1)_z = \{n\}$. Indeed the graphs $G(n_z)$ have a simple and transparent relation to the numerals they represent.

[1] The reviewer has pointed out that this does not extend to infinite sets: the supremum of the path lengths in $G(x)$ for any well-founded x is at most ω.

[2] This is a tree in the graph-theorist's sense: an undirected graph with a unique path between any pair of nodes.

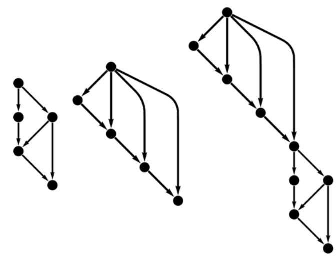

Fig. 1 For $a = \{\{1\}, 2\}$ and $b = \{0, 1, \{1\}, \{\{1\}\}\}$, this figure shows $G(a)$, $G(b)$, and $G(a +_z b)$.

Let the *Zermelo sum* of x and y be defined by

$$x +_z 0 = x, \quad x +_z y = \{x +_z r \mid r \in y\} \quad \text{when} \quad y \neq 0,$$

so that $m_z +_z n_z = (m + n)_z$ and $n_z = 1_z +_z 1_z +_z \cdots +_z 1_z = (1 + 1 + \cdots + 1)_z$.

The *Zermelo product* is obtained from addition analogously to the von Neumann case:

$$x \cdot_z y = \{x \cdot_z q +_z r \mid q \in y \land r \in x\}.$$

Then $m_z \cdot_z n_z = (m \cdot n)_z$, and the basic algebraic properties of Zermelo sums and products — associativity, left distributivity, and such like — can be proved by $\in$-induction much as was done in [6] for the von Neumann case. In the Zermelo arithmetic as in the von Neumann, each set has a unique decomposition into a sum of additively irreducible sets. Indeed, the similarity is not an accident as the algebraic properties in both cases are instances of more general results of Tarski on ordinal algebras [13].[3] For our purposes, the graph characterizations below will make some of these algebraic properties easy to see.

3 Addition

The *concatenation* of two well-founded, extensional accessible pointed graphs G and H is the graph obtained from the union of disjoint copies of G and H by identifying the source P_G of G with the sink O_H of H.

Theorem 3.1 *$G(a +_z b)$ is isomorphic to the concatenation of $G(a)$ and $G(b)$.*

Figure 1 illustrates this by an example.

Proof. For fixed a, use $\in$-induction on b. Let H be the concatenation of $G(a)$ and $G(b)$, and let ν be any child of P_H. Then $H\nu$ is the concatenation of $G(a)$ and $G(r)$ for some $r \in b$. By inductive hypothesis, $H\nu$ is isomorphic to $G(a +_z r)$. So in making H into a picture, the children of P_H must be assigned the sets $a +_z r$ for $r \in b$ and hence P_H must be assigned $a +_z b$. □

We can generalize concatenation by introducing the idea of *weighting* a node. If G and H are accessible pointed graphs, and ν is a node of H, we *attach the weight* G to ν by uniting H with an isomorphic copy of G (disjoint from anything else around), and identifying ν with P_G. So the concatenation of G and H can be obtained by attaching the weight G to O_H. Now we can adapt Theorem 3.1 to represent the Zermelo sum, even for non-extensional graphs:

Theorem 3.2 *Suppose G is a picture of a and H is a picture of b. Let J be the result of attaching the weight G to each sink of H. Then J is a picture of $a +_z b$.*

[3] Sets with either kind of ordinal addition satisfy all Tarski's axioms for ordinal algebras [13, p. 8] except the Involution Postulate, but this axiom is not needed to prove the unique decomposition. Garcia puts forth a different framework wide enough to subsume both cases, though he doesn't mention the Zermelo case [3].

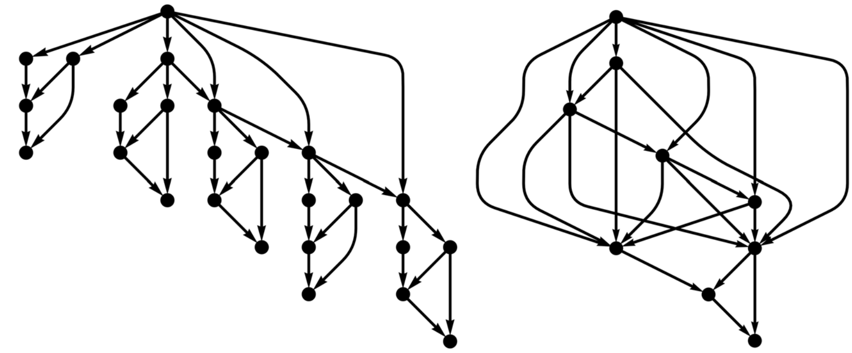

Fig. 2 A picture of $a + b$ given by Theorem 3.3, and $G(a + b)$, where a and b are as in Figure 1.

The von Neumann sum is now expressible as a graph operation:

Theorem 3.3 *Suppose G is a picture of a and H is a picture of b. Let J be the result of attaching the weight G to each node of H. Then J is a picture of $a + b$.*

Theorem 3.3 is illustrated in the first diagram of Figure 2.

Proof. For fixed a, use $\in$-induction on b. In J, any child ν of P_J must either be a node of the weight attached to P_H, in which case ν must be assigned an element of a, or else by applying the inductive hypothesis to $J\nu$, ν must be assigned $a + r$ for some element r of b. Thus P_J must be assigned $a \cup \{a + r \mid r \in b\} = a + b$. □

Theorem 3.4 will extensionalize this picture. First a reminder of some graph terminology. For sets X and Y, $K(X, Y)$, the *complete (directed) graph* from X to Y, is the directed graph consisting of edges $\nu \to \mu$ for each $\nu \in X$ and each $\mu \in Y$.

It follows from Theorem 3.1 that each node of $G(a +_z b)$ is either an element of $\mathrm{TC}(a)$ or an element of $a +_z [0, b]$. (Here we are employing standard interval notation $[0, b]$ and so on.)

Theorem 3.4 *$G(a + b)$ is isomorphic to $G(a +_z b) \cup K(a +_z (0, b], a)$.*

Thus $G(a + b)$ is obtained by concatenating $G(a)$ and $G(b)$ and festooning the result with new edges from each non-sink node of the copy of $G(b)$ to each child of the point of the copy of $G(a)$. Theorem 3.4 is illustrated in the second diagram of Figure 2.

Proof. For fixed a, let $J(r) = G(a +_z r) \cup K(a +_z (0, r], a)$. We shall show by $\in$-induction on b that the extensionalization of the weighted graph of $G(a + b)$ given by Theorem 3.3 is isomorphic to $J(b)$, and that when $J(b)$ is made into a picture of $a + b$, the node $a +_z r$ is assigned the set $a + r$.

Suppose μ is a child of the node $P = a +_z b$ in $J(b)$. The edge $P \to \mu$ is either in $G(a +_z b)$ or in the festoon from P to an element of a. In the former case μ is of form $a +_z r$ for some $r \in b$, and $J(b)\mu = J(r)$ so by the inductive hypothesis μ must be assigned the set $a + r$ when making $J(b)$ a picture. In the latter case, μ is an element s of a, so $J(b)s$ is isomorphic to $G(s)$ and μ must be assigned the set s, none of the new festoon edges having disturbed the assignment. Thus $J(b)$ is a picture of $a \cup \{a + r \mid r \in b\} = a + b$, and it is extensional. □

In [6] occurs the lift function $\lambda_a(b) = \{a + x \mid x \in b\}$, and the above can be modified to yield:

Corollary 3.5

(i) *Suppose G is a picture of a and H is a picture of b. Let J be the result of attaching the weight G to each node of H except the point. Then J is a picture of $\lambda_a(b)$.*

(ii) *$G(\lambda_a(b))$ is isomorphic to $G(a +_z b) \cup K(a +_z (0, b), a)$.*

4 Multiplication

If G is a well-founded, extensional accessible pointed graph and $\nu \to \mu$ is an edge of the accessible pointed graph H, then the operation of *replacing $\nu \to \mu$ by G* is performed as follows: remove the given edge from H, unite

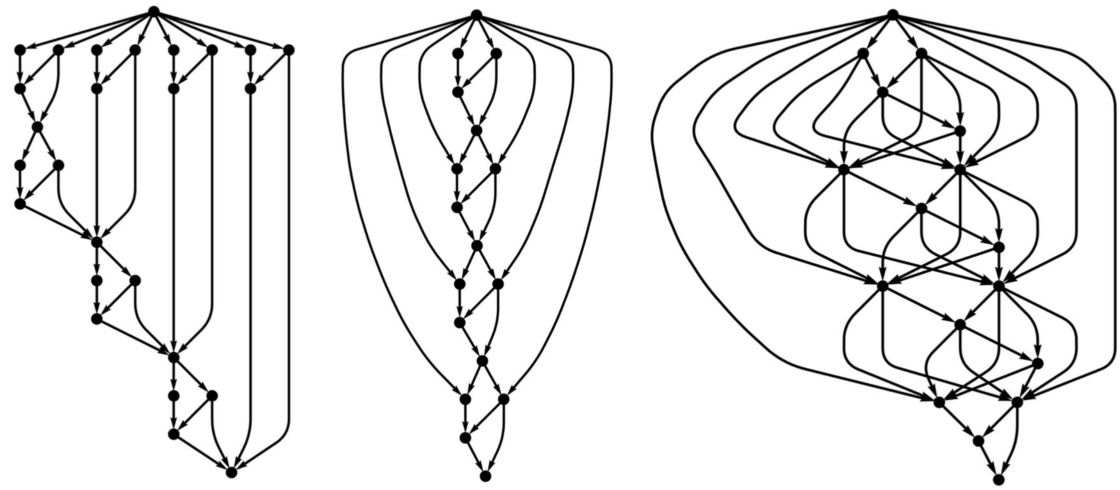

Fig. 3 A picture of $a \cdot_z b$ given by Theorem 4.1, $G(a \cdot_z b)$, and $G(a \cdot b)$, where a and b are as in Figure 1.

the rest of H with an isomorphic copy of G (disjoint from anything else around), and identify ν with P_G and μ with O_G.

Theorem 4.1 *Suppose G is an extensional picture of a and H is a picture of b. Let J be the result of replacing each edge of H by G. Then J is a picture of* $a \cdot_z b$.

Theorem 4.1 is illustrated in the first diagram of Figure 3.

P r o o f. Again we use induction on b. Let ν be any child of P_J. Then there must be, in H, some child μ of P_H such that ν is a child of P_{G^*} in the copy G^* of G that replaced the edge $P_H \to \mu$ of H. $J\nu$ is the concatenation of $J\mu$ and $G^*\nu$. The former, by inductive hypothesis, is a picture of $a \cdot_z q$ for the element q of b which was assigned to μ in making H a picture of b. The latter, by Proposition 1.2(i), must be a picture of some element r of a, since G^* is a picture of a. So, by Theorem 3.2, in J, ν must be assigned $a \cdot_z q +_z r$. Conversely any set of this form is assigned to some child of P_H. □

So $G(a \cdot_z b)$ is (isomorphic to) the extensionalization of the result of replacing each edge of $G(b)$ by $G(a)$. By chasing through the extensionalization process, we shall construct $G(a \cdot_z b)$ from $G(a)$ and $G(b)$ in Theorem 4.3 by a process which, unlike Theorem 4.1, will replace *nodes* (rather than edges) by graphs. Unlike attaching weights, a node of $G(b)$ will be identified with the sink of a graph rather than its point, so the graph will be stacked on top of the node rather than hanging down. The graphs in question will need a definition.

Definition 4.2 If G is an accessible pointed graph, let G^o be the subgraph of G induced by all nodes except the point, i.e., P_G and all edges from it are removed. Let the *orphans* of G^o be those nodes of G^o which were, in G, children of P_G.

Theorem 4.3 *$G(a \cdot_z b)$ may be constructed from $G(a)$ and $G(b)$ by the following operation: for each node ν of $G(b)$, other than the point, take a copy G^ν of $G(a)^o$, disjoint from anything else around. Now unite $G(b)$ with all the copies G^ν, identifying ν with O_{G^ν} (for each ν other than the point). Finally for each original edge $\mu \to \nu$ in $G(b)$ we remove the edge and replace it by edges from μ to each orphan of G^ν.*

This operation, while a bit awkward to state, is well suited to carrying out with pencil and paper. The second diagram of Figure 3 illustrates Theorem 4.3.

P r o o f. Suppose the process has successfully yielded an isomorphic copy of $G(a \cdot_z q)$ for each $q \in b$. Then $\bigcup_{q \in b} G(a \cdot_z q)$ is still extensional. The process is extended to b by taking this union, erecting the copy G^q of $G(a)^o$ on top of each child q of b, and attaching edges to all the orphans from a new point. It is straightforward to check that this graph J remains extensional. It is a picture of $a \cdot_z b$ because for any child ν of the point, $J\nu$ is a concatenation of some $G(a \cdot_z q)$ and a picture of some $r \in a$. □

For non-extensional G, the edge replacement in Theorem 4.1 can still be done but we need to go about it in an inductive way. Suppose G and H are well-founded accessible pointed graphs. Here is how we *inductively replace* *each edge of H by G*. Suppose the process has already reached all children of the node ν of H, i.e. for each $\nu \to \mu$ in H we have a graph $J\mu$ which is the result of inductively replacing each edge of $H\mu$ by G. To obtain $J\nu$, take the set of all edges $\nu \to \mu$ in H that begin at ν, and replace each of them by a copy of G with the weight $J\mu$ attached to each sink of G.

When G is extensional, then the result of this inductive replacement process is a picture of the same set as the result of the simpler operation in Theorem 4.1, and we can now restate Theorem 4.1 to cover non-extensional graphs:

Theorem 4.4 *Suppose G is a picture of a and H is a picture of b. Let J be the result of inductively replacing each edge of H by G. Then J is a picture of $a \cdot_z b$.*

To express the von Neumann product as an operation on graphs, we modify the edge-replacement process. We shall say that we *cumulatively replace each edge of H by G* as follows. Suppose that ν is a node of H and, inductively, for each child μ of ν in H we have a graph J^μ which is the result of cumulatively replacing each edge of $H\mu$ by G. To obtain J^ν, take the set of all edges $\nu \to \mu$ in H that begin at ν, and replace each of them by a copy of G with the weight J^μ attached to *each node, other than the point,* of G. Note that if J is the end result of this process then in fact $J^\mu = J\mu$ for each node μ of H.

Theorem 4.5 *Suppose G is a picture of a and H is a picture of b. Let J be the result of cumulatively replacing each edge of H by G. Then J is a picture of $a \cdot b$.*

Proof. Let ν be any child of the point of J. Then ν must be a child of P_{G^*} in the weighted copy G^* of G that replaced some edge $P_H \to \mu$ of H. Let q be the element of b assigned to μ when H is a picture of b. By $\in$-inductive hypothesis J^μ is a picture of $a \cdot q$. Also, in G^*, $G^*\nu$ must be a picture of r for some $r \in a$. So, in J, J^ν looks like a picture of r with weight J^μ attached to each node. By Theorem 3.3, J^ν is a picture of $a \cdot q + r$. Hence J is a picture of $\{a \cdot q + r \mid q \in b \wedge r \in a\} = a \cdot b$. □

The construction of $G(a \cdot b)$ differs from the Zermelo case (Theorem 4.3) only in the last clause, and is illustrated in the third diagram of Figure 3:

Theorem 4.6 *$G(a \cdot b)$ may be constructed from $G(a)$ and $G(b)$ by the following operation: for each node ν of $G(b)$, other than the point, take a copy G^ν of $G(a)^o$, disjoint from anything else around. Now unite $G(b)$ with all the copies G^ν, identifying ν with O_{G^ν} (for each ν other than the point). Finally for any edge $\mu \to \nu$ in $G(b)^o$ we remove the edge and replace it by edges from each node of G^μ to each orphan of G^ν. (Note that we retain edges $b \to \nu$ from the point of $G(b)$.)*

Proof. This goes much like that of Theorem 4.3. We extend the process to b from its elements by taking the union $\bigcup_{q \in b} G(a \cdot q)$, erecting the copy G^q of $G(a)^o$ on top of each child q of b, attaching edges to orphans from a new point, and also, since any $q \in b$ is no longer the point in the extended graph, adding new edges from every node of G^q to the orphans of G^s for any edge $q \to s$ of $G(b)$ with $q \in b$. The characterization of addition in Theorem 3.4 shows that any child of the point in the extended graph is a picture of $G(a \cdot q + r)$ for some $q \in b$, $r \in a$. □

5 Historical remarks

The graph representation of sets was laid out by Aczel [1]. We have followed Aczel in the "downwards" direction of the arrows in $G(x)$, whereby each arrow points from a set to its element. Using the reverse graph of $G(x)$ would of course be formally equivalent. P_G would then be the unique sink.

Since first writing this article, the present author has become acquainted with some interesting recent results of Milanič, Policriti, and Tomescu (cf. [11, 12, 14]) on graph representations of sets, also inspired by Aczel's pictures. They represent a set x by the graph of the membership relation on $\mathrm{TC}(x)$ rather than on $\mathrm{TC}(x) \cup \{x\}$, i.e., by our $G(x)^o$ (cf. Definition 4.2).

The Zermelo ordinals n_z for finite n appear in Zermelo's [15] version of the Axiom of Infinity. He calls their union "the *number sequence*". It is possible to extend them to the infinite by continuing to take unions at limit stages. When we do this, α_z has the same order type as α when ordered by $<$, although α_z is not totally ordered by $\in$. However, if we adopt this infinite extension, the operations of Zermelo arithmetic defined here do not correspond to the usual ordinal operations. For example, $1_z +_z \omega_z$ does not equal ω_z, indeed it is not even a Zermelo ordinal.

For historical remarks on the Tarski-Scott generalizations of ordinal operations to sets, cf. [6].

There have been some generalizations of arithmetic to graphs. Since an acyclic directed graph such as our $G(x)$ can be identified with a partial order, namely the transitive closure of the graph relation, which for $G(x)$ is our $<$, this can be considered a special case of generalizing arithmetic to partial orderings. Tarski [13] defined and proved the basic algebraic properties of ordinal addition and multiplication even more generally for relation types.

In 1942, before Tarski's book, Birkhoff [2] provided definitions of ordinal addition and multiplication of partially ordered sets which specialize to our Zermelo addition and multiplication given the identification mentioned in the previous paragraph.

More recently, Khan, Bhutani and Kahrobaei [4, 5] defined addition and multiplication on a class of finite directed graphs with two distinguished nodes. Their definitions again generalize our Zermelo operations on the graphs $G(x)$, although they consider only the finite case; many of the algebraic properties they prove are consequences of Tarski's general theory.

6 Isomorphic embeddings

This section generalizes the results of Section 3 to illuminate a bit of the structure of $\in$-embeddings. We now review some notation and results from [9]. $f : x \overset{\in}{\hookrightarrow} y$, or f is an *$\in$-embedding* from x to y, means that f is a monomorphism from $\langle x, \in\rangle$ to $\langle y, \in\rangle$, where the $\in$ relation is assumed restricted as appropriate.

Suppose $g : x \to V$ where x is transitive. Define $g_+ : x \to V$ by $g_+(u) = g_+``u \cup g(u)$. The following generalizes results in Section 5 of [9]. There the author was concerned with hereditarily finite functions but the proof uses $\in$-induction so applies to all well-founded sets.

Theorem 6.1 *Suppose x is transitive and $f : x \to V$. Then f is an $\in$-embedding iff $f = g_+$ for some $g : x \to V$ such that*

$$\forall u \in x\, (g(u) \cap g_+``x = 0). \tag{1}$$

Furthermore, for g and h satisfying (1), $g_+ = h_+$ *iff* $g = h$.

In [9] it was pointed out that the function $x \mapsto a + x$ is an $\in$-embedding, of form g_+ where g is the constant function to a. Likewise $x \mapsto a +_z x$ is an $\in$-embedding obtained from $g(0) = a$, $g(u) = 0$ for $u \neq 0$.

6.1 can be reformulated in graph terms thus:

Theorem 6.2 *Let $g : \mathrm{TC}(x) \cup \{x\} \to V$, and let G_+ be the graph obtained by attaching the weight $G(g(u))$ to each node u of $G(x)$. Then*:

(i) *G_+ is well-founded iff $\forall u \leq x\, (g(u) \cap g_+``x = 0)$.*

(ii) *Suppose $\forall u \leq x\, (g(u) \cap g_+``x = 0)$. Then for each $u \leq x$, G_+u is a picture of $g_+(u)$.*

Notice that $\forall u \leq x\, (g(u) \cap g_+``x = 0)$ is just (1) with x replaced by $\mathrm{TC}(x) \cup \{x\}$.

So under the conditions of (ii), G_+ is a picture of $g_+(x)$ with each node u of $G(x)$ being assigned (when considered as a node of G_+) the set $g_+(u)$, and the embedding of $G(x)$ into G_+ is a graph monomorphism.

In these circumstances, we call g a *weighting function*. Aczel has a similar construction [1, pp. 14ff], with what we call a weight being called a *set of labels*. In the presence of the Anti-Foundation Axiom, any weighting function (not necessarily satisfying (1)) gives rise to a set.

Again adapting arguments from [9, Lemma 4.7 and Theorem 4.8] to the infinite case, one can easily produce many weighting functions satisfying (1), and thereby $\in$-embeddings. Thus if $f : x \overset{\in}{\hookrightarrow} V$, x is transitive, and $y \subseteq x$ (so that $x \cup \{y\}$ is transitive), then there is a proper class of extensions of f to embeddings $x \cup \{y\} \overset{\in}{\hookrightarrow} V$. For the rest of this section, we restrict our attention to some of the simpler ones.

Let x be transitive. We shall say that a function $g : x \to V$ is a *small weighting function* if $\forall u \in x\; g(u) \subseteq \mathrm{TC}(g(0))$. It is easy to see that a small weighting function satisfies (1). In this situation, we shall call the corresponding g_+ a *small $\in$-embedding*. For a given set a, what can we say about the small $\in$-embeddings f such that $f(0) = a$? The functions $x \mapsto a + x$ and $x \mapsto a +_z x$ are among them and the latter is the "smallest" of them in a sense which is made precise in this generalization of Theorem 3.4:

Theorem 6.3 *Let a be a set and $f : x \to V$ with x transitive. Then f is a small $\in$-embedding with $f(0) = a$ iff for each $u \leq x$, $G(f(u))$ is isomorphic to $G(a +_z u) \cup H$ for some subgraph H of $K(a +_z (0, u], \mathrm{TC}(a))$.*

P r o o f. Suppose f has small weighting function g and $f(0) = a$. Inductively, let $b \in x$ and suppose that for each $u \in b$, $G(f(u))$ is isomorphic to

$$J(u) = G(a +_z u) \cup H(u)$$

with $H(u)$ a subgraph of $K(a +_z (0, u], \mathrm{TC}(u))$, and that when this latter graph pictures $f(u)$, the node $a +_z v$ is assigned the set $f(v)$. We can then form a picture of $f(b)$ as follows. Take the union $\bigcup_{u \in b} J(u)$ which is equal to $G(a +_z (0, b)) \cup H^*$ where H^* is a subgraph of $K(a +_z (0, b), \mathrm{TC}(a))$. Now add a new point P and new edges from P to $f(u)$ for each $u \in b$ and from P to each element of $g(b)$. The point P must be assigned the set $f"b \cup g(b) = f(b)$. This picture of $f(b)$ is still extensional and so is isomorphic to $G(f(b))$. It has the form required, $G(a +_z b) \cup H(b)$, where $H(b)$ is H^* with all the new edges from the point to the elements of $g(b)$ added in so is a subgraph of $K(a +_z (0, b], \mathrm{TC}(a))$, since g is small.

Conversely, suppose for each $u \leq x$, $G(f(u))$ is isomorphic to $J = G(a +_z u) \cup H(u)$ with $H(u)$ a subgraph of $K(a +_z (0, u], \mathrm{TC}(a))$. Show using induction on $u \leq x$ that, when J is a picture of $G(f(u))$, the node $a +_z u$ is assigned the set $f(u)$, and that if $g(u)$ is the set of those children, in J, of the node $a +_z u$ which are born of edges in the festoon, i.e., are not of form $a +_z v$, then g is a small weighting function and $f = g_+$. □

Thus if f is a small $\in$-embedding then $G(f(b))$ is isomorphic to the concatenation of $G(f(0))$ and $G(b)$ festooned in a similar way to what we saw with the case $x \mapsto a + x$ in Section 3. Since no new nodes are added, we get:

Corollary 6.4 *If $f : x \overset{\in}{\hookrightarrow} V$ is a small $\in$-embedding with x transitive then for all $u \in x$, $|\mathrm{TC}(f(u))| = |\mathrm{TC}(f(0))| + |\mathrm{TC}(u)|$.*

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