Chapter 6

Annuities

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Overview

1. Annuities
2. Perpetuities
3. Complex Cash Flow Streams

Learning objectives

1. Distinguish between an ordinary annuity and an annuity due, and calculate present and future values of each.
2. Calculate the present value of a level perpetuity and a growing perpetuity.
3. Calculate the present and future value of complex cash flow streams.
Ordinary Annuities

- An annuity is a series of equal dollar payments that are made at the end of equidistant points in time such as monthly, quarterly, or annually over a finite period of time.

- If payments are made at the end of each period, the annuity is referred to as ordinary annuity.

- Example 6.1 How much money will you accumulate by the end of year 10 if you deposit $3,000 each for the next ten years in a savings account that earns 5% per year?
The Future Value of an Ordinary Annuity

- **The time line:** $i = 5\%$

<table>
<thead>
<tr>
<th>Time</th>
<th>Cashflow:</th>
<th>FV</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3000</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>3000</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>...</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>3000</td>
<td></td>
</tr>
</tbody>
</table>

We want to know the future value of the 10 cash flows.

We can compute the future value of each cash flow and sum them together:

$$3000(1.05)^9 + 3000(1.05)^8 + \ldots + 3000 = \boxed{37,733.68}$$
The Future Value of an Ordinary Annuity

- The earlier cash flows have higher future values because they have more years to earn interest.
- Year 1 cash flow can earn 9 years of interest.
- Year 10 cash flow does not earn any interest.
The Future Value of an Ordinary Annuity

- Since the annuity cash flow has a strong pattern, we can also compute the future value of the annuity using a simple formula:

\[
FV_n = PMT \left[ \frac{(1 + i)^n - 1}{i} \right]
\]

- $FV_n$ = FV of annuity at the end of $n$th period.
- $PMT$ = annuity payment deposited or received at the end of each period.
- $i$ = interest rate per period
- $n$ = number of periods for which annuity will last.
Example 6.1

$3,000 for 10 years at 5% rate. Use the formula

\[ FV_n = PMT \left[ \frac{(1 + i)^n - 1}{i} \right] \]

- FV = $3,000 \left\{ \left[ (1 + .05)^{10} - 1 \right] \div (.05) \right\} 
  = $3,000 \left\{ [0.63] \div (.05) \right\} 
  = $3,000 \{12.58\} 
  = $37,733.68
Solving for PMT in an Ordinary Annuity

Instead of figuring out how much money you will accumulate (i.e. FV), you may like to know how much you need to save each period (i.e. PMT) in order to accumulate a certain amount at the end of n years.

In this case, we know the values of n, i, and $FV_n$ in the formula $FV_n = PMT \left[ \frac{(1 + i)^n - 1}{i} \right]$, and we need to determine the value of PMT.

$\text{PMT} = \frac{FV_n}{\left[ ((1+i)^n-1)/i \right]}$. 

The formula for solving for PMT in an ordinary annuity is given by $FV_n = PMT \left[ \frac{(1 + i)^n - 1}{i} \right]$. This formula allows you to calculate the periodic payment required to achieve a desired future value in an annuity.
Examples

Example 6.2: Suppose you would like to have $25,000 saved 6 years from now to pay towards your down payment on a new house. If you are going to make equal annual end-of-year payments to an investment account that pays 7%, how big do these annual payments need to be?

How much must you deposit in a savings account earning 8% interest in order to accumulate $5,000 at the end of 10 years?

If you can earn 12% on your investments, and you would like to accumulate $100,000 for your child’s education at the end of 18 years, how much must you invest annually to reach your goal?

Verify the answers: 3494.89; 345.15; 1793.73
The *Present Value* of an Ordinary Annuity

- The present value of an ordinary annuity measures the value today of a stream of cash flows occurring in the future.

- Example: What is the value today or lump sum equivalent of receiving $3,000 every year for the next 30 years if the interest rate is 5%?

- If I know its future value, I can compute its present value.

  $$ PV = \frac{FV_n}{(1+i)^n}, \text{ where } FV_n = PMT\left[\frac{(1+i)^n-1}{i}\right] $$

  $$ = PMT\left[\frac{1}{i}\right] $$

  $$ \text{Present Value} = PMT\left[\frac{1 - \frac{1}{(1+i)^n}}{i}\right] $$

For the example, $FV=199,316.54$. $PV=46,117.35$. 
One can also compute the PV of each cash flow and sum them up.

**Figure 6.2**

Timeline of a Five-Year, $500 Annuity Discounted Back to the Present at 6 Percent

To find the present value of an annuity, discount each cash flow back to the present separately and then add them. In this example, we simply add up the present value of five future cash flows of $500 each to find a present value of $2,106.18.

\[
i = 6\%
\]

<table>
<thead>
<tr>
<th>Time Period</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>Years</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash Flow</td>
<td></td>
<td>$500</td>
<td>$500</td>
<td>$500</td>
<td>$500</td>
<td>$500</td>
<td></td>
</tr>
</tbody>
</table>

\[471.70 \times (1.06) \div (1.06)^2 \div (1.06)^3 \div (1.06)^4 \div (1.06)^5\]

\[PV \text{ of Annuity} = $2,106.18\]
The Present and Future Values of an Ordinary Annuity

\[ FV_n = PMT \left[ \frac{(1 + i)^n - 1}{i} \right] \]

Present Value = \[ PMT \left[ \frac{1 - \frac{1}{(1 + i)^n}}{i} \right] \]

\[ FV_n = PV(1 + i)^n \]

- It is important that "n" and "i" match. If periods are expressed in terms of number of monthly payments, the interest rate must be expressed in terms of the interest rate per month.

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The Present Value of an Ordinary Annuity

- Your grandmother has offered to give you $1,000 per year for the next 10 years. What is the present value of this 10-year, $1,000 annuity discounted back to the present at 5%?
STEP 1: Picture the problem

We can use a timeline to identify the cash flows from the investment as follows:

\[ i = 5\% \]

<table>
<thead>
<tr>
<th>Time Period</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>Years</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash Flow</td>
<td></td>
<td>$1,000</td>
<td>$1,000</td>
<td>$1,000</td>
<td>$1,000</td>
<td>$1,000</td>
<td>$1,000</td>
<td>$1,000</td>
<td>$1,000</td>
<td>$1,000</td>
<td>$1,000</td>
<td></td>
</tr>
</tbody>
</table>

Sum up the present values of all the cash flows to find the present value of the annuity.

STEP 2: Decide on a solution strategy

In this case we are trying to determine the present value of an annuity, and we know the number of years, the dollar value that is received at the end of each year, and the number of years the annuity lasts. We also know that the discount rate is 5 percent. We can use Equation (6–2b) to solve this problem.

- Verify the answer: 7721.73;
Checkpoint 6.2: *Check Yourself*

What is the present value of an annuity of $10,000 to be received at the end of each year for 10 years given a 10 percent discount rate?

Answer: $61,445.67
Amortized Loans

- An **amortized loan** is a loan paid off in equal payments – consequently, the loan payments are an annuity.

- **Examples**: Home mortgage loans, Auto loans

- In an amortized loan, the *present value* can be thought of as the amount borrowed, $n$ is the number of periods the loan lasts for, $i$ is the interest rate per period, and *payment* is the loan payment that is made.
Example 6.5  Suppose you plan to get a $9,000 loan from a furniture dealer at 18% annual interest with annual payments that you will pay off in over five years. What will your annual payments be on this loan?

PMT = PV / [((1-(1+i)^n))/i] = $2,878.00.
The Loan Amortization Schedule:

*How interest and principal are accounted for?*

<table>
<thead>
<tr>
<th>Year</th>
<th>Amount Owed on Principal at the Beginning of the Year (1)</th>
<th>Annuity Payment (2)</th>
<th>Interest Portion of the Annuity (3) = (1) × 18%</th>
<th>Repayment of the Principal Portion of the Annuity (4) = (2) − (3)</th>
<th>Outstanding Loan Balance at Year end, After the Annuity Payment (5) = (1) − (4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$9,000</td>
<td>$2,878</td>
<td>$1,620.00</td>
<td>$1,258.00</td>
<td>$7,742.00</td>
</tr>
<tr>
<td>2</td>
<td>$7,742</td>
<td>$2,878</td>
<td>$1,393.50</td>
<td>$1,484.44</td>
<td>$6,257.56</td>
</tr>
<tr>
<td>3</td>
<td>$6,257.56</td>
<td>$2,878</td>
<td>$1,126.36</td>
<td>$1,751.64</td>
<td>$4,505.92</td>
</tr>
<tr>
<td>4</td>
<td>$4,505.92</td>
<td>$2,878</td>
<td>$811.07</td>
<td>$2,066.93</td>
<td>$2,438.98</td>
</tr>
<tr>
<td>5</td>
<td>$2,438.98</td>
<td>$2,878</td>
<td>$439.02</td>
<td>$2,438.98</td>
<td>$0.00</td>
</tr>
</tbody>
</table>
The Loan Amortization Schedule

*How interest and principal are accounted for?*

- We can observe the following from the table:
  - Size of each payment remains the same.
  - However, Interest payment declines each year as the amount owed declines and more of the principal is repaid.
Many loans such as auto and home loans require monthly payments. This requires converting $n$ to number of months and computing the monthly interest rate.

**Example 6.6** You have just found the perfect home. However, in order to buy it, you will need to take out a $300,000, 30-year mortgage at an annual rate of 6 percent. What will your monthly mortgage payments be?

- $n=30 \times 12 = 360$. $i = 6\%/12 = 0.5\%$.
- $PMT = 300000/[(1 - 1.005^{-360})/0.005] = \text{ $1798.65$}$
Let’s say that exactly ten years ago you took out a $200,000, 30-year mortgage with an annual interest rate of 9 percent and monthly payments of $1,609.25. But since you took out that loan, interest rates have dropped. You now have the opportunity to refinance your loan at an annual rate of 7 percent over 20 years. You need to know what the outstanding balance on your current loan is so you can take out a lower-interest-rate loan and pay it off. If you just made the 120th payment and have 240 payments remaining, what’s your current loan balance?

What will be your new monthly payment if you can do the refinancing?
Checkpoint 6.3: Analysis

- Double check the payment: PV=200,000, n=360, 
i=0.09/12=0.0075.
  - PMT=PV/[(1-1.0075^{-360})/0.0075]=1609.245

- The remaining principal can be computed as the present value of the remaining payments under the existing interest rate (9%).
  Remaining balance=PV = 1609.245[ (1-(1.0075)^{-240})/(0.0075)]
  =$ 178,859.49

- Now we can compute the new monthly payment on the remaining balance with a new rate i=0.05/12= 0.00583
  - PMT=178859.49/[(1-1.00583^{-240})/0.00583]= $1,386.69.
  - A monthly saving of $222.55 (=1609.25-1386.69).
Let’s assume you took out a $300,000, 30-year mortgage with an annual interest rate of 8%, and monthly payment of $2,201.29. Since you have made 15 years worth of payments, there are 180 monthly payments left before your mortgage will be totally paid off. How much do you still owe on your mortgage?

- **Hint:** The remaining balance is essentially the present value of remaining payments under the existing rate.
- **Verify the answer:** $230,344.29
Annuities Due

- **Annuity due** is an annuity in which all the cash flows occur at the beginning of the period. For example, rent payments on apartments are typically annuity due as rent is paid at the beginning of the month.

- Computation of future/present value of an annuity due requires compounding the cash flows for one additional period, beyond an ordinary annuity.

- \[ FV_{n}(\text{annuity due}) = PMT \left( \frac{(1 + i)^n - 1}{i} \right) (1 + i) \]

- \[ PV(\text{annuity due}) = PMT \left( \frac{1 - \frac{1}{(1 + i)^n}}{i} \right) (1 + i) \]
Examples

- **Example 6.1** where we calculated the future value of 10-year ordinary annuity of $3,000 earning 5% to be $37,734. What will be the future value if the deposits of $3,000 were made at the beginning of the year i.e. the cash flows were annuity due?
  - Just compound the future value for the ordinary annuity for one more period: \( FV = 37734 \times 1.05 = 39,620.7 \)

- **Checkpoint 6.2** where we computed the PV of 10-year ordinary annuity of $10,000 at a 10% discount rate to be equal to $61,446. What will be the present value if $10,000 is received at the beginning of each year i.e. the cash flows were annuity due?
  - Just compound the PV of the ordinary annuity for one more period: \( PV = 61446 \times 1.1 = 67,590.6 \)
Perpetuities

- A **perpetuity** is an annuity that continues forever or has no maturity. For example, a dividend stream on a share of preferred stock. There are two basic types of perpetuities:
  - **Growing perpetuity** in which cash flows grow at a constant rate, $g$, from period to period.
  - **Level perpetuity** in which the payments are constant rate from period to period.

- Even if the cash flows are infinite, present values can be finite if the discount rate is higher than the growth rate.
Present Value of a Level Perpetuity

\[
\text{Present Value} = PMT \left[ \frac{1}{i} \frac{1 - (1 + i)^n}{(1 + i)^n} \right]
\]

with \( n = \text{infinity} \)

\[
= \frac{PMT}{i}
\]

- \( PMT = \) level (constant) payment per period.
- \( i = \) rate per period.
**Examples**

- **Example 6.6** What is the present value of $600 perpetuity at 7% discount rate?
  
  \[ PV = \frac{PMT}{i} \]
  
  \[ PV = \frac{600}{0.07} = 8751.43. \]

- If you decide to rent an apartment with a fixed rent of $2,000 per month and live there forever (subletting it to your children after you die), how much is this apartment worth if the mortgage rate is 6% per year (Ignore tax, liquidity and other concerns).
  
  - The present value of paying $2,000 per month forever at 6% rate per year is: \[ PV = \frac{2000}{0.06/12} = 400,000. \]
  - 200 times your rent is about the house value.
Checkpoint 6.4

The Present Value of a Level Perpetuity

- What is the present value of a perpetuity of $500 paid annually discounted back to the present at 8 percent?

- What is the present value of stream of payments equal to $90,000 paid annually and discounted back to the present at 9 percent?

Verify: 6250; 1,000,000
Present Value of a Growing Perpetuity

- In growing perpetuities, the periodic cash flows grow at a constant rate each period.

- The present value of a growing perpetuity can be calculated using a simple mathematical equation:

\[ PV = \frac{PMT_{\text{period 1}}}{i - g} \]

- \( i \) -- rate per period, \( g \) -- growth per period,
- \( PMT_{\text{period 1}} \) -- payment at the end of the first period.
The Present Value of a Growing Perpetuity

What is the present value of a perpetuity stream of cash flows that pays $500 at the end of year one but grows at a rate of 4% per year indefinitely? The rate of interest used to discount the cash flows is 8%.

What if the growth rate is 6%?

What if the growth rate is 9%?
Checkpoint 6.5: Answers

- PV = 500 / (0.08 - 0.04) = 500 / 0.04 = 12,500
- PV = 500 / (0.08 - 0.06) = 500 / 0.02 = 25,000
- When growth rate is faster than discount rate, the present value is infinite -- You can no longer use the formula.
The cash flows streams in the business world may not always involve one type of cash flows. The cash flows may have a mixed pattern. For example, different cash flow amounts mixed in with annuities.

For example, figure 6-4 summarizes the cash flows for Marriott.
Complex Cash Flow Streams (cont.)

**Figure 6.4**

**Present Value of Single Cash Flows and an Annuity ($ value in millions)**

- **Time Period**:
  - **Years**: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10

- **Cash Flow**:
  - Time Period 0: $500
  - Time Period 1: $200
  - Time Period 2: -$400
  - Time Period 3: $500
  - Time Period 4: $500
  - Time Period 5: $500
  - Time Period 6: $500
  - Time Period 7: $500
  - Time Period 8: $500
  - Time Period 9: $500
  - Time Period 10: $500

**Calculations**:

- **Total present value** = $2,657.39

**Notes**:

- **Beginning of year 4 or End of year 3**
- **Rate of return (i)** = 6%

- Present values:
  - $471.70
  - 178.00
  - 335.85
  - 2,343.54
  - 2,791
In this case, we can find the present value of the project by summing up all the individual cash flows by proceeding in four steps:

1. Find the present value of individual cash flows in years 1, 2, and 3.
2. Find the present value of ordinary annuity cash flow stream from years 4 through 10.
3. Discount the present value of ordinary annuity (step 2) back three years to the present.
4. Add present values from step 1 and step 3.
Checkpoin 6.6

The Present Value of a Complex Cash Flow Stream

What is the present value of cash flows of $500 at the end of years through 3, a cash flow of a negative $800 at the end of year 4, and cash flows of $800 at the end of years 5 through 10 if the appropriate discount rate is 5%?
PV of 3x5000 = $500 \times \frac{1 - 1.05^{-3}}{0.05} = 1361.62

PV of (-800) = \frac{-800}{1.05^4} = -658.16

Year 4 value of 6x800 = 800 \times \frac{1 - 1.05^{-6}}{0.05} = 4060.55

PV = \frac{4060.55}{1.05^4} = 3340.63

Total PV = 1361.62 - 658.16 + 3340.63 = 4044.09
Checkpoint 6.6: Check Yourself

What is the present value of cash flows of $300 at the end of years 1 through 5, a cash flow of negative $600 at the end of year 6, and cash flows of $800 at the end of years 7-10 if the appropriate discount rate is 10%?
Steps

- Group the cash flow into three types, all with $i=10\%$
  1. $300$ from year 1 to 5
  2. $-600$ at year 6
  3. $800$ from year 7-10

- Find PV for each group:
  1. $PV=300\left[\frac{(1-1.1^{-5})}{0.1}\right]=1137.24$
  2. $PV=-600/1.1^6=-338.68$
  3. $PV=\frac{800\left[\left(1-1.1^{-4}\right)/0.1\right]}{1.1^6}=1431.44$ (two steps here)

- Total PV = 2300.00