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Addition and multiplication of sets

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Ordinal addition and multiplication can be extended in a natural way to all sets. I survey the structure of the sets under these operations. In particular, the natural partial ordering associated with addition of sets is shown to be a tree. This allows us to prove that any set has a unique representation as a sum of additively irreducible sets, and that the non-empty elements of any model of set theory can be partitioned into infinitely many submodels, each isomorphic to the original model. Also any model of set theory has an isomorphic extension in which the empty set of the original model is non-empty. Among other results, the relations between the arithmetical operations and the transitive closure and the adductive hierarchy are elucidated.

1 Introduction

The binary addition and multiplication operations on von Neumann ordinals have natural extensions to the universe of all sets of set theory. These universal operations $x + y$ and $x \cdot y$ on sets can be defined inductively on y by

$$x + y = x \cup \{x + z \mid z \in y\} \quad \text{and} \quad x \cdot y = \{x \cdot q + r \mid q \in y \wedge r \in x\}.$$

These definitions were first given by Alfred Tarski and Dana Scott, respectively.

The universal addition and multiplication are associative, and satisfy other algebraic properties such as the left cancellation and left distributive properties. Of course neither addition nor multiplication of ordinals is commutative; this non-commutativity extends, for the universal operations, to the finite sets.

Moreover, addition and multiplication of sets preserve cardinalities and ranks in the following sense: let $|x|$ denote the cardinality of the set x. I shall show that

$$|x + y| = |x| +_{\mathrm{c}} |y| \quad \text{and} \quad |x \cdot y| = |x| \cdot_{\mathrm{c}} |y|$$

where $+_{\mathrm{c}}$ and $\cdot_{\mathrm{c}}$ denote cardinal addition and cardinal multiplication. And if $\varrho(x)$ is the rank of x in the cumulative hierarchy, then

$$\varrho(x + y) = \varrho(x) + \varrho(y) \quad \text{and} \quad \varrho(x \cdot y) = \varrho(x) \cdot \varrho(y).$$

I shall show that the preservation of cardinalities extends to transitive closures:

$$|\mathrm{TC}(x + y)| = |\mathrm{TC}(x)| +_{\mathrm{c}} |\mathrm{TC}(y)| \quad \text{and} \quad |\mathrm{TC}(x \cdot y)| = |\mathrm{TC}(x)| \cdot_{\mathrm{c}} |\mathrm{TC}(y)|.$$

The universal $+$ induces a partial ordering of the universe of sets: $x \trianglelefteq y \leftrightarrow \exists z(x + z = y)$. I shall show that this partial ordering is a tree, and hence that any set has a unique representation as the sum of an ordinal-length sequence of additively irreducible sets. Consequences include a way of partitioning the non-empty sets of any

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model of set theory into infinitely many disjoint submodels, each of which is isomorphic to the original model (Theorem 3.21). In Section 4 submodels are produced which are isomorphic to the original model even in the extended language which includes a symbol for addition.

Because any model of set theory is generated in a straightforward way from its additively irreducible sets, the concept of set addition endows models with an unexpected regularity which adds interest to the open problem of characterizing the additively irreducible sets in terms of the membership relation, as well as some other questions discussed at the end of Section 4 below. How far does elementary number theory, or the arithmetic of the ordinals, carry across to the sets? And perhaps some statement about all sets could be proven by proving it for the additive irreducibles and showing closure under addition?

The definitions and results of this paper do not use the axiom of infinity, and are framed so that they apply equally well to Zermelo-Fraenkel set theory and to the theory of the hereditarily finite sets, with the exception of Section 5 which supposes the negation of the axiom of infinity. In this last section I shall show that addition of sets preserves adductive ranks, but multiplication only partially does so.

1.1 Historical remarks

Since first obtaining these results I have learned from Dana Scott that the definition of addition of sets was stated by Tarski in 1955 [15], and Scott (unpublished) discovered the definition of multiplication around that time. Scott proved many of the properties of these operations laid out in Sections 3 and 4 below. The core of the results of Section 3 are proven in a typescript of Scott from about 1965/66. This draft chapter of an unpublished book on set theory by Montague, Scott, and Tarski also shows that addition of sets is equivalent to a special case of ordinal addition of relation types in the sense of Tarski [16], whence follow some of the general algebraic properties of addition of sets.

Later, but independently, Narciso Garcia rediscovered the arithmetical operations on sets. In [5] he gives without proofs many of the basic algebraic properties. In [6], [7], and his thesis [4] he extends these notions to study generalized sums, generalized products, and exponentiation, which had also been found earlier by Scott (unpublished), and which I do not cover in the present article.

I showed how to define set addition and multiplication and prove their properties in the case of the theory of the hereditarily finite sets in [10], and some of the earlier development below is a generalization of results in [10].

1.2 Other versions of arithmetic operations on sets

For the hereditarily finite sets, there exist alternative notions of addition and multiplication of sets constituting an inverse of Ackermann's interpretation [1] of finite set theory in arithmetic. Those definitions (their existence was conjectured by E. W. Beth [2] and established in 1964 by Jan Mycielski [11]) are not equivalent to the definitions studied in this paper. In particular, Mycielski's operations are commutative, and the corresponding ordering is total, while our $\trianglelefteq$ defined above is partial.

Yet another kind of addition of sets, also due to Scott, is a "game addition" defined inductively by

$$x \oplus y = \{u \oplus y \mid u \in x\} \cup \{x \oplus v \mid v \in y\}.$$

This commutative operation is a special case of addition of games in the sense of Berlekamp, Conway and Guy (see [3, p. 32]).

A simple example illustrates how the various kinds of addition differ: let $+_{\mathrm{m}}$ denote Mycielski's addition, and let 1 as usual denote the von Neumann ordinal $\{0\}$. Then $\{1\} +_{\mathrm{m}} 1 = \{0, 1\} = 2$, whereas $\{1\} \oplus 1 = \{\{1\}\}$, and for the addition which will henceforth be the concern of this paper, $\{1\} + 1 = \{1, \{1\}\}$.

2 Preliminaries

My starting point is Zermelo-Fraenkel set theory ZF without Axiom of Choice. Ordinals will be the usual von Neumann ordinals. The empty set will be denoted by 0, and the set-theoretic difference by $x \setminus y$.

Except in the last section, the results will apply equally well both to ZF and to Finite Set Theory which is ZF with the axiom of infinity replaced by its negation. For this reason the base theory will be the common ground of these theories which is ZF $\setminus \{\mathrm{Inf}\}$, viz. ZF with the axiom of infinity omitted.

Finite Set Theory is equivalent (see [12, 10]) to Flavio Previale's theory PS, a set-theoretic analogue of first order Peano arithmetic PA. PS may be formulated simply [10] in a language with a symbol for *the binary adduction operator* $[x; y]$, whose interpretation in terms of the usual language for set theory is $[x; y] = x \cup \{y\}$. So if α is an ordinal, $\alpha + 1 = [\alpha; \alpha]$. For convenience, this notation is extended to

$$[x; y_1, \dots, y_n] = [\dots[[x; y_1]; y_2]; \dots; y_n] = x \cup \{y_1, y_2, \dots, y_n\}.$$

The axioms of PS are the universal closures of

$$[0; x] \neq 0, \quad [x; y, y] = [x; y], \quad [x; y, z] = [x; z, y], \quad [x; y, z] = [x; y] \leftrightarrow [x; z] = x \vee z = y$$

together with the induction schema

$$\varphi(0) \wedge \forall xy(\varphi(x) \wedge \varphi(y) \to \varphi([x; y])) \to \forall x \varphi(x)$$

for first order φ with parameters. This form of induction was introduced by Steven Givant and Alfred Tarski [8] in 1977, modifying an earlier form given in 1924 by Tarski [14]. It was foreshadowed in discussions of finiteness by Zermelo [19] and Whitehead and Russell [18]. S. Świerczkowski [13] presents detailed development of a theory HF which is similar to, and equivalent to, PS and is adapted from Tarski and Givant [17].

The membership relation and the adduction operator can each be defined in terms of the other, so we may consider ourselves as working in a language based on either of these; I shall avail myself of both although I shall stick to the usual language with $\in$. If $\mathcal{M}$ is a structure for set theory, I shall write M for its domain.

The ordinals of a model of PS are convertible into a model of PA with $[x; x]$ as successor function, and conversely any model of PA gives rise via Wilhelm Ackermann's [1] coding of sets to a corresponding model of PS.

$\in$-induction is a theorem of both PS and ZF (for PS see [12], also [13]), therefore of ZF $\setminus \{\mathrm{Inf}\}$:

Theorem 2.1 ($\in$-induction) *For first order φ with parameters,*

$$\mathsf{ZF} \setminus \{\mathrm{Inf}\} \vdash \forall x((\forall y \in x)\varphi(y) \to \varphi(x)) \to \forall x \varphi(x).$$

So proofs and definitions by $\in$-induction are a convenient way to establish properties of both ZF and PS at the same time. A special case of $\in$-induction is induction by rank in the cumulative hierarchy.

$x < y$ denotes the statement that x is an element of $\mathrm{TC}(y)$, the transitive closure of y. Recall that $\mathrm{TC}(y)$ is defined $\in$-inductively by $\mathrm{TC}(y) = y \cup \bigcup\{\mathrm{TC}(u) \mid u \in y\}$. It will be helpful to bear in mind that $x < y$ iff for some $z \in y$, $x \leq z$, and that by the foundation axiom $x < y$ implies $x \not\supseteq y$.

Section 3 will consider addition of sets and use it to establish that the non-empty elements of any model $\mathcal{M}$ of ZF $\setminus \{\mathrm{Inf}\}$ can be partitioned into infinitely many submodels each isomorphic to $\mathcal{M}$, that the partial order on $\mathcal{M}$ associated with addition is a tree, and that any element of $\mathcal{M}$ can be uniquely expressed as the sum of a sequence of additively irreducible sets.

Section 4 will study multiplication. The main work here will be in showing that multiplication preserves cardinalities and that the left cancellation property holds for multiplication. Multiplication also furnishes submodels of $\mathcal{M}$ which are isomorphic to $\mathcal{M}$ not only with regard to the membership relation, but also with regard to addition.

Section 5 is about the finite case and shows that addition preserves ranks in the adductive hierarchy, but multiplication only partially does so.

3 Addition of sets

In this section and the next, the base theory will be ZF $\setminus \{\mathrm{Inf}\}$. In [10], a definition of addition of sets was given for the finite case (PS), and Definition 3.1 is equivalent, in the finite case, to the definition given there. Likewise the basic properties of addition, Propositions 3.3, 3.4, and 3.5 proven below, generalize the results of [10, §5] from PS to ZF $\setminus \{\mathrm{Inf}\}$.

The *sum* $x+y$ of two sets is conveniently defined in an $\in$-inductive definition on y jointly with another binary operation, the *lift*, denoted $\lambda_x(y)$:

Definition 3.1 $\lambda_x(y) = \{x+z \mid z \in y\}$ and $x+y = x \cup \lambda_x(y)$.

Here are some immediate facts:

Lemma 3.2

(i) $\lambda_x(\bigcup y) = \bigcup \lambda_x(y)$. *In particular,* $\lambda_x(y) \cup \lambda_x(z) = \lambda_x(y \cup z)$.

(ii) $\lambda_x(0) = 0$, $x+0 = x$, $\lambda_x(1) = \{x\}$, *and* $x+1 = [x;x]$.

(iii) $x + [y;z] = [(x+y);(x+z)]$.

(iv) *If* x *and* y *are ordinals, then* $x+y$ *under this definition agrees with the usual addition of ordinals.*

As is well-known, addition of ordinals is not commutative. I pointed out in [10] that the non-commutativity of addition of sets is manifested even in the finite sets, for example $1 + \{1\} \neq \{1\} + 1$.

Proposition 3.3 *The universal closures of the following are provable in* $\mathsf{ZF} \setminus \{\mathrm{Inf}\}$*:*

(i) $0 + x = \lambda_0(x) = x$.

(ii) $x + (y+z) = (x+y)+z$.

(iii) $x + y \not< x$.

(iv) $\mathrm{TC}(x) \cap \lambda_x(y) = 0$. (*Hence* $x \cap \lambda_x(y) = 0$.)

(v) $\lambda_x(\lambda_y(z)) = \lambda_{x+y}(z)$.

Proof.

(i) By $\in$-induction on x.

(ii) By $\in$-induction on z:

$$\begin{aligned}(x+y)+z &= (x+y) \cup \lambda_{x+y}(z)\\ &= (x+y) \cup \{(x+y)+u \mid u \in z\}\\ &= x \cup \lambda_x(y) \cup \{x+(y+u) \mid u \in z\} && \text{(using induction hypothesis)}\\ &= x \cup \lambda_x(y) \cup \{y+u \mid u \in z\} && \text{(by Lemma 3.2(i))}\\ &= x \cup \lambda_x(y+z) = x+(y+z).\end{aligned}$$

(iii) By $\in$-induction on y. The foundation axiom gives the case $y = 0$. If $y \neq 0$, let $z \in y$. Then $x+z \in x+y$, so if $x+y < x$, then $x+z < x$ contradicting the induction hypothesis.

(iv) is immediate from (iii), and (v) follows from (ii). □

We can now prove the left cancellation property of addition:

Proposition 3.4 $\mathsf{ZF} \setminus \{\mathrm{Inf}\}$ *proves the universal closures of the following:*

(i) $\lambda_x(y) = \lambda_x(z) \leftrightarrow y = z$.

(ii) $x+y = x+z \leftrightarrow y = z$.

(iii) $x+y \in x+z \leftrightarrow y \in z$.

Proof. For any particular x and y, (ii) follows from (i) and Proposition 3.3(iv). To prove (i), fix x and prove by $\in$-induction on y that $\forall z(\lambda_x(y) = \lambda_x(z) \rightarrow y = z)$. The case $y = 0$ is immediate from Lemma 3.2(ii). If $y \neq 0$ and $\lambda_x(y) = \lambda_x(z)$, take an arbitrary $u \in y$. Then $x+u \in \lambda_x(z)$ so there must exist $v \in z$ such that $x+u = x+v$. By induction hypothesis $u = v$ and hence $u \in z$. Thus $y \subseteq z$, and similarly $z \subseteq y$.

For (iii), the implication from right to left is in the definition of addition. If $x+y \in x+z = x \cup \lambda_x(z)$, then by Proposition 3.3(iii), $x+y \in \lambda_x(z)$. As in the proof of (i), it follows that $y \in z$. □

It is now apparent, from Proposition 3.4 and Proposition 3.3(iv), that addition preserves cardinalities:

Proposition 3.5 $\mathsf{ZF} \setminus \{\mathrm{Inf}\}$ *proves the universal closures of*

(i) $|\lambda_x(y)| = |y|$,

(ii) $|x+y| = |x| +_{\mathrm{c}} |y|$.

We now look at the transitive closure of the sum.

Proposition 3.6 $\mathrm{TC}(x+y) = \mathrm{TC}(x) \cup \lambda_x(\mathrm{TC}(y))$.

This proposition can be expressed more familiarly as

$$z < x + y \leftrightarrow z < x \vee (\exists v < y)(z = x + v).$$

Proof of Proposition 3.6. By $\in$-induction on y:

$$\begin{aligned}\mathrm{TC}(x+y) &= (x+y) \cup \textstyle\bigcup_{z \in x+y} \mathrm{TC}(z)\\ &= x \cup \lambda_x(y) \cup \textstyle\bigcup_{z\in x} \mathrm{TC}(z) \cup \bigcup_{u \in y} \mathrm{TC}(x+u)\\ &= x \cup \textstyle\bigcup_{z\in x} \mathrm{TC}(z) \cup \lambda_x(y) \cup \bigcup_{u\in y}(\mathrm{TC}(x) \cup \lambda_x(\mathrm{TC}(u)))\\ &= \mathrm{TC}(x) \cup \lambda_x(y) \cup \textstyle\bigcup_{u\in y} \lambda_x(\mathrm{TC}(u))\\ &= \mathrm{TC}(x) \cup \lambda_x(y \cup \textstyle\bigcup_{u\in y} \mathrm{TC}(u)) \qquad \text{(using Lemma 3.2(i))}\\ &= \mathrm{TC}(x) \cup \lambda_x(\mathrm{TC}(y)).\end{aligned}$$ □

Corollary 3.7 $|\mathrm{TC}(x+y)| = |\mathrm{TC}(x)| +_{\mathrm{c}} |\mathrm{TC}(y)|$.

Proof. This follows from Proposition 3.6 by virtue of Proposition 3.3(iv) and Proposition 3.5(i). □

The next corollary of Proposition 3.6 will be used in Section 4:

Corollary 3.8 *If* $y \neq 0$*, then* $\mathrm{TC}(\lambda_x(y)) = \mathrm{TC}(x) \cup \lambda_x(\mathrm{TC}(y))$.

Proof.

$$\begin{aligned}\mathrm{TC}(\lambda_x(y)) &= \lambda_x(y) \cup \textstyle\bigcup_{u\in y} \mathrm{TC}(x+u)\\ &= \lambda_x(y) \cup \textstyle\bigcup_{u\in y}(\mathrm{TC}(x) \cup \lambda_x(\mathrm{TC}(u)))\\ &= \mathrm{TC}(x) \cup \lambda_x(\mathrm{TC}(y)).\end{aligned}$$ □

If $\mathcal{M} = \langle M, \in\rangle \vDash \mathsf{ZF} \setminus \{\mathrm{Inf}\}$ and $a \in M$, let $\lambda_a(M)$ denote $\{a + x \mid x \in M\}$ and $\lambda_a(\mathcal{M}) = \langle \lambda_a(M), \in\rangle$, where by a slight abuse of notation that last "$\in$" stands for the restriction of $\mathcal{M}$'s membership relation to $\lambda_a(M)$. (I shall persist in similar slight abuses below.) Proposition 3.4 shows that:

Proposition 3.9 *The function* $x \longmapsto a + x$ *is an isomorphism between* $\mathcal{M}$ *and* $\lambda_a(\mathcal{M})$.

Note that for $a \neq 0$, $\lambda_a(M)$ is a proper subset of M, so that $\mathcal{M}$ is isomorphic to a proper submodel of itself, via an isomorphism which is definable in $\mathcal{M}$. This holds even when $\mathcal{M}$ is a standard model of ZF, or the standard model $\langle V_\omega, \in\rangle$ of PS, in which case it is tempting to call $\lambda_a(\mathcal{M})$ a *substandard model*.

Further, by identifying $\mathcal{M}$ with $\lambda_a(\mathcal{M})$, we have an isomorphism between $\mathcal{M}$ and an *inward extension* of $\mathcal{M}$ in which the empty set of $\mathcal{M}$ becomes non-empty in the extension. This shows that in any model of set theory the empty set should be considered as provisional in nature, in as much as the model has extensions in which its (originally) empty set has elements, indeed as many elements as any given set of the original model.

It is worth pointing out that when $a \neq 0$, although $\lambda_a(M)$ is closed under addition in the sense of $\mathcal{M}$, the function of Proposition 3.9 is *not* an isomorphism between $\langle M, +^{\mathcal{M}}\rangle$ and $\langle \lambda_a(M), +^{\mathcal{M}}\rangle$. In other words, $+^{\lambda_a(\mathcal{M})}$ is not the restriction of $+^{\mathcal{M}}$ to $\lambda_a(M)$ because, working in $\mathcal{M}$, $(a + x) + (a + y) \neq a + (x + y)$. In Section 4 we shall see a submodel which is isomorphic for both $\in$ and $+$.

I denote by $\varrho(x)$ *the rank of* x in the usual cumulative hierarchy, so $\varrho(x) = \sup\{\varrho(y) + 1 \mid y \in x\}$. Addition also preserves ranks:

Proposition 3.10 ($\mathsf{ZF} \setminus \{\mathrm{Inf}\}$) $\varrho(x + y) = \varrho(x) + \varrho(y)$.

Proof. We may assume $y \neq 0$, and $\in$-inductively on y,

$$\varrho(x+y) = \sup\{\varrho(u) + 1 \mid u \in x \cup \{x + z \mid z \in y\}\} = \sup\{\varrho(x) + \varrho(z) + 1 \mid z \in y\}.$$ □

There is a natural partial order associated with addition:

Definition 3.11 $x \trianglelefteq y \leftrightarrow \exists z(x + z = y)$.

Equivalently, $x \trianglelefteq y \leftrightarrow y \in \lambda_x(M)$. From the associativity of addition one sees that:

Lemma 3.12

(i) $\trianglelefteq$ *is transitive.*

(ii) $x \trianglelefteq y \leftrightarrow \lambda_y(M) \subseteq \lambda_x(M)$.

x and y are $\trianglelefteq$-*incomparable* if neither $x \trianglelefteq y$ nor $y \trianglelefteq x$.

Lemma 3.13 (ZF $\setminus \{\text{Inf}\}$) *For any a, b, c, d, if $a + b = c + d$, then a and c are $\trianglelefteq$-comparable, i. e., $a \trianglelefteq c$ or $c \trianglelefteq a$.*

P r o o f. Assume that a and c are $\trianglelefteq$-incomparable. I prove by $\in$-induction on x that the formula

$$\varphi(x) : \forall z(a + x \neq c + z)$$

holds for all x. $\varphi(0)$ follows from the assumption. So suppose $x \neq 0$ and

(1) $$(\forall y \in x)\varphi(y).$$

We need to show $\varphi(x)$. Assume, hoping for a contradiction, that for some z,

(2) $$a + x = c + z.$$

Since a and c are $\trianglelefteq$-incomparable, $z \neq 0$. Also $x \neq 0$ so choose $v \in x$. It follows that

$$a + v \in a + x = c + z = c \cup \lambda_c(z).$$

C a s e 1: $a + v \in \lambda_c(z)$. Then for some $w \in z$, $a + v = c + w$, contradicting (1).

C a s e 2: $a + v \in c$. First I note that $a \subseteq c$. Let $u = x \setminus \{v\}$. So $x = [u; v]$. By (2), $a \cup \lambda_a(x) = c \cup \lambda_c(z)$. So

$$\lambda_c(z) \subseteq \lambda_a(x) = [\lambda_a(u); (a + v)].$$

Since $a + v \in c$ and $c \cap \lambda_c(z) = 0$, we have $\lambda_c(z) \subseteq \lambda_a(u)$. Choose $e \in z$. Then $c + e \in \lambda_c(z) \subseteq \lambda_a(u)$. Hence for some $y \in u$, $c + e = a + y$, contradicting (1). □

Lemma 3.13 can be restated:

Proposition 3.14 *If x and y are $\trianglelefteq$-incomparable elements of $\mathcal{M} \models$ ZF $\setminus \{\text{Inf}\}$, then $\lambda_x(M) \cap \lambda_y(M) = 0$.*

Proposition 3.15 ZF $\setminus \{\text{Inf}\}$ *proves the universal closures of*

(i) $x \trianglelefteq y \to x \subseteq y$,

(ii) $x \trianglelefteq y \to x \leq y$.

P r o o f. $x \subseteq x + z$ is immediate from the definition of $+$, and $x \leq x + z$ follows from Proposition 3.6. □

Example 3.16 The relation $\trianglelefteq$ is stronger than the conjunction of $\subseteq$ and $\leq$: let $a = [3; [0; 1]] = \{0, 1, 2, \{1\}\}$. Then $2 \subset a$ and $2 < a$ but $2 \ntrianglelefteq a$.

On the other hand, when restricted to the ordinals the relations $\in$, $\subset$, $<$ and $\triangleleft$ all agree.

Proposition 3.17 (ZF $\setminus \{\text{Inf}\}$) *For any x, the $\trianglelefteq$-predecessors of x are well ordered by $\trianglelefteq$. Thus*

$$\textit{if } \mathcal{M} \models \text{ZF} \setminus \{\text{Inf}\}, \textit{ then } \langle M, \trianglelefteq^{\mathcal{M}} \rangle \textit{ is a tree.}$$

P r o o f. It follows from Lemma 3.13 that the $\trianglelefteq$-predecessors of x are totally ordered. By Proposition 3.15(ii), this order, like $<$, is well founded. □

Definition 3.18 x is *additively irreducible* if $x \neq 0$ and

$$\forall yz(x = y + z \to y = 0 \lor z = 0).$$

Any two distinct additively irreducible sets are $\trianglelefteq$-incomparable.

Proposition 3.19 $(\mathsf{ZF}\setminus\{\mathrm{Inf}\})$ *For all $x \neq 0$ there exist unique y and z such that $x = y + z$ and y is additively irreducible.*

Proof. y is the $\trianglelefteq$-minimum element of $\{u \mid u \trianglelefteq x \wedge u \neq 0\}$. z is unique by left cancellation (Proposition 3.4(ii)). □

If $(a_\xi)_{\xi<\beta}$ is a sequence of sets, we can define the sum $\Sigma_{\xi<\beta} a_\xi$ of the sequence by taking unions at limit stages. The well ordered $\trianglelefteq$-predecessors of a non-empty a can be used to write a as such a sum of additively irreducible sets, with Lemma 3.13 guaranteeing the uniqueness of this sum.

Proposition 3.5 can be extended to:

Proposition 3.20 *For any sequence $(a_\xi)_{\xi<\beta}$, $|\Sigma_{\xi<\beta} a_\xi| = \Sigma_{\xi<\beta} |a_\xi|$.*

I now summarize the preceding development:

Theorem 3.21 *Let $\mathcal{M} \vDash \mathsf{ZF} \setminus \{\mathrm{Inf}\}$.*

(i) *$\langle M, \trianglelefteq^{\mathcal{M}} \rangle$ is a tree. The immediate successors of a in the tree are the infinitely many elements $a + b$ with b additively irreducible.*

(ii) *$\{\lambda_a(\mathcal{M}) \mid a$ is an additively irreducible element of $\mathcal{M}\}$ partitions $M \setminus \{0\}$ into infinitely many disjoint isomorphic copies of $\mathcal{M}$.*

(iii) *Any element a of M is uniquely expressible in $\mathcal{M}$ as a sum $a = \Sigma_{\xi<\beta} a_\xi$, where each a_ξ is additively irreducible.*

Definition 3.22 For any set a let $a = \Sigma_{\xi<\delta(a)} a_\xi$ be its unique decomposition into additive irreducibles. The ordinal $\delta(a)$ is *the degree of additive reducibility of a*.

$\delta(a)$ can also be characterized as the level of a in the tree $\langle M, \trianglelefteq^{\mathcal{M}} \rangle$. Scott called this ordinal *the length of a*. a is additively irreducible iff $\delta(a) = 1$, and it is easy to see that:

Proposition 3.23 *$\delta(a + b) = \delta(a) + \delta(b)$, and more generally $\delta(\Sigma_{\xi<\beta} a_\xi) = \Sigma_{\xi<\beta} \delta(a_\xi)$.*

In view of all this it would be interesting to provide a characterization of the additively irreducible sets. Some remarks in this direction now ensue.

By Proposition 3.5(ii), all singletons are additively irreducible, a fact which also follows from the next proposition if we recall that $\{x\} = \lambda_x(1)$.

Proposition 3.24 *If $x \neq 0$ and $y \neq 0$, then $\lambda_x(y)$ is additively irreducible.*

Proof. Suppose $\lambda_x(y) = s + t$, where $s, t \neq 0$. Then $s \subseteq \lambda_x(y)$ and so by Proposition 3.3(iv), $s \cap x = 0$. Now pick $w \in t$. Then $s + w \in \lambda_s(t) \subseteq \lambda_x(y)$. Therefore $s + w = x + v$ for some $v \in y$. By Lemma 3.13, $s \trianglelefteq x \vee x \trianglelefteq s$. But if neither s nor x is empty and one is a subset of the other, their intersection cannot be empty, a contradiction. □

Not every additively irreducible set is of the form $\lambda_x(y)$ with $x \neq 0$: consider $\{0, \{1\}\}$. The only additively irreducible ordinal is 1: of course, for any infinite ordinal α, $\alpha = 1 + \alpha$. The decomposition of the ordinal α given by Theorem 3.21(iii) is simply $\alpha = \Sigma_{\xi<\alpha} 1$. In particular, all non-zero ordinals are in $\lambda_1(M)$.

Proposition 3.25 $(\mathsf{ZF} \setminus \{\mathrm{Inf}\})$ *If $a \neq 1$ is a non-empty set of ordinals, then a is additively irreducible iff $0 \notin a$.*

Proof. On the one hand, if $0 \in a$, then $a = 1 + \{\xi^- \mid \xi \in a \setminus \{0\}\}$, where ξ^- is defined to be $\xi - 1$ if $\xi \neq 0$ is finite and ξ if ξ is infinite. And on the other hand, if $a = b + c$ with $b, c \neq 0$, then I shall prove that b is transitive from which it will follow that $0 \in b \subseteq a$. So suppose b is not transitive. There must be some $\xi \in \eta \in b$ such that $\xi \notin b$. Choose $x \in c$. Then

$$b + x \in \lambda_b(c) \subseteq a$$

so $b + x$ is an ordinal and in particular is transitive. But $\xi \in \mathrm{TC}(b)$ and hence by Proposition 3.3(iv), $\xi \notin \lambda_b(x)$. So $\xi \notin b + x$. Since $\xi \in \eta \in b + x$ we have shown that $b + x$ is not transitive, a contradiction. □

The proof of Proposition 3.25 shows that any non-empty set a of ordinals can be decomposed as $a = \beta + c$, where β is an ordinal and either $c = 0$ or c is an additively irreducible set of ordinals with $0 \notin c$. Also the conclusion of Proposition 3.25 holds if $a \subseteq 1 \cup \lambda_1(M)$, $a \neq 0$, $a \neq 1$, and every element of a is transitive.

A final remark on the difficulty of enumerating the additively irreducible sets. There can be no ZF-definable well-ordering of the class of additively irreducible sets, since every set is a sum of a sequence of additive irreducibles and so the lexicographic ordering on these sequences would induce a definable well-ordering of the universe.

4 Multiplication of sets

Definition 4.1 The *product* $x \cdot y$ of two sets is defined $\in$-inductively on y by

$$x \cdot y = \bigcup\{\lambda_{x \cdot u}(x) \mid u \in y\}.$$

Equivalently, $x \cdot y = \{x \cdot u + r \mid u \in y \wedge r \in x\}$. As with addition in the preceding section, this definition generalizes the definition given for the finite case in [10], and Lemma 4.2 and Proposition 4.3 generalize the basic properties of multiplication proven for PS in [10] to ZF $\setminus \{\mathrm{Inf}\}$.

Lemma 4.2

(i) $x \cdot 0 = 0$, $x \cdot 1 = x$, $x \cdot \{1\} = \lambda_x(x)$, *and* $x \cdot 2 = x + x$.

(ii) $x \cdot [y; z] = (x \cdot y) \cup \lambda_{x \cdot z}(x)$.

(iii) $x \cdot (y + 1) = x \cdot y + x$.

(iv) *If x and y are ordinals, then $x \cdot y$ under this definition agrees with the usual product of ordinals.*

Proposition 4.3 *The universal closures of the following are provable in* ZF $\setminus \{\mathrm{Inf}\}$*:*

(i) $0 \cdot x = 0$.

(ii) $1 \cdot x = x$.

(iii) $x \cdot \bigcup y = \bigcup(x \cdot y)$. *In particular,* $x \cdot (y \cup z) = x \cdot y \cup x \cdot z$.

(iv) $x \cdot \lambda_y(z) = \lambda_{x \cdot y}(x \cdot z)$.

(v) *Left distributivity:* $x \cdot (y + z) = x \cdot y + x \cdot z$.

(vi) *Multiplication is associative.*

Proof. (i) and (ii) are straightforward $\in$-inductions, and (iii) is immediate from the definition.

To prove (iv) and (v), first note that for any particular x, y, z, (iv) implies (v) because by (iii),

$$x \cdot (y \cup \lambda_y(z)) = x \cdot y \cup x \cdot \lambda_y(z)$$

and given (iv) this latter set is equal to $x \cdot y \cup \lambda_{x \cdot y}(x \cdot z)$. (iv) is now proven by $\in$-induction on z:

$$\begin{aligned} x \cdot \lambda_y(z) &= \bigcup\{\lambda_{x \cdot u}(x) \mid u \in \lambda_y(z)\} \\ &= \bigcup\{\lambda_{x \cdot (y+v)}(x) \mid v \in z\} \\ &= \bigcup\{\lambda_{x \cdot y + x \cdot v}(x) \mid v \in z\} && \text{(by induction hypothesis)} \\ &= \bigcup\{\lambda_{x \cdot y}(\lambda_{x \cdot v}(x)) \mid v \in z\} && \text{(by Proposition 3.3(v))} \\ &= \lambda_{x \cdot y}(\bigcup\{\lambda_{x \cdot v}(x) \mid v \in z\}) && \text{(by Lemma 3.2(i))} \\ &= \lambda_{x \cdot y}(x \cdot z). \end{aligned}$$

For (vi), $\in$-inductively on z,

$$\begin{aligned} (x \cdot y) \cdot z &= \bigcup\{\lambda_{(x \cdot y) \cdot u}(x \cdot y) \mid u \in z\} \\ &= \bigcup\{\lambda_{x \cdot (y \cdot u)}(x \cdot y) \mid u \in z\} && \text{(by induction hypothesis)} \\ &= \bigcup\{x \cdot \lambda_{y \cdot u}(y) \mid u \in z\} && \text{(by (iv))} \\ &= x \cdot \bigcup\{\lambda_{y \cdot u}(y) \mid u \in z\} && \text{(by (iii))} \\ &= x \cdot (y \cdot z). \end{aligned}$$

□

Multiplication of sets is not commutative even for finite sets, for example $2 \cdot \{1\} \neq \{1\} \cdot 2$.

Multiplication preserves ranks:

Proposition 4.4 (ZF $\setminus \{\mathrm{Inf}\}$) $\varrho(x \cdot y) = \varrho(x) \cdot \varrho(y)$.

Proof. Inductively on y, with the aid of Proposition 3.10:

$$\begin{aligned}\varrho(x \cdot y) &= \sup\{\varrho(x \cdot u + r) + 1 \mid u \in y \wedge r \in x\} \\ &= \sup\{\varrho(x) \cdot \varrho(u) + \varrho(r) + 1 \mid u \in y \wedge r \in x\}.\end{aligned}$$ □

Establishing the left cancellation property for multiplication, and the cardinality of the product, will need a series of results:

Lemma 4.5 (ZF $\setminus \{\mathrm{Inf}\}$) *If $y \neq 0$, then $x \leq x \cdot y$. If, further, $x \neq 0$ and $y \neq 1$, then $x < x \cdot y$.*

Proof. We may assume $x \neq 0$. Using $\in$-induction assume $x \leq x \cdot p$ for all non-empty elements p of y. The case $y = 1$ is easy, so assume $y \neq 1$ which allows us to choose a non-empty element p of y. Also choose an element r of x. Then $x \cdot p + r \in \lambda_{x \cdot p}(x) \subseteq x \cdot y$, and the result follows because $x \leq x \cdot p$ by induction hypothesis, $x \cdot p \leq x \cdot p + r$ by Proposition 3.15(ii), and $x \cdot p + r \in x \cdot y$. □

Theorem 4.6 (ZF $\setminus \{\mathrm{Inf}\}$) *Suppose $a \cdot x = a \cdot y + b$ and $b < a$. Then $b = 0$.*

Proof. We may assume $a \neq 0$ and $x \neq 0$, and also, by Lemma 4.5, $y \neq 0$ since if $y = 0$, then we would have $a \cdot x = b < a$. The proof will be by induction on rank in the cumulative hierarchy: fix a and suppose the result proven for all $x, y \in V_\alpha$. Let $x, y \in V_{\alpha+1}$ and suppose $a \cdot x = a \cdot y + b$ and $0 < b < a$. We may further suppose that for this y, x is $\leq$-minimal such that $\exists r(0 < r < a \wedge a \cdot x = a \cdot y + r)$.

Since $a \cdot y < a \cdot x$, for some element p of $a \cdot x$, $a \cdot y \leq p$, in particular $p \notin a \cdot y$. Pick $u \in x$ and $c \in a$ such that $p = a \cdot u + c$. Since $a \cdot x = a \cdot y \cup \lambda_{a \cdot y}(b)$ and $p \notin a \cdot y$, it follows that $p \in \lambda_{a \cdot y}(b)$. So for some $d \in b$,

$$p = a \cdot y + d = a \cdot u + c.$$

Lemma 3.13 informs us that $a \cdot y$ and $a \cdot u$ are $\trianglelefteq$-comparable.

Case 1: For some e, $a \cdot u = a \cdot y + e$. Then $a \cdot y + d = a \cdot y + e + c$ and by left cancellation (see Proposition 3.4(ii)), $e + c = d \in b$. Hence $e \trianglelefteq d \in b < a$, so $e < a$. Suppose that $e = 0$, i. e., $a \cdot u = a \cdot y$. Then

$$\lambda_{a \cdot y}(a) = \lambda_{a \cdot u}(a) \subseteq a \cdot x = a \cdot y \cup \lambda_{a \cdot y}(b).$$

By Proposition 3.3(iv), $\lambda_{a \cdot y}(a) \subseteq \lambda_{a \cdot y}(b)$. It follows (using left cancellation again) that $a \subseteq b$, which contradicts $b < a$. Thus $e \neq 0$.

So Case 1 entails $a \cdot u = a \cdot y + e$ and $0 < e < a$, but this contradicts the minimality supposition on x. So we must have:

Case 2: For some f, $a \cdot y = a \cdot u + f$. By left cancellation, $f + d = c$, hence $f \leq c \in a$. As in Case 1, $f \neq 0$, so $0 < f < a$. I now repeat the entire argument with x and y replaced by y and u respectively: so for this u we now suppose that y is $\leq$-minimal such that $\exists z(0 < z < a \wedge a \cdot y = a \cdot u + z)$. If not, replace y and f by y^* and f^*, with $y^* \leq y$ and y^* $\leq$-minimal satisfying this condition. The same reasoning shows that for some $v \in y^* \leq y$ and $0 < g < a$, $a \cdot u = a \cdot v + g$. Now the induction hypothesis is contradicted. □

Theorem 4.7 (ZF $\setminus \{\mathrm{Inf}\}$) *Suppose $r < a$, $s < a$, and $a \cdot x + r = a \cdot y + s$. Then $x = y$ and $r = s$.*

Proof. Fix a and hypothesize inductively that the result holds for $x, y \in V_\alpha$. Suppose $x, y \in V_{\alpha+1}$ and

$$a \cdot x + r = a \cdot y + s$$

with $r, s < a$. We may assume $x \neq 0$ and $y \neq 0$, because if (say) $x = 0$ and $y \neq 0$, then $a \cdot y + s < a$ contradicting Lemma 4.5. Let u be any element of x. Then

$$a \cdot u + r \in \lambda_{a \cdot u}(a) \subseteq a \cdot x \subseteq a \cdot y + s = a \cdot y \cup \lambda_{a \cdot y}(s).$$

C a s e 1: $a \cdot u + r \in \lambda_{a \cdot y}(s)$. Then for some $t \in s$,

$$a \cdot u + r = a \cdot y + t.$$

Now I appeal again to Lemma 3.13.

S u b c a s e 1.1: For some c, $a \cdot u = a \cdot y + c$. By left cancellation, $c + r = t$, so $c \leq t \in s < a$, hence $c < a$. By an argument parallel to that in Case 1 in the proof of Theorem 4.6, $c = 0$ is impossible because it entails

$$\lambda_{a \cdot y}(a) = \lambda_{a \cdot u}(a) \subseteq a \cdot x \subseteq a \cdot y \cup \lambda_{a \cdot y}(s)$$

which in turn implies $\lambda_{a \cdot y}(a) \subseteq \lambda_{a \cdot y}(s)$ and $a \subseteq s$, contradicting $s < a$. Hence $0 < c < a$, contradicting Theorem 4.6.

S u b c a s e 1.2: For some d, $a \cdot y = a \cdot u + d$. Then $r = d + t$ and $d \leq r < a$. Again, $d = 0$ is ruled out so $0 < d < a$ contradicting Theorem 4.6.

I have shown Case 1 to be impossible so we are left with:

C a s e 2: $a \cdot u + r \in a \cdot y$. So for some $v \in y$ and $e \in a$, $a \cdot u + r = a \cdot v + e$. By induction hypothesis $u = v$ and hence $u \in y$.

Since u was an arbitrary element of x, it follows that $x \subseteq y$, and symmetrically $y \subseteq x$, so $x = y$. Left cancellation of addition gives $r = s$. □

Considering the special case of Theorem 4.7 where $r = 0$, and recalling that $0 < a$ for any non-empty a, we obtain the left cancellation property for multiplication:

Corollary 4.8 (ZF $\setminus$ {Inf}) *Let* $a \neq 0$. *Then* $a \cdot x = a \cdot y \leftrightarrow x = y$.

Corollary 4.9 *Let* $0 \neq a \in M$, *where* $\mathcal{M} \vDash \mathsf{ZF} \setminus \{\mathrm{Inf}\}$. *Then the map* $\pi_a : x \longmapsto a \cdot x$ *is an isomorphism*

$$\pi_a : \langle M, +^{\mathcal{M}} \rangle \cong \langle \{a \cdot x \mid x \in M\}, +^{\mathcal{M}} \rangle.$$

P r o o f. It follows from left distributivity and left cancellation that $a \cdot x + a \cdot y = a \cdot z \leftrightarrow x + y = z$. □

Some more corollaries of Theorem 4.7:

Corollary 4.10 (ZF $\setminus$ {Inf}) *If* $x \neq y$, *then*

$$\lambda_{a \cdot x}(\mathrm{TC}(a)) \cap \lambda_{a \cdot y}(\mathrm{TC}(a)) = 0.$$

It follows that $\lambda_{a \cdot x}(a) \cap \lambda_{a \cdot y}(a) = 0$.

Corollary 4.11 (ZF $\setminus$ {Inf}) *If* $a \cdot x + r \in a \cdot y$ *and* $r < a$, *then* $x \in y$ *and* $r \in a$.

P r o o f. By the hypothesis, $a \cdot x + r = a \cdot v + s$ for some $v \in y$ and $s \in a$. Now apply Theorem 4.7. □

Taking $r = 0$ in Corollary 4.11 we obtain:

Corollary 4.12 $a \cdot x \in a \cdot y \leftrightarrow x \in y \wedge 0 \in a$.

Corollary 4.13 *Let* $a \in M$, *where* $\mathcal{M} \vDash \mathsf{ZF} \setminus \{\mathrm{Inf}\}$. *Then the map* $\pi_a : x \longmapsto a \cdot x$ *is an isomorphism*

$$\pi_a : \langle M, \in \rangle \cong \langle \{a \cdot x \mid x \in M\}, \in \rangle$$

iff $0 \in a$.

Theorem 4.14 (ZF $\setminus$ {Inf}) $|x \cdot y| = |x| \cdot_{\mathrm{c}} |y|$.

P r o o f. By Corollary 4.10 the union $x \cdot y = \bigcup \{\lambda_{x \cdot u}(x) \mid u \in y\}$ is a disjoint one, and by Proposition 3.5(i) each $\lambda_{x \cdot u}(x)$ has cardinality $|x|$. □

Next I pin down the transitive closure of the product:

Proposition 4.15 $\mathrm{TC}(x \cdot y) = \{x \cdot u + r \mid u < y \wedge r < x\}$.

More familiarly, $z < x \cdot y \leftrightarrow (\exists q < y)(\exists r < x)(z = x \cdot q + r)$.

P r o o f o f P r o p o s i t i o n 4.15. We may assume $x \neq 0$ and prove by $\in$-induction on y that

$$\mathrm{TC}(x \cdot y) = \bigcup\{\lambda_{x\cdot u}(\mathrm{TC}(x)) \mid u \in \mathrm{TC}(y)\}.$$

Noting that the union and transitive closure operators commute, we have:

$$\begin{aligned}\mathrm{TC}(x \cdot y) &= \textstyle\bigcup_{u\in y} \mathrm{TC}(\lambda_{x\cdot u}(x)) \\ &= \textstyle\bigcup_{u\in y}(\mathrm{TC}(x\cdot u) \cup \lambda_{x\cdot u}(\mathrm{TC}(x))) && \text{(by Corollary 3.8)} \\ &= \textstyle\bigcup_{u\in y}(\bigcup_{z\in \mathrm{TC}(u)} \lambda_{x\cdot z}(\mathrm{TC}(x)) \cup \lambda_{x\cdot u}(\mathrm{TC}(x))) && \text{(by induction hypothesis)} \\ &= \textstyle\bigcup_{u\in \mathrm{TC}(y)} \lambda_{x\cdot u}(\mathrm{TC}(x)).\end{aligned}$$

□

It follows, using Proposition 3.5(i) and Theorem 4.7, that:

Corollary 4.16 $|\mathrm{TC}(x \cdot y)| = |\mathrm{TC}(x)| \cdot_{\mathrm{c}} |\mathrm{TC}(y)|$.

Returning to the map π_a of Corollaries 4.9 and 4.13: while it is an isomorphism between $\langle M, \in, +^{\mathcal{M}}\rangle$ and $\langle \lambda_a(M), \in, +^{\mathcal{M}}\rangle$ (if $0 \in a$), it does not preserve multiplication, i. e., it is not an isomorphism between the models $\langle M, \cdot^{\mathcal{M}}\rangle$ and $\langle \lambda_a(M), \cdot^{\mathcal{M}}\rangle$ (unless $a = 1$). The next result shows that it also preserves $\trianglelefteq$, a fact which might seem obvious given the preservation of addition but is not automatic because of the existential quantifier in the definition of $\trianglelefteq$ from $+$.

Theorem 4.17 (ZF $\setminus$ {Inf}) *Let $a \neq 0$. Then $a \cdot x \lhd a \cdot y \leftrightarrow x \lhd y$.*

P r o o f. The right to left implication follows from left distributivity (Proposition 4.3(v)): if $x + r = y$, then

$$a \cdot x + a \cdot r = a \cdot y.$$

In the other direction, I prove by $\in$-induction on y that $\forall x(a \cdot x \lhd a \cdot y \to x \lhd y)$. Suppose $a \cdot y = a \cdot x + d$ and $d \neq 0$. I claim that

(3) $$\lambda_{a\cdot x}(d) = a \cdot (y \setminus x).$$

To prove this claim, first suppose $s \in a \cdot (y \setminus x)$, say $s = a \cdot v + b$, $v \in y \setminus x$, and $b \in a$. Since $s \in a \cdot y$, either $s \in a \cdot x$ or $s \in \lambda_{a\cdot x}(d)$. But if $s \in a \cdot x$, then $s = a \cdot u + c$, $u \in x$, $c \in a$, and Theorem 4.7 gives $u = v$ which is impossible because $v \in y \setminus x$, $u \in x$. I conclude that $s \in \lambda_{a\cdot x}(d)$.

Conversely, suppose $t \in \lambda_{a\cdot x}(d)$. Then $t \in a \cdot y$ and so $t = a \cdot v + b$, $v \in y$, $b \in a$. But $v \notin x$ because if $v \in x$, then $t \in a \cdot x$ which is impossible because $a \cdot x \cap \lambda_{a\cdot x}(d) = 0$. So $v \in y \setminus x$ and $t \in a \cdot (y \setminus x)$.

With (3) established, let v be any element of $y \setminus x$. Let $b \in a$. Then by (3), we have $a \cdot v + b = a \cdot x + s$ for some $s \in d$. By Lemma 3.13, $a \cdot v$ and $a \cdot x$ are $\trianglelefteq$-comparable.

C a s e 1: $a \cdot v \lhd a \cdot x$, say $a \cdot x = a \cdot v + e$ with $e \neq 0$. By left cancellation $e + s = b$ and so $e \leq b \in a$. Hence $0 < e < a$. But this is impossible by Theorem 4.6, so we must have, for any $v \in y \setminus x$:

C a s e 2: $a \cdot x \trianglelefteq a \cdot v$. By induction hypothesis and Corollary 4.8, $x \trianglelefteq v$.

For each $v \in y \setminus x$ let j_v be the unique set such that $x + j_v = v$.

Now let $r = \{j_v \mid v \in y \setminus x\} \neq 0$. Then $y \setminus x = \lambda_x(r)$, and $\lambda_{a\cdot x}(d) = a \cdot \lambda_x(r)$ by (3), $a \cdot \lambda_x(r) = \lambda_{a\cdot x}(a \cdot r)$ by Proposition 4.3(iv). It follows that $d = a \cdot r$. Now

$$a \cdot y = a \cdot x + a \cdot r = a \cdot (x + r)$$

and Corollary 4.8 gives $y = x + r$, i. e., $x \lhd y$. □

Corollary 4.18 *Let $0 \neq a \in M$, where $\mathcal{M} \vDash$ ZF $\setminus$ {Inf}. Then the map π_a is an isomorphism between the models $\langle M, \trianglelefteq^{\mathcal{M}}\rangle$ and $\langle\{a \cdot x \mid x \in M\}, \trianglelefteq^{\mathcal{M}}\rangle$.*

Example 4.19 Multiplication does not preserve degrees of additive reducibility (see Definition 3.22): for instance, $\delta(2) = 2$ and $\delta(\{1\}) = 1$ but $\delta(2 \cdot \{1\}) = \delta(\{2, 3\}) = 1$ (by Proposition 3.25). This also follows from Proposition 3.24, which implies in fact that for any $a, x, y \neq 0$, $a \cdot \lambda_x(y) = \lambda_{a\cdot x}(a \cdot y)$ is additively irreducible. But note that it follows from Theorem 4.17 that $\delta(a \cdot b) \geq \delta(b)$.

Question 4.20 *If $b \neq 1$ is additively irreducible, does it follow that $a \cdot b$ is additively irreducible for any non-empty a?*

More generally, how far can one push through elementary arithmetic (of natural numbers or of ordinals) to the sets? The division algorithm (for given x and y, producing q and r such that $y = x \cdot q + r$ and $r < x$) does not carry over as it stands to sets, although one can get a weak version, where $r \not\geq x$. In the case of the hereditarily finite sets: what do prime sets look like? Not every set of prime cardinality is prime, e. g. $\{2, 3\} = 2 \cdot \{1\}$ is composite. On the other hand, there are sets of composite cardinality which are prime, e. g. $\{0, 1, 2, 4\}$. Is there a version of unique factorization into prime factors?

5 The finite case: adductive ranks

The adductive hierarchy was introduced in [9] for the hereditarily finite sets, the standard model of PS:

$$A_0 = \{0\}, \quad A_{n+1} = A_n \cup \{[x; y] \mid x, y \in A_n\}.$$

This definition can be formalized in PS and hence the hierarchy is defined in a non-standard $\mathcal{M} \models \mathsf{PS}$.

The adductive hierarchy is defined above only for finite (albeit possibly non-standard) ordinals n. The natural way to extend the hierarchy to infinite ordinals is to take unions at limit stages, so that $A_\omega = \bigcup_{n\in\omega} A_n = V_\omega$, V_ω being the set of hereditarily finite sets. But the generation process then stalls because

$$\{[x; y] \mid x, y \in A_\omega\} = A_\omega$$

so that $A_\alpha = A_\omega$ for all $\alpha > \omega$. The adductive hierarchy is essentially a finitist notion.

Accordingly, in this section I shall take the base theory to be PS. Recall that PS is equivalent to

$$\mathsf{ZF} \setminus \{\mathrm{Inf}\} \cup \{\neg\mathrm{Inf}\}.$$

The results below are provable in the weaker theory $\mathsf{ZF} \setminus \{\mathrm{Inf}\}$, when restricted to finite ordinals and hereditarily finite sets, but are more simply formulated for PS. As Jan Mycielski has pointed out to me, PS is interpretable in $\mathsf{ZF} \setminus \{\mathrm{Inf}\}$, which suggests that the axiom $\neg\mathrm{Inf}$ is in any case quite weak.

There is a natural rank function associated with the adductive hierarchy:

Definition 5.1 *The adductive rank of* a, denoted $\alpha(a)$, is the least n such that $a \in A_n$.

The adductive rank is only weakly linked to the degree of additive reducibility $\delta(a)$ (Definition 3.22): it is easy to see that

$$\delta(a) \leq |a| \leq \alpha(a),$$

but for any set a recall that $\delta(\{a\}) = 1$, whereas if $\alpha(a) = n$, then $\alpha(\{a\}) = n + 1$. On the other hand, if n is a finite ordinal, then $\alpha(n) = \delta(n) = n$.

This section is devoted to investigating the extent to which addition and multiplication preserve adductive ranks. Basic properties of adductive ranks follow from [9, Lemma 2.3]:

Lemma 5.2 (PS)

(i) $a \subseteq b \to \alpha(a) \leq \alpha(b)$.

(ii) $a \in b \to \alpha(a) < \alpha(b)$.

Proposition 5.3 (PS) $a \in A_m \wedge b \in A_n \to a + b \in A_{m+n}$.

P r o o f. For given $a \in A_m$, we use induction on n. The case $n = 0$ is easy. If $b \in A_{n+1}$, say $b = [u; v]$ for some $u, v \in A_n$, then

$$a + b = a + [u; v] = [(a + u); (a + v)]$$

and by induction hypothesis both $a + u$ and $a + v$ are in A_{m+n}, so $a + b \in A_{m+n+1}$. □

Corollary 5.4 (PS) $a \in A_m \wedge b \in A_n \to \lambda_a(b) \in A_{m+n}$.

Proof. $\lambda_a(b) \subseteq a + b$ so this follows from Proposition 5.3 and Lemma 5.2(i). □

Lemma 5.5 (PS)

(i) *If $x \in A_{n+1}$, then x has at most one element of adductive rank n.*

(ii) *If $x \in A_{n+1}$ and $v \in x$ has an adductive rank which is maximal among elements of x, then $x \setminus \{v\} \in A_n$.*

Proof.

(i) If $x = [u; v]$ with $u, v \in A_n$, then by Lemma 5.2(ii), v is the only candidate for an element of x not in A_{n-1}. (For a more general result see [9, Lemma 4.4].)

(ii) Inductively on n. The case $n = 0$ is easy because the only non-empty element of A_1 is 1. Let x, v be as hypothesized and since $x \in A_{n+1}$, let $x = [s; t]$ with $s, t \in A_n$. If $t = v$, then we are done so assume $t \neq v$. If $\alpha(t) = n$, then by (i), t would have higher adductive rank than v. Hence $t \in A_{n-1}$. Since v has maximal adductive rank among elements of s, by induction hypothesis $s \setminus \{v\} \in A_{n-1}$ and hence

$$x \setminus \{v\} = [(s \setminus \{v\}); t] \in A_n.$$ □

Theorem 5.6 (PS) $\alpha(a + b) = \alpha(a) + \alpha(b)$.

Proof. Fix a with $\alpha(a) = m$. Suppose, as an induction hypothesis on n, that for all $i \leq n$ and for all y,

$$\alpha(a + y) = m + i \to \alpha(y) = i.$$

We need to prove this for $i = n + 1$. So suppose

$$\alpha(a + b) = m + n + 1.$$

We need to show $\alpha(b) = n + 1$. By Proposition 5.3 it is enough to show $b \in A_{n+1}$. Let $a + b = [u; v]$ with v of maximum adductive rank among the elements of $a + b$ and $v \notin u$. (Note that the existence of such v is due to our finitary context: even if we are working in $\mathsf{ZF} \setminus \{\mathrm{Inf}\}$, n is finite.) By Lemma 5.2(ii), $\alpha(v) \leq m + n$. And by Lemma 5.5(ii), $\alpha(u) \leq m + n$. In fact at least one of $\alpha(u)$ and $\alpha(v)$ is equal to $m + n$. Also

$$v \in a + b = a \cup \lambda_a(b).$$

If $v \in a$, then $\alpha(v) < m$ by Lemma 5.2(ii). Noting that $b \neq 0$, pick $y \in b$: then $a + y \in a + b$. But $a + y \supseteq a$ so by Lemma 5.2(i), $\alpha(a + y) \geq m$ contradicting the maximality of $\alpha(v)$.

So $v \in \lambda_a(b)$, say $v = a + y$, $y \in b$. Then $a + (b \setminus \{y\}) = u$. Note that $u \supseteq a$ so $\alpha(u) \geq m$. If $\alpha(u) = m + i$, then by induction hypothesis $\alpha(b \setminus \{y\}) = i \leq n$. But also

$$v \in \lambda_a(b) \subseteq a + b \in A_{m+n+1}$$

and so by Lemma 5.2(ii), $v \in A_{m+n}$. By the induction hypothesis, $y \in A_n$. It follows that

$$b = [(b \setminus \{y\}); y] \in A_{n+1}.$$

The above induction on n needs to be started off with $n = 0$. So suppose $\alpha(a + b) = \alpha(a) = m$. We need to show $b = 0$. Suppose not: let $y \in b$. Then $a + b \ni a + y \supseteq a$ so $\alpha(a + b) > m$, a contradiction. □

We shall need this generalization of Proposition 5.3:

Lemma 5.7 (PS) *If a and b are in A_m and $y \in A_n$, then $a \cup \lambda_b(y) \in A_{m+n}$.*

Proof. For fixed m, suppose the result is true for n (as usual $n = 0$ is easy). So suppose $y = [u; v]$ for some $u, v \in A_n$. Then

$$a \cup \lambda_b(y) = a \cup \lambda_b([u; v]) = a \cup [\lambda_b(u); (b + v)] = [(a \cup \lambda_b(u)); (b + v)].$$

(The last equality uses the definition of union in [10, §4].) Both $a \cup \lambda_b(u)$ and $b + v$ are in A_{m+n}, the former by induction hypothesis and the latter by Proposition 5.3. So $a \cup \lambda_b(y) \in A_{m+n+1}$. □

Proposition 5.8 (PS) $a \in A_m \wedge b \in A_n \rightarrow a \cdot b \in A_{mn}$.

P r o o f. Again by induction on n with $n = 0$ easy. If $b = [u; v]$ with $u, v \in A_n$, then

$$a \cdot b = a \cdot [u; v] = a \cdot u \cup \lambda_{a \cdot v}(a)$$

and since by induction hypothesis $a \cdot u$ and $a \cdot v$ are in A_{mn}, it follows from Lemma 5.7 that

$$a \cdot b \in A_{mn+m} = A_{m(n+1)}. \qquad \square$$

Corollary 5.9 (PS) $\alpha(a \cdot b) \leq \alpha(a) \cdot \alpha(b)$.

Example 5.10 When a and b are not finite ordinals, we cannot hope to sharpen the inequality in Corollary 5.9 to an equality: for $\alpha(\{1\}) = 2$ and $\alpha(\{\{1\}, 2\}) = 4$ but $\alpha(\{1\} \cdot \{\{1\}, 2\}) = 7$. Indeed,

$$\{1\} \cdot x = \{\{1\} \cdot u + 1 \mid u \in x\}$$

and so

$$\{1\} \cdot \{\{1\}, 2\} = \{\{1\} \cdot \{1\} + 1, \{1\} \cdot 2 + 1\} = \{\{\{1\} + 1\} + 1, \{1, \{1\} + 1\} + 1\}$$

and each of the two elements of this set can be seen to be in A_5.

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