Chapter 5

*Time Value of Money*

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Overview

1. Using Time Lines
2. Compounding and Future Value
3. Discounting and Present Value
4. Making Interest Rates Comparable
Learning Objectives

1. Construct cash flow timelines to organize your analysis of time value of money problems and learn three techniques for solving time value of money problems.

2. Understand compounding and calculate the future value of cash flow using mathematical formulas, a financial calculator, and an Excel worksheet.

3. Understand discounting and calculate the present value of cash flows using mathematical formulas, a financial calculator, and an excel spreadsheet.

4. Understand how interest rates are quoted and how to make them comparable.
Principles Used in this Chapter

- Principle 1: Money Has a Time Value.

- The concept of time value of money – a dollar received today, other things being the same, is worth more than a dollar received a year from now, underlies many financial decisions faced in business.
Using Timelines to Visualize Cashflows

- A **timeline** identifies the timing and amount of a stream of cash flows along with the interest rate.

- A timeline is typically expressed in years, but it can also be expressed in months, days, or any other unit of time.
The 4-year timeline illustrates the following:
- The interest rate is 10%.
- A cash outflow of $100 occurs at the beginning of the first year (at time 0), followed by cash inflows of $30 and $20 in years 1 and 2, a cash outflow of $10 in year 3 and cash inflow of $50 in year 4.
Creating a Timeline

Suppose you lend a friend $10,000 today to help him finance a new Jimmy John’s Sub Shop franchise and in return he promises to give you $12,155 at the end of the fourth year. How can one represent this as a timeline? Note that the interest rate is 5%.
STEP 1: Picture the problem

A timeline provides a tool for visualizing cash flows and time:

- $i$ = rate of interest
- Time Period: 0, 1, 2, 3, 4
- Cash Flow: Year 0, Year 1, Year 2, Year 3, Year 4

STEP 2: Decide on a solution strategy

To complete the timeline we simply record the cash flows onto the template.

STEP 3: Solve

We can input the cash flows for this investment on the timeline as shown below. Time period zero (the present) is shown at the left end of the timeline, and future time periods are shown above the timeline, moving from left to right, with the year that each cash flow occurs shown above the timeline.

Keep in mind that year 1 represents the end of the first year as well as the beginning of the second year.

- $i = 5\%$
- Time Period: 0, 1, 2, 3, 4
- Cash Flow: -$10,000, $12,155

STEP 4: Analyze

Using timelines to visualize cash flows is useful in financial problem solving. From analyzing the timeline, we can see that there are two cash flows, an initial $10,000 cash outflow, and a $12,155 cash inflow at the end of year 4.
Checkpoint 5.1: Check yourself

Draw a timeline for an investment of $40,000 today that returns nothing in one year, $20,000 at the end of year 2, nothing in year 3, and $40,000 at the end of year 4.
Timeline

Time Period

Cash flow

- $40,000
  $0
  $20,000
  $0
  $40,000

i = interest rate; not given
Simple Interest and Compound Interest

- What is the difference between simple interest and compound interest?
  - **Simple interest**: Interest is earned only on the principal amount.
  - **Compound interest**: Interest is earned on both the principal and accumulated interest of prior periods.

- **Example 5.1**: Suppose that you deposit $500 in your savings account that earns 5% annual interest. How much will you have in your account after two years using (a) simple interest and (b) compound interest?
Example 5.1

- **Simple Interest**
  - Interest earned = 5% of $500 = \(0.05 \times 500 = $25\) per year
  - Total interest earned = \(25 \times 2 = $50\)
  - Balance in your savings account:
    \[= \text{Principal} + \text{accumulated interest}\]
    \[= 500 + 50 = $550\]

- **Compound interest** (assuming compounding once a year)
  - Interest earned in Year 1 = 5% of $500 = $25
  - Interest earned in Year 2 = 5% of \((500 + \text{accumulated interest})\)
    \[= 5\% \times (500 + 25) = 0.05 \times 525 = $26.25\]
  - Balance in your savings account:
    \[= \text{Principal} + \text{interest earned} = 500 + 25 + 26.25 = $551.25\]
Present Value and Future Value

- Time value of money calculations involve *Present value* (what a cash flow would be worth to you today) and *Future value* (what a cash flow will be worth in the future).

- In example 5.1, Present value is $500 and Future value is $551.25 (if the yearly compounding rate is 5%).

- The linkage between present value and future value is:

  \[ \text{Future Value} = \text{Present Value} \times (1+\text{Interest Rate per period})^{\text{Number of periods}} \]

- For annual compounding (compounding once a year),

  \[ \text{Future Value} = \text{Present Value} \times (1+\text{Annual Interest Rate})^{\text{Number of years}} \]

  *If nothing is said, assume annual compounding.*
Example 5.1: The future value of $500 in 2 years with annual compounding interest rate of 5% can be computed directly from the formula:

\[
FV_2 = PV(1+i)^2 = 500(1+0.05)^2 = 500(1.05)^2 = 551.25.
\]

Continue example 5.1 where you deposit $500 in savings account earning 5% annual interest. Show the amount of interest earned for the first five years and the value of your savings at the end of five years.

- You can do the calculation year by year
- or use the formula for future value: \( FV_n = PV(1+i)^n \)
Year by year compounding

<table>
<thead>
<tr>
<th>YEAR</th>
<th>PV or Beginning Value</th>
<th>Interest Earned (5%)</th>
<th>FV or Ending Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$500.00</td>
<td>$500*.05 = $25</td>
<td>$525</td>
</tr>
<tr>
<td>2</td>
<td>$525.00</td>
<td>$525*.05 = $26.25</td>
<td>$551.25</td>
</tr>
<tr>
<td>3</td>
<td>$551.25</td>
<td>$551.25*.05 = $27.56</td>
<td>$578.81</td>
</tr>
<tr>
<td>4</td>
<td>$578.81</td>
<td>$578.81*.05 = $28.94</td>
<td>$607.75</td>
</tr>
<tr>
<td>5</td>
<td>$607.75</td>
<td>$607.75*.05 = $30.39</td>
<td>$638.14</td>
</tr>
</tbody>
</table>
Use the Future Value Equation

- We will obtain the same answer using the future value equation: $FV = PV(1+i)^n$
  
  $$= 500(1.05)^5 = 638.14$$

- So the balance in savings account at the end of 5 years will equal $638.14. The total interest earned on the original principal amount of $500 will equal $138.14 (i.e. $638.14\,-\,500.00$).
Future value of original investment increases with time, unless interest rate is zero.

(Panel B) The Power of Time

This figure illustrates the importance of time when it comes to compounding. Because interest is earned on past interest, the future value of $100 deposited in an account that earns 8% compounded annually grows over threefold in 15 years. If we were to expand this figure to 45 years (which is about how long you have until you retire, assuming you’re around 20 years old right now), it would grow to over 31 times its initial value.
Power of Interest Rate

Figure 5.1 Future Value and Compound Interest Illustrated

(Panel C) The Power of the Rate of Interest

This figure illustrates the importance of the interest rate in the power of compounding. As the interest rate climbs, so does the future value. In fact, when we change the interest rate from 10% to 20%, the future value in 25 years increases by over 8 times, jumping from $1,083.47 to $9,539.62.

An increase in interest rate leads to an increase in future value.
Calculating the Future Value of a Cash Flow

You are put in charge of managing your firm’s working capital. Your firm has $100,000 in extra cash on hand and decides to put it in a savings account paying 7% interest compounded annually. How much will you have in your account in 10 years?
Checkpoint 5.2

- Draw the time line

![Time line diagram with a 7% interest rate, 10 years, and a cash flow of -$100,000.]

- It is a simple future value problem. Use the future value equation:
  \[ FV_{10} = PV(1+i)^{10} = 100000(1.07)^{10} = \text{196,715} \]

- Check for yourself: What’s future value in 20 years? What if the annual return (interest rate) on the cash is 12%?
  \[ \text{386,968; 964,629.} \]
FV Applications in Other Areas

- **Example 5.2** A DVD rental firm is currently renting 8,000 DVDs per year. How many DVDs will the firm be renting in 10 years if the demand for DVD rentals is expected to increase by 7% per year?

- **Example 5.3** Your annual tuition at a State University is currently $20,000. If the tuition increases by 6% annually, what will be the annual cost of attending the State University in 25 years?
FV Applications in Other Areas

- Answers:
  - Example 5.2: \( FV = 8000(1.07)^{10} \) = $15,737.21
  - Example 5.3: \( FV = 20,000 (1.06)^{25} \) = $85,837.41 per year
Banks frequently offer savings accounts that compound interest every day, month, or quarter.

More frequent compounding will generate higher interest income for the savers if the annual interest rate is the same.

Example 5.4 You invest $500 for seven years to earn an annual interest rate of 8%, and the investment is compounded semi-annually. What will be the future value of this investment?

- Use the more general formula:
  \[ FV = PV \times (1 + \text{rate per period})^{\text{Number of periods}} \]
- Rate per half year period is 8%/2=4%.
- Number of half-year periods in 7 years is 7x2=14 half years.
- \( FV = 500 \times (1 + 0.04)^{14} = 865.84 \)
Another way to write the future value equation, in terms of annual rates and years:

\[
FV_n = PV \left(1 + \frac{r}{m}\right)^{m \times n}
\]

Example 5.4 again:

\[
FV = PV(1+i/2)^{m*2} = 500(1+.08/2)^{7*2} = 500\times(1.04)^{14} = 865.84.
\]
Given the same rate and time period, the more frequent the compounding, the higher the future value.

<table>
<thead>
<tr>
<th>Table 5.2</th>
<th>The Value of $100 Compounded at Various Non-Annual Periods and Various Rates</th>
</tr>
</thead>
<tbody>
<tr>
<td>Notice the impact of shorter compounding periods is heightened by both higher interest rates and compounding over longer time periods.</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>For 1 Year at ( i ) Percent</th>
<th>( i = 2% )</th>
<th>5%</th>
<th>10%</th>
<th>15%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Compounded annually</td>
<td>$102.00</td>
<td>$105.00</td>
<td>$110.00</td>
<td>$115.00</td>
</tr>
<tr>
<td>Compounded semiannually</td>
<td>102.01</td>
<td>105.06</td>
<td>110.25</td>
<td>115.56</td>
</tr>
<tr>
<td>Compounded quarterly</td>
<td>102.02</td>
<td>105.09</td>
<td>110.38</td>
<td>115.87</td>
</tr>
<tr>
<td>Compounded monthly</td>
<td>102.02</td>
<td>105.12</td>
<td>110.47</td>
<td>116.08</td>
</tr>
<tr>
<td>Compounded weekly (52)</td>
<td>102.02</td>
<td>105.12</td>
<td>110.51</td>
<td>116.16</td>
</tr>
<tr>
<td>Compounded daily (365)</td>
<td>102.02</td>
<td>105.13</td>
<td>110.52</td>
<td>116.18</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>For 10 Years at ( i ) Percent</th>
<th>( i = 2% )</th>
<th>5%</th>
<th>10%</th>
<th>15%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Compounded annually</td>
<td>$121.90</td>
<td>$162.89</td>
<td>$259.37</td>
<td>$404.56</td>
</tr>
<tr>
<td>Compounded semiannually</td>
<td>122.02</td>
<td>163.86</td>
<td>265.33</td>
<td>424.79</td>
</tr>
<tr>
<td>Compounded quarterly</td>
<td>122.08</td>
<td>164.36</td>
<td>268.51</td>
<td>436.04</td>
</tr>
<tr>
<td>Compounded monthly</td>
<td>122.12</td>
<td>164.70</td>
<td>270.70</td>
<td>444.02</td>
</tr>
<tr>
<td>Compounded weekly (52)</td>
<td>122.14</td>
<td>164.83</td>
<td>271.57</td>
<td>447.20</td>
</tr>
<tr>
<td>Compounded daily (365)</td>
<td>122.14</td>
<td>164.87</td>
<td>271.79</td>
<td>448.03</td>
</tr>
</tbody>
</table>

\[ \text{For 1 Year at } \( i = 2\% \text{, the change is } \frac{115.00 - 102.00}{102.00} = 1.18 \]  
\[ \text{For 10 Years at } \( i = 2\% \text{, the change is } \frac{448.03 - 121.90}{121.90} = 2.67 \]
Checkpoint 5.3

Calculating Future Values Using Non-Annual Compounding Periods

You have been put in charge of managing your firm’s cash position and noticed that the Plaza National Bank of Portland, Oregon, has recently decided to begin paying interest compounded semi-annually instead of annually. If you deposit $1,000 with Plaza National Bank at an interest rate of 12%, what will your account balance be in five years?

If you deposit $50,000 in an account that pays an annual interest rate of 10% compounded monthly, what will your account balance be in 10 years?

Discounting and Present Value

- What is value today of cash flow to be received in the future?
  - The answer to this question requires computing the **present value**, i.e., the value today of a future cash flow, and the process of **discounting**, determining the present value of an expected future cash flow.
  - Since we know how to compound to get future value:
    \[ FV_n = PV(1+i)^n \]
  - We can get PV from FV:
    \[ PV = FV_n / (1+i)^n \]
  - Compound (multiply) to get future value; discount (divide) to get present value.
  - Present value is smaller than future value with positive rate.
Example 5.5 How much will $5,000 to be received in 10 years be worth today if the interest rate is 7%?

- \[ PV = \frac{FV}{(1+i)^n} = \frac{5000}{(1.07)^{10}} = \$2,541.50 \]

- To calculate present value, the interest rate is often referred to as the “discount rate.”

- The textbook version of the PV formula (for annual compounding):

\[
\text{Present Value (PV)} = \frac{\text{Future Value in year } n}{(FV_n)} \left[ \frac{1}{1 + \frac{\text{Annual Interest Rate (i)}}{\text{Number of Years (n)}}} \right]
\]
Impact of Interest Rates on PV

- If the interest rate (or discount rate) is higher (say 9%), the PV will be lower.
  \[ PV = 5000 \times \left(\frac{1}{1.09}\right)^{10} = 5000 \times 0.4224 \]
  \[ = \$2,112.00 \]

- If the interest rate (or discount rate) is lower (say 2%), the PV will be higher.
  \[ PV = 5000 \times \left(\frac{1}{1.02}\right)^{10} = 5000 \times 0.8203 \]
  \[ = \$4,101.50 \]

- Note the slight variation in which the formula is written.
Checkpoint 5.4

Solving for the Present Value of a Future Cash Flow

- Your firm has just sold a piece of property for $500,000, but under the sales agreement, it won’t receive the $500,000 until ten years from today. What is the present value of $500,000 to be received ten years from today if the discount rate is 6% annually?
  - Verify the answer: $279,197.39

- What is the present value of $100,000 to be received at the end of 25 years given a 5% discount rate?
  - Verify the answer: $29,530.28
Solving for the Number of Periods

- **Key Question**: How long will it take to accumulate a specific amount in the future?

- It is easier to solve for “n” using the financial calculator or Excel rather than mathematical formula:

- \[ FV = PV(1+i)^n \]
  - \[(FV/PV) = (1+i)^n \] \( \rightarrow \) Move PV to the left.
  - \[ \ln(FV/PV) = n \ln(1+i) \] \( \rightarrow \) Take natural logs (ln) on both sides
  - \[ n = \frac{\ln(FV/PV)}{\ln(1+i)} \] \( \rightarrow \) Move \( \ln(1+i) \) to the left, switch sides.
Example 5.6 How many years will it take for an investment of $7,500 to grow to $23,000 if it is invested at 8% annually?

\[ N = \frac{\ln(FV/PV)}{\ln(1+i)} = \frac{\ln(2300/7500)}{\ln(1.08)} = 1.12/0.077 = 14.56 \]
Checkpoint 5.5

Solving for the Number of Periods, \( n \)

Let’s assume that the Toyota Corporation has guaranteed that the price of a new Prius will always be $20,000, and you’d like to buy one but currently have only $7,752. How many years will it take for your initial investment of $7,752 to grow to $20,000 if it is invested so that it earns 9% compounded annually?

How many years will it take for $10,000 to grow to $200,000 given a 15% compound growth rate?

- Verify the answers: 11 years; 21.43 years.
Rule of 72

- Rule of 72 is an approximate formula to determine the number of years it will take to double the value of your investment.

- **Rule of 72:** \[ N = \frac{72}{\text{interest rate in percentage}} \]

- **Example 5.7** Using Rule of 72, determine how long it will take to double your investment of $10,000 if you are able to generate an annual return of 9%.
  - Exact \( N = \frac{\ln(2)}{\ln(1.09)} = \frac{0.693}{0.086} = 8.04 \)
  - Approximate \( N = \frac{72}{9} = 8. \)
Solving for Rate of Interest

**Key Question**: What rate of interest will allow your investment to grow to a desired future value?

\[ FV = PV(1+i)^n \]

- \( (FV/PV) = (1+i)^n \) \( \leftarrow \) Move PV to the left

- \( (FV/PV)^{\frac{1}{n}} = 1+i \) \( \leftarrow \) Take \((1/n)\) root on both sides

- \( i = (FV/P)^{\frac{1}{n}} - 1 \). \( \leftarrow \) Move 1 to the left, switch sides.
Example

Example 5.8 At what rate of interest must your savings of $10,000 be compounded annually for it to grow to $22,000 in 8 years?

\[ i = \left( \frac{FV}{PV} \right)^{\frac{1}{n}} - 1 = \left( \frac{22000}{10000} \right)^{\frac{1}{8}} - 1 = 0.1036 = 10.36\% . \]
Checkpoint 5.6

Solving for the Interest Rate, $i$

- Let’s go back to that Prius example in Checkpoint 5.5. Recall that the Prius always costs $20,000. In 10 years, you’d really like to have $20,000 to buy a new Prius, but you only have $11,167 now. At what rate must your $11,167 be compounded annually for it to grow to $20,000 in 10 years?

- At what rate will $50,000 have to grow to reach $1,000,000 in 30 years?

- Verify the answers: 6%; 10.5%.
Making Interest Rates Comparable

- The **annual percentage rate** (APR) indicates the amount of interest paid or earned in one year without compounding. APR is also known as the **nominal or stated interest rate**. This is the rate required by law.

- We cannot compare two loans based on APR if they do not have the same compounding period.

- To make them comparable, we calculate their equivalent rate using an annual compounding period. We do this by calculating the **effective annual rate (EAR)**.
Linking APR to EAR

- APR is the quoted annual rate with a pre-specified compounding frequency. [Let $m$ be the number of compounding periods per year for this APR.]

- EAR is the effective annual rate at an annual compounding frequency. [One compounding per year]

- The two should generate the same amount of money in one year:

  \[(1+\text{APR}/m)^m=(1+\text{EAR}) \rightarrow \text{EAR}=(1+\text{APR}/m)^m-1.\]
Example 5.9 Calculate the EAR for a loan that has a 5.45% quoted annual interest rate compounded monthly.

- Monthly compounding implies 12 compounding per year. $m=12$.
- \[ \text{EAR} = (1 + \frac{\text{APR}}{m})^m - 1 = (1 + \frac{0.0545}{12})^{12} - 1 \]

\[ = 1.0558 - 1 = \text{0.0558} \text{ or } 5.59\% \]
Checkpoint 5.7

Calculating an EAR or Effective Annual Rate

- Assume that you just received your first credit card statement and the APR, or annual percentage rate listed on the statement, is 21.7%. When you look closer you notice that the interest is compounded daily. What is the EAR, or effective annual rate, on your credit card?

- What is the EAR on a quoted or stated rate of 13% that is compounded monthly?

- Verify the answers: 24.23%; 13.80%.
Continuous Compounding

- When the time intervals between when interest is paid are infinitely small, we call it continuous compounding. In this case, future value and present value is linked as:
  - \( FV = PV \, e^{rt} \)
  - \( r \) is the continuous compounding rate, \( t \) is number of years.
  - \( E \) is the “natural number” 2.71828

- Continuous compounding rate is linked to EAR as
  - \( EAR = e^r - 1 \), again obtained by matching the one-year future value from the two compounding frequencies.
Example

Example 5.10 What is the EAR on your credit card with continuous compounding if the APR is 18%?

EAR = \( e^{0.18} - 1 \)

\[
= 1.1972 - 1
\]

\[
= .1972 \text{ or } 19.72\%
\]