

Binomial Trees

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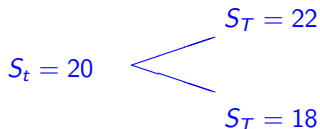
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(Hull chapter: 11, 17)

A simple binomial model

- **Observation:** The current stock price (S_t) is \$20.
- **Binomial model assumption:** In 3 months, the stock price is either \$22 or \$18 (no dividend for now).
- **Observation:** The current continuously compounded interest rate r (at 3 month maturity) is 12%. (crazy number from the book).



Comments:

- Once we make the binomial assumption, the actual probability of reaching either node (going up or down) no longer matters.
- We have enough information (we have made enough assumption) to price options that expire in 3 months.
- **Remember:** For derivative pricing, what matters is the list of possible scenarios, but not the actual probability of each scenario happening.

A 3-month call option

Consider a 3-month call option on the stock with a strike of \$21.

- **Backward induction:** Given the terminal stock price (S_T), we can compute the option payoff at each node, $(S_T - K)^+$.
- The zero-coupon bond price with \$1 par value is: $1 \times e^{-.12 \times .25} = \0.9704 .

$$\begin{array}{l} B_t = 0.9704 \\ S_t = 20 \\ c_t = ? \end{array} \begin{array}{l} \nearrow \\ \searrow \end{array} \begin{array}{l} B_T = 1 \\ S_T = 22 \\ c_T = 1 \\ \\ B_T = 1 \\ S_T = 18 \\ c_T = 0 \end{array}$$

Two angles:

- **Replicating:** See if you can replicate the call's payoff using stock and bond. \Rightarrow If the payoffs equal, so does the value.
- **Hedging:** See if you can hedge away the risk of the call using stock. \Rightarrow If the hedged payoff is riskfree, we can compute the present value using riskfree rate.

Replicating

$$\begin{array}{l} B_t = 0.9704 \\ S_t = 20 \\ c_t = ? \end{array} \left\{ \begin{array}{l} B_T = 1 \\ S_T = 22 \\ c_T = 1 \\ B_T = 1 \\ S_T = 18 \\ c_T = 0 \end{array} \right.$$

- Assume that we can **replicate** the payoff of 1 call using Δ share of stock and D par value of bond, we have

$$1 = \Delta 22 + D1, \quad 0 = \Delta 18 + D1.$$

- Solve for Δ : $\Delta = (1 - 0)/(22 - 18) = 1/4$. $\Delta = \text{Change in } C/\text{Change in } S$, a **sensitivity measure**.
- Solve for D : $D = -\frac{1}{4}18 = -4.5$ (borrow).
- Value of option = value of replicating portfolio
 $= \frac{1}{4}20 - 4.5 \times 0.9704 = 0.633$.

Hedging

$$\begin{array}{l} B_t = 0.9704 \\ S_t = 20 \\ c_t = ? \end{array} \left\{ \begin{array}{l} B_T = 1 \\ S_T = 22 \\ c_T = 1 \\ \\ B_T = 1 \\ S_T = 18 \\ c_T = 0 \end{array} \right.$$

- Assume that we can **hedge** away the risk in 1 call by shorting Δ share of stock, such as the hedged portfolio has the same payoff at both nodes:
 $1 - \Delta 22 = 0 - \Delta 18$.
- Solve for Δ : $\Delta = (1 - 0)/(22 - 18) = 1/4$ (the same):
 $\Delta = \text{Change in } C / \text{Change in } S$, a **sensitivity measure**.
- The hedged portfolio has riskfree payoff: $1 - \frac{1}{4}22 = 0 - \frac{1}{4}18 = -4.5$.
- Its present value is: $-4.5 \times 0.9704 = -4.3668 = 1c_t - \Delta S_t$.
- $c_t = -4.3668 + \Delta S_t = -4.3668 + \frac{1}{4}20 = 0.633$.

One principle underlying two angles

- If you can **replicate**, you can **hedge**:
Long the option contract, short the replicating portfolio.
- The replication portfolio is composed of stock and bond.
- Since bond only generates parallel shifts in payoff and does not play any role in offsetting/hedging risks, it is the stock that really plays the hedging role.
- The **optimal hedge ratio** when hedging options using stocks is defined as the ratio of option payoff change over the stock payoff change — **Delta**.
- Hedging derivative risks using the underlying security (stock, currency) is called *delta hedging*.
 - ▶ To hedge a long position in an option, you need to short delta of the underlying.
 - ▶ To replicate the payoff of a long position in option, you need to long delta of the underlying.

The limit of delta hedging

- Delta hedging completely erases risk under the binomial model assumption: The underlying stock can only take on **two** possible values.
- Using two (independent) securities can **span** the two possible realizations.
 - ▶ We can use stock and bond to replicate the payoff of an option.
 - ▶ We can also use stock and option to replicate the payoff of a bond.
 - ▶ One of the 3 securities is redundant and can hence be priced by the other two.
- What will happen for the hedging/replicating exercise if the stock can take on **3** possible values three months later, e.g., (22, 20, 18)?
 - ▶ It is impossible to find a Δ that perfectly hedge the option position or replicate the option payoff.
 - ▶ We need another instrument (say another option) to do perfect hedging or replication.

Pricing more options on the same tree

$$\begin{array}{l} B_t = 0.9704 \\ S_t = 20 \end{array} \left\{ \begin{array}{l} B_T = 1 \\ S_T = 22 \\ \\ B_T = 1 \\ S_T = 18 \end{array} \right.$$

- Reminder: The value of a 3-month call option with a strike \$21 is \$0.633.
- Price another 3-month call option with a strike of \$20.
- Price 3-month call options with strikes of \$22,\$23,\$24,\$25,\$26,...
- Price a 3-month put option with a strike of \$21.
- Price 3-month put options with strikes of \$18, \$17, \$16, ...
- Do you see the limit of a one-step binomial tree? How do you get around it?

The 3rd angle: risk-neutral valuation

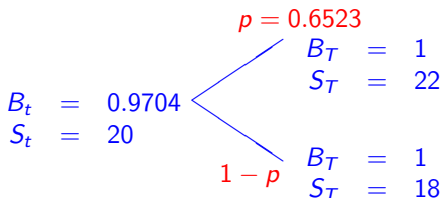
$$\begin{array}{l} B_t = 0.9704 \\ S_t = 20 \end{array} \begin{array}{l} \nearrow^p \\ \searrow_{1-p} \end{array} \begin{array}{l} B_T = 1 \\ S_T = 22 \\ \\ B_T = 1 \\ S_T = 18 \end{array}$$

- Compute a set of artificial probabilities for the two nodes ($p, 1 - p$) such that the stock price equals the expected value of the terminal payment, discounted by the riskfree rate.

$$20 = 0.9704 (22p + 18(1 - p)) \Rightarrow p = \frac{20/0.9704 - 18}{22 - 18} = 0.6523$$

- Since we are discounting risky payoffs using riskfree rate, it is as if we live in an artificial world where people are risk-neutral. Hence, ($p, 1 - p$) are the risk-neutral probabilities.
- These are not really probabilities, but more like unit prices: $0.9704p$ is the price of a security that pays \$1 at the up state and zero otherwise. $0.9704(1 - p)$ is the unit price for the down state.
- To exclude arbitrage, the same set of unit prices (risk-neutral probabilities) should be applicable to all securities, including options.

Pricing options with risk-neutral valuation

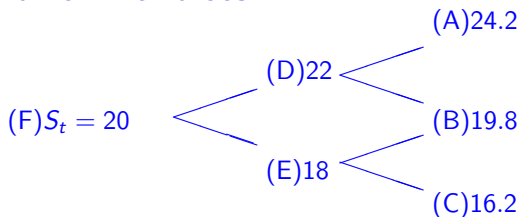


- Once we know the risk-neutral probabilities, it is easy to compute the present value of different options:
 - ▶ Call at $K = 21$: $c_t = 0.9704(1 \times 0.6523 + 0 \times (1 - 0.6523)) = 0.633$.
 - ▶ Put at $K = 21$: $p_t = 0.9704(0 \times 0.6523 + 3 \times (1 - 0.6523)) = 1.012$.
 - ▶ Call at $K = 20$: $c_t = 0.9704(2 \times 0.6523 + 0 \times (1 - 0.6523)) = 1.266$.
 - ▶ Put at $K = 20$: $c_t = 0.9704(0 \times 0.6523 + 2 \times (1 - 0.6523)) = 0.6748$.
- It seems worthwhile to compute the risk-neutral probabilities.
- Analogously, if we know the unit price of each state, it is easy to compute the value of any “state-contingent” claims, which are just portfolios of the unit state prices.

More on risk-neutral probabilities

- Actual probabilities are real-life chances of each event happening. It is also referred to as physical, statistical, objective probabilities.
 - ▶ We can estimate the actual probabilities using historical data (e.g., histogram).
- “Risk-neutral” probabilities are artificial (not real) probabilities that match the observed security prices. They are a mixture of *subjective* probabilities (people’s expectation, right or wrong), and risk preferences (hate or love risk).
 - ▶ We can estimate the risk-neutral probabilities from the currently observed prices on different securities.
 - ▶ As said earlier, they are really just prices, up to a discount.
 - ▶ When we compute “risk-neutral probabilities” from the stock price and then use the probabilities to price options, we are really just compute the value of the option *relative to the stock value*, with no forecasting involved.
- The two types of probabilities do not need to be the same. They normally aren’t.

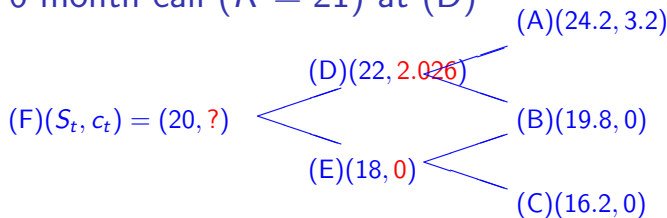
Multiple step binominal trees



[Assume each step is 3 month, $r = 12\%$.]

- Consider pricing a 6-month call option with $K = 21$.
- Backward induction: Starting at expiry, we know the payoff of the call: 3.2 at (A), 0 at (B), 0 at (C).
- We can compute the option value at node (D) the same as before on a one-step binomial model, using any of the three angles (replication, hedging, risk-neutral valuation).
- We can do the same on (E).
- Given the option values at (D) and (E), we have a one-step binomial model again to obtain value at (F).

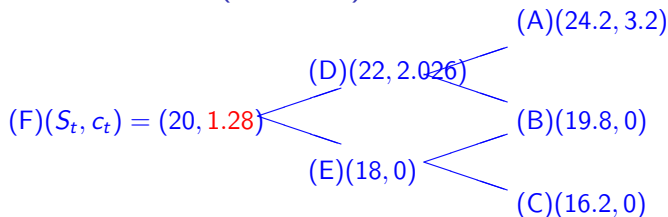
Valuing 6-month call ($K = 21$) at (D)



- Hedging: $\Delta_D = (3.2 - 0)/(24.2 - 19.8) = 0.7273$.
 - Value of hedged portfolio:

$$P_D = 0.9704(1 \times 0 - 0.7273 \times 19.8) = -13.9738.$$
 - Option value $c_D = -13.9738 + 0.7273 \times 22 = 2.026$.
- Replication: $\Delta_D = 0.7273$. $D_D = -\Delta 19.8 = -14.400$.
 - $c_D = 0.7273 \times 22 - 14.400 \times 0.9704 = 2.026$.
- Risk-neutral probability: $p_D = \frac{22/0.9704 - 19.8}{24.2 - 19.8} = 0.6523$.
 - $c_D = 0.9704(0.6523 \times 3.2 + (1 - 0.6523)0) = 2.026$.
- $c_E = 0$ since payoffs are zero at both states.

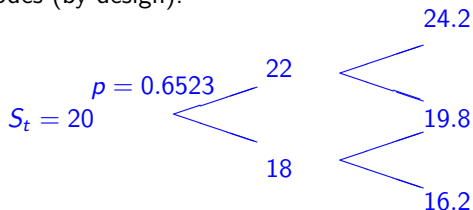
Valuing 6-month call ($K = 21$)



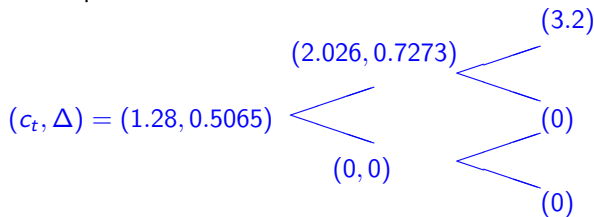
- Hedging: $\Delta_F = (2.026 - 0)/(22 - 18) = 0.5065$.
 - ▶ Value of hedged portfolio:
 $P_D = 0.9704(1 \times 0 - 0.5065 \times 18) = -8.847$.
 - ▶ Option value $c_D = -8.847 + 0.5065 \times 20 = 1.283$.
- Replication: $\Delta_D = 0.5065$. $D_D = -\Delta 18 = -9.117$.
 - ▶ $c_D = 0.5065 \times 20 - 9.117 \times 0.9704 = 1.283$.
- Risk-neutral probability: $p_D = \frac{20/0.9704 - 18}{22 - 18} = 0.6523$.
 - ▶ $c_D = 0.9704(0.6523 \times 2.026 + (1 - 0.6523)0) = 1.282$.

The complete tree

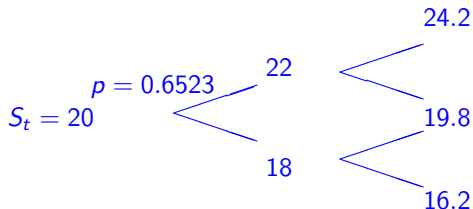
- The complete stock price tree: The risk-neutral probability of going up is 0.6523 at all nodes (by design).



- The call option tree: The delta of the option at each node is useful: It tells us how much the underlying stock we need to short to hedge against the risk of the option.

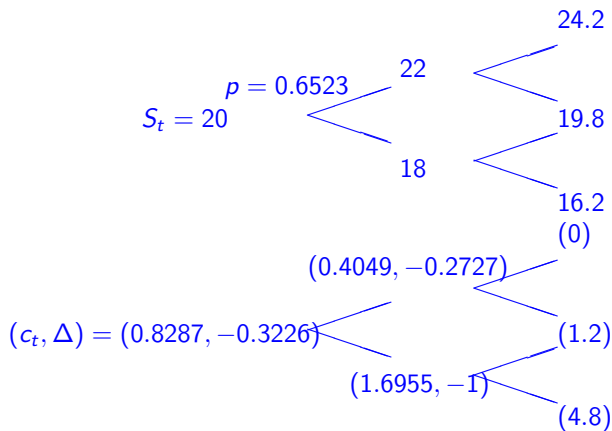


Pricing other options on the tree

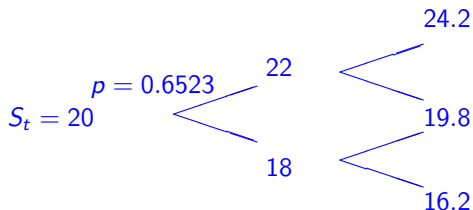


- Once we have the risk-neutral probabilities, it is easy to compute the value at each node of a derivative with known payoffs.
 - ▶ Exercise: price a 6-month put option with $K = 21$.
- Even if we do not use the delta for valuation, it is still important to figure out the deltas of the option at each node for hedging.

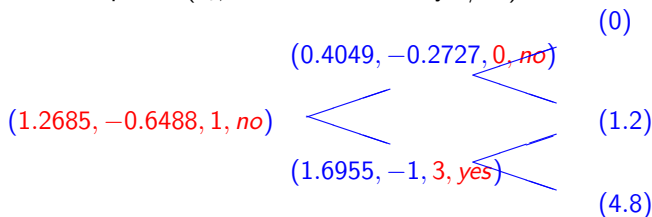
Exercise: 6-month put with $K = 21$



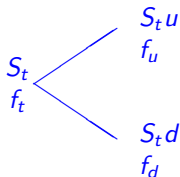
What if the put is American?



- At each node, compare the continuation value with the exercise value $((K - S)^+)$, and exercise if the exercise value is larger.
- Each node reports: $(c_t, \Delta, \text{exercise value, yes/no})$

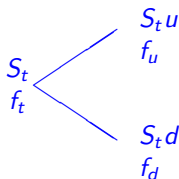


General setup and notation



- S_t denotes the current stock price at time t (The textbook often set $t = 0$).
- f_t denotes the time- t price of a derivative security.
- u denotes the proportional increase in stock price at the “up” node.
- d denotes the proportional decrease in stock price at the “down” node.
- (f_u, f_d) denotes the derivative value at the “up” and “down” nodes.

Pricing under the general setup



- Hedging: $\Delta = \frac{f_u - f_d}{S_t(u-d)}$.
- Value of the hedged portfolio: $P = e^{-r\Delta t}(f_u - \Delta S_t)u = f_t - \Delta S_t$.
- Derivative value: $f_t = \Delta S_t + e^{-r\Delta t}(f_u - \Delta S_t) = [pf_u - (1-p)f_d]e^{-r\Delta t}$, with $p = \frac{e^{r\Delta t} - d}{u-d}$ being the risk-neutral probability.
- $e^{r\Delta t}p$ is the one-period up-state price, $e^{r\Delta t}(1-p)$ is the one-period down-state price.
- Δt denotes the time step.

Calibrating the binominal tree

- So far, we have taken the tree (S_t, u, d) as given and proceed to price derivatives.
- One way to set up the tree is to use (u, d) to match the stock return volatility (Cox, Ross, Rubinstein):

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = 1/u.$$

σ — the annualized return volatility (standard deviation).

- ▶ The pricing is usually more accurate with more steps (smaller Δt), and hence more nodes.
- ▶ The choice of time steps depend on precision requirement.
- The risk-neutral probability becomes: $p = \frac{a-d}{u-d}$ where
 - ▶ $a = e^{r\Delta t}$ for non-dividend paying stock.
 - ▶ $a = e^{(r-q)\Delta t}$ for dividend paying stock.
 - ▶ $a = e^{(r-r_f)\Delta t}$ for currency.
 - ▶ $a = 1$ for futures.

Estimating the return volatility σ

- To determine the tree (S_t, u, d) , we set

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = 1/u.$$

- The only un-observable or “free” parameter is the return volatility σ .
 - ▶ Mean return does not matter for derivative pricing.
 - ▶ S_t can be observed from the stock market.

- We can estimate the return volatility using the time series data:

$$\hat{\sigma} = \sqrt{\frac{1}{\Delta t} \left[\frac{1}{N-1} \sum_{t=1}^N (R_t - \bar{R})^2 \right]}. \quad \leftarrow \frac{1}{\Delta t} \text{ is for annualization.}$$

- **Implied volatility:** More appropriately, we can estimate the volatility (σ) by matching observed option prices.
- Under the current model assumption, the two approaches should generate similar estimates on average, but they differ in reality (for reasons that we'll discuss later).

Practical issues with binominal tree pricing

- The option prices obtained from the binominal tree often shows a zigzag behavior across strikes.
- One way to smooth it up is to price the options using two trees, one using N steps and the other using $(N + 1)$ steps. Then, simply averaging the value from the two trees with generate much better/smoother results.
- Yet another simple approach is to replace the last step value by the BSM model analytical value.
 - ▶ Even if the option is American, replacing the last step with the European value does not hurt much.
 - ▶ The smoothing effect can be even better than the averaging approach.